# Formal ontologies for space and time

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**Abstract.** We propose an ontological theory that is powerful enough to describe both complex spatio-temporal processes (occurrents) and the enduring entities (continuants) that participate in such processes. For this purpose we distinguish between meta-ontology and token ontologies. Token ontologies fall into two major categories: ontologies of type SPAN and ontologies of type SNAP. These represent two complementary perspectives on reality and result in distinct though compatible systems of categories. The meta-ontological level then describes the relationships between the different token ontologies. In a SNAP (snapshot) ontology we have enduring entities such as substances, qualities, roles, functions as these exist to be inventoried at a given moment of time. In a SPAN ontology we have perduring entities such as processes and their parts and aggregates. We argue that both kinds of ontological theory are required, together with the meta-ontology which joins them together, in order to give a non-reductionistic account of both static and dynamic aspects of the geospatial world.

## 1 Introduction

We propose a formal ontological theory that is powerful enough to contain the resources to describe both complex spatio-temporal processes and the enduring entities which participate therein. The theory we have in mind is formal in the sense that it is designed to serve as a re-usable module that can be applied in a variety of material domains. It comprehends two major categories: ontologies of type SNAP and ontologies of type SPAN. As we shall see, these ontologies represent inventories of reality, comparable to the division familiar in the discipline of geography between geographic *objects* (cities, mountains, etc.) and geographic *processes* (erosion, migration, etc.). SNAP and SPAN reflect two distinct perspectives on reality and result in distinct though compatible systems of categories.

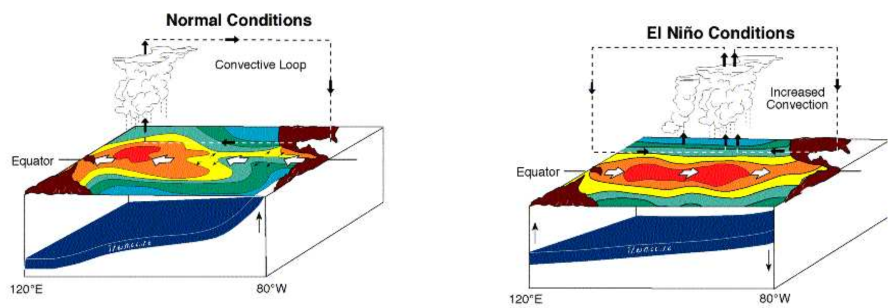


**Fig. 1.** Normal conditions in the tropical pacific (left) and El Nino conditions (right). (From the University of Illinois WW2010 Project.)

As running example we will use the El Niño phenomenon [oCNOA]. Consider Figure 1. Normal weather conditions in the tropical Pacific are shown in the left part, El

Niño conditions are shown in the right part. Trade winds normally drive the surface waters of the tropical Pacific westward. The surface water then becomes progressively warmer because of its longer exposure to the sun. El Niño occurs when the trade winds weaken, allowing the warmer waters of the western Pacific to migrate eastward and eventually to reach the coast of South America. This migration process is shown in Figure 2. The cool nutrient-rich sea water normally found along the coast of Peru is replaced by warmer water depleted of nutrients, resulting in a dramatic reduction in marine fish and plant life.

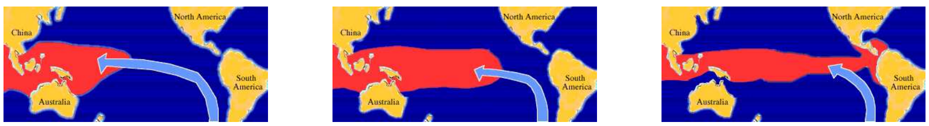


**Fig. 2.** Warm waters of the western Pacific migrating eastward to the South American coast. (From the University of Illinois WW2010 Project.9

## 2 SNAP and SPAN ontologies

In this section we introduce the SNAP and SPAN ontologies informally by means of examples. A formal theory is given in the second part of the paper.

### 2.1 Two categories of entities

Entities in reality fall onto two basic categories:

– *enduring* entities, which exist in full at every instant in time at which they exist at all;

– *perduring* entities, which unforld themselves through time and never exist in full at any single moment in time.

Enduring entities are: you, your car, lakes, mountains, the sea water along the coast of Peru, the body of warm water marked red in Figure 2, etc.

Perduring entities are also often referred to as *processes*. Examples of processes are: your life, processes of erosion or drainage, changes in air pressure, movements of water across the Pacific. Figure 2 shows different stages of the process of migration of the warmer waters of the western Pacific towards the South American coast. Another process is the gradual weakening of the trade winds represented by the shorter and shorter arrows. The winds themseves, however, are enduring entities; they are that which remains the same during the process of weakening of intensity.

The two groups of entities are related to each other in the sense that processes *depend* on the enduring entities which participate in them. For example, your life cannot exist without you; erosion cannot occur without a surface to be eroded. There is no migration of a body of water without the body of water and no weakening of the trade winds without the trade winds. We will discuss such relationships in subsequent sections. First we must focus on the categorical systems formed by enduring and perduring entities, drawing on ideas first set out in [Gre03] and [BS03a].

### 2.2 SNAP ontologies

SNAP ontologies recognize only enduring entities. Examples of very simple and narrowly focused SNAP ontologies are given in Figure 2. Enduring entities exist in full at every moment in time at which they exist at all. Every part of you exists in this moment and if there is an El Niño condition now then it exists in full now. Enduring entities may of course change, but they yet remain the same entity. You may gain weight, but still you remain the same person. An El Niño condition may get stronger or weaker, but still it remains the same El Niño condition.

It follows that enduring entities do not have temporal parts or phases. Thus your childhood is not a part of you but a part of your life. The latter is a process and therefore has temporal parts, as we shall see below. The beginning of an El Niño condition is a part of the course of this condition but not of the condition itself. Enduring entities have no temporal parts but they do have spatial parts. For example, your hand is a part of you, the right front wheel of your car is a part of your car, the summit is a part of the mountain.

SNAP entities are entities recognized by a SNAP ontology and thus they are enduring entities. They fall into three major categories: (i) substances, their fiat parts and aggregates; (ii) places and spatial environments; and (iii) entities, like qualities, powers, roles, functions, conditions, which depend on (i) and (ii).

Substances are maximally connected entities, i.e., they have connected bona fide boundaries [SV00]. For example you are a substance and the planet Earth is a substance. Neither neither your nose nor Mount Everest nor the United States are substances. Mount Everest is a fiat part of that substance we call the planet Earth. Your nose is a fiat part of you. The United States is a fiat part of the surface of the Earth. Aggregates of substances too, are not substances. Examples of aggregates are: your family, the collection of people living in a certain neighborhood, the Iraqi Republican Guard, and so forth.

Aggregates can also be aggregates of fiat entities or a mixture of substances and fiat entities. That part of the Earth affected by El Niño is an example of an aggregate of this kind. It consists of parts of the pacific ocean, parts of the atmosphere, parts of continents, etc. all of which are demarcated in highly specific, fiat ways. Other examples of aggregates involving fiat entities are environments and niches [SV99].

Places are endurants which are *co-located* with substances, their fiat parts, and aggregates. Malta is an island, i.e., a substance but also a place where people spend their vocations. Mount Everest is a mountain, i.e., a fiat part of the planet Earth, but co-located with it there is also a place where people go for mountaineering. The equatorial Pacific is a fiat part of the Pacific ocean but also the place where El Niño conditions occur.

Dependent entities are entities which cannot exist without some other entity or entities they upon which they depend. SNAP entities which depend on substances, their fiat parts and aggregates, as well as on places, are qualities, powers, roles, functions, etc. We call these QPR entities for short. The quality of being of such-and-such-a height cannot exist without something that is of that height. The latter might be a substance as in the case of your height, or it might be a part of a substance, as in the case of the height of Mount Everest. The quality of being red cannot exist without something that is red; a role such as being-a-trade-wind, cannot exist without some body of air with certain qualities; the quality of being a body of water of a certain temperature cannot exist without the body of water, and so on. The quality of being a crowded place depends on the underlying place, as also on the substances (people) who form the crowd.

### 2.3 SPAN ontologies

Perduring entities are inventarized in SPAN ontologies. For example the processes that are involved (at a certain level of granularity) in the course of an El Niño phenomenon are shown in the right part of Figure 1. (In this example we can see that it is hard to draw a picture of processes without also drawing the enduring entities which participate in them.) Every entity recognized by a SPAN ontology is such as to evolve or unfold itself in time. This means that, leaving aside the instantaneous boundaries (beginnings and endings) of processes, a SPAN entity never exists in full at any single instant of time. Rather it exists in its successively unfolding phases or stages. The course of an el Niño phenomenon unfolds over time. It usually starts before Christmas and lasts for several weeks thereafter.

Every perduring entity is located at a certain four-dimensional spatio-temporal region. The beginning and the ending of the temporal extent of this region coincides with the beginning and the ending of the perduring entity located within it. Since perduring entities are extended in time it follows that they have temporal parts. The process of your life, for example, has your childhood and your adulthood as temporal parts.

### 2.4 Three- and four-dimensionalist roots

The underlying idea in all of the above is that we can associate ontologies with the ways we humans project onto reality, i.e., with the different perspectives we take when describing or perceiving reality. SNAP and SPAN ontologies reflect precisely the perspectives of an instantaneous inventory (for example of the *stocks* in your warehouse) and of a temporally extended survey (for example of the *flow* of goods in and out of your warehouse over a given period in time). In the case of El Niño a SNAP ontology is an inventory of all substances, fiat parts, and aggregates together with their qualities, roles, functions, etc. existing at a single moment in time. A SPAN ontology is an inventory of all processes – such as weakening of trade winds, migration of warm water, etc. – existing within a certain spatio-temporal region.

We can indeed define an ontology as an inventory of those entities existing in reality which are visible from a certain perspective. The SPAN perspective then corresponds to a popular position in contemporary analytic metaphysics which is called four-dimensionalism. The SNAP perspective similarly corresponds to one or other of the no less popular positions called three-dimensionalism or presentism.

The four-dimensionalist holds that all entities in reality are four-dimensional worms extended in space and time á la Minkowski. (See [Sid01] for further discussion.) He takes a view of the world as consisting exclusively of variously demarcated and variously qualitatively filled spatio-temporal worms. Prominent four-dimensionalists are: Amstrong [Ams80], Carnap [Car67], Cartwright [Car75], and Lewis [Lew].

An important aspect of four-dimensionalism is the thesis that time is just another dimension, in addition to and analogous to the three spatial dimensions. We can think of the four-dimensionalist ontology as what results when reality is described from the perspective of a god-like observer spanning the whole of reality from beginning to end and from one spatial extremity to the other. Human beings take this stance, for example, when they view the world through the lenses of the theory of relativity. SPAN ontologies span the four-dimensional plenum in this way; hence reality in such ontologies is described atemporally.

The acceptance of the possibility of such *atemporal* descriptions of reality, however, need not mean that the existence of a special *temporal* dimension of spatio-temporal entities is denied. Rather, from our present perspective, it means merely that certain aspects of this temporal dimension – above all its subdivision into past, present, and future – are *traced over*. Also traced over in a view of the world as consisting exclusively of spatio-temporal worms is the existence of *enduring* entities such as people (whose identity survives changes such as the gain or loss of molecules and cells), or plans (whose identity is preserved through the different stages of their fulfillment).

