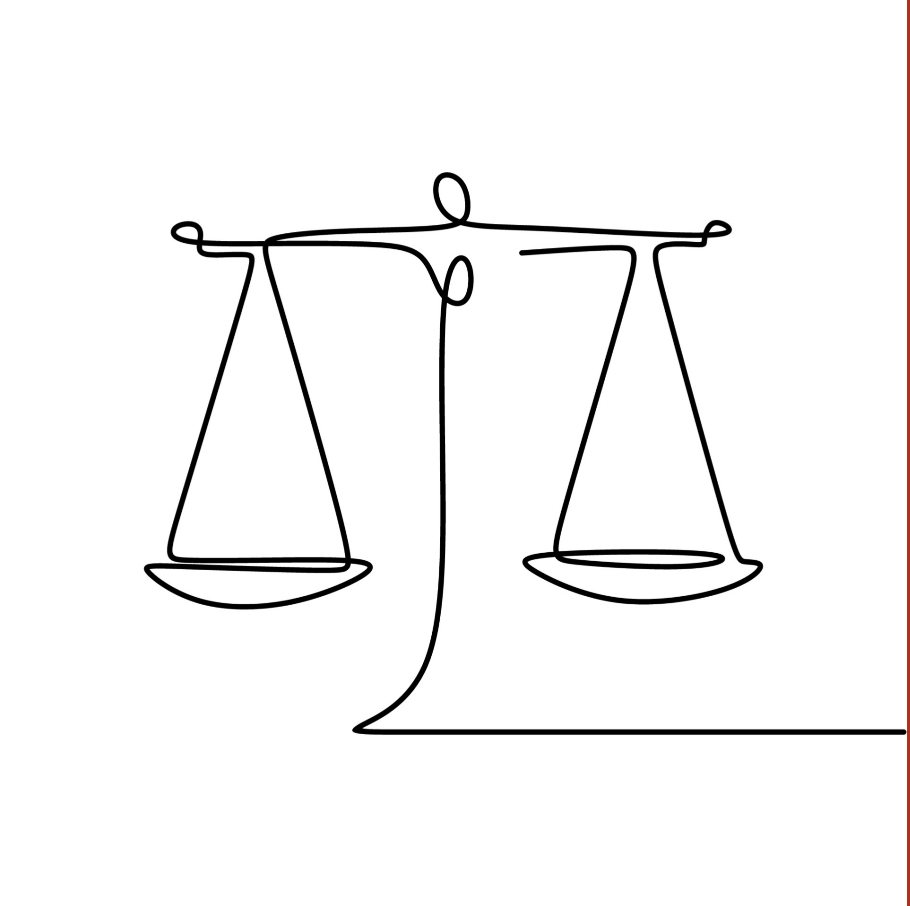
librationist cum classical theories of sets



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Frode Alfson Bjørdal

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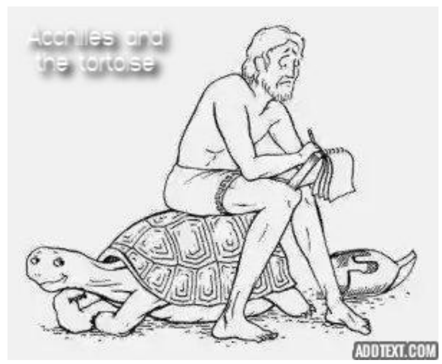
Acchilles and
the tortoise
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To my godson, Jon LW, 2/11/2012,

& my dear friend, nephew Jon VL, 2/17/1979 – 7/22/2011.

Contents

1 Preface

One cannot see it all from one point of view!

The investigations, which led to the *librationist set theories*, began in the spring of 1993. It was a struggle to build upon imprecise thoughts to express beliefs precise enough to be useful. But the author pressed on, as the investigations were very addictive, and began giving talks already in 1996, and publishing unfinished ideas already in 1997, with (Bjørdal 1998). Motivation to persist with the work was always found, so the hard work continued over all those years, in between other rather taxing tasks.

The problems with the paradoxes in set theory are of great importance not only when one attempts to find useful and philosophically reasonable foundational theories for the formal sciences, but also when one seeks to account for a variety of important problems in metaphysics, epistemology and other areas of philosophy.

Paradoxes, are important in metaphysics. As stressed in (Grim 1991), thinking according to the contemporary book has the awkward consequence that one must think that the world is not complete. It will be seen that librationism is not according to the current book, for it has, as shown in §24.6, the result that there are only denumerably many objects in the world; importantly, the *validity* of Cantor's arguments for uncountability are not challenged.

We can see the more comprehensive relevance of the paradoxes rather directly from the fact that modal logics are very important philosophical tools for reasoning about ethics, knowledge, and other central philosophical concepts. But threats of paradox undermine the use of reasonable modal logics, with more than just a minimum of linguistic resources and plausible closure principles. This is on account of such limitative results as were discovered by (Montague 1963), and investigated further by others, like (Friedman and Sheard 1987), and (Cantini 1996). A takeaway is for example that if a modal logic is expressive enough to license the inference from *Smith ought to sell his house* to *there is something Smith ought to do*, then Russell like paradoxes arise.

So paradoxes are pervasive in philosophy. The standard way to evade Montague like limits is to put restrictions upon the linguistic resources. Librationist resolutions are preferable, as such restrictions are not needed.

The focus in this essay will be upon the paradoxes in the context of mathematics, and so foremostly in theories of sets, and related theories, as category theory. A central result is that the librationist set theoric extension $\mathfrak{T}\mathcal{HR}(\mathbf{D})$ of £ accounts for **Neumann-Bernays-Gödel** set theory with the **Axiom of Choice** and **Tarski's Axiom**. Moreover, $\mathfrak{T}$ succeeds with defining an impredicative manifestation set $\mathbf{W}$, *die Welt*, so that $\mathfrak{T}\mathcal{H}(\mathbf{W})$ accounts for Quine's **New Foundations**. Nevertheless, the points of view developed support the view that the truth-paradoxes and the set-paradoxes have common origins, so that the librationist resolutions of the set theoretic paradoxes are at the same time resolutions of the truth theoretic paradoxes. Both the librationist resolutions of the set theoretic paradoxes and the truth theoretic paradoxes have non-trivial philosophical implications: librationist set theories have the consequence that there are no absolutely uncountable sets, and librationist truth theories allow the use of syntactical modalities in ways which circumvent limitations as those of (Montague 1963), and a truth predicate which is useful for more precise philosophical discourse.

2 Acknowledgements

It was the best of times. It was the worst of times.

Charles Dickens

Several *lectures* on librationism were given, and *essays* published, over the years, and it was at times a worry. too much material had seen the light of day.

The first lecture, *Truth in Perspectives*, was held for the Scandinavian Logic Symposium in Uppsala, Sweden, in 1996. The next, *Towards a Foundation for Type-Free Reasoning*, was held in Villa Lanna, Prague, published as an essay in (Bjørdal 1998), fifteen years before (Bjørdal 2012) which introduced the neologism *librationism*. (Bjørdal 2005), (Bjørdal 2006) and (Bjørdal 2011) were published in the last interval.

During some semesters as from 2013 a number of seminaries were held, on a variety of topics, for master and doctorate students of Philosophy, as *Professor Colaborador Voluntário*, at Programa de Pós-Graduação em Filosofia na Universidade Federal do Rio Grande do Norte, in Natal, Brasil: In the second semester of 2013 the seminary *The librationist Foundation for Reasoning* was offered, and in the second semester of 2014 the seminary *A teoria libracionista das coisas*.

Talks were held on the topic for annual LOGICA congresses, under the auspices of the Czech Academy of Sciences, in the Czech republic, in 1997, 2004, 2005 and 2010. Associated papers were published in the LOGICA Yearbook series as (Bjørdal 1998), (Bjørdal 2005), (Bjørdal 2006) and (Bjørdal 2011).

Other lectures were held for the Logic Colloquia, under the auspices of the Association of Symbolic Logic: in Barcelona, 2011 – Helsinki, 2014 – Stockholm, 2017 and Prague, 2019; for World Congresses on Paraconsistent Logic: Melbourne, 2008 and Kolkata, 2014; for International Conferences on Non-classical Mathematics: Guangzhou, 2011 and Vienna, 2014; for Sociedade Brasileira de Lógica: Petrópolis, 2014 – Pirenópolis, 2017 and Salvador, 2022; for World Congresses on Logic, Methodology, and Philosophy of Science: Helsinki 2014 and Prague, 2019; and for the World Congresses on Universal Logic in Lisbon, 2010, Rio de Janeiro, 2013 and Istanbul, 2015. The Kolkata lecture was reworked and published as (Bjørdal 2015).

Lectures were as well delivered for the conference on Logic, Reasoning and Rationality at Centre for Logic & Philosophy of Science, Ghent University, Belgium, in 2010; for the Steklov Mathematical Institute at the Moscow division of the Russian Academy of Sciences's, in 2014, and for the Euler International Mathematical Institute at the Saint Petersburgh division of Steklov Mathematical Institute, in 2015.

Most importantly, I delivered many lectures for the Seminary in Logic under the auspices of the Mathematics Department at the University of Oslo, where I have had the honor to talk about these matters ever since I started to teach at its Philosophy Department, in 1996. Those seminaries were, and are very useful, and I have learned a lot from participating there with colleagues and advanced students.

All encounters were important for the mathematical and philosophical maturation of the author, and resistance was usually useful. No one shares responsibility for undetected errors, so the author will not now risk the names of others by thanking them especially.

3 Introduction

> *Nur wenn man nicht auf den Nutzen nach aussen sieht, sondern in der Mathematik selbst auf das Verhältnis der unbenutzten Teile, bemerkt man das andere und eigentliche Gesicht dieser Wissenschaft. Es ist nicht zweckbedacht, sondern unökonomisch und leidenschaftlich. [...] Die Mathematik ist Tapferkeitsluxus der reinen Ratio, einer der wenigen die es heute gibt.*
>
> Robert Musil, in *Der mathematische Mensch*,
> Mitteilungen der Deutschen Mathematiker-Vereinigung, No°20, page 50, 1912.

It is presupposed that $A \wedge \neg A$ is a contradiction, and that a theory is inconsistent just if it has contradictory theses. As per §15, £ is consistent and not contradictory. So the librationist points of view are not dialetheic, for *dialetheism* is canonically characterized, in (Priest, Berto, and Weber 2022), as a view which takes some contradictions to be true. Moreover, £ is not a paraconsistent point of view, as the latter are not conservative in the sense of Definition 15.3.13. Librationism is instead, as per Definition 15.3.15,taken toof-fer a *classic*, *extraclassical* and *extracoherent* point of view. To complete the distinction, take librationism to offer a *bialethic* point of view, and not a dialetheic one.

It will be showns in §28 that Librationism meets a challenge which it is difficult to see can be met if one presupposes that contradictions, as $p \wedge \neg p$, are true, viz. to offer an account of what a true sentence p says, in a paradoxical situation, which its true negation $\neg p$ does not say in that situation.

A remark on designator is called for. One might hold that a theory is not a set theory if it does not presuppose exactly the same linguistic resources as the language of set theory *according to the current book*, so that *LSAT* is understood to be the language of first order logic plus the symbol $\in$. This tenet is not abided by here, and it is instead presupposed that set theoretic reality should be investigated with such rescources which best reveal it. As will become clear, *set abstracts* are used, and these are not eliminable, due to the fact that £ is a highly non-extensional theory. The symbol $\in$, however, *is* eliminable, by means of apposition.

As £, pronounced as "libra", with additional assumptions, amounting to a definition of librationist system $\mathfrak{P}$, pronounced as "pounds", interprets classical set theory **NBG** and extensions, given yet other assumptions, it would seem disingenious indeed, to hold that £ and $\mathfrak{P}$ are not themselves set theories.

4 Librationism and its formal language

> Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk.
>
> Leopold Kronecker

4.1 Nomenclature

(Bjørdal 2012, p. 323) states *"Librationism takes its name from the word "libration", which the reader is asked to look up if unfamiliar."* *Lunar librations* were an inspiration.

After the publication of (Bjørdal 2012), £ was used to denote the librationist foundational system. It will be indicated, as in §§25–27, when additional assumptions are made, which result in the use of $\mathfrak{T}$ for an extension of £.

The pound sign £ is most prevalently used for the currency of Great Britain. It derives from Latin *libra pondo*. *Pondo* is an adverb which means *by weight*. *Libra* was used for the Roman pound - which was about 327 g, but also for scales and balances. Such scales were an attribute to the Greek Goddess for Divine Justice, Themis, and for her daughter Dike, who was the Goddess for Human justice. The roles of the attributes were thought to be the weighing of the consequences of acts to find balance, and, therefore, justice. The Goddess corresponding to Themis and Dike in the Roman religion, according to the *interpretatio Romana* was the blinded *Justitia*, also referred to as *Lady Justice*, as well had a scale as attribute.

In the context of librationism, £ may be taken to symbolize the weighing and gauging of balances between sentences, and perhaps most interestingly, from the librationst points of view, in the case of sentences which are *incompatible* or *complementary*, in the sense of Definitions 15.4.1 and 15.4.3.

4.2 Numeralism - the chiffer standpoint

The *chiffers* are the *numbers-of-the-meta-language*. The *ordinal* chiffers are defined à la von Neumann by means of the meta mathematical *variety theory*, which one may take to be the set-theory-of-the-meta-language. One must carefully distinguish ordinal chiffers from corresponding ordinal numbers of the *set theories* expressed, and accounted for, by the object language. The *ciphers* are *numerals-of-the-meta-language*, denoting finite order chiffers. In the case of finite chiffers we underline the denoting cipher to contrast with numerals of the set theories accounted for in the object language. So $\underline{0}$ e.g. denotes the ordinal chiffer Zero. The *natural* chiffers are the finite ordinal chiffers, and the variety of *counting* chiffers is the variety of the natural chiffers minus $\underline{0}$. The *integer* chiffers are the natural chiffers extended with the negative counterparts of the counting chiffers.

The chiffer standpoint presupposed here is stronger than the point of view presupposed by (Gödel 1931), which was that formulas, and expressions akin, may be *correlated* via a coding with numerals denoting natural numbers. For the symbols and expressions of £

are taken to *be* counting chiffers, and their syntactical manipulations are accounted for by the variety theory presupposed.

4.3 The inclusion of abstracts

The inclusion of abstracts is a trait shared with (Gandy 1959), and with contributions to the literature on non-classical set theories, including some which were at the time called *property theories*[1], as e.g. (Gilmore, 1974), and theories discussed by (Cantini, 1996), and others, where abstracts were used because the principle of extensionality fails.

4.4 "=" and "∈" are not primitive in £

The formal language of £ is Polish, and without symbols for identity or membership.

A Polish ↓-connective is used, as per Definition 4.5.4.3. The membership relation can be defined by means of apposition of terms, because there as a consequence of the Polish policy are no parentheses in the formal language of £.

§11 shows that the identity of a and b can be delineated adequately by the statement that b is an element of all sets that have a as an element, as in Definition 11.1.2.

4.5 Metalinguistic conventions

4.5.1. Definition Symbols of the meta language:

(1) Σ is the existential quantifier.

(2) Π is the universal quantifier.

(3) $\sim$ is negation.

(4) & is conjunction.

(5) ⓡ is disjunction.

(6) $\Rightarrow$ is for implication.

(7) $\Leftrightarrow$ is for bi-implication.

[1]It seems that the term "property theory", despite seemingly having an origin with Kurt Gödel, became unfortunate. The opening sentence of Roger Myhill's article *Paradoxes*, in Synthese 60 (1984), 129-143, is: "Gödel said to me more than once "There never were any set-theoretic paradoxes, but the property-theoretic paradoxes are still unresolved"; and he may well have said the same thing in print."

Remarks as this may have had such influence that some authors later used the term "property-theory", for non-extensional set theories, which attempt to give more type-free accounts that approximate naive abstraction in dealing with the paradoxes.

Nevertheless, there are now so many non-extensional set theories in the literature, beyond attempts to deal with the paradoxes, that it seems unreasonable to consider them *property theoretic*, as opposed to *set theoretic*.

Was Gödel aware of the contribution in (Scott 1961), or did he study (Friedman 1973). (Shapiro 1985) is another witness to modern research into set theories without extensionality.

(8) $[x\colon \dots]$ is the set notation for use in the metalanguage.

(9) ε is the metalanguage symbol for membership.

4.5.2. Definition (Other metamathematical symbols)

(1) $==$ is for metamathematical identification and definition.

(2) $\alpha, \beta, \gamma, \delta, \dots$ are for ordinal numbers of the metalanguage.

(3) $<, \leq, \geq,$ and $>$ are the orderings on the ordinal numbers of the meta language.

(4) μ is for the least operator of the metalanguage.

4.5.3. Definition (The finite order chiffers, and their integers)

(1) $\Omega == 0, 1, 2 \dots$ is the term for the finite order chiffers, i.e. the natural chiffers.

(2) $\Omega_+ == 1, 2, 3...$ is the term for the positive natural numbers, i.e. *the counting chiffers*.

(3) $\Omega_- == -1, -2, -3 \dots$ is the term for the negative natural numbers.

(4) $\Omega^\pm == 0, 1, -1, 2, -2 \dots$ is the term for the integer chiffers.

(5) $\Omega^- == 0, -1, -2 \dots$ is the term for the integers which are not positive.

4.5.4. Definition (The symbols, their ciphers and chiffers)

(1) •

(2) v̈

(3) ↓

(4) ∀

(5) ς

(6) c̈

(7) #

are the symbols, which stand for the chiffers denoted by the bijective base-2 ciphers 1, 21, 221, 2221, 22221, 222221 and 2222221, respectively.

4.5.5. Definition (Bijective base-2 cipher strings)

(1) Let $n_0, n_1, n_2, n_3, \dots$ be base-2 cipher strings.

(2) $\ell(n_0) == \lfloor log_2(n_0 + 1) \rfloor$ invokes the floor function $\lfloor\ \rfloor$, and defines the length of the bijective base-2 cipher needed to express chiffer n_0.

(3) Concatenation $\frown$ is the function given by $n_0 \frown n_1 == n_0 \cdot 2^{\ell(n_1)} + n_1$.

(4) We know that $\frown$, so defined, is associative.

(5) $n_0 \frown n_1$ is taken to be the denotatum of the apposition $n_0\, n_1$.

(6) Just the ciphers 1 and 2 are the cipher strings of length 1.

(7) If σ_0 is a cipher string of length n and cipher string σ_1 has length 1, then cipher string $\sigma_0\,\sigma_1 = \sigma_0 \frown \sigma_1$ has length $n + 1$.

4.5.6. Exercise: σ is a bijective base 2 cipher string just if it is a bijective base-2 cipher.

4.5.7. Definition (Expressionforms)

(1) A string of symbols from Definition 4.5.4.1-7, is *an expression* in *symbolic form*, just if it is formed according to the formation rules in §§4.5.10-4.5.12.

(2) An expression is in cipher form just if a bijective base-2 cipher which corresponds with the symbolic form via coding of ciphers into symbols, as in Definition 4.5.4.

(3) As symbols are identified with ciphers, cipher forms of expressions are canonical.

4.5.8. Definition (Predicates for terms, formulas, sentences and expressions.)

$\mathbf{Ve}(\mathrm{v}), \mathbf{Ct}(\mathrm{c}), \mathbf{Tm}(\mathrm{a}), \mathbf{Fa}(\mathrm{A}), \mathbf{Se}(\mathrm{B}), \mathbf{En}(\mathrm{X})$ are written to state that v is a variable, c is a constant, a is a term, A is a formula, B is a sentence and X is an expression.

4.5.9. Definition (The underlines)

To remind that expressions in the last analysis are chiffers, denoted by ciphers, we in the remainder of this section underline, and write <u>expression</u>, <u>variable</u>, <u>constant</u>, <u>term</u>, <u>formula</u>, <u>sentence</u>, and <u>expression</u>. To ease the reading, the underlines will not be used as from the next section.

4.5.10. Definition (<u>Variables</u>)

(1) v̈ is a <u>variable</u>.

(2) A <u>variable</u> succeeded by • is a <u>variable</u>.

(3) v_0 is variable v̈, and v_{n+1} is variable $\ddot{\mathrm{v}}_n \frown \bullet$.

(4) Nothing else is a <u>variable</u>.

