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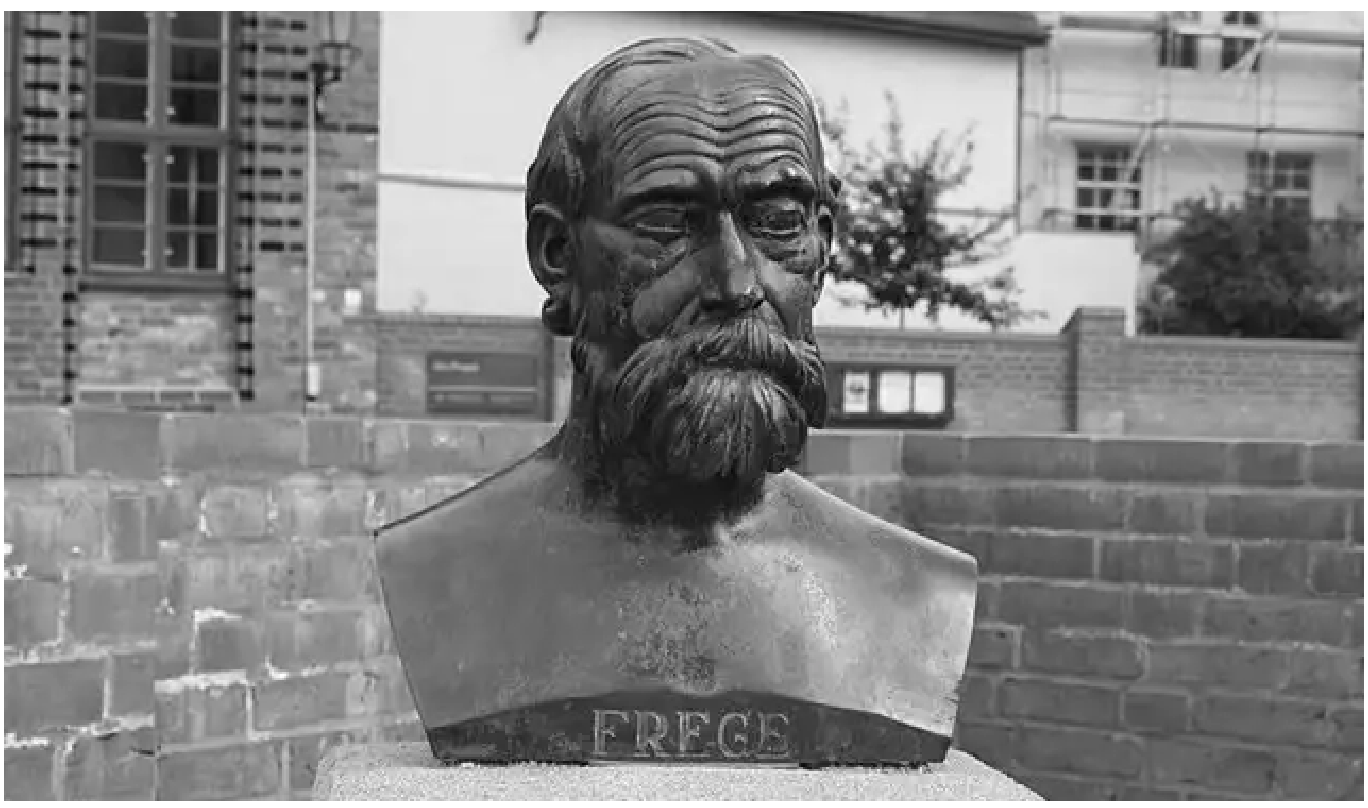
Frege's Concept Of Natural Numbers

A very brief introduction to Frege's philosophy of mathematics.



Bronze bust of Gottlob Frege (1848–1925) by Karl-Heinz Appelt (1940–2013). Image downloaded from Wikimedia Commons and edited by me.

Empirism and Rationalism — The Two Most Important Points of View in the Philosophy of Mathematics

These are the two most basic distinctions in the philosophy of mathematics. So we can roughly say that any philosopher of mathematics has to take a stand on these two currents of thought, and Gottlob Frege (1848–1925) did so. He discussed Mill's empiricist ideas and Kant's rationalist ideas about the nature of mathematics and developed a new concept for the natural numbers in his *The Foundations of Arithmetic: A logico-mathematical inquiry into the concept of number* (Translated by J. L. Austin. Ed. Harper Torchbooks. New York, 1980).