In order to take account of such enduring entities, we admit a second type of ontological view – the SNAP perspective – which we can think of as being analogous to the taking of instantaneous snapshots of reality in such a way as to apprehend all enduring entities existing at a given time [Gea66,Zem70].

From the SNAP perspective time is an index which we assign to the inventories of the world we take at different moments – one index per ontology. Thus the succession of times is itself outside the scope of each SNAP ontology. This indexing – which we can make explicit by writing $\mathrm{SNAP}_{t_1}$, $\mathrm{SNAP}_{t_2}$, etc. – occurs not within the ontology itself but rather on a meta-level. For this reason there is no representation of the flow of time in SNAP.

### 2.5 Cross-categorical relations

We argue that in order to do justice to the complex nature of reality we need *both* four- and three-dimensional views, both SNAP and SPAN, both synchrony and diachrony, simultaneously. Both the endurant entity which is a certain El Niño condition exists as also does the process of its development over time. We can take snapshots of an El Niño condition at different times and we can track the course of its development over time (Figure 2), and for a complete inventory of reality both sorts of views are needed.

We cannot simply glue SNAP and SPAN ontologies together, since the respective types of entities exist in time in different (and as it were orthogonal) ways. Rather we have to establish in painstaking fashion the different sorts of relationships between them. These fall into three major families: dependence (of processes on substances), participation (of substances in processes), and realization (of roles, functions, plans in processes).

The three-dimensionalist view gives us access to enduring entities such as a particular El Niño condition, the land and water body and air masses involved, etc., as well as certain relations between them, together with their qualities, powers, roles, functions, and so forth. The four-dimensionalist view gives us access to the processes the substances participate in, to changes in their qualities over time, to the execution of functions, the realization of plans, and so on.

Consider again Figure 1, which depicts endurants and perdurants involved in an El Niño condition. The figure as a whole refers to a complex system of processes involving the movement of large bodies of water and air and their unfolding over a certain time-interval. At the same time it presupposes that there is some spatial environment within which the process unfolds itself through time. An ontology describing the entities referred to by this figure thus needs to have the resources to describe both complex spatio-temporal processes and the enduring entities which participate therein.

# 3 Formal ontology of endurants and perdurants

We now provide a formal theory which describes the notions discussed above in more formal terms. An extended discussion of the strategy underlying formal theories of this kind can be found in [Gre03]. An alternative approach can be found in [ANC$^+$].

## 3.1 Entities and regions

In this section we provide an ontology of spatio-temporal particulars. The world of particulars has a mereological structure which satisfies the axioms of general extensional mereology (GEM) [Sim87]. For the purposes of this paper it is sufficient to state that the part-of relation $x \leq y$ as axiomatized by GEM is a partial ordering, i.e., it is reflexive, antisymmetric, and transitive (P1–3).

We then distinguish two mutually exclusive classes of particulars: entities $(Ent\ x)$ and regions $(Reg\ x)$ (E1–2). We also demand that every part of an entity is an entity (E3) and that every part of a region is a region (R1).

(P1) $x \leq x$

(P2) $x \leq y \wedge y \leq x \rightarrow x = y$

(P3) $x \leq y \wedge y \leq z \rightarrow x \leq z$

(E1) $Ent\ x \vee Reg\ x$

(E2) $\neg(Ent\ x \wedge Reg\ x)$

(E3) $(Ent\ x \wedge y \leq x) \rightarrow Ent\ y$

(R1) $(Reg\ x \wedge y \leq x) \rightarrow Reg\ y$

Here and in all that follows we omit leading universal quantifiers.

The relationship between entities and regions is established by the relation of location. $L\ xy$ holds if and only if $x$ is located at $y$. For example, you are located, at any given moment of your life, in a certain region of space. For an extended discussion of the notion of location and its axiomatization see [CV99]. For the purposes of this paper it is sufficient to state that the second argument of the location relation is always a region (L1) and that location is a (partial) functional relation, i.e., every entity is located at a single region (L2). We also demand that if $x$ is located at $y$ then for all of its parts $u$ there exists a region $v$ such that $v$ is a part of $y$ and $u$ is located at $v$ (L3). Finally we demand that every region is located at itself (L4).

(L1) $L\ xy \rightarrow Reg\ y$

(L2) $(L\ xy \wedge L\ xz) \rightarrow y = z$

(L3) $L\ xy \rightarrow (u \leq x \rightarrow (\exists v)(v \leq y \wedge L\ uv))$

(L4) $Reg\ x \rightarrow L\ xx$

We then introduce definitions to capture the distinction between abstract and concrete entities: A concrete entity is an entity that is located at some region (DCE). An abstract entitiy is an entity that does is not located at some region (DAE). From this it immediately follows that no entity can be abstract as well as concrete (Th2). Using (E3) we then can prove that every part of a concrete entity is a concrete entity (Th2). Finally we add an axiom to the effect that every entity is either abstract or concrete (E5).

(DCE) $CE\ x := Ent\ x \wedge (\exists y)L\ xy$

(DAE) $AE\ x := Ent\ x \wedge \neg(\exists y)L\ xy$

(E5) $Ent\ x \rightarrow (AE\ x \vee CE\ x)$

(Dl) $l\ x := CE\ x \wedge (\exists! y)L\ xy$

(Th1) $\neg(CE\ x \wedge AE\ x)$

(Th2) $(CE\ x \wedge y \leq x) \rightarrow CE\ y$

For convenience we introduce the notion of a total function, $l$, that allows us to refer to the location $l\ x$ of the concrete entity $x$ (Dl). Here $(\exists! y)\Phi y$ is an abbreviation for $(\exists y)(\Phi y \wedge (\forall z)(\Phi z \rightarrow y = z))$.

### 3.2 Space, time, and spacetime

We now introduce the primitive constants of $\mathcal{S}$, $\mathcal{T}$, and $\mathcal{ST}$ which are interpreted as SPACE, TIME, and SPACETIME. We then define a spatial region, $SR\ x$, as a region which is part of SPACE (DSR) and add similar definitions for temporal and spatio-temporal regions ($TR\ x$ and $STR\ x$ in DTR and DSTR). We then demand that SPACE is a spatial region (R2), TIME is a temporal region (R3), and SPACETIME is a spatio-temporal region (R4). We then demand that spatial, temporal, and spatio-temporal regions are pairwise disjoint domains (R5–7). We can then prove that there is no region which is part of SPACE as well as part of TIME (Th3), that there is no region which is part of SPACE as well as part of SPACETIME (Th4), and that there is no region which is part of TIME as well as part of SPACETIME (Th5). From the (P3), (R1) and (DSR) it follows immediately that every part of a spatial region is a spatial region (Th6) and similarly for temporal and spatio-temporal regions (Th7–8).

(DSR) $SR\,x := Reg\,x \wedge x \leq \mathcal{S}$

(DTR) $TR\,x := Reg\,x \wedge x \leq \mathcal{T}$

(DSTR) $STR\,x := Reg\,x \wedge x \leq \mathcal{ST}$

(R2 ) $SR \quad \mathcal{S}$

(R3 ) $TR \quad \mathcal{T}$

(R4 ) $STR \quad \mathcal{ST}$

(R5) $\neg(TR\,x \wedge SR\,x)$

(R6) $\neg(TR\,x \wedge STR\,x)$

(R7) $\neg(SR\,x \wedge STR\,x)$

(Th3) $\neg(Reg\,x \wedge x \leq \mathcal{S} \wedge x \leq \mathcal{T})$

(Th4) $\neg(Reg\,x \wedge x \leq \mathcal{S} \wedge x \leq \mathcal{ST})$

(Th5) $\neg(Reg\,x \wedge x \leq \mathcal{T} \wedge x \leq \mathcal{ST})$

(Th6) $(SR\,x \wedge y \leq x) \rightarrow SR\,y$

(Th7) $(TR\,x \wedge y \leq x) \rightarrow TR\,y$

(Th8) $(STR\,x \wedge y \leq x) \rightarrow STR\,y$

Each moment in time is associated with a partition of TIME into two jointly exhaustive and mutually disjoint temporal regions: past and future. At the formal level we use the notion of a pair $(p, f)$ in order to refer to the moment in time that is associated with past $p$ and future $f$. The pair $(p, f)$ then is a ssociated with a moment in time if and only if the temporal regions $p$ and $f$ partition TIME in the appropriate way (DM).

(DM) $M\ (p, f) := TR\,p \wedge TR\,f \wedge p + f = \mathcal{T} \wedge \neg(\exists z)(z \leq p \wedge z \leq f)$

Here we use the notation $p + f$ in order to refer to the mereological sum of $p$ and $f$. For convenience we write $t_{pf}$ to refer to the unique moment in time $M\ (p, f)$ defined by the partition formed by $p$ and $f$.

We then introduce a total order between time moments using axioms (TO1–4). (TO5) ensures that every moment has a succeeding moment which is such that its past contains the past of its predecessor as a proper part.

If we want to identify our sequence of successive moments with more common conceptions of the time-line as a sequence $t_{-\infty} \preceq \ldots \preceq t_i \preceq \ldots \preceq t_{+\infty}$ indexed by real or rational numbers, then we might identify $t_{-\infty}$ with $t_{\_\mathcal{T}}$ and $t_{+\infty}$ with $t_{\mathcal{T}\_}$, where (_) indicates that there is no region to fill the corresponding slot.

(TO1) $t_{pf} \preceq t_{pf}$

(TO2) $t_{p_1 f_1} \preceq t_{p_2 f_2} \wedge t_{p_2 f_2} \preceq t_{p_3 f_3} \to t_{p_1 f_1} \preceq t_{p_3 f_3}$

(TO3) $t_{p_1 f_1} \preceq t_{p_2 f_2} \wedge t_{p_2 f_2} \preceq t_{p_1 f_1} \to t_{p_1 f_1} = t_{p_2 f_2}$

(TO4) $t_{p_1 f_1} \preceq t_{p_2 f_2} \vee t_{p_2 f_2} \prec t_{p_1 f_1}$

(TO5) $M(p_1, f_1) \to (\exists (p_2, f_2))(M\ (p_2, f_2) \wedge t_{p_1 f_1} \prec t_{p_2 f_2} \wedge p_1 < p_2)$

So far our axioms admit two kinds of incomapatible mathematical models. In the first interpretation, $\mathcal{M}_1$, we identify $\mathcal{T}$ with the positive part of $\Re^1$, $\mathcal{S}$ with $\Re^3$, and $\mathcal{ST}$ with the pairs of subsets thereof, i.e., $\mathcal{ST} = \mathcal{PT} \times \mathcal{PS}$. Temporal and spatial regions then are interpreted regular closed subsets[1] of $\mathcal{T}$ and $\mathcal{S}$ and spatio-temporal regions are interpreted as pairs thereof. The parthood relation is interpreted as subset relation between regular sets or pairs thereof. We have $(t_1, s_1) \leq (t_2, s_2)$ if and only if $t_1, t_2, s_1, s_2$ are regular sets with $t_1 \subseteq t_2$ and $s_1 \subseteq s_2$, $s_1 \leq s_2$ if and only if $s_1$ and $s_2$ are regular closed sets and $s_1 \subseteq s_2$, and so on. This view was taken for example in [Bit02].