(5) <u>Variables</u> are <u>terms</u>.

4.5.11. Definition (<u>Primitive constants</u>)

(1) $\ddot{c}$ is a <u>primitive constant</u>.

(2) A <u>primitive constant</u> succeeded by • is a <u>primitive constant</u>.

(3) c_0 is constant c̈, and c_{n+1} is constant $\ddot{\mathrm{c}}_n \frown \bullet$.

(4) Nothing else is a <u>primitive constant</u>.

(5) <u>Primitive constants</u> are <u>terms</u> without free <u>variables</u>, and so, per 4.5.16, <u>constants</u>.

4.5.12. Definition a_i for arbitrary <u>term</u> and A_i for arbitrary <u>formula</u>:

(1) If a_0 and a_1 are <u>terms</u>, $a_1 a_0$ is a <u>formula</u>.

(2) If A_0 and A_1 are formulas, $\downarrow A_0 A_1$ is a formula.

(3) If A_0 is a formula and v_0 is a variable, $\forall v_0 A_0$ is a formula.

(4) If A_0 is a formula and v_0 is a variable, $\varsigma v_0 A_0$ is a term.

(5) Nothing else is a term or a formula.

(6) Just terms and formulas are *expressions.*

4.5.13. Definition Suppressing subscripts:

When possible, $a, b, \ldots$ is written for $a_0, a_1, \ldots$, while $v, w, \ldots$ are for $v_0, v_1, \ldots$, and $m, n, \ldots$ for $n_0, n_1, \ldots$, along with $\mathrm{A}, \mathrm{B}, \ldots$ instead of A with subscripts. Other letters, or letter-like symbols, may be used for special terms, or formulas.

4.5.14. Definition Binders, binds, ties and scopes:

(1) In $\forall v \mathrm{A}$, $\forall$ is the *binder*. v is the *bind* of A and the *tie* of $\forall$. A is the *scope* of $\forall$.

(2) In $\varsigma v \mathrm{A}$, ς is the *binder*. v is the *bind* of A and the *tie* of ς. A is the *scope* of ς.

4.5.15. Definition Free and bound variables:

(1) A variable occurrence in a formula, or term, is bound, just if it is a bind, or it is in the scope of a binder with another occurrence as tie.

(2) Variable occurrences in a formula, or term, are *free* if not bound.

(3) A variable is free in a formula, or term, just if an occurrence is.

(4) A variable is bound in a formula, or term, just if an occurrence is.

4.5.16. Definition Sentences and constants:

(1) A term without free variables is a *constant*.

(2) A formula without free variables is a sentence.

4.5.17. Definition Substitution: If $\mathbf{En}(\mathrm{X})$, a is a term and v is a variable, X^a_v is the expression obtained by substituting all free occurrences of v in $\mathbf{X}$ with term a.

4.5.18. Definition Substitutability: Term a is substitutable for variable v in A just if A is atomic, or A is $\uparrow BC$ and a is substituable for v in both B and C, or A is $\forall w B$ and v is not free in B, or, w does not occur in a and a is substitutable for v in B.

4.5.19. Definition Postfixed variable vector notation:

$$\mathcal{E}(v, w, x) \text{ signifies that variables } v, w \text{ and } x \text{ are free in } \mathcal{E}.$$

4.5.20. Presentation resolve: $\mathrm{A}(v, w, a)$ may be written for $\mathrm{A}(v, w, x)^a_x$.

4.5.21. Definition Prefixed variable vector notation: Occasionally $\forall\vec{\mathbb{v}}\mathrm{A}$ is used for a sentence which either is A, or for some $n > 0$ and variables $\mathbb{v}_0 \ldots \mathbb{v}_{n-1}$, $\forall\vec{v}\mathrm{A}$ is $\forall\mathbb{v}_0 \ldots \forall\mathbb{v}_{n-1}\mathrm{A}$.

4.5.22. Definition Parentheses, and defined operators for the object language:

(1) Delimiters for punctuation: (,), [,], ...

(2) $\neg\mathrm{A} \Longrightarrow\!\downarrow \mathrm{AA}$

(3) $(\mathrm{A} \wedge \mathrm{B}) \Longrightarrow\!\downarrow \neg\mathrm{A}\neg\mathrm{B}$

(4) $(\mathrm{A} \vee \mathrm{B}) \Longrightarrow \neg \downarrow \mathrm{AB}$

(5) $(\mathrm{A} \rightarrow \mathrm{B}) \Longrightarrow (\neg\mathrm{A} \vee \mathrm{B})$

(6) $(\mathrm{A} \leftrightarrow \mathrm{B}) \Longrightarrow (\mathrm{A} \rightarrow \mathrm{B}) \wedge (\mathrm{B} \rightarrow \mathrm{A})$

(7) $\exists v\mathrm{A} \Longrightarrow \neg\forall v\neg\mathrm{A}$

(8) $a \in b \Longrightarrow ba$

(9) $\{v|\mathrm{A}\} \Longrightarrow \varsigma v\mathrm{A}$

4.5.23. Definition Notation for binders restricted to set b:

(1) A^b and a^b signifiy that all variables bound in A and a are restricted to b.

(2) v^b is v.

(3) $(c \in d)^b$ is $c^b \in d^b$.

(4) $\neg A^b$ is $\neg(A^b)$, $(A \wedge B)^b$ is $(A^b \wedge B^b)$, and so on for other connectives.

(5) $\{v|A\}^b \Longrightarrow \{v|v \in b \wedge A^b\}$.

(6) $(\forall v)A^b \Longrightarrow (\forall v)(v \in b \rightarrow A^b)$.

(7) $(\forall\vec{v})A^b \Longrightarrow$ is the *sentence* given by the least $n \geq 0$ such that

$$\Big(n > 0 \ \&\ (\forall v_0 \ldots \forall v_{n-1})(v_0 \in b \wedge \ldots \wedge v_{n-1} \in b \rightarrow A^b)\Big)$$

ⓡ

$$\Big(n = 0 \ \&\ A^b\Big).$$

5 Semantics

> Development of mathematics resembles a fast revolution of a wheel: sprinkles of water are flying in all directions. Fashion – it is the stream that leaves the main trajectory in the tangential direction. The streams of epigone works draw most attention, and they constitute the main mass, but inevitably disappear after a while because they parted with the wheel. To remain on the wheel, it is necessary to apply the effort in the direction perpendicular to the main stream.
>
> Vladimir Igorevich Arnold

The underlying theory of the meta language is *variety theory* $\Sigma_3\mathrm{KP}\Omega$, which is Kripke-Platek variety theory, with Σ_3-collection, and the variety Ω of natural chiffers. Care should be taken to not confuse the varieties of the meta language used to introduce £ with the sets £ postulates the existence of.

5.1 On expression names, and their extension

5.1.1. Definition Expression names:

(1) If X is an expression, then $\ulcorner\mathrm{X}\urcorner$ is its name.

(2) The semantic values of expression names are accounted for in §22.

5.2 Fairs

5.2.1. Definition: $\Vdash$ is a function from *initials*, i.e. initial sets $\Xi, \Xi', \Xi'', \ldots$ of formulas, and ordinals, to sets of formulas. For any ordinal α, and formula A, we write $(\Xi, \alpha) \Vdash \mathrm{A}$ for $\mathrm{A}\varepsilon \Vdash (\Xi, \alpha)$.

5.2.2. Definition Fairs: Initial Ξ is *fair*, or *a fair*, just if for all formulas A and B:

(1) $(\Xi, \alpha) \Vdash\downarrow \mathrm{AB}$ just if neither $(\Xi, \alpha) \Vdash \mathrm{A}$ nor $(\Xi, \alpha) \Vdash \mathrm{B}$.

(2) $(\Xi, \alpha) \Vdash \forall v\mathrm{A}(v)$ just if $(\Xi, \alpha) \Vdash \mathrm{A}^b_v$ for all b substitutable for v in A .

(3) $\alpha == 0 \Rightarrow \Vdash (\Xi, \alpha) == \Xi$, so $(\Xi, 0) \Vdash \mathrm{A} \Leftrightarrow \mathrm{A}\varepsilon\Xi$, and $(\Xi, 0) \Vdash \neg\mathrm{A} \Leftrightarrow \mathrm{A}\not\varepsilon\Xi$.

(4) $\alpha > 0 \Rightarrow ((\Xi, \alpha) \Vdash \mathcal{T}\ulcorner\mathrm{A}\urcorner \Leftrightarrow \mathbf{Se}(\mathrm{A})\ \&\ \Sigma\gamma(\gamma < \alpha\ \&\ \Pi\delta(\gamma \leq \delta < \alpha \Rightarrow (\Xi, \delta) \Vdash \mathrm{A})))$.

(5) $(\Xi, \alpha) \Vdash \mathrm{A} \Rightarrow \mathbf{Fa}(\mathrm{A})$.

(6) $\Pi\Xi[(\Xi, \alpha) \Vdash \mathcal{T}\ulcorner\mathrm{A}\urcorner \vee \mathcal{T}\ulcorner\neg\mathrm{A}\urcorner]$ ⓡ $\big(\Pi\Xi[(\Xi, \alpha) \Vdash \neg\mathcal{T}\ulcorner\neg\mathrm{A}\urcorner] \Leftrightarrow \Pi\Xi[(\Xi, \alpha) \Vdash \neg\mathcal{T}\ulcorner\mathrm{A}\urcorner]\big)$.

5.2.3. Remark: Diagonalization in £ is not as diagonalization in *Peano arithmetic*, nor as in the associated modal provability logic GL with precisely $\Box(\Box p \to p) \to \Box p)$ as characteristic axiom. For such reasons a symbolization $\ulcorner\mathrm{A}\urcorner$ distinct from $\ulcorner\mathrm{A}\urcorner$ is used for the name of formula, or sentence A. The diagonalization construction in £ allows quantifying into named contexts, so that $\forall x \exists y \mathcal{T} \ulcorner x = y \urcorner$ is a well formed sentence.

5.2.4. Presentation resolve:

When the context allows, we write $\mathcal{T}\mathrm{A}$ as an abbreviation for $\mathcal{T}\ulcorner \mathrm{A} \urcorner$.

5.2.5. Remark: If formula A has one variable free, there may be different initials Ξ and Ξ' such that $(\Xi, 0) \Vdash \mathrm{A}$ and $(\Xi', 0) \Vdash \neg\mathrm{A}$. So quantifying over initials, as in §5.4, is like quantifying over interpretations to define tautologicality of formulas in logics.

5.2.6. Theorem (Omega standard)

$$(\Xi, \alpha) \Vdash \exists v \mathrm{A} \Leftrightarrow (\Xi, \alpha) \Vdash \mathrm{A}^b_v \text{ for some } b \text{ substitutable for } v \text{ in } \mathrm{A}.$$

Proof:

$$(\Xi, \alpha) \Vdash \exists v \mathrm{A}$$

$$\Updownarrow$$

$$(\Xi, \alpha) \Vdash \neg\forall v \neg\mathrm{A}$$

$$5.2.2.1 \Updownarrow$$

$$(\Xi, \alpha) \nVdash \forall v \neg\mathrm{A}$$

$$5.2.2.2 \Updownarrow$$

$$\text{for a } b \text{ substitutable for } v \text{ in } \mathrm{A}, (\Xi, \alpha) \nVdash \neg\mathrm{A}^b_v$$

$$5.2.2.1 \Updownarrow$$

$$\text{for a } b \text{ substitutable for } v \text{ in } \mathrm{A}, (\Xi, \alpha) \Vdash \mathrm{A}^b_v$$

□

5.3 Closure

5.3.1. Definition Cover, stabilization and closure:

(1) $\mathrm{IN}(\alpha, \mathrm{A}, \Xi)$ just if $\Pi\beta(\alpha \leq \beta \Rightarrow (\Xi, \beta) \Vdash \mathcal{T}\mathrm{A})$.

(2) $\mathrm{OUT}(\alpha, \mathrm{A}, \Xi)$ just if $\Pi\beta(\alpha \leq \beta \Rightarrow (\Xi, \beta) \nVdash \mathcal{T}\mathrm{A})$.

(3) $\mathrm{IN}(\mathrm{A}, \Xi)$ just if $\Sigma\alpha \mathrm{IN}(\alpha, \mathrm{A}, \Xi)$.

(4) $\mathrm{OUT}(\mathrm{A}, \Xi)$ just if $\Sigma\alpha \mathrm{OUT}(\alpha, \mathrm{A}, \Xi)$.

(5) $\mathrm{STAB}(\mathrm{A}, \Xi)$ just if $\mathrm{IN}(\mathrm{A}, \Xi)$ ⓡ $\mathrm{OUT}(\mathrm{A}, \Xi)$.

(6) $\mathrm{UNSTAB}(\mathrm{A}, \Xi)$ just if $\sim \mathrm{STAB}(\mathrm{A}, \Xi)$.

(7) α *covers* Ξ just if: $\mathrm{IN}(\mathrm{A}, \Xi) \Rightarrow \mathrm{IN}(\alpha, \mathrm{A}, \Xi)$.

(8) α *stabilizes* Ξ just if α covers Ξ, and $(\Xi, \alpha) \Vdash \mathcal{T}\mathrm{A} \Rightarrow \mathrm{IN}(\mathrm{A}, \Xi)$.

(9) The *closure* ordinal ♀ is the least stabilizing ordinal.

5.3.2. Theorem (Herzberger 1980)

There is a closure ordinal.

Proof: Assume first that $\mathrm{IN}(\mathrm{A}, \Xi)$, to presuppose

5.3.3. Definition: $\mathsf{h}(\mathrm{A}) = \mu\alpha(\mathrm{IN}(\alpha, \mathrm{A}, \Xi))$.

1. We first show that there is a covering ordinal:

   We have

$$\Pi\mathrm{A}(\mathrm{IN}(\mathrm{A}, \Xi) \Rightarrow \Sigma\beta(\beta = \mathsf{h}(\mathrm{A}))). \tag{5.3.4}$$

   So

$$\Pi\mathrm{A}\Sigma\beta(\mathrm{IN}(\mathrm{A}, \Xi) \Rightarrow \beta = \mathsf{h}(\mathrm{A})). \tag{5.3.5}$$

   Π_2–collection and quantifier rules give us

$$\Pi\mathrm{B}\Sigma\mathrm{Y}\Pi\mathrm{A}(\mathrm{A}\varepsilon\mathrm{B} \Rightarrow \Sigma\beta(\beta\varepsilon\mathrm{Y}\&(\beta = \mathsf{h}(\mathrm{A})))). \tag{5.3.6}$$

   Instantiate with $\mathrm{B} = [\mathrm{A} : \mathrm{IN}(\mathrm{A}, \Xi)]$ to obtain

$$\Sigma\mathrm{Y}\Pi\mathrm{A}(\mathrm{IN}(\mathrm{A}, \Xi) \Rightarrow \Sigma\beta(\beta\varepsilon\mathrm{Y}\&(\beta = \mathsf{h}(\mathrm{A})). \tag{5.3.7}$$

   Let Z be a witness for (5.3.7), and define the least covering ordinal by means of Π_2–separation,

$$\varkappa = [\nu : \nu\varepsilon\mathrm{Z}\ \&\ \mathrm{Ordinal}(\nu)\ \&\ \Sigma\mathrm{A}(\mathrm{IN}(\mathrm{A}, \Xi)\ \&\ \nu = \mathsf{h}(\mathrm{A}))]. \tag{5.3.8}$$

   Π_2-collection was invoked in the step from (5.3.5) to (5.3.6), and as Π_n-collection implies Σ_{n+1} collection for Kripke–Platek theories, this justifies the choice of an underlying variety theory at least as strong as $\Sigma_3\mathrm{KP}\Omega$ for the meta language.[2]

[2](Welch 2011) shows that $\mathrm{KP} + \Sigma_3$-Determinacy is sufficient for the semantics for a commensurate system AQI (*Arithmetical Quasi Induction*) introduced in (Burgess 1986), and (Hachtman 2019) shows this equivalent to $\mathrm{KP} + \Pi^1_2$-Monotone Induction. So a Σ_3-admissible ordinal is not necessary, but it may be needed for the proof we use, which connects the coding of the formal language with the natural chiffers of the meta theory. Welch has pointed out in private communication that a Σ_2-admissible ordinal, without further assumptions, can be proven to be insufficient.

2. We next prove that there is a stabilizing ordinal:

   Let $[f(n)\colon n\varepsilon\Omega]$, by an adaptation of Cantor's pairing function, be an enumeration of all elements of $\mathrm{UNSTAB}(\Xi)$, where each element recurs infinitely often, so that if $\mathrm{B}=f(m)$ and $m \prec n\varepsilon\Omega$, then there is a natural number o, $n \prec o\varepsilon\Omega$, such that $f(o) = \mathrm{B}$. Let $g(0) = \varkappa$ and $g(n+1) =$ the least $\nu > g(n)$ such that

   $$(\Xi, \nu) \Vdash f(n) \Leftrightarrow (\Xi, g(n)) \nVdash f(n)$$

   Let ß $= [\gamma\colon \Sigma m \Sigma \nu (m\varepsilon\Omega \ \&\ \nu = g(m) \ \&\ \gamma\varepsilon\nu)]$. It is obvious that ß is a limit ordinal which covers Ξ. It is also clear that if $m \prec n\varepsilon\Omega$ then $g(m) \prec g(n)$. Since ß covers Ξ, it suffices to show that $(\Xi, ß) \Vdash \mathcal{T}\mathrm{B}$ entails that B is in $\mathrm{STAB}(\Xi)$, to establish that ß stabilizes Ξ.

   Suppose $(\Xi, ß) \Vdash \mathcal{T}\,\mathrm{B}$. It follows that

   a) $\Sigma\nu\Pi\xi(\nu \leq \xi \prec ß \Rightarrow (\Xi, \xi) \Vdash \mathrm{B})$

   Since g is increasing with ß as its range, we will for some natural number $m\varepsilon\Omega$ have that $\nu \leq g(m) \prec ß$, so that

   b) $\Pi\xi(g(m) \leq \xi \prec ß \Rightarrow (\Xi, \xi) \Vdash \mathrm{B})$

   Suppose $\mathrm{B} \notin \mathrm{STAB}(\Xi)$. By our enumeration of unstable elements where each term recurs infinitely often, we have that $\mathrm{B} = f(n)$ for some natural number n, $m \prec n \in \Omega$. It follows that $g(m) \prec g(n) \prec ß$. From a) and b) we can infer that $(\Xi, g(n)) \Vdash \mathrm{B}$, since we supposed that $(\Xi, ß) \Vdash \mathcal{T}\mathrm{B}$. From the construction of function g, $(\Xi, g(n{+}1)) \nVdash \neg\mathrm{B}$, contradicting b). It follows that $(\Xi, ß) \Vdash \mathcal{T}\mathrm{B}$ only if $\mathrm{B} \in \mathrm{STAB}(\Xi)$, so ß stabilizes Ξ.

3. The proof finishes with an appeal to Definition 5.3.1.9. □

5.4 The range of librationist satisfaction

5.4.1. Definition Satisfaction and satiation:

(1) Initial Ξ maximally satisfies A just if $(\Xi, ♀) \Vdash \mathcal{T}\mathrm{A}$.

(2) Initial Ξ optimally satisfies A just if $(\Xi, ♀) \Vdash \mathrm{A}$.

(3) Initial Ξ plainly satisfies A just if $(\Xi, ♀) \Vdash \neg\mathcal{T}\neg\mathrm{A}$.

(4) Initial Ξ minorly satisfies A just if $(\Xi, ♀) \Vdash \neg\mathcal{T}\neg\mathrm{A} \wedge \neg\mathcal{T}\mathrm{A}$.

5.4.2. Definition:

(1) A is *maximally* satisfied just if for all initials Ξ, $(\Xi, ♀) \Vdash \mathcal{T}\mathrm{A}$.

(2) A is *optimally* satisfied just if for all initials Ξ, $(\Xi, ♀) \Vdash \mathrm{A}$.

(3) A is *plainly* satisfied just if for all initials Ξ, $(\Xi, ♀) \Vdash \neg\mathcal{T}\neg A$.

(4) A is *minorly* satisfied just if for all initials Ξ, $(\Xi, ♀) \Vdash \neg\mathcal{T}\neg A \wedge \neg\mathcal{T}A$.