Frege's concepts for each natural number

I recently asked in a text here for Cantor's Paradise: what is a number? Frege would answer me thusly:

> *something objective. If we say 'The North Sea is 10,000 square miles in extent' then neither by 'North Sea' nor by '10,000' do we refer to any state of or process in our minds: on the contrary, we assert something quite objective,*

> *which is independent of our ideas and everything of the sort (FREGE, 1980, p.34 / §26).*

But although he is speaking of objectivity and rejecting the processes in our minds, he was not an empiricist at all. For him, each concept for the natural numbers should be constructed with only mathematical and logical objects. Mostly because according to him,

> *For a proposition to be true is just not the same thing as for it to be thought (FREGE, 1980, §77 / p.91).*

That is, unlike purely psychological (and empirical) objects, only because something is thought or exists that doesn't mean it is true in logic and mathematics.

For Frege, the mathematical and logical objects would have a different nature — they would be purely analytical, and the best way to classify a proposition, according to him, as synthetic or analytic, would be to examine the justification that makes such a proposition true or false. That is, if a proposition needs empirical elements in order to be true it's synthetic and if it doesn't it's analytic. He says:

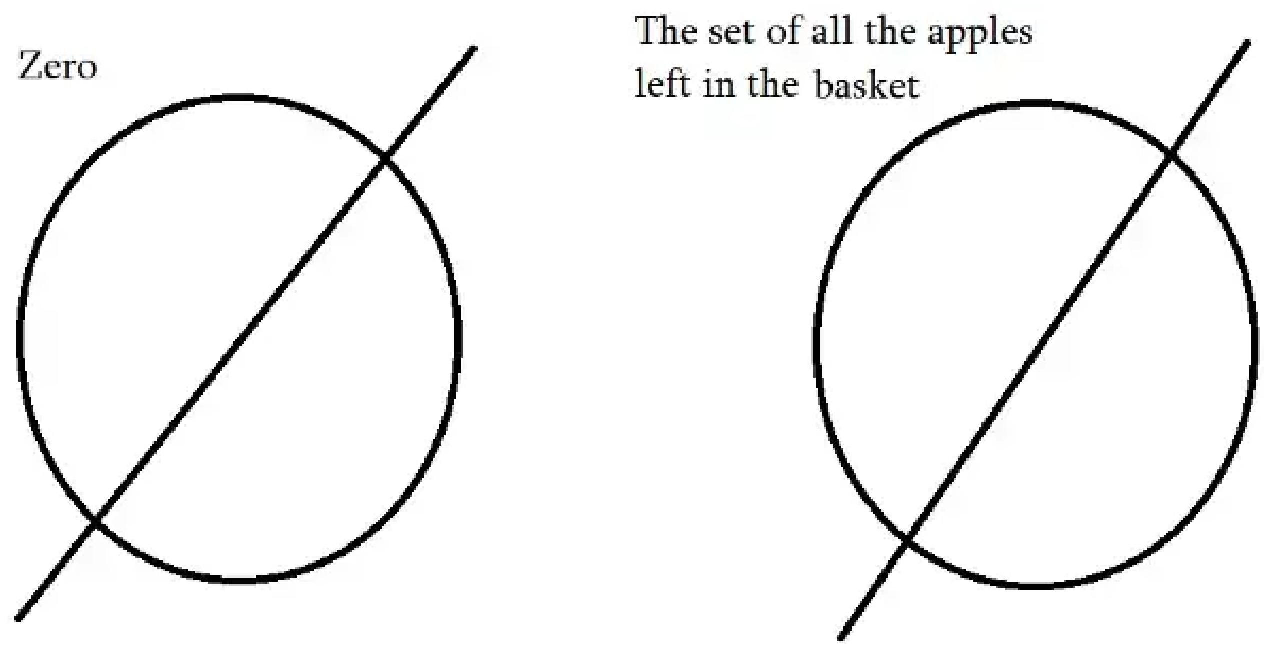
> *these distinctions between a priori and a posteriori, synthetic and analytic, concern, as I see it, not the content of the judgement but the justification for making the judgement (FREGE, 1980, p.3 / §3).*

In case you are interested in this distinction scroll down to the last section, where I discuss it further.

Now let's see how Frege elaborated the concepts of the numbers 0, 1, and of all the other natural numbers:

The concept for the natural number "zero"

The concept of "zero" would be a set that has no elements identical to itself, and it would behave as a "number" when compared to other sets that have no elements belonging to it.



Created by the author.

So when you say “there is no apple left in the basket” you are comparing, according to Frege’s philosophy of mathematics, the “set of all the apples left in the basket” with Frege’s concept of zero.