In the second interpretation, $\mathcal{M}_2$, we identify $\mathcal{ST}$ with $\Re^4$ assuming a non-negative temporal dimension, i.e., the set $ST = \{(x, y, z, t) \in \Re^4 \ \mid t \geq 0\}$. If $(x, y, z, t)$ is a spatio-temporal coordinate then $(x, y, z)$ is its spatial projection and $t$ is its temporal

[1] Regular closed sets are sets which are identical to the closure of their interior.

projection. This generalizes in the obvious way to sets of coordinates. The interpretation of $\mathcal{T}$, denoted *T*, then is the temporal projection of the interpretation of *ST*. Given *ST* and its temporal projection *T* then for every $t \in T$ there is a time-indexed space $S_i$. Consequently, the interpretation of SPACE a time-indexed *layered* structure in the sense of [Don03]. The parthood relation is interpreted as the subset relation among regular sets of tuples of matching arity.

In $\mathcal{M}_2$ spatio-temporal regions are interpreted as regular closed subsets of $\Re^4$. Consequently $x$ is a spatio-temporal region if and only if it is a regular closed subset of *ST*. We also write $\{(x, y, z, t) \mid \Phi(x, y, z, t)\}$ in order to denote the spatio-temporal region which corresponds to the set of coordinate quadtuples which satisfy the open formula $\Phi$. $x$ is a region of space if and only if $x$ is a regular subset of *S*and we write $\{(x, y, z) \mid \Phi(x, y, z)\} \subseteq S_i$. Correspondingly, $x$ is a region of time if and only if $x$ is a regular subset of *T*and we write $\{t \mid \Phi(t)\} \subseteq T$.

In both models the temporal ordering, $\preceq$, is interpreted as the ordering of the non-negative $\Re^1$.

### 3.3 Spatio-temporal projection

The relationship between SPACETIME, TIME, and SPACE and regions thereof is now established by the notions of temporal and spatial projection. Consider Figure 3, where $z$ is a two-dimensional spatio-temporal region with one spatial and one temporal dimension. We can think of it as representing the process of shrinking of a one dimensional line segment over the temporal interval between $t_1$ and $t_2$ where, $t_1$ marks the temporal beginning and $t_2$ marks the temporal ending of $z$.

The temporal projection (*tpr*) of the spatio-temporal region $z$ is the interval bounded by $t_1$ and $t_2$. At each moment $t$ in time during this interval the spatial projection (*spr*) of $z$ yields a line segment of a certain length.

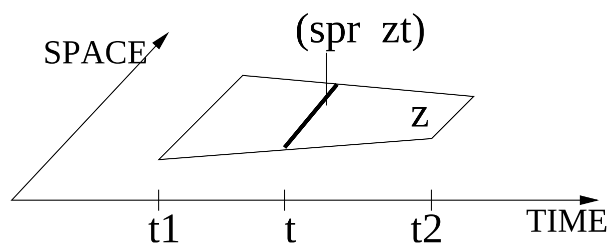


**Fig. 3.** Illustrations of $(spr\ zt)$ and $(Occurs\ xt)$ .

In order to formalize the above intuitions we introduce a primitive binary relation *Tpr* $xy$ and a primitive ternary relation *Spr* $xyt$. The former is intended to hold if and only if $x$ is the temporal projection of $y$ and the latter is intended to hold if and only if $x$ is the spatial projection of the spatio-temporal region $y$ at moment $t$. Given the interpretation in model $\mathcal{M}_2$ these intuitions about spatial and temporal projection correspond to the notion of projection defined on sets of quadtuples discussed above.

Formally we now demand that every spatio-temporal region has a unique temporal projection (DTpr1). For convinience we introduce a functional notation *tpr* in the obvious way (DTpr2).

(DTpr1) $STR\,x \rightarrow (!\exists y)TPr\;yx$

(DTpr2) $\mathit{tpr}\;x = y := \mathit{Tpr}\;xyt$

(Spr1) $(STR\,y \wedge t = (p,f) \wedge M\;(p,f) \wedge$

$(\exists u)(TR\,u \wedge u \leq p \wedge u \leq \mathit{tpr}\;y) \wedge (\exists v)(TR\,v \wedge v \leq f \wedge v \leq \mathit{tpr}\;y)) \rightarrow$

$(\exists x)((\mathit{Spr}\;xyt \wedge (\forall z)(\mathit{Spr}\;zyt \rightarrow x = z))$

(DSpr) $\mathit{spr}\;yt = x := \mathit{Spr}\;xyt$

Let $t = (p,f)$ be a moment in time characterized by the past $p$ and the future $f$ and let $y$ be a spatio-temporal region such that there are parts of the temporal projection of $y$ in the past and other parts of the temporal projection of $y$ in the future relative to $t$. We then demand that there exists a unique $x$ such that *Spr* $xyt$ holds (Spr1). Again, for convinience we introduce a functional notation *spr* in the obvious way (DSpr). Consider Figure 3 and note that *Spr1* is false for $t_1$ and $t_2$ since a point in time can not have a tempoal region as a part.

Now consider the interpretation of *tpr* and *spr* in the model $\mathcal{M}_1$. Given a spatio-temporal tuple $(x_t, x_s)$ *tpr* is just interpreted as the projection onto the first component of the tuple. *spr* is interpreted in a similar way as the second component of the tuple whenever $t$ falls within the range of the temporal interval $x_t$.

Using the notion of spatial projection we now can distinguish beween the models $\mathcal{M}_1$ and $\mathcal{M}_2$. If we prefer the interpretation in $\mathcal{M}_1$ then we need to add an axiom to the effect that the spatial projection of $\mathcal{ST}$ for any moment in time is $\mathcal{S}$ (Spr2–1). If we prefer the interpretation in $\mathcal{M}_2$ then we need to add an axiom to the effect that the spatial projection of $\mathcal{ST}$ at every moment of time $t$ yields a space with a specific index $\mathcal{S}_i$ (Spr2–2). $\mathcal{ST}$ then is $\mathcal{T}$ plus a corresponding sequence of indexed spaces.

(Spr2–1) $(\mathit{spr}\ \mathcal{ST}\ t) = \mathcal{S}$

(Spr2–2) $t_1 \neq t_2 \rightarrow (\mathit{spr}\ \mathcal{ST}\ t_1) \neq \mathit{spr}\ \mathcal{ST}\ t_2)$

We here leave open the question of which model to prefer.

### 3.4 Spatial and temporal parts

We say that $x$ is a *spatial* part of $y$ if and only if $x$ is a part of $y$ and all parts of $y$ are located at spatial regions (DSP). Similarly we say that $x$ is a *temporal* part of $y$ if and only if $x$ is a part of $y$ and all parts of $y$ are located at spatio-temporal regions (DTP). We then can prove that all parts of a spatial region are spatial parts (Th9) and that all parts of a spatio-temporal region are temporal parts (Th10). We can also prove that $x$ is a spatial region if and only if it is a region which has only spatial parts (Th11). Similarly it holds that $x$ is a spatio-temporal region if and only if it is a region which has only temporal parts (Th12). Finally we prove that nothing can be a spatial part as well as a spatio-temporal part of some whole (Th13).

(DSP) $SP\,xy := x \leq y \wedge (\forall u)(u \leq x \rightarrow (\exists z)(L\,uz \wedge SR\,z))$

(DTP) $TP\,xy := x \leq y \wedge (\forall u)(u \leq x \rightarrow (\exists z)(L\,uz \wedge STR\,z))$

(Th) $SR\,x \rightarrow (\forall y)(y \leq x \rightarrow SP\,yx)$

(Th) $STR\,x \rightarrow (\forall y)(y \leq x \rightarrow TP\,yx)$

(Th) $SR\,x \leftrightarrow (Reg\,x \wedge (\forall z)(z \leq x \rightarrow SP\,zx))$

(Th) $STR\,x \leftrightarrow (Reg\,x \wedge (\forall z)(z \leq x \rightarrow TP\,zx))$

(Th) $\neg(\exists y)(SP\,yx \wedge TP\,yx)$

### 3.5 Endurants and Perdurants

We now define an endurant $x$ as an entity which has only spatial parts (DEnd). A perdurant is an entity $x$ which has only temporal parts (DPerd).

(DEnd) $End\,x := Ent\,x \wedge (\forall y)(y \leq x \rightarrow SP\,yx)$

(DPerd) $Perd\,x := Ent\,x \wedge (\forall y)(y \leq x \rightarrow TP\,yx)$

(End1) $(\exists x)(End\,x)$

We can then prove that endurants and perdurants are concrete entities and that nothing can be an endurant and a perdurant (Th14–16). If we add an axiom to the effect that there exists an endurant (End1) – corresponding to our intuition that we ourselves are endurants – then we can prove that concrete entities, spatial regions, and thus regions in general also exist (Th17–19). We can prove also that every part of an endurant is an endurant (Th20) and that every part of a perdurant is a perdurant (Th21).

(Th14) $End\, x \to CE\ x$

(Th15) $Perd\, x \to CE\ x$

(Th16) $\neg(End\, x \wedge Pred\, x)$

(Th17) $(\exists x)(CE\ x)$

(Th18) $(\exists x)(SR\ x)$

(Th19) $(\exists x)(Reg\ x)$

(Th20) $(End\, x \wedge y \leq x) \to End\, y$

(Th21) $(Perd\, x \wedge y \leq x) \to Perd\, y$

We now define the predicate *Occurs* $xt$ with the interpretation that the perdurant $x$ occurs at time $t$. Consider again Figure 3. Here the perdurant $z$ (the process of shrinking a line segment) occurs at all times in the interval between $t_1$ and $t_2$. The predicate *Occurs* $xt$ is defined to hold if and only if the fact that $x$ is a perdurant and $t$ is a moment in time implies that $x$ the temporal projection of $x$ has a part which is part of the past with respect to $t$ and that the temporal projection of $x$ has a temporal part which is part of the future with respect to $t$. Formally we define:

(DOccurs) *Occurs* $xt := (\forall(p,f))((Perd\, x \wedge t = (p,f) = \ \wedge M\ (p,f)) \to$

$((\exists u)(TR\ u \wedge u \leq p \wedge u \leq (\textit{tpr}\ x)) \wedge (\exists v)(TR\ v \wedge v \leq f \wedge v \leq (\textit{tpr}\ x))))$

### 3.6 Top-level categories of endurants

We distinguish three disjoint classes of endurants: the class of substances with their fiat parts and aggregates – SPA entities for short; places $Pl\ x$; and entities which depend on SPA entities or on places – the QPR entities (CEnd1–4). We leave open the question whether or not the three categories exhaust the category of endurants. (Thus geographic fields may constitute a fourth category of endurants.)

(CEnd1) $SPA\,x \rightarrow End\,x$

(CEnd2) $Pl\,x \rightarrow End\,x$

(CEnd3) QPR $x \rightarrow End\,x$

(CEnd4) $\neg(SPA\,x \wedge \text{QPR}\,x) \wedge \neg(SPA\,x \wedge Pl\,x) \wedge \neg(Pl\,x \wedge \text{QPR}\,x)$

(CEnd5) $Pl\,x \rightarrow (\exists y) SPA\,y \wedge l\,x = l\,y$

We do not have the ressources within our present mereological framework to give a definition of substance or of fiat part, since both notions rest on the notion of boundary, whose treatment calls for the tools of mereotopology. For an extended discussion of those notions see [SV00] and [Smi01]. Due to space limitations we also have to omit a discussion of aggregates and QPR entities. We refer the reader to [Gre03], [Smi99], [SG], and [Grept].

Places are endurants which are *co-located* with substances, their fiat parts, and aggregates (CEnd5).