5.4.3. Definition Notation:

(1) $\overset{M}{\vDash} A$ — A is a *maxim*, as it is maximally satisfied.

(2) $\overset{O}{\vDash} A$ — A is an *optimum*, as it is optimally satisfied.

(3) $\vDash A$ — A is a *plain*, as it is plainly satisfied.

(4) $\overset{m}{\vDash} A$ — A is a *minor*, as it is minorly satisfied.

5.5 Relations between maxims, optima, plains and minors

5.5.1. Definition of tautologies, antologies, determinates, standards:

(1) A is a tautology just if $\vDash A$.

(2) A is an antology just if $\overset{M}{\vDash}\neg A$.

(3) A is a *determinate* just if $(\vDash A \;\&\; \nvDash\neg A)$ ⓡ $(\nvDash A \;\&\; \vDash\neg A)$.

(4) A is an *indeterminate* just if $\nvDash A \;\&\; \nvDash\neg A$.

(5) A is a *standard*, or a standard, just if $\overset{M}{\vDash} (\mathcal{T}\ulcorner A\urcorner \vee \mathcal{T}\ulcorner\neg A\urcorner)$.

5.5.2. Exercise Show that $(\Xi, \alpha) \Vdash \{x|x \in x\} \in \{x|x \in x\} \leftrightarrow \mathcal{T}\{x|x \in x\} \in \{x|x \in x\}$.

5.5.3. Exercise Notice that $\{x|x \in x\} \in \{x|x \in x\}$ is maximally satisfied by Ξ, or $\{x|x \in x\} \notin \{x|x \in x\}$ is maximally satisfied by Ξ, and that $\{x|x \in x\} \in \{x|x \in x\}$ consequently is indefinite, in the sense of Definition 5.6.1.3.

5.5.4. Postulate

(1) $\overset{O}{\vDash} A \Rightarrow\; \vDash A$

(2) $\overset{M}{\vDash} A \Rightarrow \overset{O}{\vDash} A$

(3) $\overset{m}{\vDash} A \Leftrightarrow\; \vDash A \;\&\; \vDash\neg A$

(4) $\overset{M}{\vDash} A \Leftrightarrow\; \vDash A \;\&\; \nvDash\neg A$

Proof:
(1): Use 7.1.6.

(2): $\overset{M}{\vDash} A \Rightarrow \overset{O}{\vDash} \mathcal{T}\ulcorner A\urcorner$, given Definitions 5.4.2 and 5.4.3. So if $\overset{M}{\vDash} A$, $\overset{O}{\vDash} \mathcal{T}\ulcorner A\urcorner$ holds. Also, $\overset{O}{\vDash} \mathcal{T}\ulcorner A\urcorner \rightarrow A$, which is 7.1.6. Thus, $\overset{M}{\vDash} A$ only if $\overset{O}{\vDash} A$.

(3): By Definitions 5.4.2.3 and 5.4.2.4.

(4): Leftwards - $\vDash$ A & $\nvDash$A $\Rightarrow \overset{O}{\vDash}$ $(\mathcal{T}\mathrm{A} \vee \mathcal{T}\neg\mathrm{A})$ on account of Postulate 6.3.1. So for all initials Ξ, $(\Xi, ♀) \Vdash (\mathcal{T}\mathrm{A} \vee \mathcal{T}\neg\mathrm{A})$. As $\vDash$ A, so that for all initials Ξ, $(\Xi, ♀) \Vdash \neg\mathcal{T}\neg\mathrm{A}$, for all initials Ξ, $(\Xi, ♀) \Vdash \mathcal{T}\mathrm{A}$. So $\overset{M}{\vDash}$ A.

Rightwards - Given $\overset{M}{\vDash}$ A, for all initials Ξ, $(\Xi, ♀) \Vdash \mathcal{T}\mathrm{A}$, so that by Definition 5.2.2.4, for all Ξ, $\Sigma\gamma(\gamma \prec \alpha$ & $\Pi\delta(\gamma \leq \delta \prec \alpha \Rightarrow (\Xi, \delta) \Vdash \mathrm{A}))$. Consequently,

$$\Pi\Xi\big[\Pi\gamma(\gamma \prec \alpha \;\&\; \Sigma\delta(\gamma \leq \delta \prec \alpha \Rightarrow (\Xi, \delta) \Vdash \mathrm{A}))\big],$$

so for all Ξ, $(\Xi, ♀) \Vdash \neg\mathcal{T}\neg\mathrm{A}$, and so $\vDash$ A. Moreover, $\overset{M}{\vDash}$ A & $\vDash \neg$A clearly entails absurdity, so $\overset{M}{\vDash}$ A $\Rightarrow$ $\nvDash\neg$A. In sum, $\overset{M}{\vDash}$ A $\Rightarrow$ $\vDash$ A & $\nvDash\neg$A. Finish by joining the directions. □

5.5.5. Theorem: There are just minor and maximal tautologies. Optimal tautologies are either minor or maximal, and minor tautologies are not maximal tautologies.

Proof: This follows from Theorems 5.5.4.3 and 5.5.4.4. □

5.6 Orthodoxy, definiteness and paradoxicality

5.6.1. Definition:

(1) A is *orthodox* just if $\overset{O}{\vDash}$ $\forall\vec{v}(\mathcal{T}\mathrm{A} \vee \mathcal{T}\neg\mathrm{A})$.

(2) Set a is orthodox just if $x \in a$ is orthodox.

(3) A is *definite* just if $\vDash$ A or $\vDash \neg$A.

(4) A is *apocryphal* just if orthodox and *indefinite*.

(5) Set a is apocryphal just if $b \in a$ is apocryphal for some set b.

5.6.2. Remark: Some definite sentences are determinate, and some are indeterminate.

5.6.3. Remark: Set $s \Longleftarrow \{v|v \in v\}$ is apocryphal. For sentence $s \in s$ is apocryphal, by cause of its orthodoxy and the fact that it is indefinite because $\nvDash s \in s$ and $\nvDash s \notin s$.

5.6.4. Definition: Formula A is paradoxical just if not orthodox. Given Definition 5.6.1.1, this is the case just if $\overset{O}{\nvDash}$ $\forall\vec{v}(\mathcal{T}\mathrm{A}(\vec{v}) \vee \mathcal{T}\neg\mathrm{A}(\vec{v}))$; so there is, given Definition 5.4.2.2, an initial Ξ such that $(\Xi, ♀) \Vdash \exists\vec{v}(\neg\mathcal{T}\mathrm{A}(\vec{v}) \wedge \neg\mathcal{T}\neg\mathrm{A}(\vec{v}))$. By adapting Theorem 14.1.1, we find a vector $\vec{a}$ for instantiation so that $(\Xi, ♀) \Vdash (\neg\mathcal{T}\mathrm{A}(\vec{a}) \wedge \neg\mathcal{T}\neg\mathrm{A}(\vec{a}))$

5.6.5. Definition: Sentence A is paradoxical just if not orthodox, just if $\overset{O}{\nvDash}$ $(\mathcal{T}\mathrm{A} \vee \mathcal{T}\neg\mathrm{A})$; so there is, given Definition 5.4.2.2, an initial Ξ such that $(\Xi, ♀) \Vdash (\neg\mathcal{T}\mathrm{A} \wedge \neg\mathcal{T}\neg\mathrm{A})$.

5.6.6. Definition: Set a is paradoxical just if not orthodox just if $\overset{O}{\nvDash}$ $\forall x(\mathcal{T}x \in a \vee \mathcal{T}x \notin a)$; so, given Definition 5.4.2.2, there is an initial Ξ such that $(\Xi, ♀) \Vdash \exists x(\neg\mathcal{T}x \in a \wedge \neg\mathcal{T}x \notin a)$. Consequently, given Theorem 14.1.1, for some term b, $(\Xi, ♀) \Vdash (\neg\mathcal{T}b \in a \wedge \neg\mathcal{T}b \notin a)$.

5.6.7. FACT: The proof of Theorem 6.3.2 shows that $\overset{\mathtt{O}}{\vDash}(\mathcal{T}\mathrm{A} \vee \mathcal{T}\neg\mathrm{A}) \Rightarrow \overset{\mathtt{M}}{\vDash}(\mathcal{T}\mathrm{A} \vee \mathcal{T}\neg\mathrm{A})$, so Definition 5.6.1.1 entails that A is orthodox just if $\overset{\mathtt{M}}{\vDash}(\mathcal{T}\mathrm{A} \vee \mathcal{T}\neg\mathrm{A})$. But the latter should not be used for defining orthodoxy, as the induced revision of Definition 5.6.4 would not give the intended extension for the term 'paradoxical'.

5.7 The non-triviality assumptions

5.7.1. DEFINITION: A logical theory is *trivial* just if all of its sentences are derivable.

The assumption that there *are* fair initials for variants of £ amounts to assuming that the system under consideration is not trivial, and, consequently, consistent. It follows from (Bjørdal 2012), under the assumption that $\Sigma_3\mathrm{KP}_\omega$ is consistent, that the empty set is a fair intial for £ simpliciter. So if $\Sigma_3\mathrm{KP}_\omega$ is consistent, then £ is not trivial.

§25 shows that $\mathfrak{T}\mathcal{H}$ & $\mathcal{R}\mathbf{D}$ has an account of $\mathbf{NBGC} + \mathbf{TA}$ if $\mathbf{NBGC} + \mathbf{TA}$ is consistent.

6 Maxims

$\{(x,y)|y \geq f(x)\}$

6.1 Axioms & warrants, theorems & proofs.

6.1.1. Definition:

(1) A *warrant* of an axiom is a semantic demonstration of it from Definition 5.2.2.

(2) A *proof* of a theorem is a demonstration of it from axioms and other theorems.

6.2 The relations between the varieties of theses

§5.4 gave the semantic distinctions between *maximally, optimally, plainly* and *minorly* true. The corresponding syntactic notions are *maxim, optimum, plain* and *menor*. Here "plain" is used as a noun, and occasionally as an adjective. The word "menor" is a variant of "minor", according to the Oxford English Dictionary, and the term "minor" is here used as an adjective, while "menor" is used as a noun.

6.2.1. Postulate of the soundness: £ is sound for all theses of the treatise, as that is checked individually. So

$$\overset{\mathsf{M}}{\vdash} \mathrm{A} \Rightarrow \overset{\mathsf{M}}{\vDash} \mathrm{A};\ \overset{\mathsf{O}}{\vdash} \mathrm{A} \Rightarrow \overset{\mathsf{O}}{\vDash} \mathrm{A};\ \vdash \mathrm{A} \Rightarrow \vDash \mathrm{A} \text{ and } \overset{m}{\vdash} \mathrm{A} \Rightarrow \overset{m}{\vDash} \mathrm{A}.$$

6.2.2. Axiom Relations between maxims, optima, plains and menors:

(1) $\overset{\mathsf{M}}{\vdash} \mathrm{A} \Rightarrow \overset{\mathsf{O}}{\vdash} \mathrm{A}$

(2) $\overset{\mathsf{O}}{\vdash} \mathrm{A} \Rightarrow \vdash \mathrm{A}$

(3) $\overset{m}{\vdash} \mathrm{A} \Leftrightarrow \vdash \mathrm{A}\ \&\ \vdash \neg\mathrm{A}$

(4) $\overset{\mathsf{M}}{\vdash} \mathrm{A} \Leftrightarrow \vdash \mathrm{A}\ \&\ \nvdash \neg\mathrm{A}$

6.3 Arbitration

6.3.1. Postulate (Optimal arbitration)

$$\overset{\mathsf{O}}{\vDash} (\mathcal{T}\ulcorner\mathrm{A}\urcorner \vee \mathcal{T}\ulcorner\neg\mathrm{A}\urcorner)\ ⓡ\ (\vDash \mathrm{A} \Leftrightarrow \vDash \neg\mathrm{A}).$$

Warrant of 6.3.1: Definition 5.2.2.6 states that

$$\Pi\Xi[(\Xi,\alpha) \Vdash \mathcal{T}\ulcorner\mathrm{A}\urcorner \vee \mathcal{T}\ulcorner\neg\mathrm{A}\urcorner]\ ⓡ\ \big(\Pi\Xi[(\Xi,\alpha) \Vdash \neg\mathcal{T}\ulcorner\neg\mathrm{A}\urcorner] \Leftrightarrow \Pi\Xi[(\Xi,\alpha) \Vdash \neg\mathcal{T}\ulcorner\mathrm{A}\urcorner]\big).$$

The right disjunct of Definition 5.2.2.6 amounts to $\vDash \mathrm{A} \Leftrightarrow \vDash \neg\mathrm{A}$, given Definitions 5.4.2.3 and 5.4.3.3, and $\overset{\mathsf{O}}{\vDash} (\mathcal{T}\ulcorner\mathrm{A}\urcorner \vee \mathcal{T}\ulcorner\neg\mathrm{A}\urcorner)$ is entailed by the left disjunct of Definition 5.2.2.6 via Definitions 5.4.2.2 and 5.4.3.2. ∎

6.3.2. Theorem (Maximal arbitration)

$$\overset{\mathsf{M}}{\vDash} (\mathcal{T}\ulcorner A\urcorner \vee \mathcal{T}\ulcorner \neg A\urcorner) \circledR (\vDash A \Leftrightarrow \vDash \neg A).$$

Proof: Confer Fact 5.6.7. It suffices to prove $\overset{\mathsf{O}}{\vDash} (\mathcal{T}A \vee \mathcal{T}\neg A) \Rightarrow \overset{\mathsf{M}}{\vDash} (\mathcal{T}A \vee \mathcal{T}\neg A)$. Make use of a disjunctive syllogism to obtain $\overset{\mathsf{O}}{\vDash} (\mathcal{T}\mathcal{T}\ulcorner A\urcorner \vee \mathcal{T}\mathcal{T}\ulcorner \neg A\urcorner)$ with Postulate 7.1.1 from $\overset{\mathsf{O}}{\vDash} (\mathcal{T}\ulcorner A\urcorner \vee \mathcal{T}\ulcorner \neg A\urcorner)$, and use theorem $\overset{\mathsf{O}}{\vdash} \mathcal{T}\mathcal{T}\ulcorner B\urcorner \rightarrow \mathcal{T}\ulcorner(\mathcal{T}\ulcorner B\urcorner \vee C)\urcorner$ and disjunctive syllogism with $\overset{\mathsf{O}}{\vDash} (\mathcal{T}\mathcal{T}\ulcorner A\urcorner \vee \mathcal{T}\mathcal{T}\ulcorner \neg A\urcorner)$ to obtain $\overset{\mathsf{O}}{\vDash} \mathcal{T}\ulcorner\mathcal{T}\ulcorner A\urcorner \vee \mathcal{T}\ulcorner \neg A\urcorner\urcorner$. Use Definitions 5.4.2.1, 5.4.2.2, 5.4.3.1 and 5.4.3.2 and the results noted to conclude $\overset{\mathsf{M}}{\vDash} \mathcal{T}\ulcorner A\urcorner \vee \mathcal{T}\ulcorner \neg A\urcorner$ from $\overset{\mathsf{O}}{\vDash} \mathcal{T}\ulcorner\mathcal{T}\ulcorner A\urcorner \vee \mathcal{T}\ulcorner \neg A\urcorner\urcorner$. □

6.4 Logic maxims

6.4.1. Postulate Classical logic maxims:

(1) $\overset{\mathsf{M}}{\vdash} A \rightarrow (B \rightarrow A)$

(2) $\overset{\mathsf{M}}{\vdash} (A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$

(3) $\overset{\mathsf{M}}{\vdash} (\neg B \rightarrow \neg A) \rightarrow (A \rightarrow B)$

(4) $\overset{\mathsf{M}}{\vdash} \forall x(A \rightarrow B) \rightarrow (\forall xA \rightarrow \forall xB)$

(5) $\overset{\mathsf{M}}{\vdash} A \rightarrow \forall vA$, provided v is not free in A

(6) $\overset{\mathsf{M}}{\vdash} \forall vA \rightarrow A^b_v$, provided b is substitutable for v in A

(7) If $\overset{\mathsf{M}}{\vdash} \Gamma$ belongs to (6.4.1.1–6.4.1.6), then so does $\overset{\mathsf{M}}{\vdash} \forall v\Gamma$.

6.4.2. Remark: The role of a maximal inference mode, which allows the deduction from $\overset{\mathsf{M}}{\vdash} (A \rightarrow B)$ and $\overset{\mathsf{M}}{\vdash} A$ to $\overset{\mathsf{M}}{\vdash} B$, is played by mode 9.2.5.

6.4.3. Remark: An induction, upon 6.4.1.7 and 9.2.5, proves *generalization* is a derived inference mode. Compare the proof of Theorem 45.4 of (Hunter 1971, pp. 174–175).

6.5 Maxims on truth

6.5.1. Definition Russell's paradoxical set:

$$r \Longleftrightarrow \{x | x \notin x\}$$

6.5.2. Postulate Truth maxims:

(1) $\overset{\mathsf{M}}{\vdash} \mathcal{T}(A \rightarrow B) \rightarrow (\mathcal{T}A \rightarrow \mathcal{T}B)$

(2) $\overset{\mathsf{M}}{\vdash} \mathcal{T}A \rightarrow \neg\mathcal{T}\neg A$

(3) $\overset{\mathsf{M}}{\vdash} (\mathcal{T}r \in r \vee \mathcal{T}r \notin r) \rightarrow (\mathcal{T}A \vee \mathcal{T}\neg A)$

(4) $\overset{\mathsf{M}}{\vdash} \mathcal{T}A \vee \mathcal{T}\neg A \vee (\mathcal{T}\neg\mathcal{T}\neg B \rightarrow \mathcal{T}B)$

(5) $\overset{\mathsf{M}}{\vdash}$ $\mathcal{T}A \vee \mathcal{T}\neg A \vee (\mathcal{T}B \rightarrow \mathcal{T}\mathcal{T}B)$

(6) $\overset{\mathsf{M}}{\vdash}$ $\mathcal{T}(\mathcal{T}A \rightarrow A) \rightarrow (\mathcal{T}A \vee \mathcal{T}\neg A)$

(7) $\overset{\mathsf{M}}{\vdash}$ $\mathcal{T}(\mathcal{T}A \rightarrow \mathcal{T}\mathcal{T}A) \rightarrow (\mathcal{T}A \vee \mathcal{T}\neg A)$

(8) $\overset{\mathsf{M}}{\vdash}$ $\exists v\mathcal{T}\mathrm{A} \rightarrow \mathcal{T}\exists v\mathrm{A}$.

(9) $\overset{\mathsf{M}}{\vdash}$ $\mathcal{T}\forall v\mathrm{A} \rightarrow \forall v\mathcal{T}\mathrm{A}$

(10) $\overset{\mathsf{M}}{\vdash}$ $\forall u(a \in u \rightarrow b \in u) \rightarrow (A^a_v \rightarrow A^b_v)$, for a and b both substitutable for v in A.