The concept for the natural number “one”:

As for number 1, Frege points out that the concepts of two adjacent numbers must relate somehow. In section §79 of *The Foundations of Arithmetics* (1884), Frege says:

> *in order to prove that after every Number in the series of natural numbers a Number directly follows, we must produce a concept (of number) to which this latter Number belongs (FREGE, 1980, p.92 / §79).*
> ***The text in parenthesis was added by me.***

In order to satisfy this, he states that, following the crescent order of the natural numbers, the set that works as the concept of a natural number must contain the sum of the “extensions” of all the concepts of all the smallest natural numbers. So the number that is contained by the concept of 1, and which, therefore, helps compose the Fregean definition of the “number 1” is its smaller adjacent number — the number zero. Frege says,

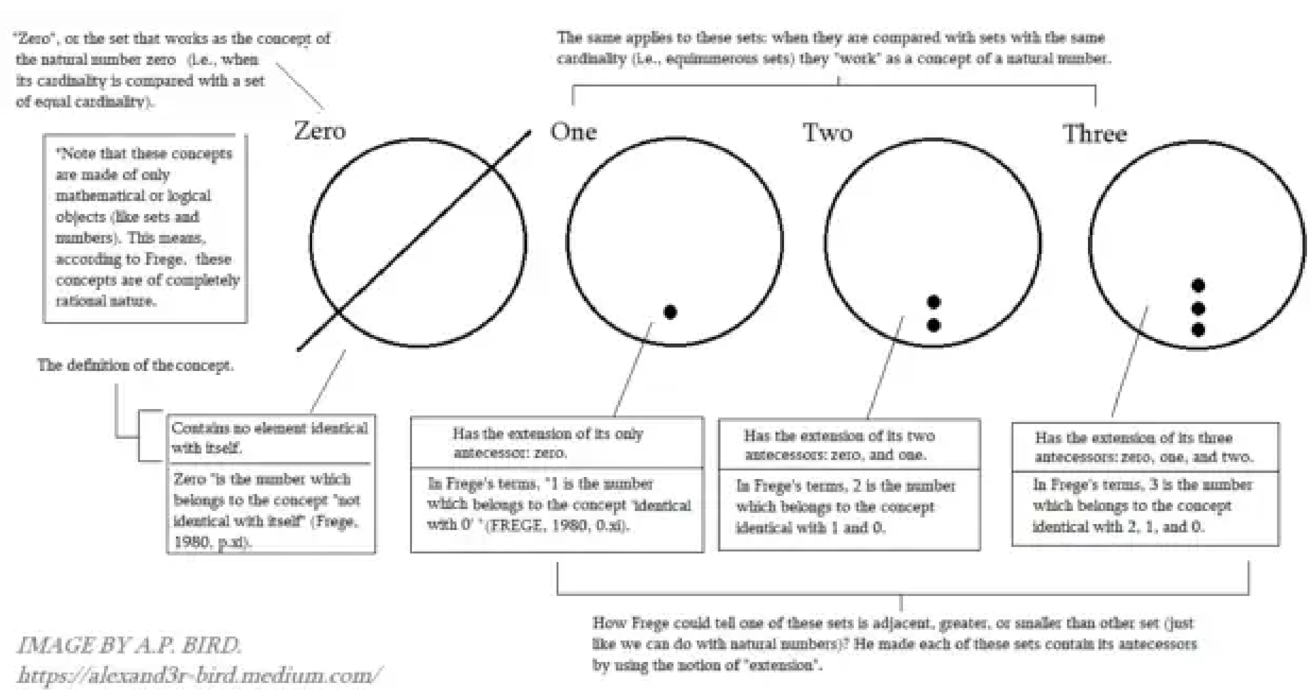
> *1 is the Number which belongs to the concept “identical with 0" (FREGE, 1980, p.xi).*

Note that this makes the concept of the “natural number 1” contains only one element (zero), and such a concept permits us to identify what is the “natural number 1" in purely analytic terms.

Then, when you compare Frege’s set for the natural number 1 with another set that has only one element belonging to it you are successfully using Frege’s concept of the natural number 1 as a natural number. Or, as I say, you would be using Frege’s concept for the natural number 1 as a classification of another set.

Frege’s concepts for all the other natural numbers:

The concepts of the remaining numbers are also obtained according to the “extension” of all the “preceding numbers” as well. See image.



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Three Important Notions: Cardinality, Extention, and Equinumerosity

In order to have a more complete understanding of these concepts let’s look at the concepts of *cardinality, extension,* and *equinumerosity*. These three concepts are taken by Frege as follows:

Cardinality refers to the possibility of finding the cardinal number of a set, which would be measured (defined) according to the sum of elements that belong to a set. In Cantor’s Set Theory, this notion (cardinality) is, for example, what allows us to say that some infinities are bigger than others.