### 3.7 Lives of endurants

*LifeOf* is a binary relation between an endurant $x$ and a perdurant $y$: the life of $x$ (LifeOf1). Every endurant $x$ has a unique perdurant as its life $y$ (LifeOf2). If *LifeOf* $yx$ holds then at every moment in time $t$ at which the perdurant $y$ occurs the endurant $x$ is located at the spatial projection of the location of $y$ at $t$ (LifeOf3). We then can prove that *LifeOf* is a non-reflexive relation (Th22) and – since endurants exist – that perdurants and temporal regions exist (Th23–24).

(LifeOf1) $\mathit{LifeOf}\ xy \rightarrow (End\ x \wedge Perd\ y)$ (Th22) $\neg \mathit{LifeOf}\ xx$

(LifeOf2) $End\ x \rightarrow (\exists! y)(Perd\ y \wedge \mathit{LifeOf}\ yx)$ (Th23) $(\exists x)(Perd\ x)$

(LifeOf3) $\mathit{LifeOf}\ xy \rightarrow (\mathit{Occurs}\ xt \rightarrow (\mathit{spr}\ (l\ x)t) = (l\ y))$ (Th24) $(\exists x)(STR\ x)$

# 4 Ontologies

We now consider ontologies as subjects of study from a meta-theoretical perspective. We call those ontologies which are the targets of our consideration *token ontologies* and write $\mathcal{O}\ \omega$ in order to signify that $\omega$ is a token ontology. Specific representations of token ontologies are maps, figures, lists of names, category trees, partonomies, etc. Consequently, everything that is said about token ontologies holds in particular also for maps. In this section however we abstract from specific forms of representation of token ontologies.

## 4.1 Token ontologies

A token ontology is an abstract entity (O1) and so are all its parts (O2). Therefore ontologies are disjoint from the domain of concrete entities (Th25, Th26). We introduce the notion of a constituent of an ontology signified by the binary predicate $Co$. We also introduce the notion of ontological projection $\mathit{\Pi}\ xy$ in order to signify that the constituent $x$ projects (ontologically) onto $y$. Hereby the relation of ontological projection between a constituent $x$ and target $y$ is similar to the relation between the name 'Mount Everest' and the corresponding mountain. A constituent of an ontology $\omega$ is a part of $\omega$ which projects upon or refers to something (DCo) that is not itself a constituent of this ontology (O3). It follows that projection is non-reflexive (Th27). We require that no ontology is empty (O4).

(O1) $\mathcal{O}\,\omega \rightarrow AE\,\omega$

(O2) $(\mathcal{O}\,\omega \wedge x \leq \omega) \rightarrow AE\,x$

(DCo) $Co\,x\omega := (\mathcal{O}\,\omega \wedge x \leq \omega \wedge (\exists y)\mathit{\Pi}\,xy)$

(O3) $\mathit{\Pi}\,xy \rightarrow (Co\,x\omega \wedge \neg Co\,y\omega)$

(O4) $\mathcal{O}\,\omega \rightarrow (\exists x)Co\,x\omega$

(Th25) $\neg(\mathcal{O}\,x \wedge CE\,x)$

(Th26) $\neg(\mathcal{O}\,\omega \wedge x \leq \omega \wedge CE\,x)$

(Th27) $\neg\mathit{\Pi}\,xx$

Notice that the binary relation of (ontological) projection $\mathit{\Pi}$ whose first parameter is a constituent of an ontology (an abstract entity) has nothing to do with the notion of spatial projection *spr* whose first parameter is a spatio-temporal region.)

If a token ontology is represented as a map, then the constituents of the ontology are represented as features in the map. Consider the map in the left part of Figure 2. Constituents of this (token) ontology are: China, Australia, North America, South America, the Pacific ocean, the body of warm water, and the trade winds represented by the arrow.

We now continue to consider token ontologies as collections of abstract entities that have a particular projective relationship to external entities. We postpone the question about the exact nature of the projective relationship and the question about the structure of token ontologies until Section 5. For the moment it will be sufficient to assume that such a projective relationship exists and that the constituents of token ontologies are structured in an appropriate manner.

An ontology $\omega$ acknowledges an entity $x$ if and only if there is some constituent of the ontology which projects onto $x$ (DAckn). A concrete ontology is an ontology such that all its constituents project onto concrete entities (DCO).

$$\text{(DOAckn)}\ \mathit{Ackn}\ \omega x := (\exists y)(Co\ y\omega \wedge \Pi\ yx)$$

$$\text{(DCO)}\quad CO\ \omega := \mathcal{O}\ \omega \wedge (\forall x)(\forall y)(Co\ x\omega \rightarrow (\Pi\ xy \rightarrow CE\ y))$$

### 4.2 SNAP and SPAN token ontologies

A SNAP ontology is such that its constituents project onto things which have only spatial parts (DSnap); a SPAN ontology is such that its constituents project onto things with temporal parts (DSpan). It follows that no ontology is of type SNAP and of type SPAN (Th28).

$$\text{(DSnap)}\ SNAP\ \omega := \mathcal{O}\ \omega \wedge (\forall x)(\forall y)(Co\ x\omega \rightarrow (\Pi\ xy \rightarrow (\forall z)(z \leq y \rightarrow SP\ zy)))$$

$$\text{(DSpan)}\ SPAN\ \omega := \mathcal{O}\ \omega \wedge (\forall x)(\forall y)(Co\ x\omega \rightarrow (\Pi\ xy \rightarrow (\forall z)(z \leq y \rightarrow TP\ zy)))$$

$$\text{(Th28)}\quad \neg(SNAP\ \omega \wedge SPAN\ \omega)$$

A SNAP entity is an entity which is acknowledged by a SNAP ontology (DSnapEnt) and a SPAN entity is an entity which is acknowledged by a SPAN ontology (DSpanEnt). We now prove that a SNAP entity is an endurant or a spatial region (Th29) and that a SPAN entity is a perdurant or a spatio-temporal region (Th30).

$$\text{(DSnapEnt)}\ SnapEnt\ x := (\exists \omega)(\mathit{Ackn}\ \omega x \wedge SNAP\ \omega)$$

$$\text{(DSpanEnt)}\ SpanEnt\ x := (\exists \omega)(\mathit{Ackn}\ \omega x \wedge SPAN\ \omega)$$

$$\text{(Th29)}\quad SnapEnt\ x \rightarrow (End\ x \vee SR\ x)$$

$$\text{(Th30)}\quad SpanEnt\ x \rightarrow (Perd\ x \vee STR\ x)$$

SNAP ontologies acknowledge enduring entities like places and spatial environments as well as spatial regions. It is however an important aspect of the present framework that there is a clear-cut distinction between regions on one hand and places or spatial environments on the other. Regions are, if you like, *abstract*, places and environments are *domesticated* spatial entities.

### 4.3 Indexed SNAP ontologies

Every SNAP ontology has a *unique* temporal index (SnapI1). An index of a SNAP ontology $\omega$ is a moment of time $t$, where $t$ is identified with the corresponding partition of $\mathcal{T}$ into past ($p$) and future ($f$) (SnapI2). Every constituent $x$ of a SNAP ontology $\omega$ has a corresponding life $y$ which is occurring (*inter alia*) at the time index of $\omega$ (SnapI3). Figure 2 shows SNAP ontologies with successive time indexes.

(SnapI1) $SNAP\ \omega \rightarrow (\exists t)t = ind(\omega) \wedge (\forall t_1)((t_1 = ind(\omega)) \rightarrow t = t_1)$

(SnapI2) $ind(\omega) = t \rightarrow (SNAP\ \omega \wedge (\exists p)(\exists f)(t = (p, f) \wedge M\ (p, f)))$

(SnapI3) $ind(\omega) = t \rightarrow (SNAP\ \omega \wedge (\forall x)(Co\ x\omega \rightarrow (\exists y)(\mathit{LifeOf}\ yx \wedge \mathit{Occurs}\ yt)))$

### 4.4 Relationships between SNAP and SPAN entities

Since SNAP entities are endurants (Th 26) and SPAN entities are perdurants (Th 27) and since every endurant has a unique perdurant as its life (LifeOf1–3), it follows that there exist complex cross-ontological relationships between SNAP and SPAN entities. As discussed in Section 2.5 there is a whole system of such relationships. It is important that these cross-ontological relationships also belong to the meta-level and are thus outside the scope of token ontologies.

In order to formalize the relationships between SNAP and SPAN entities we need to define matching pairs $MP\ (\omega_1, \omega_2)$ of SNAP and SPAN ontologies which characterize ontologies with compatible content or constituents. We demand that if $(\omega_1, \omega_2)$ is a matching pair then $\omega_1$ is an ontology of type SPAN and $\omega_2$ of type SNAP (MP1). All SPAN entities in $\omega_1$ need to be occurring at the temporal index of $\omega_2$ (MP2). The life of every entity acknowledged by $\omega_2$ must be acknowledged by $\omega_1$ (MP3).

$$(MP1)\ MP\,(\omega_1, \omega_2) \rightarrow (SPAN\ \omega_1 \wedge SNAP\ \omega_2)$$

$$(MP2)\ MP\,(\omega_1, \omega_2) \rightarrow (SpanEnt\ x\omega_1 \rightarrow \mathit{Occurs}\ x(ind\ \omega_2))$$

$$(MP3)\ MP\,(\omega_1, \omega_2) \rightarrow (\forall x)(\mathit{Ackn}\ \omega_2 x \rightarrow (\exists y)(\mathit{Ackn}\ \omega_1 y \wedge \mathit{LifeOf}\ yx))$$

In the two diagrams in Figure 1 certain aspects of the underlying matching pairs of SNAP and SPAN ontologies are represented. Associated with the migration process (SPAN) indicated by the white arrows is an underlying body of water (SNAP). Corresponding to the movement of air from East to West (SPAN) there are trade winds (SNAP) signified by black arrows.

The discussion of other aspects such, as the compatibility of levels of granularity, needs to be omitted here. For further discussion see [BS03a] and [RBpt].

## 5 Directly depicting ontologies and their hierarchical structure

In the previous section we characterized constituents of a token ontology as abstract entities which project onto something that is not a constituent of this ontology. In this section we concentrate on the structures formed by constituents of ontologies together with their projective relation to the entities in their target domains. We will show that token ontologies form granular partitions in the sense of [BS03b].

As a first example we will use a SNAP token ontology $\omega$ whose constituents target parts of the body of some human being named Tom. Tom's body is subdivided into head, torso, and limbs, which are subdivided further into: arms, legs, and so on. One possible representation of this token ontology $\omega$ is given in the left part of Figure 4. As a second example we use a map of the subdivision of the United States into states, a part of which is shown in the right part of the figure.

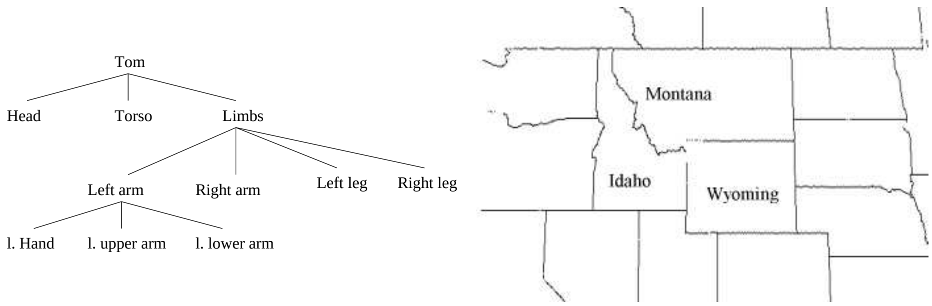


**Fig. 4.** Hierarchical subdivision of the human body (left) and map of parts of the United States (right).

We now focus on two aspects of such token ontologies: (a) the way their constituents form *hierarchical structures* like the ones we know from partonomies and category trees; and (b) the way their constituents *project* onto their target domains (i.e. onto the collection of entities targeted by the constituents of an ontology).