(11) $\overset{\mathsf{M}}{\vdash}$ $\mathfrak{D}(\{x|\mathrm{A}\}) \rightarrow (\forall x\mathcal{T}\mathrm{A} \rightarrow \mathcal{T}\forall x\mathrm{A})$

6.6 Warrants of truth maxims

Warrant 6.5.2.1: Suppose $(\Xi,\gamma) \Vdash \mathcal{T}(\mathrm{A} \rightarrow \mathrm{B})$ and $(\Xi,\gamma) \Vdash \mathcal{T}\mathrm{A}$. It follows that for some ordinal δ and any ordinal ϵ such that $\delta \leq \epsilon \prec \gamma$, $(\Xi,\epsilon) \Vdash (\mathrm{A} \rightarrow \mathrm{B})$ and $(\Xi,\epsilon) \Vdash \mathrm{A}$. So, on account of Definition 5.2.2.1, $(\Xi,\epsilon) \Vdash \mathrm{B}$, and, consequently, $(\Xi,\gamma) \Vdash \mathcal{T}\mathrm{B}$. So for any ordinal γ, $(\Xi,\gamma) \Vdash \mathcal{T}(\mathrm{A} \rightarrow \mathrm{B}) \rightarrow (\mathcal{T}\mathrm{A} \rightarrow \mathcal{T}\mathrm{B})$. $(\Xi,♀) \Vdash \mathcal{T}(\mathcal{T}(\mathrm{A} \rightarrow \mathrm{B}) \rightarrow (\mathcal{T}\mathrm{A} \rightarrow \mathcal{T}\mathrm{B}))$ is a consequence of this, so $\overset{\mathsf{M}}{\vDash}$ $\mathcal{T}(\mathrm{A} \rightarrow \mathrm{B}) \rightarrow (\mathcal{T}\mathrm{A} \rightarrow \mathcal{T}\mathrm{B})$. ∎

Warrant 6.5.2.2: Assume $(\Xi,\gamma) \nVdash (\mathcal{T}\mathrm{A} \rightarrow \neg\mathcal{T}\neg\mathrm{A})$. It follows that $(\Xi,\gamma) \Vdash (\mathcal{T}\mathrm{A} \wedge \mathcal{T}\neg\mathrm{A})$. As a consequence, $(\Xi,\gamma) \Vdash \mathcal{T}\mathrm{A}$ and $(\Xi,\gamma) \Vdash \mathcal{T}\neg\mathrm{A}$. It follows that for some ordinal δ and any ordinal ϵ such that $\delta \leq \epsilon \prec \gamma$, $(\Xi,\epsilon) \Vdash \mathrm{A}$ and $(\Xi,\epsilon) \Vdash \neg\mathrm{A}$. But that is impossible. ∎

Warrant 6.5.2.3: The postulate's maxim somewhat extends (Bjørdal 2012). Let an ordinal δ be *monogamous* just if a successor ordinal, so $(\Xi,\delta) \Vdash \mathcal{T}\mathrm{B}$ just if $(\Xi,\delta) \Vdash \neg\mathcal{T}\neg\mathrm{B}$, for any sentence B. $\overset{\mathsf{M}}{\vDash}$ $(\mathcal{T}\mathrm{r} \in \mathrm{r} \vee \mathcal{T}\mathrm{r} \notin \mathrm{r}) \rightarrow (\mathcal{T}A \vee \mathcal{T}\neg A)$ holds simply because monogamous ordinals are monogamous ordinals. ∎

Warrant 6.5.2.4: Let an ordinal γ be reflected, just if $(\Xi,\gamma) \Vdash \mathcal{T}B$, provided $(\Xi,\gamma) \Vdash \mathcal{T}\neg\mathcal{T}\neg\mathrm{B}$. Any limit ordinal λ is reflected, for if B holds at all ordinals as from some ordinal μ below λ according to Ξ, then also $\neg\mathcal{T}\neg\mathrm{B}$ holds at all ordinals as from μ below λ according to Ξ. So limit ordinals are reflected, and successor ordinals are monogamous, in the sense of Postulate 6.5.2.3. The content of 6.5.2.4 is thus that all ordinals are reflected or monogamous, as for a monogamous successor ordinal δ, $((\Xi,\delta) \Vdash (\mathcal{T}\mathrm{A} \vee \mathcal{T}\neg\mathrm{A})$, and if δ is a reflected limit ordinal, $(\Xi,\delta) \Vdash (\mathcal{T}\neg\mathcal{T}\neg\mathrm{B} \rightarrow \mathcal{T}\mathrm{B})$. In either case, 6.5.2.4 is warranted. ∎

Warrant 6.5.2.5: Let an ordinal γ be *transitive* just if for any A,

$$\exists\theta(\theta \prec \gamma \;\&\; \Pi\xi(\theta \leq \xi \Rightarrow (\Xi,\xi) \Vdash \mathrm{A})) \Rightarrow \exists\theta(\theta \prec \gamma \;\&\; \Pi\xi(\theta \leq \xi \Rightarrow (\Xi,\xi) \Vdash \mathcal{T}\mathrm{A})).$$

Precisely limit ordinals are transitive ordinals.

The content of Postulate 6.5.2.5 is that ordinals are transitive, or monogamous, in the sense of Warrant 6.5.2.3. But that is true, as all ordinals larger than 0 are successor ordinals or limit ordinals. So 6.5.2.5 has been warranted. ∎

Warrant 6.5.2.6: At successor ordinals this holds, because there the consequent is true. Let λ be a limit ordinal, and ρ such that

$$\Pi\xi(\rho \leq \xi \prec \lambda) \Rightarrow (\Xi, \xi) \Vdash \mathcal{T}\mathrm{A} \rightarrow \mathrm{A},$$

so that $(\Xi, \lambda) \Vdash \mathcal{T}(\mathcal{T}\ulcorner\mathrm{A}\urcorner \rightarrow \mathrm{A})$. Suppose there is some ordinal $\sigma \prec \lambda$ and $\rho \leq \sigma$ such that $(\Xi, \sigma) \Vdash \mathrm{A}$. If so, $(\Xi, \lambda) \Vdash \mathcal{T}\mathrm{A}$. If there is no ordinal $\sigma \prec \lambda$ and $\rho \prec \sigma$ such that $(\Xi, \sigma) \Vdash \mathrm{A}$, then $(\Xi, \lambda) \Vdash \mathcal{T}\neg\mathrm{A}$. So 6.5.2.6 has been warranted. ∎

Warrant 6.5.2.7: Postulate 6.5.2.7 holds at all successor ordinals, as the consequent always holds there. $(\Xi, \lambda) \Vdash \mathcal{T}(\mathcal{T}\mathrm{A} \rightarrow \mathcal{T}\mathcal{T}A) \Rightarrow \Sigma\delta\Pi\epsilon(\delta \leq \epsilon \prec \lambda \Rightarrow (\Xi, \epsilon) \Vdash \mathcal{T}\mathrm{A} \rightarrow \mathcal{T}\mathcal{T}A)$ if λ is a limit ordinal. But all ordinals ϵ in the interval from and including δ and less than λ will have a successor $\epsilon+1$ which is also in the interval, so also $(\Xi, \epsilon+1) \Vdash \mathcal{T}\mathrm{A} \rightarrow \mathcal{T}\mathcal{T}A$. But the latter statement has the consequence that $(\Xi, \epsilon) \Vdash \mathrm{A} \rightarrow \mathcal{T}A$. So we have established that for any limit λ, $(\Xi, \lambda) \Vdash \mathcal{T}(\mathcal{T}\mathrm{A} \rightarrow \mathcal{T}\mathcal{T}A) \rightarrow \mathcal{T}(\mathrm{A} \rightarrow \mathcal{T}A)$. Given postulate 6.5.2.2 and contraposition, we obtain that $(\Xi, \lambda) \Vdash \mathcal{T}(\mathcal{T}\mathrm{A} \rightarrow \mathcal{T}\mathcal{T}A) \rightarrow \mathcal{T}(\mathcal{T}\neg\mathrm{A} \rightarrow \neg\mathrm{A})$. At this point is only takes postulate 6.5.2.6 to finish the warrant. ∎

Warrant 6.5.2.8: Suppose $(\Xi, \gamma) \Vdash \exists v\mathcal{T}\mathrm{A}$. On account of Definition 5.2.2.2, $(\Xi, \gamma) \Vdash \mathcal{T}\mathrm{A}^b_v$ for a b substitutable for v in A. So, on account of Definition 5.2.2.4 it follows that for an ordinal δ and any ordinal ϵ such that $\delta \leq \epsilon \prec \gamma, (\Xi, \epsilon) \Vdash \mathrm{A}^b_v$ for a b substitutable for v in A. So on account of Definition 5.2.2.2, again, for an ordinal δ and any ordinal ϵ such that $\delta \leq \epsilon \prec \gamma, (\Xi, \epsilon) \Vdash \exists v\mathrm{A}$. So on account of Definition 5.2.2.4, $(\Xi, \gamma) \Vdash \mathcal{T}\exists v\mathrm{A}$. ∎

Warrant 6.5.2.9: Let ordinal γ be such that $(\Xi, \gamma) \Vdash \mathcal{T}\forall v\mathrm{A}$. There is, consequently, an ordinal δ such that for any ordinal ϵ fulfilling $\delta \leq \epsilon \prec \gamma, (\Xi, \epsilon) \Vdash \forall v\mathrm{A}$. So either $\gamma = \delta + 1 = \epsilon + 1$ or γ is a limit ordinal such that $(\Xi, \epsilon) \Vdash \forall v\mathrm{A}$ for all ordinals ϵ such that $\delta \leq \epsilon \prec \gamma$. In either case, $(\Xi, \epsilon) \Vdash \forall v\mathrm{A}$ holds at any ϵ smaller than γ and at least as large as δ. It follows from Definition 5.2.2.2, that $(\Xi, \epsilon) \Vdash \mathrm{A}^b_v$, at any ϵ smaller than γ and at least as large as δ, for all b substitutable for v in A. So $(\Xi, \gamma) \Vdash \mathcal{T}\mathrm{A}^b_v$, for all b substitutable for v in A. So from Definition 5.2.2.2, again, $(\Xi, \gamma) \Vdash \forall v\mathcal{T}\mathrm{A}$. So $(\Xi, \beta) \Vdash \mathcal{T}\forall v\mathrm{A} \rightarrow \forall v\mathcal{T}\mathrm{A}$ holds at any ordinal β. So $(\Xi, \female) \Vdash \mathcal{T}(\mathcal{T}\forall v\mathrm{A} \rightarrow \forall v\mathcal{T}\mathrm{A})$, and consequently $\overset{\mathsf{M}}{\vDash} \mathcal{T}\forall v\mathrm{A} \rightarrow \forall v\mathcal{T}\mathrm{A}$. ∎

Warrant 6.5.2.10: The warrant is in the proof of Theorem 11.2.1.5. ∎

Warrant 6.5.2.11: – Notice that 6.5.2.11 is the *Barcan postulate* for orthodox formulas.

Assume

$$\overset{\mathsf{M}}{\nvDash} \mathfrak{D}(\{x|\mathrm{A}\}) \rightarrow (\forall x\mathcal{T}\mathrm{A} \rightarrow \mathcal{T}\forall x\mathrm{A}).$$

It follows, by Definitions 5.2.2 and 5.4.3, that for some fair function Ξ':

$$(\Xi', \female) \Vdash \neg\mathcal{T}(\mathfrak{D}(\{x|\mathrm{A}\}) \rightarrow (\forall x\mathcal{T}\mathrm{A} \rightarrow \mathcal{T}\forall x\mathrm{A})). \tag{6.6.1}$$

Definition 5.2.2.4 has the consequence:

$$\Pi\gamma(\gamma \prec \female \Rightarrow \Sigma\delta\big(\gamma \leq \delta \prec \female \ \& \quad (\Xi', \delta) \Vdash \mathfrak{D}(\{x|A\}) \ \wedge \forall x \mathcal{T} A \wedge \neg \mathcal{T} \forall x A\big)) \tag{6.6.2}$$

Case 1/2 - δ is a limit: Suppose

$$(\Xi', \delta) \Vdash \mathfrak{D}(\{x|A\}) \wedge \ \forall x \mathcal{T} A \wedge \neg \mathcal{T} \forall x A. \tag{6.6.3}$$

Then, for all constants c, and all ordinals ψ larger than a ξ smaller than δ,

$$(\Xi', \psi) \Vdash \mathfrak{D}(\{x|A\}) \wedge A_x^c,$$

so as well

$$(\Xi', \psi) \Vdash \mathfrak{D}(\{x|A\}) \wedge \forall x A.$$

Also, however,

$$(\Xi', \delta) \Vdash \neg \mathcal{T} \forall x A,$$

so that for some $\psi \leq \phi \leq \delta$,

$$(\Xi', \phi) \Vdash \neg A_x^c.$$

So

$$(\Xi', \delta) \Vdash \mathfrak{D}(\{x|A\}) \wedge \ \forall x \mathcal{T} A \wedge \neg \mathcal{T} \forall x A$$

cannot hold at a limit ordinal δ.

Case 2/2 - $\delta = \gamma + 1$ is a successor. Suppose

$$(\Xi', \delta) \Vdash \mathfrak{D}(\{x|A\}) \wedge \ \forall x \mathcal{T} A \wedge \neg \mathcal{T} \forall x A.$$

Then

$$(\Xi', \gamma) \Vdash \exists x \neg A,$$

so that there, by Theorem 5.2.6, is a constant c for which

$$(\Xi', \gamma) \Vdash \neg A_x^c.$$

However, as

$$(\Xi', \delta) \Vdash \forall x \mathcal{T} A,$$

also

$$(\Xi', \gamma) \Vdash A_x^c.$$

So

$$(\Xi', \delta) \Vdash \mathfrak{D}(\{x|\mathrm{A}\}) \wedge \forall x \mathcal{T} A \wedge \neg \mathcal{T} \forall x \mathrm{A}$$

cannot hold at a successor ordinal δ.

Cases 1/2 and 2/2 entail that for any ordinal β, $(\Xi, \beta) \Vdash \mathfrak{D}(\{x|\mathrm{A}\}) \rightarrow (\forall x \mathcal{T}\mathrm{A} \rightarrow \mathcal{T}\forall x\mathrm{A})$. So $(\Xi, ♀) \Vdash \mathcal{T}(\mathfrak{D}(\{x|\mathrm{A}\}) \rightarrow (\forall x \mathcal{T}\mathrm{A} \rightarrow \mathcal{T}\forall x\mathrm{A}))$, and so $\overset{\mathsf{M}}{\vDash} \mathfrak{D}(\{x|\mathrm{A}\}) \rightarrow (\forall x \mathcal{T}\mathrm{A} \rightarrow \mathcal{T}\forall x\mathrm{A})$. That warrants Postulate 6.5.2.11's posit of $\overset{\mathsf{M}}{\vdash} \mathfrak{D}(\{x|\mathrm{A}\}) \rightarrow (\forall x \mathcal{T}\mathrm{A} \rightarrow \mathcal{T}\forall x\mathrm{A})$. ∎

6.6.4. Remark: The semantic justification for some of the maxims of Postulates 6.5.2.1 – 6.5.2.11 can be lifted from (Bjørdal 2012)(340–341).

6.6.5. Remark: Postulates 6.5.2.6 and 6.5.2.7 originate with (Turner 1990).

6.6.6. Remark: The maxims of Postulates 6.5.2.7 and 6.5.2.8 were not included in (Bjørdal 2012), as the author thought they were both derivable. The warrant of Postulate 6.5.2.8 shows that this was correct for its maxim schema, but the warrant of Postulate 6.5.2.7 suggests that Postulate 6.5.2.3 is needed for its semantical justification.

6.6.7. Remark: Although the converses of Postulates 6.5.2.5 and 6.5.2.6 hold at limit ordinals, they are not maxims, for we may at a sucessor σ have that

$$(\Xi, \sigma) \Vdash (\mathcal{T}\neg A \vee \mathcal{T}A) \wedge \neg\mathcal{T}(\mathcal{T}A \rightarrow A),$$

and it happen for $\{x|x \notin x\} \in \{x|x \notin x\}$ at σ or $\sigma+1$. This contrasts with Remark 69.3.1.(ii) in (Cantini 1996)(396).

6.6.8. Exercise: Let A be *deferent* just if for all fairs Ξ, $(\Xi, ♀) \Vdash \mathcal{T}\mathrm{A}$ ⓡ $(\Xi, ♀) \Vdash \mathcal{T}\neg\mathrm{A}$. Show that just deferent formulas are orthodox.

6.6.9. Exercise: Prove that $\overset{\mathsf{O}}{\vDash} \forall\vec{v}(\mathcal{T}\mathrm{A} \vee \mathcal{T}\neg\mathrm{A}) \Rightarrow \overset{\mathsf{M}}{\vDash} \forall\vec{v}(\mathcal{T}\mathrm{A} \vee \mathcal{T}\neg\mathrm{A})$.

Remark on Exercise 6.6.9: Defining a formula A as orthodox just if $\overset{\mathsf{M}}{\vDash} \forall\vec{v}(\mathcal{T}\mathrm{A} \vee \mathcal{T}\neg\mathrm{A})$, instead of using Definition 5.6.1.1, is not advisable. For defining a formula as paradoxical just if not orthodox, as in Definition 5.6.4, would then induce an unacceptable extension for the term “paradoxical”.

7 Optima

Pax optima *rerum, quas homini novisse datum est: pax una triumphis innumeris potior: pax, custodire salutem et cives aequare potens.*

Silius Italicus

We have, as in Theorem 2 of (Bjørdal 2012, p. 342):

7.1. Postulate:

(1) $\overset{O}{\vDash} \mathcal{T}A \leftrightarrow \mathcal{T}\mathcal{T}A$

(2) $\overset{O}{\vDash} \mathcal{T}\neg\mathcal{T}\neg A \leftrightarrow \mathcal{T}A$

(3) $\overset{O}{\vDash} \mathcal{T}(\mathcal{T}A \rightarrow \mathcal{T}B) \rightarrow \mathcal{T}(A \rightarrow B)$

(4) $\overset{O}{\vDash} \mathcal{T}(A \rightarrow \mathcal{T}A) \leftrightarrow \mathcal{T}(\mathcal{T}A \rightarrow A)$

(5) $\overset{O}{\vDash} \forall x\mathcal{T}A(x) \rightarrow \mathcal{T}\forall xA(x)$

(6) $\overset{O}{\vDash} \mathcal{T}\ulcorner A\urcorner \rightarrow A$

It is left as an exercise to warrant the optimal tautologies.

8 Plains

Pure mathematics is, in its way, the poetry of logical ideas.

Albert Einstein

8.1. Postulate Plains:

(1) $\vdash \mathcal{T}\ulcorner A \urcorner \to A$.

(2) $\vdash A \to \mathcal{T}A$.

(3) $\vdash \mathcal{T}\exists v A \to \exists v \mathcal{T}A$.

(4) $\vdash \forall v \mathcal{T}A \to \mathcal{T}\forall v A$.

(5) $\mathrm{r} \in \mathrm{r}$.

(6) $\mathrm{r} \notin \mathrm{r}$.

8.2. Remark: Instances of the plains in Postulates 8.1.1 and 8.1.2 may be maxims or minors. There are minor instances of Postulates 8.1.1 and 8.1.2 on account of Russell's paradoxical set, here denoted as in 8.1.5 and 8.1.6 by the r of Definition 6.5.1 on page 22. In the case of the *attestor* schema of Postulate 8.1.3, the failure of some maximal versions follow from Corollary 14.1.5. The failure of the maximality for all instances of Postulate 8.1.4 is shown in §14.3.

8.3. Exercise: As regards Postulates 8.1.5 and 8.1.6, prove that £ has $\vdash \mathrm{r} \in \mathrm{r}$ and $\vdash \mathrm{r} \notin \mathrm{r}$.

9 Inference modes

Recall that the valid inference from A *to* $\mathcal{T}\ulcorner\mathrm{A}\urcorner$ *is the number theoretic and meta theoretic principle that for all fairs* Ξ*, if* A *is in the variety of formula numbers* $\vdash(\Xi, ♀)$*, then also* $\mathcal{T}\ulcorner\mathrm{A}\urcorner$ *is in* $\vdash(\Xi, ♀)$*.*

9.1 The simple inference modes

Only $\neg$, $\mathcal{T}$ and one occurence of a formula variable are allowed in the formulas in the antecedent and the consequent of the simple inference modes. Moreover, $\mathcal{T}$ may only occur once in the antecedent, and in the consequent.

9.1.1. Postulate simple thetical inference modes:

1 $\vdash A \Rightarrow \vdash \mathcal{T}A$

2 $\vdash A \Rightarrow \vdash \neg\mathcal{T}\neg A$

3 $\vdash \neg A \Rightarrow \vdash \mathcal{T}\neg A$

4 $\vdash \neg A \Rightarrow \vdash \neg\mathcal{T}A$

5 $\vdash \mathcal{T}A \Rightarrow \vdash A$

6 $\vdash \mathcal{T}A \Rightarrow \vdash \neg\mathcal{T}\neg A$

7 $\vdash \mathcal{T}\neg A \Rightarrow \vdash \neg A$

8 $\vdash \mathcal{T}\neg A \Rightarrow \vdash \neg\mathcal{T}A$

9 $\vdash \neg\mathcal{T}A \Rightarrow \vdash \neg A$

10 $\vdash \neg\mathcal{T}A \Rightarrow \vdash \mathcal{T}\neg A$

11 $\vdash \neg\mathcal{T}\neg A \Rightarrow \vdash A$

12 $\vdash \neg\mathcal{T}\neg A \Rightarrow \vdash \mathcal{T}A$

The corresponding valid, simple *maximal* inference modes of Postulate 9.1.2 can be justified by the valid simple *thetical* inference modes in Postulate 9.1.1 on account of the syntactical correlate of Theorem 5.5.4.4, which says that $\vdash^{\mathsf{M}} A$ just if $\vdash A$ & $\nvdash \neg A$. The inference mode of Postulate 9.1.2.1 is for example a consequence of the conjunction of the modes provided by Postulates 9.1.1.1 and 9.1.1.9. The other dependencies are straightforward to establish.