The notion of **equinumerosity** indicates the possibility of equivalence between sets with the same cardinality. This notion is understood by Frege as follows:

> *The (cardinal) Number which belongs to the concept F is identical (equal, or equinumerous) with the (cardinal) Number which belongs to the concept G, if there exists a relation which correlates one to one the objects falling under F with those falling under G (FREGE, 1980, p.xi).*
> ***The text in parenthesis was added by me.***

In section §67, Frege also offers a more extensive explanation of the notion of “*equinumerosity*”:

> *If line a is parallel to line b, then the extension of the concept “line parallel to line a” is identical with the extension of the concept “line parallel to line b”; and conversely, if the extensions of the two concepts just named are identical, then a is parallel to b. (…) To apply this to our own case of Number, we must substitute for lines or triangles concepts, and for parallelism or similarity the possibility of correlating one to one the objects which fall under the one concept*

> *with those which fall under the other. For brevity, I shall, when this condition is satisfied, speak of the concept F being equal (or equinumerous) to the concept G (FREGE, 1980, p.79 / §67).*
> ***The text in parenthesis was added by me.***

As for the concept of "**extension**", I would say that it is the key concept that connects the concepts of *cardinality* and *equinumerosity* with *quantification*. Frege uses the term "extension" to relate "functions" and "objects" in a way that, for him, the extension of a concept (or set) F would be "the course-of-values of a concept F" (ZALTA, 2020).

But what does *course-of-values* mean? The x and y axes of a function on the Cartesian plane, for example, can both contain the ascending order of the natural numbers (or, in other words, contain the "*course-of-values*" of the natural numbers).

Frege's Rationalism and His Arguments Against Stuart Mill's Empiricism

As we saw in the last sections, Frege employs only analytical elements in his definitions. He does not appeal to empiricist notions like Mill explicitly does. According to Mill, mathematical calculations are a consequence of observing facts. He says,

> *(…) the calculations do not follow from the definition itself but from the observed matter of fact (MILL apud FREGE,1980, p.10 / §7).*

To which Frege answers:

> *On Mill's view we could actually not put 1,000,000 = 999,999 + 1 unless we had observed a collection of things split up in precisely this peculiar way (FREGE, 1980, p.10–11 / §7).*

And Frege also indicates in section §8 of The Foundations of Arithmetic (1884–1903) that this strict empiricist view of arithmetic would find it difficult to explain the numbers "zero" and "one". After all, what would be the "physical facts" that would give empirical support to the natural numbers 0 and 1?

Are Frege's Concepts for the Natural Numbers the Most Adequate?

Frege used new formal resources (sets, and propositional functions) to create new logico-philosophical schemes. In a sense, his work was deeper than Peano's who also tried to formulate the foundations for arithmetical reasoning (see Peano's five axioms).

However, Frege also faced problems. Russell's paradox and Godel's theorems made Cantor's Set Theory, and the entire logicist project looks flawed.

In fact, as I show in another text here in Cantor’s Paradise, even before Gödel’s Incompleteness Theorems, there were already strong arguments against Frege’s project for the foundation of mathematics. Poincaré (1854-1912), for example, pointed out weaknesses in formalism and logicism, and as Gray admits, Poincaré “turned out to be right” in some of his criticisms.

A few notes on the “Frege vs Kant” debate

Study and understand Kant’s philosophy of mathematics is no ordinary task. I recommend the article [1], the first chapters of Frege’s The Foundations of Arithmetic (1884), and this last section may help you a little too.

[1] Sutherland, Daniel. *Kant’s Philosophy of Mathematics and the Greek Mathematical Tradition*. Rev. The Philosophical Review Vol. 113, №2, p.157–201, (2004).

Frege did not ground his philosophy of mathematics on concrete magnitudes, as did Stuart Mill, nor did he agree with the Kantian idea that mathematical propositions have a synthetic nature. Frege opted, as we saw, for a purely analytical foundation for mathematics.

However, it isn't simple to say, for example, that “space and time” (the pure forms of intuition, according to Kant) would not help to constitute Frege’s concept of number. After all, Frege, for example, uses the concept of “extension” to achieve his concepts of natural numbers (as we saw a few sections ago), and “extension” is a concept extremely close to the idea of space. Frege even uses this term in geometric reasoning when he says:

> *If line a is parallel to line b, then the extension of the concept ‘line parallel to line a’ is identical with the extension of the concept ‘line parallel to line b’ (FREGE, 1980, p.79 / §67).*

For this reason, I think Frege’s relationship with Kantian ideas deserves more attention.