### 5.1 Hierarchical structure

At the abstract level we enforce the tree structure by defining a specific partial order among constituents of a token ontology. This corresponds to theory A in the framework of granular partitions set forth in [BS03b].

We introduce a subcell relation $\subseteq$ which holds among constituents of a single ontology (OA1). Consider the left part of Figure 4. Here the constituents are the nodes of the tree with their resprctive labels; the subcell relation holds wherever an edge connects such nodes. We then have a structure which satisfies the following axioms:

(OA1) $x \subseteq y \to (Co\, x\omega \wedge Co\, y\omega)$

(OA2) $x \subseteq x$

(OA3) $(x \subseteq y \wedge y \subseteq x) \to x = y$

(OA4) $(x \subseteq y \wedge y \subseteq z) \to x \subseteq y$

(Droot) $root\ \omega := (\exists! x)(\forall y)(y \subseteq x)$

(OA5) $x \subseteq y \to x \leq y$

(OA6) $\omega = root\ \omega$

(DISubcell) $x \overline{\subset} y := x \subset y \wedge \neg \exists z : x \subset z \wedge z \subset y$

(OA7) $Co\, y\omega \to (y = \omega) \vee (\exists x_1) \ldots (\exists x_n)(y \overline{\subset} x_1 \overline{\subset} \ldots \overline{\subset} x_n \wedge x_n = \omega)$

(OA8) $(x \subseteq y \wedge x \subseteq z) \to (y \subseteq z \vee z \subseteq y)$

The subcell relation $\subseteq$ is reflexive, antisymmetric, and transitive (OA2–4) and it corresponds to mereological parthood in the sense that if $x$ is a subcell of $y$ then $x$ is also a part of $y$ (OA5). Notice, however, that the converse does not in general hold since the mereological sum of two constituents of an ontology is not necessarily a constituent of that ontology. There is a unique maximal element or root in every token ontology which has as subcells all their constituents of the ontology (OA6). Every constituent is connected to the root through a finite chain of intermediate subcells (DISubcell, OA7). And finally there is no partial overlap among constituents in the sense that if one constituent is a subcell of two others then of the latter one must be a subcell of the other (OA8).

Consider the right part of Figure 4. Here the hierarchy is rather flat. We have one root cell – the United States – and one subcell for every state. But still – it satisfies (OA1–8).

### 5.2 Projection onto reality

The projective relationship between constituents of a token ontology and the entities in its target domain is complex. In the context of this paper we focus on ontologies with particularly well-defined projection relations. For a more general approach and an extensive discussion of the axioms below see [BS03b].

Consider the left part of Figure 4. Here the projection is given by the obvious interpretation of the labels as depicting body parts. Consider the right part of Figure 4. Here the projection is such that the constituent labeled 'Montana' projects onto the state of Montana, and so on.

From axiom (O3) we know that every constituent of a token ontology projects onto something that is not a constituent of this ontology. We now focus on concrete token ontologies, i.e., token ontologies of type SNAP or SPAN, and add further axioms. We demand that the projection relation $\Pi$ is a mapping (OB1) which is one-one (OB2), i.e., every constituent projects onto one entity in the target domain and each entity in the target domain is targeted by at most one constituent. One can easily verify that this is the case for map representations of token-ontologies. We also demand that $\Pi$ be an order homomorphism (OB3) and that wherever the inverse of $\Pi$ is defined then this inverse is an order homeomorphism also (OB4). This insures that token ontologies do not distort the mereological structure of their target domains. This means that the partonomic structure of the human body is indeed the way it is depicted in the left part of Figure 4.

(OB1) $(\Pi\ xy \wedge \Pi\ xz) \rightarrow y = z$

(OB2) $(\Pi\ xy \wedge \Pi\ zy) \rightarrow x = z$

(OB3) $(x_1 \subseteq x_2 \wedge \Pi\ x_1 y_1 \wedge \Pi\ x_2 y_2) \rightarrow y_1 \leq y_2$

(OB4) $(y_1 \leq y_2 \wedge \Pi\ x_1 y_1 \wedge \Pi\ x_2 y_2) \rightarrow x_1 \subseteq x_2$

(OB5) $\exists x_1, y_1, z_1 : (\Pi\ x_1 x_2 \wedge \Pi\ y_1 y_2 \wedge \Pi\ z_1 z_2 \wedge x_2 \leq y_2 \wedge x_2 \leq z_2)$

$\rightarrow (y_2 \leq z_2 \vee z_2 \leq y_2)$

(OB5) ensures that $\Pi$ also preserves the tree structure, i.e., that there is no partial overlap among the targeted entities. Thus in the right part of Figure 4, (OB4) ensures that distinct states do not overlap.

## 6 Conclusions

The theory outlined above contains the resources to describe both complex spatio-temporal processes and the enduring entities which participate therein. At the formal level we distinguished a meta-level and a level of token ontologies. At the meta-level we have abstract SPACE, TIME, and SPACETIME, as well as the formal relations that connect token ontologies together. We distinguished two major categories of token ontologies: ontologies of type SPAN and ontologies of type SNAP. These ontologies represent orthogonal inventories of reality; they presuppose different perspectives on reality and result in distinct though compatible systems of categories.

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# Appendix

## Entities and regions

$(Th1) \vdash (\forall x)\neg(AE\ x \wedge CE\ x)$

| | | |
|---|---|---|
| 1 | $(\exists x)(AE\ x \wedge CE\ x)$ | *assumption* |
| 2 | $AE\ x \wedge CE\ x$ | |
| 3 | $AE\ x$ | 2 *simpl* |
| 4 | $Ent\ x \wedge \neg(\exists y)L\ xy$ | 3 $DAE$ |
| 5 | $\neg(\exists y)L\ xy$ | 4 *simpl* |
| 6 | $CE\ x$ | 2 *simpl* |
| 7 | $Ent\ x \wedge (\exists y)L\ xy$ | 3 $DCE$ |
| 8 | $(\exists y)L\ xy$ | 7 *simpl* |
| 9 | $\neg(\exists y)L\ xy \wedge (\exists y)L\ xy$ | 5, 8 *conj* |
| 10 | $\neg(\exists x)(AE\ x \wedge CE\ x)$ | $1-9$ $IP$ |

$(Th2) \vdash (CE\ x \wedge y \leq x) \rightarrow CE\ y$

| | | |
|---|---|---|
| 1 | $CE\ x \wedge u \leq x$ | *assumption* |
| 2 | $CE\ x$ | 1 *simpl* |
| 3 | $Ent\ x \wedge (\exists y)L\ xy$ | 3 $DCE$ |
| 4 | $(\exists y)L\ xy$ | 3 *simpl* |
| 5 | $L\ xy$ | |
| 6 | $L\ xy \rightarrow (u \leq x \rightarrow (\exists v)(v \leq y \wedge L\ uv))$ | $L3\ UI$ |
| 7 | $u \leq x \rightarrow (\exists v)(v \leq y \wedge L\ uv)$ | $5, 6\ MP$ |
| 8 | $u \leq x$ | 1 *simpl* |
| 9 | $(\exists v)(v \leq y \wedge L\ uv)$ | $7, 8\ MP$ |
| 10 | $v \leq y \wedge L\ uv$ | |
| 11 | $L\ uv$ | 10 *simpl* |
| 12 | $(\exists v)L\ uv$ | $11\ EG$ |
| 13 | $(\exists v)L\ uv$ | $10 - 12\ EI$ |
| 14 | $(\exists v)L\ uv$ | $5 - 13\ EI$ |
| 15 | $Ent\ x \wedge u \leq x$ | $3, 8$ *conj* |
| 16 | $Ent\ x \wedge u \leq x \rightarrow Ent\ u$ | $E3\ UI$ |
| 17 | $Ent\ u$ | $15, 16\ MP$ |
| 18 | $Ent\ u \wedge (\exists v)L\ uv$ | $14, 17$ *conj* |
| 19 | $CE\ u$ | $18\ DCE$ |
| 20 | $(CE\ x \wedge u \leq x) \rightarrow CE\ u$ | $1 - 20\ CP$ |
| 21 | $(\forall x)(\forall y)((CE\ x \wedge y \leq x) \rightarrow CE\ y)$ | $20\ UG$ |

$(Th3) \vdash \neg(Reg\, x \wedge x \leq \mathcal{S} \wedge x \leq \mathcal{T})$

| | | |
|---|---|---|
| 1 | $(\exists x)(Reg\, x \wedge x \leq \mathcal{S} \wedge x \leq \mathcal{T})$ | $assumption$ |
| 2 | $Reg\, x \wedge x \leq \mathcal{S} \wedge x \leq \mathcal{T}$ | |
| 3 | $Reg\, x \wedge x \leq \mathcal{S}$ | $2\ simpl$ |
| 5 | $SR\, x$ | $4\ DSR$ |
| 6 | $Reg\, x \wedge x \leq \mathcal{T}$ | $2\ simpl$ |
| 8 | $TR\, x$ | $6\ DTR$ |
| 9 | $SR\, x \wedge TR\, x$ | $5, 8\ Conj$ |
| 10 | $(\exists x)(SR\, x \wedge TR\, x)$ | $9\ EG, 1-10\ EI$ |
| 11 | $\neg(\exists x)(SR\, x \wedge TR\, x)$ | $R5, QN$ |
| 12 | $(\exists x)(SR\, x \wedge TR\, x) \wedge \neg(\exists x)(SR\, x \wedge TR\, x)$ | $10, 11\ conj$ |
| 13 | $\neg(\exists x)(Reg\, x \wedge x \leq \mathcal{S} \wedge x \leq \mathcal{T})$ | $1-12\ IP$ |
| 14 | $\neg(Reg\, x \wedge x \leq \mathcal{S} \wedge x \leq \mathcal{T})$ | $13\ QN$ |

$(Th6) \vdash (\forall x)(\forall y)(SR\, x \wedge y \leq x \rightarrow SR\, y)$

| | | |
|---|---|---|
| 1 | $SR\, x \wedge y \leq x$ | $assumption$ |
| 2 | $SR\, x$ | $1\, simpl$ |
| 3 | $Reg\, x \wedge x \leq \mathcal{S}$ | $2\, DSR$ |
| 4 | $y \leq x$ | $1\, simpl$ |
| 5 | $y \leq x \wedge x \leq \mathcal{S}$ | $4, 3\, conj$ |
| 6 | $y \leq \mathcal{S}$ | $P3, UI, 3\, MP$ |
| $6a$ | $Reg\, x$ | $3\, simpl$ |
| $6b$ | $Reg\, x \wedge y \leq x$ | $6a, 4\, conj$ |
| $6c$ | $Reg\, y$ | $6c, R1\, MP$ |
| $6d$ | $Reg\, y \wedge y \leq \mathcal{S}$ | $6c, 6\, conj$ |
| 7 | $SR\, y$ | $6\, DSR$ |
| 8 | $SR\, x \wedge y \leq x \rightarrow SR\, y$ | $1 - 7\, CP$ |
| 9 | $(\forall x)(\forall y)(SR\, x \wedge y \leq x \rightarrow SR\, y)$ | $8\, UG$ |