9.1.2. Postulate simple maximal inference modes:

1 $\vdash^{\mathsf{M}} A \Rightarrow \vdash^{\mathsf{M}} \mathcal{T}A$

2 $\vdash^{\mathsf{M}} A \Rightarrow \vdash^{\mathsf{M}} \neg\mathcal{T}\neg A$

3 $\vdash^{\mathsf{M}} \neg A \Rightarrow \vdash^{\mathsf{M}} \mathcal{T}\neg A$

4 $\vdash^{\mathsf{M}} \neg A \Rightarrow \vdash^{\mathsf{M}} \neg\mathcal{T}A$

5 $\vdash^{\mathsf{M}} \mathcal{T}A \Rightarrow \vdash^{\mathsf{M}} A$

6 $\vdash^{\mathsf{M}} \mathcal{T}A \Rightarrow \vdash^{\mathsf{M}} \neg\mathcal{T}\neg A$

7 $\vdash^{\mathsf{M}} \mathcal{T}\neg A \Rightarrow \vdash^{\mathsf{M}} \neg A$

8 $\vdash^{\mathsf{M}} \mathcal{T}\neg A \Rightarrow \vdash^{\mathsf{M}} \neg\mathcal{T}A$

9 $\vdash^{\mathsf{M}} \neg\mathcal{T}A \Rightarrow \vdash^{\mathsf{M}} \neg A$

10 $\vdash^{\mathsf{M}} \neg\mathcal{T}A \Rightarrow \vdash^{\mathsf{M}} \mathcal{T}\neg A$

11 $\vdash^{\mathsf{M}} \neg\mathcal{T}\neg A \Rightarrow \vdash^{\mathsf{M}} A$

12 $\vdash^{\mathsf{M}} \neg\mathcal{T}\neg A \Rightarrow \vdash^{\mathsf{M}} \mathcal{T}A$

9.2 Involved inference modes

9.2.1. Postulate Quantificational thetical modes:

1 $\vdash \forall v\mathcal{T}A \Rightarrow \vdash \mathcal{T}\forall vA$

2 $\vdash \neg\mathcal{T}\forall vA \Rightarrow \vdash \neg\forall v\mathcal{T}A$

3 $\vdash \mathcal{T}\exists vA \Rightarrow \vdash \exists v\mathcal{T}A$

9.2.2. Postulate The Barcan mode: Postulates 9.2.1.1 and 9.2.1.2 justify

$$\vdash^{\mathsf{M}} \forall v\mathcal{T}A \Rightarrow \vdash^{\mathsf{M}} \mathcal{T}\forall vA.$$

9.2.3. Remark: Quantificational thetical mode 9.2.1.3 does not enter such a combination as do 9.2.1.1 and 9.2.1.2, for

$$\vdash \neg\exists v\mathcal{T}\mathrm{A} \Rightarrow \vdash \neg\mathcal{T}\exists v\mathrm{A}$$

is *not* a valid mode schema; so neither is $\vdash^{\mathsf{M}} \mathcal{T}\exists v\mathrm{A} \Rightarrow \vdash^{\mathsf{M}} \exists v\mathcal{T}\mathrm{A}$. This is clarified in the limitative results of Theorem 14.1.3, its Corollary 14.1.5, and Theorem 14.1.5 in §14.1.

9.2.4. Postulate Thetical distributive modes:

1 $\vdash^{\mathsf{M}} (A \rightarrow B) \Rightarrow (\vdash A \Rightarrow \vdash B)$.

2 $\vdash^{\mathsf{M}} (A \rightarrow B) \Rightarrow (\vdash \neg B \Rightarrow \vdash \neg A)$.

3 $\vdash (A \rightarrow B) \Rightarrow (\vdash^{\mathsf{M}} A \Rightarrow \vdash B)$.

9.2.5. Postulate The maxim mode:

$$\vdash^{\mathsf{M}} (A \rightarrow B) \Rightarrow (\vdash^{\mathsf{M}} A \Rightarrow \vdash^{\mathsf{M}} B).$$

9.2.6. Remark:

Postulate 9.2.5 is entailed by Postulates 9.2.4.1 and 9.2.4.2.

9.2.7. Postulate Complex modes:

1 $\vdash^{\mathsf{M}} \mathcal{T}A \Rightarrow \vdash^{\mathsf{M}} \mathcal{T}(\mathcal{T}A \leftrightarrow A) \wedge \mathcal{T}(\mathcal{T}\neg A \leftrightarrow \neg A)$ (The Tarski mode)

2 $\vdash^{\mathsf{M}} \mathcal{T}A \rightarrow A \Rightarrow \vdash^{\mathsf{M}} \mathcal{T}A \vee \mathcal{T}\neg A$

3 $\vdash^{\mathsf{M}} \mathcal{T}\neg\mathcal{T}\neg A \Rightarrow \vdash^{\mathsf{M}} \mathcal{T}A$

4 $\vdash^{\mathsf{M}} \mathcal{T}(\mathcal{T}A \rightarrow \mathcal{T}B) \Rightarrow \vdash^{\mathsf{M}} \mathcal{T}(A \rightarrow B)$

5 $\vdash A \;\&\; \vdash B \Rightarrow \vdash \neg\mathcal{T}\neg A \wedge \neg\mathcal{T}\neg B$

6 $\vdash^{\mathsf{M}} \mathfrak{O}(A(x)) \Rightarrow (\vdash^{\mathsf{M}} \exists x A \Rightarrow \vdash^{\mathsf{M}} A^a_x$ for some a substitutable for x in A).

7 $\vdash^{\mathsf{M}} A^a_v$ for any constant $a \Rightarrow \vdash^{\mathsf{M}} \forall v A$

Warrant 9.2.7.1: Clearly $\vdash^{\mathsf{M}} \mathcal{T}A \Rightarrow \vdash^{\mathsf{M}} \mathcal{T}(A \wedge \mathcal{T}A) \wedge \mathcal{T}(A \wedge \neg\mathcal{T}\neg A)$. It is librationistically derivable that $\vdash^{\mathsf{M}} \mathcal{T}((A \wedge \mathcal{T}A) \rightarrow \mathcal{T}(A \leftrightarrow \mathcal{T}A))$ and $\vdash^{\mathsf{M}} \mathcal{T}((A \wedge \neg T\neg A) \rightarrow \mathcal{T}(\neg A \leftrightarrow \mathcal{T}\neg A))$, so Postulate 9.2.5 suffices to finish. ∎

Proof: (9.2.7.4) Suppose $(\Xi, \text{♀}) \Vdash \mathcal{T}(\mathcal{T}A \rightarrow \mathcal{T}B)$. (i) Let ρ be be a ordinal as from which $\mathcal{T}A \rightarrow \mathcal{T}B$ holds, so that

$$\Pi\xi(\rho \leq \xi \prec \text{♀} \Rightarrow (\Xi, \xi) \Vdash (\mathcal{T}A \rightarrow \mathcal{T}B).$$

Thus $(\Xi, \rho+1) \Vdash (\mathcal{T}A \rightarrow \mathcal{T}B)$, and therefore $(\Xi, \rho) \Vdash (A \rightarrow B)$. Consequently, succeeding successors will have $\mathcal{T}A \rightarrow \mathcal{T}B$ and $A \rightarrow B$. (ii) Let limit ordinal $\lambda \prec \text{♀}$, above ρ, have $\mathcal{T}A \rightarrow \mathcal{T}B$, and $A \rightarrow B$ below, as from ρ. As $\lambda \prec \text{♀}$, from the assumption on ρ, $(\Xi, \lambda) \Vdash (\mathcal{T}A \rightarrow \mathcal{T}B)$. As $(\Xi, \lambda+1) \Vdash (\mathcal{T}A \rightarrow \mathcal{T}B)$, also $(\Xi, \lambda) \Vdash (A \rightarrow B)$. (iii) By a repetition of (i) and (ii) it follows that $A \rightarrow B$ holds as from ρ below ♀, so that $(\Xi, \text{♀}) \Vdash \mathcal{T}(A \rightarrow B)$. □

Proof: (9.2.7.6) This is established on page 47, in the proof of Theorem 14.2.1. □

10 Alethic comprehension

If one, per impossibile, *could have used naive comprehension for truth, and for abstraction, the alethic comprehension principle would have been true. Fortunately, one cannot justify the opposite entailment from alethic comprehension to naive comprehension.*

10.1. Postulate Alethic comprehension without parameters:

$$\vdash^{\mathsf{M}} \forall x(x \in \{y|\mathrm{A}\} \leftrightarrow \mathcal{T}\ulcorner \mathrm{A} \urcorner^{x}_{y}),$$ where x is substitutable for y in A.

10.2. Theorem Alethic comprehension with parameters from $\vec{v}$:

$$\vdash^{\mathsf{M}} \forall \vec{v} \forall x(x \in \{y|\mathrm{A}\} \leftrightarrow \mathcal{T}\ulcorner \mathrm{A} \urcorner^{x}_{y}),$$ where x is substitutable for y in A.

Proof: Appeal to 9.2.7.7 and Postulate 10.1. □

11 The theory of identity

To be is that there is something in all your essences.

A streamlining of sections 4 and 5 of (Bjørdal 2012, pp. 342–345) is obtained from the inference modes 9.2.7.1 – 9.2.7.4, and as a result £ does not, as e.g. the comparable systems studied by (Cantini 1996), need additional axiomatic principles for having a well behaved notion of identity in this section, or natural number in §13.

11.1 Co-essentiality

In a lasting contribution (Whitehead and Russell 1927) improves upon *Leibniz' law*, as a bi-conditional corresponding with Definition 11.1.2 is Principa Mathematica's theorem *13.101, proven via its predicative Definition *13.1 and its *Axiom of Reducibility* *12.1.

It bears mentioning that the second Principia author published the thorough monograph (Russell 1900) on Leibniz, though this does not establish that he contributed theorem *13.101. For a cursory reading of that historical treatise suggests that Russell did not make such discoveries while writing that text.

We define the identity relation by means of a notion of *co-essentiality*, which is similar to the relation named *membership congruency* by Abraham A. Fraenkel and Yehoshua Bar-Hillel, and discussed in (A. A. Fraenkel and Bar-Hillel 1973, p. 27), though not used in the previous edition (A. A. Fraenkel and Bar-Hillel 1958).

11.1.1. Definition: Sets a and b are *co-essential* just if $\forall u(a \in u \rightarrow b \in u)$.

The term "co-essentiality" is coined from (Forster 2019), which relates that (Hailperin 1944) "gave the first of a number of finite axiomatisations of NF now known. Many of them exploit the function $x \mapsto \{y|y \in x\}$ which is injective and total and is an $\in$ -isomorphism. This function was known to Whitehead, who suggested to Quine that $\{y|x \in y\}$ should be called the "essence" of x (a terminology clearly suggested by a view of sets as properties-in-extension)." Incidentally, Quine was Whitehead's student while doing his doctorate at Harvard, but Quine obtained his doctorate twelve years before the publication of (Hailperin 1944).

11.1.2. Definition Identity via co-essentiality:

$$a = b \;=\!=\; \forall u(a \in u \rightarrow b \in u)$$

Notice that the *definiens* in Definition 11.1.2 is a conditional, and not a biconditional.

The justification for the analogous definition $*13 \cdot 01$ in Principia Mathematica, will not justify Definition 11.1.2. For the symmetry of Definition 11.1.2, is in £ shown by the proof of 11.2.1.4 as follows, and it does not appeal to *predicativity* or the Axiom of reducibility, as in the proof of $*13{\cdot}01$ by (Whitehead and Russell 1927).

11.1.3. Lemma: $\vdash^{\mathsf{M}} \mathcal{T}(\forall u(a \in u \to b \in u) \to \mathcal{T}\forall u(a \in u \to b \in u))$.

Proof: Suppose $\vdash^{\mathsf{M}} \forall u(a \in u \to b \in u)$. By instantiation we have:

$$\vdash^{\mathsf{M}} \forall u(a \in u \to b \in u) \to \\ (a \in \{v|\forall u(a \in u \to v \in u)\} \to b \in \{v|\forall u(a \in u \to v \in u)\}).$$

But $\vdash^{\mathsf{M}} a \in \{v|\forall u(a \in u \to v \in u)\}$, so that

$$\vdash^{\mathsf{M}} \forall u(a \in u \to b \in u) \to b \in \{v|\forall u(a \in u \to v \in u)\}.$$

Finish with Alethic Comprehension and Postulate 9.1.2.1. □

11.1.4. Lemma: $$\vdash^{\mathsf{M}} \mathcal{T}(a = b \to \mathcal{T}a = b)$$

Proof: Use Definition 11.1.2 and Lemma 11.1.3. □

11.1.5. Lemma: $$\mathcal{T}(\mathcal{T}\ulcorner a \neq b \urcorner \to a \neq b)$$

Proof: Use Lemma 11.1.4, Postulates 6.5.2.2, 9.1.2.1 and logic. □

11.1.6. Lemma:

$$\vdash^{\mathsf{M}} \mathcal{T}(\mathcal{T}\ulcorner\neg\forall u(a \in u \to b \in u)\urcorner \to \neg\forall u(a \in u \to b \in u))$$

Proof: Use Lemma 11.1.5 and Definition 11.1.2. □

11.1.7. Lemma:

$$\vdash^{\mathsf{M}} \mathcal{T}((\mathcal{T}\forall u(a \in u \to b \in u)) \to \forall u(a \in u \to b \in u))$$

Proof: Combine Lemma 11.1.6 with Postulate 7.1.4 to obtain

$$\vdash^{\mathsf{M}} \mathcal{T}(\neg\forall u(a \in u \to b \in u) \to \mathcal{T}\neg\forall u(a \in u \to b \in u)).$$

An instance of Theorem 6.5.2.2 is

$$\vdash^{\mathsf{M}} \mathcal{T}(\mathcal{T}\neg\forall u(a \in u \to b \in u) \to \neg\mathcal{T}\forall u(a \in u \to b \in u)).$$

A hypothetical syllogism and contraposition now suffices to finish the proof. □

11.2 The adequacy of identity as co-essentiality

11.2.1. Theorem (Orthodoxy, equivalence and fungibility)

(1) $\vdash^{\mathsf{M}} \mathcal{T} a = b \vee \mathcal{T} a \neq b$ Orthodoxy

(2) $\vdash^{\mathsf{M}} a = a$ Reflexivity

(3) $\vdash^{\mathsf{M}} a = b \wedge b = c \rightarrow a = c$ Transitivity

(4) $\vdash^{\mathsf{M}} a = b \rightarrow b = a$ Symmetry

(5) $\vdash^{\mathsf{M}} a = b \rightarrow (A^{a}_{v} \rightarrow A^{b}_{v})$, with a and b substitutable for v in A. Fungibility

Proof:

1. Use Lemma 11.1.5 and Postulate 6.5.2.6.
2. Trivial
3. Trivial, given Definition 11.1.2
4. Clearly,

   $$\vdash^{\mathsf{M}} \forall v(a \in v \rightarrow b \in v) \rightarrow (a \in \{w|\forall v(w \in v \rightarrow a \in v)\} \rightarrow (b \in \{w|\forall v(w \in v \rightarrow a \in v)\}.$$

   But
   $$\vdash^{\mathsf{M}} a \in \{w|\forall v(w \in v \rightarrow a \in v)\},$$

   so that by alethic comprehension,

   $$\vdash^{\mathsf{M}} \forall v(a \in v \rightarrow b \in v) \rightarrow \mathcal{T}\forall v(b \in v \rightarrow a \in v). \tag{11.2.2}$$

   An instance of Lemma 11.1.4 states:

   $$\vdash^{\mathsf{M}} \forall v(b \in v \rightarrow a \in v) \rightarrow \mathcal{T}\forall v(b \in v \rightarrow a \in v). \tag{11.2.3}$$

   By invoking 7.1.4 on equation 11.2.3 we obtain

   $$\vdash^{\mathsf{M}} \mathcal{T}\ulcorner\forall v(b \in v \rightarrow a \in v)\urcorner \rightarrow \forall v(b \in v \rightarrow a \in v). \tag{11.2.4}$$

   Finish with a hypothetical syllogism with equations 11.2.2 and 11.2.4, and lastly an appeal to co-essentiality Definition 11.1.2.

5. The promissory note issued in sentence Warrant 6.3.W10 of Postulate 6.5.2.10 on page 24 of §6 is satisfied, and the mentioned Postulate is warranted.

   Suppose, for a and b substitutable for v in A, and fair function Ξ,

   $$(\Xi, \female) \nVdash \mathcal{T}(\forall u(a \in u \rightarrow b \in u) \rightarrow (A^{a}_{v} \rightarrow A^{b}_{v})).$$

On account of the validity of the mode of 9.2.7.3 we get

$$(\Xi, ♀) \nVdash \mathcal{T}\neg\mathcal{T}(\forall u(a \in u \to b \in u) \wedge A^a_v \wedge \neg A^b_v).$$

It follows from Definition 5.2.2.1 that

$$(\Xi, ♀) \Vdash \neg\mathcal{T}\neg\mathcal{T}(\forall u(a \in u \to b \in u) \wedge A^a_v \wedge \neg A^b_v).$$

On account of Postulate 6.5.2.1,

$$(\Xi, ♀) \Vdash \neg\mathcal{T}\neg(\mathcal{T}\forall u(a \in u \to b \in u) \wedge \mathcal{T}A^a_v \wedge \mathcal{T}\neg A^b_v).$$

On account of the tautologicality of Lemma 11.1.7, we get

$$(\Xi, ♀) \Vdash \neg\mathcal{T}\neg(\forall u(a \in u \to b \in u) \wedge \mathcal{T}A^a_v \wedge \neg\mathcal{T}A^b_v).$$

From alethic comprehension and existential generalization we obtain

$$(\Xi, ♀) \Vdash \neg\mathcal{T}\neg(\forall u(a \in u \to b \in u) \wedge \exists u(a \in u \wedge b \notin u)),$$

which is absurd. So Postulate 6.5.2.10 is tautological, and we are done. □

12 Alphabetologicality

That the universe was formed by a fortuitous concourse of atoms, I will no more believe than that the accidental jumbling of the alphabet would fall into a most ingenious treatise of philosophy.

Jonathan Swift

Postulates 12.1 and 12.2 express, given Definition 12.3, that identity is an equivalence relation which is neutral with respect to alphabetological variants.

12.1. Postulate The Lindenbaum-Tarski closure for identity: If classical logic proves that $\forall x(A(x) \leftrightarrow B(x))$, then

$$\vdash^{\mathsf{M}} \{x|A(x)\} = \{x|B(x)\}.$$

12.2. Postulate Alphabetical indifference:

$$\{x|A(x)\} = \{x|B(x)\} \rightarrow \{x|A(x)\} = \{y|B(x)^y_x\},$$

where y is substitutable for x in B.

12.3. Definition Alphabetologicality: Two sets are *alphabetological variants* of each other just if they are identical on account of Postulates 12.1 and 12.2.

Postulates 12.1 and 12.2 compensate somewhat for the loss of extensionality in £, as per §21, and secure such theorems as:

$$\vdash^{\mathsf{M}} \{x|A(x)\} = \{y|A(y) \wedge \exists z(B(z) \vee \neg B(z))\}.$$

13 Arithmetic

The numbers may be said to rule the whole world of quantity, and the four rules of arithmetic may be regarded as the complete equipment of the mathematician.

James C. Maxwell

13.1. Definition

(1) $\varnothing = \{x | x \neq x\}$

(2) $a' = \{x | x = a \vee x \in a\}$

(3) $\omega = \{x | \forall y(\varnothing \in y \wedge \forall z(z \in y \to z' \in y) \to x \in y)\}$

13.2. Theorem

(1) $\vdash^{\mathsf{M}} \varnothing \in \omega$

(2) $\vdash^{\mathsf{M}} \forall x(x \in \omega \to x' \in \omega)$

(3) ω is orthodox

(4) $\vdash^{\mathsf{M}} \forall y(\varnothing \in y \wedge \forall z(z \in y \to z' \in y) \to \forall x(x \in \omega \to x \in y))$

(5) $\vdash^{\mathsf{M}} \mathrm{A}(\varnothing) \wedge \forall x(\mathrm{A}(x) \to \mathrm{A}(x')) \to \forall y(y \in \omega \to \mathrm{A}(y))$

Proof:

1. Combine alethic comprehension and the fact that

$$\vdash^{\mathsf{M}} \mathcal{T}\forall y(\varnothing \in y \wedge \forall z(z \in y \to z' \in y) \to \varnothing \in y)$$

2. This follows from alethic comprehension and

$$\vdash^{\mathsf{M}} \forall x(\mathcal{T}(\forall y(\varnothing \in y \wedge \forall z(z \in y \to z' \in y) \to x \in y)) \to \mathcal{T}(\forall y(\varnothing \in y \wedge \forall z(z \in y \to z' \in y) \to x' \in y))).$$

3. From logic:

$$\vdash^{\mathsf{M}} \varnothing \in \omega \wedge \forall x(x \in \omega \to x' \in \omega) \to (\forall y(\varnothing \in y \wedge \forall x(x \in y \to x' \in y) \to a \in y) \to a \in \omega).$$

By combining 1 and 2 we have

$$\vdash^{\mathsf{M}} \forall y(\varnothing \in y \wedge \forall x(x \in y \to x' \in y) \to a \in y) \to a \in \omega).$$

Postulates 6.5.2.1 and 9.1.2.1, and alethic comprehension, give us

$$\vdash^{\mathsf{M}} a \in \omega \to \mathcal{T}a \in \omega.$$

9.2.5 along with Postulates 6.5.2.1, 6.5.2.2 and 6.5.2.6 give us

$$\vdash^{\mathsf{M}} \mathcal{T}a \in \omega \vee \mathcal{T}a \notin \omega$$

As a was arbitrary, $\vdash^{\mathsf{M}} \forall x(\mathcal{T}x \in \omega \vee \mathcal{T}x \notin \omega)$, and the proof is finished.

4. Immediate, given 3, as it is equivalent with

$$\vdash^{\mathsf{M}} \forall x(x \in \omega \to \forall y(\varnothing \in y \wedge \forall z(z \in y \to z' \in y) \to x \in y)).$$

5. For the following, compare (Cantini 1996, p. 356).

13.3. Definition: Let, for arbitrary sentence $\mathrm{A}(x)$,

$$\mathrm{A}'(x) =\!= \mathrm{A}(\varnothing) \wedge \forall x(\mathrm{A}(x) \to \mathrm{A}(x')) \to \mathrm{A}(x)).$$

By logic,

$$\vdash^{\mathsf{M}} \mathrm{A}'(\varnothing) \;\&\; \vdash^{\mathsf{M}} \forall x(\mathrm{A}'(x) \to \mathrm{A}'(x')).$$

The inference mode of Postulate 9.1.2.1 and Postulate 6.5.2.10 entail

$$\vdash^{\mathsf{M}} \mathcal{T}\mathrm{A}'(\varnothing) \;\&\; \vdash^{\mathsf{M}} \forall x\mathcal{T}(\mathrm{A}'(x) \to \mathrm{A}'(x')).$$

By quantifier distribution and Postulate 6.5.2.1 we get

$$\vdash^{\mathsf{M}} \mathcal{T}\mathrm{A}'(\varnothing) \;\&\; \vdash^{\mathsf{M}} \forall x(\mathcal{T}\mathrm{A}'(x) \to \mathcal{T}\mathrm{A}'(x')).$$

Alethic comprehension gives us

$$\vdash^{\mathsf{M}} \varnothing \in \{y|\mathrm{A}'(y)\} \;\&\; \vdash^{\mathsf{M}} \forall x(x \in \{y|\mathrm{A}'(y)\} \to x' \in \{y|\mathrm{A}'(y)\}).$$

Adjunction gives us

$$\vdash^{\mathsf{M}} \varnothing \in \{y|\mathrm{A}'(y)\} \wedge \forall x(x \in \{y|\mathrm{A}'(y)\} \to x' \in \{y|\mathrm{A}'(y)\}).$$

4 and the inference of mode 9.2.5 give us

$$\vdash^{\mathsf{M}} \forall x(x \in \omega \to x \in \{y|\mathrm{A}'(y)\}).$$

From 3 and 9.2.7.1 we have

$$\vdash^{\mathsf{M}} \forall x(\mathcal{T}x \in \omega \to x \in \omega),$$

so that

$$\vdash^{\mathsf{M}} \forall x(\mathcal{T}x \in \omega \to x \in \{y|\mathrm{A}'(y)\}).$$

Alethic comprehension gives us

$$\vdash^{\mathsf{M}} \forall x(\mathcal{T}x \in \omega \to \mathcal{T}\mathrm{A}'(x)),$$

which, combined with 9.2.7.4 establishes

$$\vdash^{\mathsf{M}} \forall x(x \in \omega \to \mathrm{A}'(x))$$

Finish with an appeal to Definition 13.3, and rearrangement. □

14 Shortcomings and redresses

If all problems seem resolved, look in another direction!

§6 is supplemented with negative results, which to a large degree depend upon §11.

14.1 Shortcoming related to existential instantiation

Despite the important Theorem 5.2.6, which justifies

14.1.1. Theorem

$$(\Xi,\alpha) \Vdash \exists x(\neg\mathcal{T}\mathrm{A} \wedge \neg\mathcal{T}\neg\mathrm{A}) \xRightarrow{\text{for some term } a} (\Xi,\alpha) \Vdash (\neg\mathcal{T}\mathrm{A}^a_x \wedge \neg\mathcal{T}\neg\mathrm{A}^a_x),$$

and consequently

14.1.2. Theorem Optimal existential instantiation:

$$\text{If } \vdash^{\mathsf{O}} \exists xA, \text{then } \vdash^{\mathsf{O}} A^a_x \text{ for some term } a$$

There is, nevertheless, as pointed to in Remark 9.2.3, the following limitative result:

14.1.3. Theorem (Maximal lack of existential instantiation)

$$\text{It may happen that} \vdash^{\mathsf{M}} \exists xA, \text{ and for no term } a, \vdash^{\mathsf{M}} A^a_x.$$

Proof: As the proof of Theorem 14.1.5. □

14.1.4. Corollary: Maximal existential instantiation, in the form

$$\vdash^{\mathsf{M}} \exists xA \Rightarrow \Sigma a\big(\vdash^{\mathsf{M}} A^a_x\big), \text{ is not valid.}$$

14.1.5. Theorem: The inference mode $\vdash^{\mathsf{M}} \mathcal{T}\exists xA \Rightarrow \vdash^{\mathsf{M}} \exists x\mathcal{T}A$ is not valid.

Proof: Let A be $(x = \varnothing \leftrightarrow \mathrm{r} \in \mathrm{r})$. Obviously, $\overset{\mathsf{M}}{\vDash} \mathcal{T}\exists x\mathrm{A}$ holds. Suppose that $\overset{\mathsf{M}}{\vDash} \exists x\mathcal{T}\mathrm{A}$. If so $(\Xi, ♀) \Vdash \mathcal{T}\exists x\mathcal{T}\mathrm{A}$, and there is an ordinal γ such that $(\Xi,\beta) \Vdash \exists x\mathcal{T}A$ holds whenever $\gamma \prec \beta \prec ♀$. Let limit ordinal λ satisfy $\gamma \prec \lambda \prec ♀$, so that $(\Xi,\lambda) \Vdash \exists x\mathcal{T}A$. On account of 5.2.2.1 and 5.2.2.2, there is a term a and an ordinal δ such that $a = \varnothing \leftrightarrow \mathrm{r} \in \mathrm{r}$ holds at all ordinals θ which satisfy $\delta \prec \theta \prec \lambda$. But this is impossible, as $r \in r$ holds at some of those ordinals, and $\mathrm{r} \notin \mathrm{r}$ holds at others, whereas identity is orthodox. □

As stated in Remark 8.2, Theorem 14.1.5 entails that the attestor schema of Postulate 8.1.3 does not hold as a maxim, for, as its proof just showed, some instances of the schema $\mathcal{T}\exists vA \rightarrow \exists v\mathcal{T}A$ are minor, i.e. paradoxical, truths.

14.2 An orthodox redress

14.2.1. Theorem The validity of 9.2.7.6 is shown, as announced on page 33:

$$\vdash^{\mathsf{M}} \mathfrak{D}(A(x)) \Rightarrow (\vdash^{\mathsf{M}} \exists x A \Rightarrow \vdash^{\mathsf{M}} A^a_x, \text{ for some } a \text{ substitutable for } x \text{ in A}).$$

Proof:

$$\text{Assume that } A(x) \text{ is orthodox, i.e. } \vdash^{\mathsf{M}} \mathcal{T}A(x) \vee \mathcal{T}\neg A(x). \tag{14.2.2}$$

By soundness,

$$\vdash^{\mathsf{M}} \exists x A \Rightarrow \models^{\mathsf{M}} \exists x A, \text{so for all fair functions } \Xi, (\Xi, \text{♀}) \Vdash \mathcal{T}\exists x A. \tag{14.2.3}$$

$$\text{As ♀ is a stabilising ordinal, } (\Xi, \text{♀}) \Vdash \exists x A. \tag{14.2.4}$$

$$\text{Given Definition 5.2.2 and Theorem 5.2.6, for a } a, (\Xi, \text{♀}) \Vdash A^a_x. \tag{14.2.5}$$

$$\text{As } A(x) \text{ is orthodox, } (\Xi, \text{♀}) \Vdash \mathcal{T}A^a_x. \tag{14.2.6}$$

$$\text{So } \models^{\mathsf{M}} A^a_x. \tag{14.2.7}$$

$$\text{So 9.2.7.6 is valid.} \tag{14.2.8}$$

□

14.3 The Barcan failure

As mentioned in Remark 8.2, it will be shown that the Barcan schema, in Postulate 8.1.4, does not hold as a maxim, but only as a thesis.

The precursor to this negative result, in a truth theoretic context, is *McGee's paradox*, in (McGee 1985), which we adapt to our context. Compare (Cantini 1996, pp. 380–382) and (Bjørdal 2012, p. 537).

First we decide upon some notions:

14.3.1. Definition: For r in 14.3.1.5, recall Definition 6.5.1:

(1) $a' \Longleftarrow \{x|x \in a \vee x = a\}$.

(2) $\{a, b\} \Longleftarrow \{x|x \in a \vee x \in b\}$.

(3) $\{a\} \Longleftarrow \{a, a\}$.

(4) $a_\omega \Longleftarrow \{u|\forall x(\langle\varnothing, a\rangle \in x \wedge \forall y, z(\langle y, z\rangle \in x \to \langle y', \{v|v \in z\}\rangle) \to u \in x)\}$.

(5) $\mathrm{t} \Longleftarrow \{x|x = r \wedge x \notin x \wedge \neg\mathcal{T}x \in x\}$.

(6) Use $\overline{0}$, $\overline{1}$, $\overline{2}$, …for the members of ω.

(7) Let $\mathrm{t}_{\overline{0}} \Longleftarrow \mathrm{t}$ and $\mathrm{t}_{\overline{n+1}} \Longleftarrow \{v|v \in \mathrm{t}_{\overline{n}}\}$.

(8) $\mathrm{B}(\mathrm{t}_{\bar{i}}) =\!= \exists w(\langle w, \mathrm{t}_{\bar{i}}\rangle \in \mathrm{t}_\omega) \to r \notin \mathrm{t}_{\bar{i}}$

(9) $\mathrm{B}(x) =\!= \exists w(\langle w, x\rangle \in \mathrm{t}_\omega) \to r \notin x$

14.3.2. Lemma: For any a, a_ω is orthodox.

Proof: Adapt the proof of Theorem 13.2.3. □

14.3.3. Lemma:

$$(\Xi, \lambda) \Vdash r = r \wedge \mathrm{r} \notin \mathrm{r} \wedge \neg \mathcal{T}\mathrm{r} \in \mathrm{r} \text{ just if } \lambda \text{ is a limit.}$$

Proof: For any successor ordinal $\chi + 1$, $(\Xi, \chi + 1) \Vdash \neg\mathcal{T}\mathrm{r} \in \mathrm{r} \leftrightarrow \mathrm{r} \in \mathrm{r}$. Precisely at any limit ordinal λ, $(\Xi, \lambda) \Vdash \mathrm{r} \notin \mathrm{r} \wedge \neg\mathcal{T}\mathrm{r} \in \mathrm{r}$. □

14.3.4. Theorem

Let $\alpha \prec ♀$ be a limit ordinal, and β be $\alpha + \omega$:

1. $(\Xi, \beta) \Vdash \forall x \mathcal{T}\mathrm{B}(x)$
2. $(\Xi, \beta) \Vdash \neg\mathcal{T}\forall x \mathrm{B}(x)$.

Proof: 1. If $(\Xi, \beta) \Vdash \neg\exists w(\langle w, x\rangle \in \mathrm{t}_\omega)$, it follows that $(\Xi, \beta) \Vdash \mathcal{T}\mathrm{B}(x)$ on account of Lemma 14.3.2. If, on the other hand, $(\Xi, \beta) \Vdash \exists w(\langle w, \mathrm{t}_{\bar{i}}\rangle \in \mathrm{t}_\omega)$ we have that $(\Xi, \beta) \Vdash \mathcal{T}\mathrm{B}(\mathrm{t}_{\bar{i}})$, as there is a $\gamma \geq \alpha + i$ such that

$$\forall\delta(\alpha \prec \gamma \leq \delta \prec \beta \Rightarrow (\Xi, \delta) \Vdash \mathrm{B}(\mathrm{t}_{\bar{i}})).$$

So for any term y, $(\Xi, \beta) \Vdash \mathcal{T}\mathrm{B}(y)$, and so $\Xi(\beta) \vdash \forall x\mathcal{T}\mathrm{B}(x)$.

2. Otherwise, $(\Xi, \beta) \Vdash \mathcal{T}\forall x\mathrm{B}(x)$, and we would have $(\Xi, \delta) \Vdash \forall x\mathrm{B}(x)$ as from some ordinal δ below β and above α. Let $\delta =\!= \alpha + (n+1)$, for finite ordinal $n \geq 0$, be such an ordinal. A $(\Xi, \delta) \Vdash \mathrm{B}(\mathrm{t}_{\bar{\mathrm{n}}})$, by instantiation, this entails that $(\Xi, \alpha + (n+1)) \Vdash \mathrm{B}(t_{\bar{n}})$. As $\vDash \exists w(\langle w, \mathrm{t}_{\bar{n}}\rangle \in \mathrm{t}_\omega)$, it follows that $(\Xi, \alpha + (n+1)) \Vdash r \notin \mathrm{t}_{\bar{n}}$. As a consequence, $(\Xi, \alpha + 1) \Vdash r \notin \mathrm{t}_{\bar{0}}$. But the latter entails $(\Xi, \alpha) \Vdash (r \neq r \vee \mathrm{r} \in \mathrm{r} \vee \mathcal{T}\mathrm{r} \in \mathrm{r})$ which contradicts Lemma 14.3.3, as α is presupposed to be a limit ordinal. □

14.3.5. Theorem

$$\overset{\mathsf{M}}{\nvDash} \forall x\mathcal{T}\mathrm{B}(x) \to \mathcal{T}\forall x\mathrm{B}(x).$$

Proof: Theorem 14.3.4 with Definition 5.2.2 entail that for some β,

$$(\Xi, \beta) \nVdash \forall x\mathcal{T}\mathrm{B}(x) \to \mathcal{T}\forall x\mathrm{B}(x).$$

It follows that

$$(\Xi, ♀) \nVdash \mathcal{T}(\forall x\mathcal{T}\mathrm{B}(x) \to \mathcal{T}\forall x\mathrm{B}(x)),$$

and an appeal to Definition 5.4.3.1 finishes the proof. □

14.3.6. THEOREM:

$$\nvdash^{\mathsf{M}} \forall x\mathcal{T}\mathrm{B}(x) \to \mathcal{T}\forall x\mathrm{B}(x)$$

Proof: Appeal to soundness, i.e. in the case

$$\vdash^{\mathsf{M}} \forall x\mathcal{T}\mathrm{B}(x) \to \mathcal{T}\forall x\mathrm{B}(x) \Rightarrow \models^{\mathsf{M}} \forall x\mathcal{T}\mathrm{B}(x) \to \mathcal{T}\forall x\mathrm{B}(x),$$

and Theorem 14.3.5. □

14.3.7. THEOREM: For some formula A,

$$\vdash \forall x\mathcal{T}\mathrm{A} \to \mathcal{T}\forall x\mathrm{A} \;\;\&\;\; \vdash \forall x\mathcal{T}\mathrm{A} \land \neg\mathcal{T}\forall x\mathrm{A}$$

Proof: Let $\mathrm{B}(x)$ in Theorem 14.3.6 be A, and combine with Postulate 8.1.4. □

14.4 £ is omega-consistent

Recall that Theorem 5.2.6 states that £ is omega-consistent.

(McGee 1985) famously isolated a rudimentary theory of truth which is consistent but ω-inconsistent. (Friedman and Sheard 1987) proposed a more substantial a theory of truth, which inherits the ω-inconsistency property. (Halbach 1994) studied the Friedman and Sheard logic, and found that its proof-theoretic strength is the same as the theory of ramified analysis for all finite levels.

Given Theorem 14.3.7, an essential ingredient in the proof of McGee's negative result fails in £, viz. the statement that

$$\forall x(x \in \omega \to \mathcal{T}\ulcorner\mathrm{A}(x)\urcorner) \to \mathcal{T}\ulcorner\forall x(x \in \omega \to \mathrm{A}(x)\urcorner) \tag{14.4.1}$$

in (McGee 1985, p. 399). Notice that $\vdash^{\mathsf{M}} \forall x(x \in \omega \to \mathcal{T}\ulcorner\mathrm{A}(x)\urcorner) \leftrightarrow \forall x\mathcal{T}\ulcorner x \in \omega \to \mathrm{A}(x)\urcorner$, so 14.4.1 follows from the Barcan-formula whose thesishood is denied by Theorem 14.3.7. Moreover, exceptions to 14.4.1 in £ follow from Theorem 5.2.6 and Mcgee's argument.

14.5 More orthodox redresses

Theorem 14.2.1 (Orthodox existential instantiation)

$$\vdash^{\mathsf{M}} \mathfrak{D}(A(x)) \Rightarrow (\vdash^{\mathsf{M}} \exists xA \Rightarrow \vdash^{\mathsf{M}} A^{a}_{x} \text{ for some } a \text{ substitutable for } x \text{ in A}).$$

Proof: As on page 47. □

14.5.1. THEOREM Orthodox attestor: If $A(x)$ is orthodox, then

$$\vdash^{\mathsf{M}} \mathcal{T}\exists xA(x) \Rightarrow \vdash^{\mathsf{M}} \exists x\mathcal{T}A(x).$$

Proof: Appeal to Theorem 14.2.1, and existential generalization. □

14.5.2. THEOREM The Barcan formula holds for orthodox formulas:

$$\vdash^{\mathsf{M}} \mathfrak{O}(B(x)) \Rightarrow \vdash^{\mathsf{M}} (\forall x \mathcal{T} \mathrm{B}(x) \rightarrow \mathcal{T} \forall x \mathrm{B}(x)).$$

Proof: As on page 33. □

15 Classicalities and deviations

On a signalé beaucoup d'antinomies, et le désaccord a subsisté, personne n'a été convaincu; d'une contradiction, on peut toujours se tirer par un coup de pouce! Je veux dire par un distinguo.

Henri Poincaré

The *Grundlagenkrise* which struck the mathematical and philosophical communities as a consequence of the paradoxes, showed one could not presuppose all pretheorethically plausible comprehension principles in set theory or semantics.

In the following some facts which relate to desiderata fulfilled by £ will be expressed. The reader may compare with the desiderata of (Leitgeb 2007) and (Sheard 2003), or others, concerning theories on paradoxes. Some of the facts on desiderata follow from §14.4, §15.2 and §15.3.