**Spatial and temporal parts**

$(Th9) \vdash (\forall x)(SR\, x \to (\forall y)(y \leq x \to SP\, yx))$

$\vdash (\forall x)(\forall y)(SR\, x \to (y \leq x \to SP\, yx))$

$\vdash (\forall x)(\forall y)(SR\, x \wedge y \leq x) \to SP\, yx))$

| | | |
|---|---|---|
| 1 | $SR\, x \wedge y \leq x$ | $assumption$ |
| $1a$ | $SR\, y$ | $1, Th6\; MP$ |
| $1b$ | $Reg\, y \wedge y \leq \mathcal{S}$ | $1a, DSR$ |
| $1c$ | $Reg\, y$ | $1b\; simpl$ |
| 2 | $y \leq x$ | $1\; simpl$ |
| 3 | $L\, yy$ | $1c, L4\; MP$ |
| 4 | $L\, yy \to (\forall u)(u \leq y \to (\exists v)(v \leq y \wedge L\, uv))$ | $L3\; UI$ |
| 5 | $(\forall u)(u \leq y \to (\exists v)(v \leq y \wedge L\, uv))$ | $3, 4\; MP$ |
| 6 | $u \leq y \to (\exists v)(v \leq y \wedge L\, uv)$ | $5\; UI$ |
| 7 | $u \leq y$ | $assumption$ |
| 8 | $(\exists v)(v \leq y \wedge L\, uv)$ | $8, 4\; MP$ |
| 9 | $v \leq y \wedge L\, uv$ | |
| $9a$ | $v \leq y$ | $9\; simpl$ |
| $9b$ | $SR\, y \wedge v \leq y$ | $9a, 1a\; conj$ |
| 10 | $(SR\, y \wedge v \leq y) \to SR\, v$ | $Th6\; UI$ |
| 11 | $SR\, v$ | $9c, 10\; MP$ |
| 12 | $L\, uv \wedge SR\, v$ | |
| 13 | $(\exists v)(L\, uv \wedge SR\, v)$ | $12\; EG$ |
| 14 | $(\exists v)(L\, uv \wedge SR\, v)$ | $9 - 12\; EI$ |
| 15 | $u \leq y \to (\exists v)(L\, uv \wedge SR\, v)$ | $7 - 15\; CP$ |
| 16 | $(\forall u)(u \leq y \to (\exists v)(L\, uv \wedge SR\, v))$ | $15\; UG$ |
| 17 | $y \leq x \wedge (\forall u)(u \leq y \to (\exists v)(L\, uv \wedge SR\, v))$ | $2, 16\; conj$ |
| 18 | $SP\, yx$ | $17\; DSP$ |
| 19 | $(SR\, x \wedge y \leq x) \to SP\, yx$ | $1 - 19 CP$ |
| 20 | $(\forall x)(\forall y)((SR\, x \wedge y \leq x) \to SP\, yx)$ | $19\; UG$ |

$(Th11a) \vdash Reg\, x \wedge (\forall z)(z \leq x \rightarrow SP\, zx) \rightarrow SR\, x$

| | | |
|---|---|---|
| 1 | $Reg\, x \wedge (\forall z)(z \leq x \rightarrow SP\, zx)$ | *assumption* |
| 2 | $(\forall z)(z \leq x \rightarrow SP\, zx)$ | $1$ *simpl* |
| 3 | $x \leq x \rightarrow SP\, xx$ | $2\, UI$ |
| 4 | $SP\, xx$ | $3, P1\, MP$ |
| 5 | $x \leq x \wedge (\forall y)(y \leq x \rightarrow (\exists u)(L\, yu \wedge SR\, u))$ | $4\, DSP$ |
| 6 | $(\forall y)(y \leq x \rightarrow (\exists u)(L\, yu \wedge SR\, u))$ | $5$ *simpl* |
| 7 | $x \leq x \rightarrow (\exists u)(L\, xu \wedge SR\, u)$ | $6\, UI$ |
| 8 | $(\exists u)(L\, xu \wedge SR\, u)$ | $7, P1\, MP$ |
| 9 | $L\, xu \wedge SR\, u$ | |
| 10 | $L\, xu$ | $9$ *simpl* |
| 11 | $Reg\, x$ | $1$ *simpl* |
| 12 | $L\, xx$ | $11, L4\, MP$ |
| 13 | $L\, xx \wedge L\, xu$ | $10, 12$ *conj* |
| 14 | $x = u$ | $13, L2\, MP$ |
| 15 | $SR\, u$ | $9$ *simpl* |
| 15 | $SR\, x$ | $14, 15\, EQ$ |
| 16 | $SR\, x$ | $9 - 15\, EI$ |
| 17 | $(Reg\, x \wedge (\forall z)(z \leq x \rightarrow SP\, zx)) \rightarrow SR\, x$ | $1 - 16\, CP$ |
| 18 | $(\forall x)((Reg\, x \wedge (\forall z)(z \leq x \rightarrow SP\, zx)) \rightarrow SR\, x)$ | $17\, UG$ |

$(Th11b) \vdash SR\,x \rightarrow (Reg\,x \wedge (\forall z)(z \leq x \rightarrow SP\,zx))$

| | | |
|---|---|---|
| 1 | $SR\,x$ | *assumption* |
| 2 | $(\forall y)(y \leq x \rightarrow SP\,yx)$ | $1, Th9\,MP$ |
| 3 | $Reg\,x \wedge x \leq \mathcal{S}$ | $1\,DSR$ |
| 4 | $Reg\,x$ | $3\,simpl$ |
| 5 | $Reg\,x \wedge (\forall z)(z \leq x \rightarrow SP\,zx)$ | $2, 4\,conj$ |
| 6 | $(\forall x)Reg\,x \wedge (\forall z)(z \leq x \rightarrow SP\,zx)$ | $5\,UG$ |

$(Th11) \vdash SR\,x \leftrightarrow (Reg\,x \wedge (\forall z)(z \leq x \rightarrow SP\,zx))$

| | | |
|---|---|---|
| 1 | $(\forall x)SR\,x \rightarrow (Reg\,x \wedge (\forall z)(z \leq x \rightarrow SP\,zx))$ | $Th11b$ |
| 2 | $SR\,x \rightarrow (Reg\,x \wedge (\forall z)(z \leq x \rightarrow SP\,zx))$ | $1\,UI$ |
| 3 | $(\forall x)((Reg\,x \wedge (\forall z)(z \leq x \rightarrow SP\,zx)) \rightarrow SR\,x)$ | $Th11a$ |
| 4 | $((Reg\,x \wedge (\forall z)(z \leq x \rightarrow SP\,zx)) \rightarrow SR\,x)$ | $4\,UI$ |
| 5 | $(SR\,x \rightarrow (Reg\,x \wedge (\forall z)(z \leq x \rightarrow SP\,zx))) \wedge$ | |
| | $((Reg\,x \wedge (\forall z)(z \leq x \rightarrow SP\,zx)) \rightarrow SR\,x)$ | $2, 4\,conj$ |
| 6 | $(SR\,x \leftrightarrow (Reg\,x \wedge (\forall z)(z \leq x \rightarrow SP\,zx)))$ | $5\,Equiv$ |
| 7 | $(\forall x)(SR\,x \leftrightarrow (Reg\,x \wedge (\forall z)(z \leq x \rightarrow SP\,zx)))$ | $6\,UG$ |

$(Th13) \vdash (\forall x)(\forall y)\neg(SP\,yx \wedge TP\,yx)$

| | | |
|---|---|---|
| 1 | $(\exists x)(\exists y)(SP\,yx \wedge TP\,yx)$ | *assumption* |
| 2 | $SP\,yx \wedge TP\,yx$ | |
| 3 | $SP\,yx$ | 2 *simpl* |
| 4 | $y \leq x \wedge (\forall u)(u \leq y \rightarrow (\exists z)(L\,uz \wedge SR\,z))$ | 3 $DSP$ |
| 5 | $(\forall u)(u \leq y \rightarrow (\exists z)(L\,uz \wedge SR\,z))$ | 4 *simpl* |
| 6 | $y \leq y \rightarrow (\exists z)(L\,yz \wedge SR\,z)$ | 5 $UI$ |
| 7 | $(\exists z)(L\,yz \wedge SR\,z)$ | $6, P1\ MP$ |
| 8 | $TP\,yx$ | 2 *simpl* |
| 9 | $y \leq x \wedge (\forall u)(u \leq y \rightarrow (\exists z)(L\,uz \wedge STR\,z))$ | 3 $DTP$ |
| 10 | $(\forall u)(u \leq y \rightarrow (\exists z)(L\,uz \wedge STR\,z))$ | 9 *simpl* |
| 11 | $y \leq y \rightarrow (\exists z)(L\,yz \wedge STR\,z)$ | 10 $UI$ |
| 12 | $(\exists z)(L\,yz \wedge SR\,z)$ | $11, P1\ MP$ |
| 13 | $L\,yz \wedge SR\,z$ | |
| 14 | $L\,yw \wedge STR\,w$ | |
| 15 | $L\,yz \wedge L\,yw$ | $13, 14$ *simpl, conj* |
| 16 | $z = w$ | $15, L2\ MP$ |
| $16a$ | $SR\,z \wedge STR\,w$ | $13, 14$ *simpl, conj* |
| 17 | $SR\,z \wedge STR\,z$ | $16a\ EQ$ |
| 18 | $(\exists z)(SR\,z \wedge STR\,z)$ | 17 $EG$ |
| 19 | $(\exists z)(SR\,z \wedge STR\,z)$ | $14 - 18\ EI$ |
| 20 | $(\exists z)(SR\,z \wedge STR\,z)$ | $13 - 19\ EI$ |
| $20a$ | $(\exists z)(SR\,z \wedge STR\,z)$ | $2 - 20\ EI$ |
| 21 | $(\exists z)(SR\,z \wedge STR\,z) \wedge \neg(\exists z)(SR\,z \wedge STR\,z)$ | $20, R7$ *conj* |
| 22 | $\neg(\exists x)(\exists y)(SP\,yx \wedge TP\,yx)$ | $1 - 21\ IP$ |
| 23 | $(\forall x)\neg(\exists y)(SP\,yx \wedge TP\,yx)$ | 22 $QN$ |
| 24 | $(\forall x)(\forall y)\neg(SP\,yx \wedge TP\,yx)$ | 23 $QN$ |

**Endurants and perdurants**

$(Th16) \vdash \neg(\exists x)(End\, x \wedge Perd\, x)$

| | | |
|---|---|---|
| 1 | $(\exists x)(End\, x \wedge Perd\, x)$ | $assumption$ |
| 2 | $End\, x \wedge Perd\, x$ | |
| 3 | $End\, x$ | $2\ simpl$ |
| 4 | $Ent\, x \wedge (\forall y)(y \leq x \rightarrow SP\ yx)$ | $3\ DEnd$ |
| 5 | $(\forall y)(y \leq x \rightarrow SP\ yx)$ | $4\ simpl$ |
| 6 | $x \leq x \rightarrow SP\ xx$ | $5\ UI$ |
| 7 | $SP\ xx$ | $6, P1\ MP$ |
| 8 | $Perd\, x$ | $2\ simpl$ |
| 9 | $Ent\, x \wedge (\forall y)(y \leq x \rightarrow TP\ yx)$ | $8\ DPerd$ |
| 10 | $(\forall y)(y \leq x \rightarrow TP\ yx)$ | $9\ simpl$ |
| 11 | $x \leq x \rightarrow TP\ xx$ | $10\ UI$ |
| 12 | $TP\ xx$ | $11, P1\ MP$ |
| 13 | $SP\ xx \wedge TP\ x$ | $7, 12\ conj$ |
| 14 | $(\exists x)(SP\ xx \wedge TP\ xx)$ | $13\ EG$ |
| 15 | $(\exists x)(SP\ xx \wedge TP\ xx)$ | $2-14\ EI$ |
| 16 | $(\forall x)\neg(SP\ xx \wedge TP\ xx)$ | $Th13$ |
| 17 | $\neg(\exists x)(SP\ xx \wedge TP\ xx)$ | $16\ QN$ |
| 18 | $(\exists x)(SP\ xx \wedge TP\ xx) \wedge \neg(\exists x)(SP\ xx \wedge TP\ xx)$ | $15, 18\ conj$ |
| 17 | $\neg(\exists x)(End\, x \wedge Perd\, x)$ | $1-16\ IP$ |

$(Th14) \vdash (End\, x \rightarrow CE\, x)$

| | | |
|---|---|---|
| 1 | $End\, x$ | $assumption$ |
| 2 | $Ent\, x \wedge (\forall y)(y \leq x \rightarrow SP\, yx)$ | $DEnd$ |
| 3 | $(\forall y)(y \leq x \rightarrow SP\, yx)$ | $2\ simpl$ |
| 4 | $x \leq x \rightarrow SP\, xx$ | $3\ UI$ |
| 5 | $SP\, xx$ | $P1, 4\ MP$ |
| 6 | $x \leq x \wedge (\forall u)(u \leq x \rightarrow (\exists z)(L\, uz \wedge SR\, z))$ | $5\ DSP$ |
| 7 | $(\forall u)(u \leq x \rightarrow (\exists z)(L\, uz \wedge SR\, z))$ | $6\ simpl$ |
| 8 | $x \leq x \rightarrow (\exists z)(L\, xz \wedge SR\, z)$ | $7\ UI$ |
| 9 | $(\exists z)(L\, xz \wedge SR\, z)$ | $8, P1\ MP$ |
| 10 | $L\, xz \wedge SR\, z$ | |
| 11 | $L\, xz$ | $10\ simpl$ |
| 12 | $(\exists z)L\, xz$ | $11\ EG$ |
| 13 | $Ent\, x$ | $2\ simpl$ |
| 14 | $Ent\, x \wedge (\exists z)L\, xz$ | $12, 13\ conj$ |
| 15 | $CE\, x$ | $14\ DCE$ |
| 16 | $Ent\, x \rightarrow CE\, x$ | $1-15\ CP$ |
| 17 | $(\forall x)(Ent\, x \rightarrow CE\, x)$ | $16\ UG$ |

$(Th17) \vdash (\exists x) CE\ x$

| | | |
|---|---|---|
| 1 | $(\exists x) End\ x$ | $(End1)$ |
| 2 | $End\ x$ | |
| 3 | $(\forall x)(End\ x \rightarrow CE\ x)$ | $Th14$ |
| 4 | $End\ x \rightarrow CE\ x$ | $3\ UI$ |
| 5 | $CE\ x$ | $2, 4\ MP$ |
| 6 | $(\exists x)(CE\ x)$ | $5\ EG$ |
| 7 | $(\exists x)(CE\ x)$ | $2 - 6 EI$ |

$(Th18) \vdash (\exists x)(SR\ x)$

| | | |
|---|---|---|
| 1 | $(\exists x)(End\ x)$ | $(End\ 1)$ |
| 2 | $End\ x$ | |
| 3 | $Ent\ x \wedge (\forall y)(y \leq x \rightarrow (\exists z)(L\ yz \wedge SR\ z))$ | $Def$ |
| 4 | $(\forall y)(y \leq x \rightarrow (\exists z)(L\ yz \wedge SR\ z))$ | $3\ Simpl$ |
| 5 | $x \leq x \rightarrow (\exists z)(L\ xz \wedge SR\ z)$ | $4\ UI$ |
| 6 | $x \leq x$ | $P1$ |
| 7 | $(\exists z)(L\ xz \wedge SR\ z)$ | $5, 6\ MP$ |
| 8 | $L\ xz \wedge SR\ z$ | |
| 9 | $SR\ z$ | $8\ Simpl$ |
| 10 | $(\exists x) SR\ x$ | $9\ EG$ |
| 11 | $(\exists x) SR\ x$ | $8 - 11\ EI$ |
| 12 | $(\exists x) SR\ x$ | $2 - 11\ EI$ |

$(Th19) \vdash (\exists x)(Reg\ x)$

| | | |
|---|---|---|
| 1 | $(\exists x)(CE\ x)$ | $Th17$ |
| 2 | $CE\ x$ | |
| 3 | $(\forall x)(CE\ x \to (\exists y)(L\ xy))$ | $DCE, Simp$ |
| 4 | $CE\ x \to (\exists y)(L\ xy)$ | $3\ UI$ |
| 5 | $(\exists y)(L\ xy)$ | $2, 4\ MP$ |
| 6 | $L\ xy$ | |
| 7 | $(\forall x)(\forall y)(L\ xy \to Reg\ y)$ | $L\ 1$ |
| 8 | $L\ xy \to Reg\ y$ | $7\ UI$ |
| 9 | $Reg\ y$ | $6, 8\ MP$ |
| 10 | $(\exists x)Reg\ x$ | $9\ EG$ |
| 11 | $(\exists x)Reg\ x$ | $6-10\ EI$ |
| 12 | $(\exists x)Reg\ x$ | $2-11\ EI$ |

$(Th20) \vdash (\forall x)(\forall y)(End\,x \wedge y \leq x \rightarrow End\,y)$

| | | |
|---|---|---|
| 1 | $End\,x \wedge y \leq x$ | *assumption* |
| 1*a* | $Ent\,x \wedge y \leq x \rightarrow Ent\,y$ | $E3$ |
| 1*b* | $(\forall x, y)L\,xy \rightarrow (\forall u)(u \leq x \rightarrow (\exists v)(v \leq y \wedge L\,uv))$ | $L3$ |
| 2 | $End\,x$ | 1 *simpl* |
| 3 | $Ent\,x \wedge (\forall y)(y \leq x \rightarrow SP\,yx)$ | 2 *DEnd* |
| 4 | $Ent\,x$ | 3 *simpl* |
| 5 | $y \leq x$ | 1 *simpl* |
| 6 | $Ent\,x \wedge y \leq x$ | 4, 5 *conj* |
| 7 | $Ent\,y$ | 1*a*, 6 *MP* |
| 8 | $(\forall y)(y \leq x \rightarrow SP\,yx)$ | 3 *simpl* |
| 9 | $y \leq x \rightarrow SP\,yx$ | 8 *UI* |
| 10 | $SP\,yx$ | 9, 5 *MP* |
| 10*a* | $y \leq x \wedge (\forall u)(u \leq y \rightarrow (\exists z)(L\,uz \wedge SR\,z))$ | 10 *DSP* |
| 10*b* | $(\forall u)(u \leq y \rightarrow (\exists z)(L\,uz \wedge SR\,z))$ | 10*a* *simpl* |
| 10*c* | $u \leq y \rightarrow (\exists z)(L\,uz \wedge SR\,z)$ | 10*b* *UI* |
| 10*d* | $u \leq y$ | *assumption* |
| 10*e* | $(\exists z)(L\,uz \wedge SR\,z)$ | 10*c*, 10*d* *MP* |
| 11 | $L\,uz \wedge SR\,z$ | |
| 12 | $L\,uz$ | 11 *simpl* |
| 13 | $L\,uz \rightarrow (\forall w)(w \leq u \rightarrow (\exists v)(v \leq z \wedge L\,wv))$ | 1*b* *UI* |
| 15 | $(\forall w)(w \leq u \rightarrow (\exists v)(v \leq z \wedge L\,wv))$ | 12, 13 *MP* |
| 16 | $w \leq u \rightarrow (\exists v)(v \leq z \wedge L\,wv)$ | 15 *UI* |
| 16*a* | $w \leq u$ | *assumption* |
| 18 | $(\exists v)(v \leq z \wedge L\,wv)$ | 16, 16*a* *MP* |
| 19 | $(v \leq z \wedge L\,wv)$ | |
| 20 | $v \leq z$ | 19 *simpl* |
| 21 | $SR\,z$ | 11 *simpl* |
| 22 | $SR\,z \wedge v \leq z$ | 20, 21 *conj* |
| 23 | $(\forall x, y)(SR\,x \wedge y \leq x \rightarrow SR\,y)$ | $Th6$ |

**Lives of endurants**

$(Th22) \vdash \neg \mathit{LifeOf}\, xx$

| | | |
|---|---|---|
| 1 | $(\exists x)\mathit{LifeOf} xx$ | |
| 2 | $\mathit{LifeOf} xx$ | |
| 3 | $(\forall x)(\forall y)(\mathit{LifeOf}\, xy \rightarrow (End\, x \wedge Perd\, y)$ | $\mathit{LifeOf}1$ |
| 4 | $\mathit{LifeOf}\, xx \rightarrow (End\, x \wedge Perd\, x)$ | $3\, UI$ |
| 5 | $(End\, x \wedge Perd\, x)$ | $1, 4\, MP$ |
| 6 | $(\exists x)(End\, x \wedge Perd\, x)$ | $5\, EG$ |
| 7 | $(\exists x)(End\, x \wedge Perd\, x)$ | $2 - 6\, EI$ |
| 8 | $(\forall x)\neg(End\, x \wedge Perd\, x)$ | $Th16$ |
| 9 | $\neg(\exists x)(End\, x \wedge Perd\, x)$ | $QN$ |
| 10 | $\neg(\exists x)(End\, x \wedge Perd\, x) \wedge (\exists x)(End\, x \wedge Perd\, x)$ | $7, 9\, Conj$ |
| 11 | $\neg(\exists x)\mathit{LifeOf} xx$ | $1 - 11\, IP$ |
| 12 | $\neg(\mathit{LifeOf} xx)$ | |

$(Th23) \vdash (\exists x) Perd\ x$

| | | |
|---|---|---|
| 1 | $(\exists x) End\ x$ | $End1$ |
| 2 | $End\ x$ | |
| 3 | $(\forall x)(End\ x \rightarrow (\exists ! y)(Perd\ y \wedge \mathit{LifeOf}\ yx))$ | $\mathit{LifeOf}1$ |
| 4 | $End\ x \rightarrow (\exists ! y)(Perd\ y \wedge \mathit{LifeOf}\ yx)$ | $3\ UI$ |
| 5 | $(\exists ! y)(Perd\ y \wedge \mathit{LifeOf}\ yx)$ | $2, 4\ MP$ |
| 6 | $(\exists y)((Perd\ y \wedge \mathit{LifeOf}\ yx) \wedge (\forall z)(Perd\ z \wedge \mathit{LifeOf}\ zx) \rightarrow z = y)$ | $5\ Def$ |
| 7 | $(Perd\ y \wedge \mathit{LifeOf}\ yx) \wedge (\forall z)(Perd\ z \wedge \mathit{LifeOf}\ zx) \rightarrow z = y$ | |
| 8 | $(Perd\ y \wedge \mathit{LifeOf}\ yx)$ | $7\ \mathit{Simpl}$ |
| 9 | $Perd\ y$ | $8\ \mathit{Simpl}$ |
| 10 | $(\exists x) Perd\ x$ | $9\ EG$ |
| 11 | $(\exists x) Perd\ x$ | $7 - 10\ EI$ |
| 12 | $(\exists x) Perd\ x$ | $2 - 11\ EI$ |