15.1 Facts on desiderata met fully, or partially, by £

15.1.1. Fact: There are no type restricions imposed, and there is no language hierarchy.

15.1.2. Fact: Truth is compositional over $\vdash^{M}$, and over similar set theoretic contexts. But it is *not* compositional over $\vdash$, as there are cases such that $\vdash A$ and $\vdash B$ but not $\vdash A \wedge B$.

15.1.3. Fact: Truth is a set, and so truth is as well a predicate. So it is a consequence from the alethic comprehension principle of §10 that truth-paradoxes and set-paradoxes are treated in the same way in £.

15.1.4. Fact: On account of results in §15.3, £ is *classic* in the sense that $\vdash^{M} A$ only if classical logic does not prove $\neg A$, and if classical logic proves A then $\vdash^{M} A$. Moreover, $\vdash B$ if B is a thesis of classical logic, and if $\vdash B$ then classical logic does not prove $\neg B$.

15.1.5. Fact: £ is *unswerving* in the sense that that if A is a paradoxical sentence, then £ should have $\vdash A$ or $\vdash \neg A$, and indeed it as a rule has both.

15.1.6. Fact: The outer veridical logic of £ is the set of theses which encapsulate truth statements. $\vdash^{M} \mathcal{T}\ulcorner A \urcorner \vee \neg\mathcal{T}\ulcorner A \urcorner$ is for example an instance of *The Law of Excluded Middle* of classical logic in the *outer veridical logic* of £. The inner veridical logic of £ is the set of theses which are encapsulated by a truth predicate. $\vdash^{M} \mathcal{T}\ulcorner A \vee \neg A \urcorner$ is for example an instance of *The Law of Excluded Middle* of classical logic in the *inner verdidical logic* of £. It is a consequence of Fact 15.1.4 that the inner and outer veridical logics of £ are classical, as they should be.

15.1.7. Fact: The variety of truth conditionals summed up in Exercise 15.2.2 has the consequence that the *outer veridical* and *inner veridical* logics of $\mathcal{T}$, see Fact 15.1.6, coincide in £, in the sense of Definitions 15.2.1.7 and 15.2.1.10.

15.1.8. Fact: As related in §9, £ has novel inferential modes. The conjunction of these may seem to be an amputation of the classical inferential principle *modus ponens*. But they are in reality an extension of the classical inference rule modus ponens, as *the maxim mode* 9.2.5 serves all the purposes as modus ponens serves in classical logics, and all classical logical theses are maxims of £. The inference modes £ has beyond the maxim mode helps engender novel minor theses which are out of reach for classical logic.

15.1.9. Fact: A naive desideratum is that £ should obtain all truth-biconditionals, as in Definitions 15.2.1.1 and 15.2.1.7, with their weak counterparts, by means of the inference modes which £ endorses, as per §9. £ compensates for the fact that the statements of Definitions 15.2.1.1, 15.2.1.2, 15.2.1.3 and 15.2.1.4 are not true with the truth of the statements of Definitions 15.2.1.5 and 15.2.1.6, and with the fact that the inferential modes exhibited in Definitions 15.2.1.7 and 15.2.1.10 can be used. A consequence of this is that *revenge paradoxicalities* are not a threat. For more on this, see §18.

15.1.10. Fact: By §14.4, £ is omega-consistent, so it allows for standard interpretations.

15.2 The truth–conditionals

15.2.1. Definition:

1. `Hale` material truth adequacy: $\vdash^{\mathsf{M}} \mathcal{T}\ulcorner A \urcorner \leftrightarrow A$
2. `Hale` material truthwards adequacy: $\vdash^{\mathsf{M}} \mathcal{T}\ulcorner A \urcorner \leftarrow A$
3. `Hale` material truthly adequacy: $\vdash^{\mathsf{M}} \mathcal{T}\ulcorner A \urcorner \rightarrow A$
4. `Weak` material truth adequacy: $\vdash \mathcal{T}\ulcorner A \urcorner \leftrightarrow A$
5. `Weak` material truthwards adequacy: $\vdash \mathcal{T}\ulcorner A \urcorner \leftarrow A$
6. `Weak` material truthly adequacy: $\vdash \mathcal{T}\ulcorner A \urcorner \rightarrow A$
7. `Hale` formal truth adequacy: $\vdash^{\mathsf{M}} \mathcal{T}\ulcorner A \urcorner \Leftrightarrow \vdash^{\mathsf{M}} A$
8. `Hale` formal truthwards adequacy: $\vdash^{\mathsf{M}} \mathcal{T}\ulcorner A \urcorner \Leftarrow \vdash^{\mathsf{M}} A$
9. `Hale` formal truthly adequacy: $\vdash^{\mathsf{M}} \mathcal{T}\ulcorner A \urcorner \Rightarrow \vdash^{\mathsf{M}} A$
10. `Weak` formal truth adequacy: $\vdash \mathcal{T}\ulcorner A \urcorner \Leftrightarrow \vdash A$
11. `Weak` formal truthwards adequacy: $\vdash \mathcal{T}\ulcorner A \urcorner \Leftarrow \vdash A$
12. `Weak` formal truthly adequacy: $\vdash \mathcal{T}\ulcorner A \urcorner \Rightarrow \vdash A$

15.2.2. Exercise. £ obeys the formal and as well the weak material truthwards and truthly adequacies of Definition 15.2.1. The first four adequacies in the list fail on account of paradoxicalities

15.3 £ is *classic* and *paraclassical*, but it is not paraconsistent

Let $\mathbb{T}$ be a theory.

15.3.1. Definition: $\mathbb{T}$ is adjunctive just if $\vdash A \ \& \ \vdash B \Rightarrow \vdash A \wedge B$.

15.3.2. Definition: $\mathbb{T}$ is dejunctive just if $\vdash A \wedge B \Rightarrow \vdash A \ \& \ \vdash B$.

15.3.3. Definition: $\mathbb{T}$ is *cosistent* just if for no p, $T \vdash p$ and $T \vdash \neg p$.

15.3.4. Definition: $\mathbb{T}$ is consistent just if for no p, $T \nvdash p \wedge \neg p$

15.3.5. Definition: $\mathbb{T}$ is *contrasistent* just if it is not cosistent.

15.3.6. Definition: $\mathbb{T}$ is *contradictory* just if it is inconsistent.

15.3.7. Definition: Let τ be *classical* logic.

15.3.8. Definition A is an *antithesis* of $\mathbb{T}$ just if ¬A is a thesis of $\mathbb{T}$.

15.3.9. Definition $\mathbb{S}$ is a sedation of $\mathbb{T}$ iff no thesis of $\mathbb{S}$ is an antithesis of $\mathbb{T}$.

15.3.10. Definition $\mathbb{X}$ is an extension of $\mathbb{T}$ just if all theses of $\mathbb{T}$ are theses of $\mathbb{X}$.

15.3.11. Definition: Let τ be *classical* logic.

15.3.12. Fact.[3]

That $\mathbb{X}$ is a proper extension of $\mathbb{T}$ holds just if
$\mathbb{X}$ is an extension of $\mathbb{T}$ and $\mathbb{T}$ is not an extension of $\mathbb{X}$.

15.3.13. Definition Progressive, moderate and classic theories:

1 $\mathbb{T}$ is *progressive* just if it is a proper extension of τ.

2 $\mathbb{T}$ is *moderate* just if it is a sedation of τ.

3 $\mathbb{T}$ is *classic* just if it is progressive and moderate.

15.3.14. Definition $\mathbb{T}$ is *coherent* just if it is classic.

15.3.15. Definition $\mathbb{T}$ is *extraclassical* just if it is classic and contrasistent.

15.3.16. Definition $\mathbb{T}$ is *extracoherent* just if it is coherent and contrasistent.

15.3.17. Lemma £ is an extension of τ.

Proof: Appeal to §6.4. □

15.3.18. Lemma τ is not an extension of £.

Proof: Given the solution to Exercise 8.3, £ has the paradoxical theses $r \in r$ and $r \notin r$. But $r \in r$ and $r \notin r$ are not theses of classical logic. □

15.3.19. Lemma £ is progressive.

Proof: £ is a proper extension of τ given Fact 15.3.12, Lemma 15.3.12 and Lemma 15.3.18. An appeal to Definition 15.3.13.1 suffices to finish the proof. □

[3]For the following definition, and the notions involved here, compare with (Bjørdal 2015, p. 511).

15.3.20. Lemma £ is moderate.

Proof: If τ proves $\neg$A, $\vdash^{M}\neg$A as £ is progressive. Axiom 6.2.2.4, justified by Postulate 6.2.2.4, entails that if τ proves $\neg$A, $\vdash \neg$A & $\nvdash$ A. So a fortiori, if τ proves $\neg$A, $\nvdash$ A. By contraposition, if $\vdash$ A then τ does not prove $\neg$A. A is arbitrary, so no thesis of £ is an antithesis of classical logic τ. Consequently, £ is a sedation of τ. An appeal to Definition 15.3.13.2 finishes the proof. □

15.3.21. Theorem £ is classic.

Proof: From Definition 15.3.13.3, as £ is progressive and moderate given Lemma 15.3.19 and Lemma 15.3.18. □

15.3.22. Exercise $\mathbb{T}$ is contrasistent just if for some p, $\mathbb{T} \vdash p$ and $\mathbb{T} \vdash \neg p$.

15.3.23. Exercise If $\mathbb{T}$ is contrasistent and adjunctive then $\mathbb{T}$ is contradictory.

15.3.24. Exercise If $\mathbb{T}$ is contradictory and dejunctive then $\mathbb{T}$ is contrasistent.

15.3.25. Exercise If $\mathbb{T}$ is dejunctive and adjunctive, $\mathbb{T}$ is contradictory iff contrasistent.

15.3.26. Exercise If $\mathbb{T}$ is dejunctive and adjunctive, $\mathbb{T}$ is cosistent just if consistent.

15.3.27. Exercise Adjunction is not a valid inference mode in £.

15.3.28. Exercise £ is extraclassic and extracoherent, and so contrasistent.

15.3.29. Exercise Paraconsistent theories are not classic.

15.3.30. Theorem £ is not paraconsistent.

Proof: Given Theorem 15.3.21 and Exercise 15.3.29, paraconsistent theories are not classic. But £ is classic by Theorem 15.3.21. □

15.3.31. Remark The pairs consistency & cosistency and contrasistency & contradiction conflate in classical contexts, for classical systems are adjunctive and dejunctive.

15.3.32. Remark With proper comprehension, most paraconsistent theories are not even moderate, as then some contradiction is a thesis.

15.3.33. Remark. The well-known non-adjunctive paraconsistent logic of (Jaskowski 1999) and (Jaskowski 1948), is moderate even with liberal comprehension principles. But is it not conservative, and so not classic.

15.4 Incompatability and complementarity

15.4.1. Definition (Incompatability) The theses A and B of a consistent theory $\mathbb{T}$ are *incompatible* just if $\mathbb{T}$ proves A, B, and $\neg(A \wedge B)$.

15.4.2. Theorem (£ has incompatible theses) By the result of Exercise 8.3, £ proves $R \in R$ and $R \notin R$. But £ is conservative, given §15.3. So the theses $R \in R$ and $R \notin R$ of £ are incompatible, for given its conservativeness, £ proves $\neg(R \in R \wedge R \notin R)$.

15.4.3. Definition (Complementarity) A and $\neg A$ in a theory $\mathbb{T}$ are complementary just if they are incompatible theses of $\mathbb{T}$.

15.4.4. Corollary £ has complementary theses

16 The Liar is Russell's condition on his set

Thus mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true.

Bertrand Russell

Frank Ramsey argued, in (Ramsey 1925, p. 20), that there is an essential difference between *syntactical paradoxes* which "involve only logical or mathematical terms such as class and number", and *semantic paradoxes*, which "...are not purely logical, and cannot be stated in logical terms alone; for they all contain some reference to thought, language, or symbolism".

Ramsey considered Russell's paradox a canonical representative of syntactic paradoxes, and the Liar he considered an archetypical semantic paradox.

In (A. A. Fraenkel and Bar-Hillel 1958, p. 5), the authors adjudged:

> "Since (Ramsey 1925) it has become customary to distinguish between logical and semantic (sometimes also called syntactic or epistemological) antinomies."

It is here argued, to the contrary, that one should take paradoxes, as the Liar-paradox, to be so inextricably intertwined with set theoretical paradoxes so as not consider them to be different in kinds.

Others reached the same conclusion, but on the basis of considerations different from the ones adduced further below:

(Scott 1974)(1967) argued that the Zermelo axioms were justified by type theoretic reasoning:

> "The truth is that there is only one way of avoiding the paradoxes: namely, the use of some form of the theory of types. That was at the basis of both Russell's and Zermelo's intuitions. Indeed the best way to regard Zermelo's theory is as a simplification and extension of Russell's. (We mean Russell's *simple* theory of types, of course.) The simplification was to make the types *cumulative*." (Scott 1974)(208)

Alonzo Church, who was my teacher in a graduate seminary in logic, with an oral exam, in the spring of 1989, at UCLA, virtually equated Russell's theory of types and Alfred Tarski's resolution of the Liar paradox, in (Church 1976), as he stated:

> "In the light of this it seems justified to say that Russell's resolution of the semantical antinomies is not a different one than Tarski's but is a special case of it."(Church 1976, p. 756)

The interest of Scott's and Church's points of view, for our purposes here, is that they take Tarski's resolution of the alleged *semantic* paradoxes to be the same as Russell's or Zermelo's resolution of the allegedly syntactical, set theoretic paradoxes.

In £ there are bridge principles, as for example per Theorem 16.1 and Definition 16.4, between given, supposedly syntactical paradoxes, and supposedly semantical paradoxes.

16.1. THEOREM: There is a *liar sentence* L given by $\vdash^{\mathsf{M}} \mathrm{L} \leftrightarrow \neg\mathcal{T}\ulcorner\mathrm{L}\urcorner$.

Proof: By alethic comprehension,

$$\vdash^{\mathsf{M}} \mathrm{r} \in \mathrm{r} \leftrightarrow \mathcal{T}\ulcorner\mathrm{r} \notin \mathrm{r})\urcorner. \tag{16.2}$$

By negating both sides of the biconditional in 16.2, we get

$$\vdash^{\mathsf{M}} \mathrm{r} \notin \mathrm{r} \leftrightarrow \neg\mathcal{T}\ulcorner\mathrm{r} \notin \mathrm{r})\urcorner. \tag{16.3}$$

16.4. DEFINITION:

$$\mathrm{L} =\!\!= \mathrm{r} \notin \mathrm{r},$$

Substituting with L of Definition 16.4 in equation 16.3 gives the more canonical form for the Liar sentence:

$$\vdash^{\mathsf{M}} \mathrm{L} \leftrightarrow \neg\mathcal{T}\ulcorner\mathrm{L}\urcorner. \tag{16.5}$$

□

16.5 is resolved as Russell's paradox.

16.6. PROPOSITION: Liar sentences, and variants, with provenances from classical Greek philosophy, should be taken as given by maxims of Theorems as 16.1.

16.7. THEOREM: $\vdash \mathrm{L}, \vdash \neg\mathrm{L}, \vdash \mathcal{T}\ulcorner\mathrm{L}\urcorner, \vdash \mathcal{T}\ulcorner\neg\mathrm{L}\urcorner, \vdash \neg\mathcal{T}\ulcorner\neg\mathrm{L}\urcorner$ and $\vdash \neg\mathcal{T}\ulcorner\mathrm{L}\urcorner$.

Proof: We know that $\vdash \mathrm{r} \in \mathrm{r}$ and $\vdash \mathrm{r} \notin \mathrm{r}$, so from Definition 16.4, $\vdash \mathrm{L}$ and $\vdash \neg\mathrm{L}$. Finish with 9.2.4.1 and 9.2.4.2. □

16.8. OBSERVATION: Each element in variety $[\mathrm{L}, \mathcal{T}\ulcorner\mathrm{L}\urcorner, \neg\mathcal{T}\ulcorner\neg\mathrm{L}\urcorner]$ is incompatible with any member of $[\neg\mathrm{L}, \mathcal{T}\ulcorner\neg\mathrm{L}\urcorner, \neg\mathcal{T}\ulcorner\mathrm{L}\urcorner]$ in £, and vice versa. Moreover, each element in variety $[\mathrm{L}, \mathcal{T}\ulcorner\mathrm{L}\urcorner, \neg\mathcal{T}\ulcorner\neg\mathrm{L}\urcorner]$ is complementary to precisely one member of $[\neg\mathrm{L}, \mathcal{T}\ulcorner\neg\mathrm{L}\urcorner, \neg\mathcal{T}\ulcorner\mathrm{L}\urcorner]$ in £, and vice versa.

17 Librationist incompleteness phenomena

Kurt Gödel, when colleague John Bachall presented himself as a physicist at an Institute for Advanced Studies faculty dinner: "I don't believe in natural science."

(Regis 1988, p. 58)

It is of interest to note that the proofs of Gödel's incompleteness theorem typically appeal to the theorys *cosistency*, which conflates with its consistency in the classical frameworks which are usually presupposed. So may consistent contrasistent theories, as £, possibly finesse the limitation? Let us explore this cursorily, without commitments.

Observe first that $\mathcal{T}$ maximally obeys the Hilbert-Bernays-Löb derivability conditions in the sense that for all A and B,

1. $\vdash^{\mathsf{M}} \mathrm{A} \Rightarrow \vdash^{\mathsf{M}} \mathcal{T}\ulcorner\mathrm{A}\urcorner$.
2. $\vdash^{\mathsf{M}} \mathcal{T}\ulcorner\mathrm{A}\urcorner \rightarrow \mathcal{T}\ulcorner\mathcal{T}\ulcorner\mathrm{A}\urcorner\urcorner$.
3. $\vdash^{\mathsf{M}} \mathcal{T}\ulcorner\mathrm{A}\rightarrow\mathrm{B}\urcorner \rightarrow (\mathcal{T}\ulcorner\mathrm{A}\urcorner \rightarrow \mathcal{T}\ulcorner\mathrm{B}\urcorner)$.

Take $\vdash^{\mathsf{M}} \mathcal{T}\ulcorner\mathrm{A}\urcorner$ to express not only that A is a true maxim, but as well that A is provable as a maxim. Take therefore a thesis as $\vdash^{\mathsf{M}} \exists x\neg\mathcal{T}x$ to expresses that £ is not trivial, and $\vdash^{\mathsf{M}} \neg\exists x\mathcal{T}\ulcorner\mathcal{T}x\wedge\neg\mathcal{T}x\urcorner$ to express that £ does not prove a contradiction, or inconsistency.

Let us at this point restate 16.5: $\vdash^{\mathsf{M}} \mathrm{L}\leftrightarrow\neg\mathcal{T}\ulcorner\mathrm{L}\urcorner$.

If one supposes $\vdash^{\mathsf{M}} \mathrm{L}$ it follows that $\vdash^{\mathsf{M}} \mathcal{T}\ulcorner\mathrm{L}\urcorner$ from 1, and $\vdash^{\mathsf{M}} \neg\mathcal{T}\ulcorner\mathrm{L}\urcorner$ from 16.5. If one supposes $\vdash^{\mathsf{M}} \neg\mathrm{L}$ it follows that $\vdash^{\mathsf{M}} \mathcal{T}\ulcorner\neg\mathrm{L}\urcorner$ from 1, and $\vdash^{\mathsf{M}} \mathcal{T}\ulcorner\mathrm{L}\urcorner$ from 16.5, so that one with Postulate 6.5.2.2 has $\vdash^{\mathsf{M}} \mathcal{T}\ulcorner\neg\mathrm{L}\urcorner$ and $\vdash^{\mathsf{M}} \neg\mathcal{T}\ulcorner\neg\mathrm{L}\urcorner$. As £ is maximally adjunctive so that $[\vdash^{\mathsf{M}} \mathrm{A}\ \&\ \vdash^{\mathsf{M}} \mathrm{B}] \Rightarrow \vdash^{\mathsf{M}} \mathrm{A}\wedge\mathrm{B}$, in either case $\vdash^{\mathsf{M}} \mathcal{T}\ulcorner\neg\mathrm{L}\urcorner\wedge\neg\mathcal{T}\ulcorner\neg\mathrm{L}\urcorner$. So neither $\vdash^{\mathsf{M}} \neg\mathrm{L}$ nor $\vdash^{\mathsf{M}} \mathrm{L}$, but rather $\vdash\neg\mathrm{L}$ and $\vdash\mathrm{L}$. So the sentence $\neg\mathrm{L}$ which is *maximally incomplete*, in the sense that neither $\neg\mathrm{L}$ nor L is a maxim, is nevertheless a minor thesis.