$(Th24) \vdash (\exists x) STR\, x$

| | | |
|---|---|---|
| 1 | $(\exists x) Perd\, x$ | $Th23$ |
| 2 | $Perd\, x$ | |
| 3 | $Ent\, x \wedge (\forall y)(x \leq y \to TP\, yx)$ | $2\ DPerd$ |
| 4 | $(\forall y)(x \leq y \to TP\, yx)$ | $3\ Simpl$ |
| 7 | $x \leq x \to TP\, xx$ | |
| 8 | $TP\, xx$ | $7, P1\ MP$ |
| 9 | $x \leq x \wedge (\forall u)(u \leq x \to (\exists v)(L\, uv \wedge STR\, v))$ | $8\ DTP$ |
| 10 | $(\forall u)(u \leq x \to (\exists v)(L\, uv \wedge STR\, v))$ | $9\ simpl$ |
| 11 | $x \leq x \to (\exists v)(L\, ux \wedge STR\, v)$ | $10\ UI$ |
| 12 | $(\exists v)(L\, ux \wedge STR\, v)$ | $11, P1\ MP$ |
| 13 | $L\, ux \wedge STR\, v$ | |
| 14 | $STR\, v$ | $13\ simpl$ |
| 15 | $(\exists x) STR\, x$ | $EG\ 14$ |
| 16 | $(\exists x) STR\, x$ | $13 - 15\ EI$ |
| 17 | $(\exists x) STR\, x$ | $2 - 16\ EI$ |

$(Th25) \vdash (\forall x)\neg(\mathcal{O}\ x \wedge CE\ x)$

| | | |
|---|---|---|
| 1 | $(\exists x)(\mathcal{O}\ x \wedge CE\ x)$ | $assumption$ |
| 2 | $\mathcal{O}\ x \wedge CE\ x$ | |
| 3 | $(\forall x)(\mathcal{O}\ x \rightarrow AE\ x)$ | $O1$ |
| 4 | $\mathcal{O}\ x \rightarrow AE\ x$ | $3\ UI$ |
| 5 | $\mathcal{O}\ x$ | $2\ Simpl$ |
| 6 | $AE\ x$ | $4, 5\ MP$ |
| 7 | $CE\ x$ | $2\ simpl$ |
| 8 | $AE\ x \wedge CE\ x$ | $6, 7\ conj$ |
| 9 | $(\exists x)(AE\ x \wedge CE\ x)$ | $8\ EG$ |
| 10 | $(\exists x)(AE\ x \wedge CE\ x)$ | $2-9\ EI$ |
| 11 | $(\exists x)(AE\ x \wedge CE\ x) \wedge \neg(\exists x)(AE\ x \wedge CE\ x)$ | $10, Th1\ conj$ |
| 12 | $\neg(\exists x)(\mathcal{O}\ x \wedge CE\ x)$ | $1-11\ IP$ |
| 13 | $(\forall x)\neg(\mathcal{O}\ x \wedge CE\ x)$ | $12\ QN$ |

$(Th27) \vdash \neg \mathit{\Pi}\ xx$

| | | |
|---|---|---|
| 1 | $(\exists x)\mathit{\Pi}\ xx$ | *assumption* |
| 2 | $\mathit{\Pi}\ xx$ | |
| 3 | $(\forall x)(\forall y)(\forall \omega)(\mathit{\Pi}\ xy \rightarrow (Co\ x\omega \wedge \neg Co\ y\omega)$ | $O3$ |
| 4 | $\mathit{\Pi}\ xx \rightarrow (Co\ x\omega \wedge \neg Co\ x\omega)$ | $3\ UI$ |
| 5 | $Co\ x\omega \wedge \neg Co\ x\omega$ | $2, 4\ MP$ |
| 6 | $(\exists x)(Co\ x\omega \wedge \neg Co\ x\omega)$ | $5\ EG$ |
| 7 | $(\exists x)(Co\ x\omega \wedge \neg Co\ x\omega)$ | $2-6\ EI$ |
| 8 | $\neg(\exists x)\mathit{\Pi}\ xx$ | $1-7\ IP$ |
| 9 | $\neg \mathit{\Pi}\ xx$ | $8\ QN$ |

**SNAP and SPAN ontologies**

$(Th28) \vdash \neg(SNAP\ \omega \wedge SPAN\ \omega)$

| | | |
|---|---|---|
| 1 | $(\exists\omega)(SNAP\ \omega \wedge SPAN\ \omega)$ | $assumption$ |
| 2 | $SNAP\ \omega \wedge SPAN\ \omega$ | |
| 3 | $SNAP\ \omega$ | $3\ simpl$ |
| 4 | $\mathcal{O}\ \omega \wedge (\forall x)(\forall y)(Co\ x\omega \rightarrow (\Pi\ xy \rightarrow (\forall z)(z \leq y \rightarrow SPzy)))$ | $3\ DSnap$ |
| 5 | $(\forall x)(\forall y)(Co\ x\omega \rightarrow (\Pi\ xy \rightarrow (\forall z)(z \leq y \rightarrow SPzy)))$ | $4\ simpl$ |
| 6 | $\mathcal{O}\ \omega$ | $4\ simpl$ |
| 7 | $(\exists x)Co\ x\omega$ | $6, O4\ MP$ |
| 8 | $Co\ x\omega$ | |
| 9 | $\mathcal{O}\ \omega \wedge x \leq \omega \wedge (\exists y)\Pi\ xy$ | $8\ DCo$ |
| 10 | $(\exists y)\Pi\ xy$ | $9\ simpl$ |
| 11 | $\Pi\ xy$ | |
| 12 | $Co\ x\omega \rightarrow (\Pi\ xy \rightarrow (\forall z)(z \leq y \rightarrow SPzy))$ | $5\ UI$ |
| 13 | $\Pi\ xy \rightarrow (\forall z)(z \leq y \rightarrow SPzy)$ | $8, 12\ MP$ |
| 14 | $(\forall z)(z \leq y \rightarrow SPzy)$ | $11, 13\ MP$ |
| 15 | $y \leq y \rightarrow SPyy$ | $14\ UI$ |
| 16 | $SP\ yy$ | $15, P1\ MP$ |
| 17 | $SPAN\ \omega$ | $3\ simpl$ |
| 18 | $\mathcal{O}\ \omega \wedge (\forall x)(\forall y)(Co\ x\omega \rightarrow (\Pi\ xy \rightarrow (\forall z)(z \leq y \rightarrow TPzy)))$ | $17\ DSpan$ |
| 19 | $(\forall x)(\forall y)(Co\ x\omega \rightarrow (\Pi\ xy \rightarrow (\forall z)(z \leq y \rightarrow TPzy)))$ | $18\ simpl$ |
| 20 | $Co\ x\omega \rightarrow (\Pi\ xy \rightarrow (\forall z)(z \leq y \rightarrow TPzy))$ | $19\ UI$ |
| 21 | $\Pi\ xy \rightarrow (\forall z)(z \leq y \rightarrow TPzy)$ | $8, 20\ MP$ |
| 22 | $(\forall z)(z \leq y \rightarrow TPzy)$ | $11, 21\ MP$ |
| 23 | $y \leq y \rightarrow TPyy$ | $22\ UI$ |
| 24 | $TPyy$ | $23, P1\ MP$ |
| 25 | $SP\ yy \wedge TP\ yy$ | $16, 24\ conj$ |
| 26 | $(\exists y)(SP\ yy \wedge TP\ yy)$ | $25\ EG$ |
| 27 | $(\exists y)(SP\ yy \wedge TP\ yy)$ | $11 - 26\ EI$ |
| 28 | $(\exists y)(SP\ yy \wedge TP\ yy)$ | $2 - 27\ EI$ |
| 29 | $(\exists y)(SP\ yy \wedge TP\ yy) \wedge \neg(\exists y)(SP\ yy \wedge TP\ yy)$ | $28, Th13\ conj$ |

$(Th29) \vdash SnapEnt\, x \to (End\, x \vee SR\, x)$

| | | |
|---|---|---|
| 1 | $SnapEnt\, x$ | $assumption$ |
| 2 | $(\exists\omega)(Ackn\, \omega x \wedge SNAP\, \omega)$ | $1\, DSnapEnt$ |
| 3 | $Ackn\, \omega x \wedge SNAP\, \omega$ | |
| 4 | $Ackn\, \omega x$ | $3\, simpl$ |
| 5 | $(\exists y)(Co\, y\omega \wedge \mathit{\Pi}\ yx)$ | $9\, DAckn$ |
| 6 | $Co\, y\omega \wedge \mathit{\Pi}\ yx$ | |
| 7 | $Co\, y\omega$ | $6\, simpl$ |
| 8 | $\mathit{\Pi}\ yx$ | $6\, simpl$ |
| 9 | $SNAP\, \omega$ | $3\, simpl$ |
| 10 | $\mathcal{O}\, \omega \wedge (\forall x)(\forall y)(Co\, x\omega \to (\mathit{\Pi}\ xy \to (\forall z)(z \leq y \to SP\ zy)))$ | $9\, DSnap$ |
| 11 | $(\forall x)(\forall y)(Co\, x\omega \to (\mathit{\Pi}\ xy \to (\forall z)(z \leq y \to SP\ zy)))$ | $10\, simpl$ |
| 12 | $Co\, y\omega \to (\mathit{\Pi}\ yx \to (\forall z)(z \leq x \to SP\ zx))$ | $11\, UI$ |
| 13 | $\mathit{\Pi}\ yx \to (\forall z)(z \leq x \to SP\ zx)$ | $12, 7\, MP$ |
| 14 | $(\forall z)(z \leq x \to SP\ zx)$ | $13, 8\, MP$ |
| 15 | $(\forall z)(z \leq x \to SP\ zx)$ | $5 - 14\, EI$ |
| 16 | $(\forall z)(z \leq x \to SP\ zx)$ | $3 - 15\, EI$ |
| 17 | $Ent\, x \vee Reg\, x$ | $E1$ |
| 16 | $(\forall z)(z \leq x \to SP\ zx) \wedge (Ent\, x \vee Reg\, x)$ | $16, 17\, conj$ |
| 17 | $((\forall z)(z \leq x \to SP\ zx) \wedge Ent\, x) \vee ((\forall z)(z \leq x \to SP\ zx) \wedge Reg\, x)$ | $16\, Dist$ |
| 18 | $End\, x \vee (Reg\, x \wedge (\forall z)(z \leq x \to SP\ zx))$ | $17\, DEnd$ |
| 19 | $End\, x \vee SR\, x$ | $18, Th11\, Equiv$ |
| 20 | $SnapEnt\, x \to (End\, x \vee SR\, x)$ | $1 - 19\, CP$ |
| 21 | $(\forall x)(SnapEnt\, x \to (End\, x \vee SR\, x))$ | $21\, UG$ |