It was pointed out, by means of Exercises 5.5.2 and 5.5.3, and Definition 5.6.1.3, that neither $\vdash s\in s$ nor $\vdash s\notin s$, if $s=\{x|x\in x\}$. But this is not a genuine incompleteness, as neither $\vDash s\in s$ nor $\vDash s\notin s$.

The author does not know that there is a sentence A such that $\nvdash^{\mathsf{M}} \mathrm{A}$ and such that we should want that $\vdash^{\mathsf{M}} \mathrm{A}$, nor that there is a sentence B such that $\nvdash \mathrm{B}$ and such that we should want that $\vdash\mathrm{B}$. Certainly, if C is the statement that there is a certain inaccessible cardinal larger or equal to the first hypothetized *1-inaccesible* cardinal, it will be the case, with the assumptions made, that even $\nvdash^{\mathsf{M}} \mathrm{C}^{\mathbf{V}}$, where $\mathbf{V}$ is as in §25, and the notation $\mathrm{C}^{\mathbf{V}}$ as in Definition 4.5.23. It is not obvious to the author, however, at this point, that we should want $\vdash^{\mathsf{M}} \mathrm{C}^{\mathbf{V}}$.

But for the record, given §25 if D is the statement that there is an *inaccesible* cardinal, and that there for any *inaccessible* cardinal is a larger *inaccessible* cardinal, then $\vdash^{\mathsf{M}} D^{\mathbf{V}}$. However, those inaccessible cardinals count as *0-inaccessible* cardinals here, and the theory **NBG+TA** has a standard model in $\mathbf{V}_\zeta$ if ζ is the first *1-inaccessible* cardinal, i.e. the first regular limit of *inaccessible* cardinals.

18 The reflective theory of comprehension

Two paradoxes are better than one; they may even suggest a solution.

Edward Teller

One may as a first approximation take the reflective theory of truth comprehension that is supported by the librationist set theoryto be expressed above all by the inference modes of §9, and especially the simple inference modes, for truth. The reflective theory of set comprehension is obtained from the reflective theory of truth comprehension via alethic comprehension.

18.1 Responsible naiveté without revenge

The revenge problem is avoided as £ is unswerving, in the sense of Fact 15.1.5, and as it has complementary theses in the sense of Definition 15.4.3.

Consider the Liar sentence L of Equation 16.5. If $\vdash \mathrm{L}$, it follows that $\vdash \neg\mathcal{T}\ulcorner\mathrm{L}\urcorner$ via the equation. However, it as well follows that $\vdash \mathcal{T}\ulcorner\mathrm{L}\urcorner$ from $\vdash \mathrm{L}$ and inference mode 9.1.1.1. So $\vdash \mathcal{T}\ulcorner\mathrm{L}\urcorner$ and $\vdash \mathrm{L}$ state things as they are. Moreover, given $\vdash \neg\mathcal{T}\ulcorner\mathrm{L}\urcorner$, it follows that $\vdash \neg\mathrm{L}$ via inference mode 9.1.1.9. So $\vdash \neg\mathrm{L}$ and $\vdash \neg\mathcal{T}\ulcorner\mathrm{L}\urcorner$ state things as they are.

It is not a desirable option to prefer $\nvdash \mathrm{L}$ and $\nvdash \neg\mathrm{L}$, for $\vDash \mathrm{L}$ and $\vDash \neg\mathrm{L}$, and one should attempt to have $\vdash \mathrm{B}$ whenever $\vDash \mathrm{B}$. Moreover, it has, as discussed in Fact 15.1.5, been presupposed as desideratum that £ be unswerving, and decide paradoxical sentences.

18.2 *Argumenta ad paradoxo*

That an assumption in £ has the consequence that $\vdash \mathrm{A}$ and $\vdash \neg\mathrm{A}$ does not suffice as a proof by contradiction against the assumption. Instead, if the considerations leading to $\vdash \mathrm{A}$ and $\vdash \neg\mathrm{A}$ cannot be extended to arrive at $\vdash \mathrm{A} \wedge \neg\mathrm{A}$, they constitute an *argumentum ad paradoxo* to show that A and $\neg\mathrm{A}$ are complementary theses of £. The considerations in §18.1 are *argumenta ad paradoxo*, which justify such metamathematical statements as that L and $\neg\mathrm{L}$ are complementary theses of £.

19 The manifestation sets

There are very few theorems in advanced analysis which have been demonstrated in a logically tenable manner. Everywhere one finds this miserable way of concluding from the special to the general and it is extremely peculiar that such a procedure led to so few of the so-called paradoxes.

Niels Henrik Abel

We explain the manifestation set construction in §19.1, and will as from §25 see that it facilitates £'s ability to be extended with strong set theoretic principles. In §19.2 we show how we may obtain Quine atoms via orthodox manifestation sets. The foci in the succeeding sections will be upon *negative* results: In §19.3 we account for the *auto-combative* paradox. Next, in §20, we elucidate the virtually universal paradoxicality of power sets. Finally the failure of extensionality in £ is discussed in §21, where it is shown that all orthodox sets are distinct from, as well as co-extensional with infinitely many co-extensional and pairwise distinct orthodox sets.

19.1 The manifest construction

For the following construction, cfr. (Bjørdal 2012)(345–46), (Cantini 1996)(76), (Visser 1989)(695–96) and earlier literature referred to there. One may, plausibly, find that Roger's theorem and Kleene's second recursion theorem are related, but the proof that there are manifestation sets does not rely upon any presuppositions on computability.

19.1.1. Definition Kuratowskian ordered pairs:

$$\langle a, b\rangle == \{\{a\}, \{a, b\}\}$$

19.1.2. Definition The manifestation set $\mathbf{A}$ of formula $\mathrm{A}(v_0, v_1)$:

(1) $v\eta b == \exists v_2(v_2 = \langle v, b\rangle \wedge v_2 \in b)$

(2) $\mathfrak{a} == \{v_2 | \exists v_0, v_1(\langle v_0, v_1\rangle = v_2 \wedge \mathrm{A}(v_0, v_1)^{\{v|v\eta v_1\}}_{v_1}\}$

(3) $\mathbf{A} == \{v | v\eta\mathfrak{a}\}$

19.1.3. Theorem *Manifest comprehension,* for the manifestation set in Definition 19.1.2.3:

$$\vdash^{\mathsf{M}} \forall v(v \in \mathbf{A} \leftrightarrow \mathcal{T}\mathcal{T}\mathrm{A}(v, \mathbf{A}))$$

$$\vdash^{\mathsf{M}} \forall v(v \in \mathbf{A} \leftrightarrow \mathcal{T}\mathcal{T}\mathrm{A}(v, \mathbf{A}))$$

Proof: From Definition 19.1.2.3 and alethic comprehension,

$$\vdash^{\mathsf{M}} c \in \mathbf{A} \leftrightarrow \mathcal{T}c\eta\mathfrak{a}.$$

As a consequence of Definition 19.1.2.1 we have

$$\vdash^{\mathsf{M}} \mathcal{T} c \eta \mathfrak{a} \leftrightarrow \mathcal{T} \exists v_2 (v_2 = \langle c, \mathfrak{a} \rangle \wedge v_2 \in \mathfrak{a})$$

From the two previous steps, Definition 19.1.2.2, alethic comprehension and 9.2.5 we have

$$\vdash^{\mathsf{M}} c \in \mathbf{A} \leftrightarrow \mathcal{T} \exists v_2 (v_2 = \langle c, \mathfrak{a} \rangle \wedge \mathfrak{T} \exists v_0, v_1 (\langle v_0, v_1 \rangle = v_2 \wedge \mathrm{A}(v_0, v_1)^{\{v | v \eta v_1\}}_{v_1}))$$

It follows, by means of the theory of identity, that

$$\vdash^{\mathsf{M}} c \in \mathbf{A} \leftrightarrow \mathcal{T}\mathcal{T} \mathrm{A}(c, v_1)^{\mathbf{A}}_{v_1},$$

so that, on account of Definition 19.1.2.3 and Definition 4.5.19,

$$\vdash^{\mathsf{M}} c \in \mathbf{A} \leftrightarrow \mathcal{T}\mathcal{T} \mathrm{A}(c, \mathbf{A}).$$

Finish with universal generalization. □

19.1.4. Corollary Orthodox manifestation:

If $\mathrm{A}(v_0, v_1)$ is orthodox, $\vdash^{\mathsf{M}} \forall v \big(v \in \mathbf{A} \leftrightarrow \mathrm{A}(v, \mathbf{A})\big)$.

19.1.5. Theorem(Comprehension for orthodox manifestation set with parameters)

Some manifestation sets have parameters, so if orthodox A has the free variables in $\vec{v}$:

$$\vdash^{\mathsf{M}} \forall v \forall \vec{\mathfrak{v}} (v \in \mathbf{A} \leftrightarrow \mathrm{A}(\vec{\mathfrak{v}}, v, \mathbf{A})).$$

Proof: Adjust Definition 19.1.2. For the notation, recall Definition 4.5.21. □

19.2 Quine atoms

The most elementary Quine atom is the manifestation set **=** of formula $v_0 = v_1$. By means of manifest comprehension,

$$\vdash^{\mathsf{M}} \forall v (v \in \mathbf{=} \leftrightarrow \mathcal{T}\mathcal{T} v = \mathbf{=}). \tag{19.2.1}$$

As identity is an orthodox relation,

$$\vdash^{\mathsf{M}} \forall v (v \in \mathbf{=} \leftrightarrow v = \mathbf{=}). \tag{19.2.2}$$

As identity is an equivalence relation,

$$\overset{\mathsf{M}}{\vdash} \boldsymbol{=} = \boldsymbol{=}. \tag{19.2.3}$$

So from equations 19.2.2 and 19.2.3,

$$\overset{\mathsf{M}}{\vdash} \boldsymbol{=} \in \boldsymbol{=} \tag{19.2.4}$$

19.2.5. Exercise: Prove that there are infinitely many distinct Quine atoms.

19.3 The autocombatant

In contrast to orthodox manifestation sets, many are paradoxical. This is for example the case with the following quite heretical manifestations set $\boldsymbol{\notin}$, which generates an infinity of incompatible and complementary theses.

19.3.1. Theorem (The *autocombative truths*) $\boldsymbol{\notin}$ is the manifestation set of formula $v_0 \notin v_1$, so that:

$$\vDash \forall v(v \in \boldsymbol{\notin}) \ \& \ \vDash \forall v(v \notin \boldsymbol{\notin}).$$

Proof: On account of Theorem 19.1.3:

$$\overset{\mathsf{M}}{\vdash} \forall v(v \in \boldsymbol{\notin} \leftrightarrow \mathcal{T}\mathcal{T} v \notin \boldsymbol{\notin}),$$

so that by soundness

$$\overset{\mathsf{M}}{\vDash} \forall v(v \in \boldsymbol{\notin} \leftrightarrow \mathcal{T}\mathcal{T} v \notin \boldsymbol{\notin}).$$

If λ is any limit below the closure ordinal ♀, we will, for any term a, and any fair function Ξ, have that $(\Xi, \lambda) \Vdash a \notin \boldsymbol{\notin}$; otherwise a contradiction would follow as $a \notin \boldsymbol{\notin}$ would hold at succeeding successor ordinals σ, $\sigma + 1$ and $\sigma + 2$ below λ. Consequently, we for such a limit λ as well have that $(\Xi, \lambda + 2) \Vdash a \in \boldsymbol{\notin}$. From 5.2.2.2 we have that $(\Xi, \lambda) \Vdash \forall v(v \notin \boldsymbol{\notin})$ and $(\Xi, \lambda + 2) \Vdash \forall v(v \in \boldsymbol{\notin}))$. As a result, $(\Xi, ♀) \Vdash \neg\mathcal{T}\neg\forall v(v \in \boldsymbol{\notin}))$ and $(\Xi, ♀) \Vdash \neg\mathcal{T}\neg\forall v(v \notin \boldsymbol{\notin}))$. The proof finishes by invoking Definitions 5.4.2.3 and 5.4.3.3. □

20 Powersets are paradoxical lest as $\mathcal{P}(\{v|v = v\})$

Das Wesen der Mathematik liegt in ihrer Freiheit.
Georg Cantor

Use standard notation, so that $\vdash^{\mathsf{M}} a \subset b \leftrightarrow \forall x(x \in a \to x \in b)$, and posit

20.1. Definition The power set of a:

$$\mathcal{P}(a) \Longleftarrow \{v|v \subset a\}.$$

It turns out that a power set is paradoxical unless it is the power set of a maximally filled set b for which $\vdash^{\mathsf{M}} \forall x(x \in b)$.

20.2. Definition The universal set:

$$\mathbf{U} \Longleftarrow \{v|v = v\}$$

20.3. Theorem a is paradoxical if $\nvdash^{\mathsf{M}} \forall x(x \in a \leftrightarrow x \in \mathbf{U})$:

Proof: We use a case distincion to provide a distinct proof for the case where $\vdash^{m} \exists v(v \notin a)$.

(1) If $\vdash^{\mathsf{M}} \exists v(v \notin a)$, use the autocombatant $\boldsymbol{\notin}$, of Theorem 19.3.1, for which

$$\vdash \forall v(v \in \boldsymbol{\notin}) \ \& \ \vdash \forall v(v \notin \boldsymbol{\notin}).$$

In this case $\vdash \boldsymbol{\notin} \notin \mathcal{P}(a)$ and $\vdash \boldsymbol{\notin} \in \mathcal{P}(a)$, so $\mathcal{P}(a)$ is paradoxical.

(2) If $\vdash^{m} \exists v(v \notin a)$, $\vdash \mathbf{U} \in \mathcal{P}(a)$ and $\vdash \mathbf{U} \notin \mathcal{P}(a)$, so $\mathcal{P}(a)$ is paradoxical. □

21 Non-extensionality and Ursets

It is impossible to be a mathematician without being a poet in soul.
Sofia Kovalevskaya

The principle of extensionality's failure in type free theories is well known, and many have contributed to the deposit of knowledge.

Let us first posit

21.1. DEFINITION The principle of extensionality:

$$\vdash^{\mathsf{M}} a \stackrel{\mathsf{e}}{=} b \rightarrow a = b.$$

A particularly easy proof of the failure of the extensionality principle in £ is obtained by making use of the fact that for any limit ordinal λ,

$$\Xi(\lambda + 1) \Vdash \{v|v = v\} \stackrel{\mathsf{e}}{=} \{v|v \notin v\} \wedge \{v|v = v\} \neq \{v|v \notin v\}.$$

As a consequence, there are sets a and b such that $\nvDash a \stackrel{\mathsf{e}}{=} b \rightarrow a = b$, and so it follows, a fortiori, that $\nvDash^{\mathsf{M}}\ a \stackrel{\mathsf{e}}{=} b \rightarrow a = b$, But $\vdash^{\mathsf{M}}\ a \stackrel{\mathsf{e}}{=} b \rightarrow a = b \Rightarrow \vDash^{\mathsf{M}}\ a \stackrel{\mathsf{e}}{=} b \rightarrow a = b$ is a soundness requirement, so that $\nvdash^{\mathsf{M}}\ a \stackrel{\mathsf{e}}{=} b \rightarrow a = b$.

(Gilmore 1974) showed that a partial set theory proves that there is an orthodox set a such that $a \stackrel{\mathsf{e}}{=} \emptyset$ and $a \neq \emptyset$. (Bjørdal 2012, p. 345) relates Lev Gordeev's more concise proof of the same result as Gilmore's, in the context of Explicit Mathematics, and some on why it was published in (Beeson 1985), with acknowledgement.

Define Gordeev's set with the manifestation theorem 19.1.3, so that one may posit

21.2. DEFINITION (Via manifestation)$\forall x(x \in \dot{g} \leftrightarrow \mathcal{TT}(x = \emptyset \wedge x = \dot{g})$.

21.3. THEOREM: [Gordeev] $\dot{g}$ is (i) orthodox, so $\vdash^{\mathsf{M}}\ x \in \dot{g} \leftrightarrow (x = \emptyset \wedge x = \dot{g})$, (ii) empty and (iii) distinct from $\emptyset$.

Proof: As the proof of Theorem 4 in (Bjørdal 2012, p. 345): (i) $\dot{g}$ is orthodox, on account of the theory of identity. (ii) As $\vdash^{\mathsf{M}}\ x \in \dot{g} \rightarrow (x = \emptyset \wedge x = \dot{g})$, $\vdash^{\mathsf{M}}\ x \in \dot{g} \rightarrow \dot{g} = \emptyset$, so $\dot{g}$ is empty. (iii) $\dot{g} \neq \emptyset$, for else $\dot{g} = \{\dot{g}\}$ on account of Theorem 21.3 (i), which contradicts (ii). □

(Cantini 1996)(74), relates a proof, by Pierluigi Minari that we for *any* orthodox set a may find a *distinct* orthodox set b such that a and b are nevertheless co-extensional.

Theorem 5 (ii) in (Bjørdal 2012)(346), whose proof was left as an exercise, states the result that Minari's construction can be generalized, as in Theorem 21.4. This result appears to be the most general non-extensionality result which has been available, and the mentioned exercise is solved, by the following

21.4. Theorem: For orthodox set b, there are infinitely many pairwise distinct orthodox and co-extensional sets, which are all co-extensional with b and distinct from b.

Proof: Let orthodox υ_1 be given, and let υ_{n+1} be the manifestation set of

$$\Big(\bigwedge_{i=1}^{i=n} v_1 \in \upsilon_i \wedge \upsilon_i \neq v_1\Big) \vee \Big(\bigwedge_{i=1}^{i=n} \upsilon_i \notin \upsilon_i \wedge \bigvee_{i=1}^{i=n} \upsilon_i = v_1\Big)$$

so that, by manifest comprehension and the logic of identity,

$$\forall v(v \in \upsilon_{n+1} \leftrightarrow (\Big(\bigwedge_{i=1}^{i=n} v \in \upsilon_i \wedge \upsilon_i \neq \upsilon_{n+1}\Big) \vee \Big(\bigwedge_{i=1}^{i=n} \upsilon_i \notin \upsilon_i \wedge \bigvee_{i=1}^{i=n} \upsilon_i = \upsilon_{n+1}\Big)).$$

If $\bigvee_{i=1}^{i=n}(\upsilon_i = \upsilon_{n+1})$, it follows that $\vdash^{\mathsf{M}} \upsilon_{n+1} \in \upsilon_{n+1} \leftrightarrow \upsilon_{n+1} \notin \upsilon_{n+1}$, which is impossible. So $\bigwedge_{i=1}^{i=n}(\upsilon_i \neq \upsilon_{n+1})$. Clearly, $\bigwedge_{i=1}^{i=n}(\upsilon_i \overset{\mathsf{e}}{=} \upsilon_{n+1})$. The process can be iterated, so we are done. □

Theorem 22.7 extends Theorem 21.4 to ordinal limit chiffers larger than ω.

22 Names and sets of Urelemente to transfinite orders

Os números são as regras dos seres, e a matemática é o regulamento do mundo.

Francisco Gomes Teixeira

22.1. Definition:

(1) Let $\upsilon_1 = \dot{g}$, as in Definition 21.2.

(2) For any $n \in \Omega_+$, $\ulcorner n \urcorner$ is the υ_n of Theorem 21.4.

(3) $\upsilon^\omega \Longleftarrow \{x | \forall y(\upsilon_0 \in y \land \forall z(\upsilon_z \in y \to \upsilon_{z+1} \in y) \to x \in y)\}$.

22.2. Theorem: [For manifestation set υ_ω]

$$\vdash^{\mathsf{M}} \forall x(x \in \upsilon_\omega \leftrightarrow (x = \upsilon_\omega \land \exists y(y \in \upsilon^\omega \land x = y))).$$

22.3. Remark: υ_ω in Theorem 22.2 is orthodox as υ^ω in Definition 22.1.3 is orthodox.

22.4. Theorem: (i) υ_ω is empty. (ii) $\vdash^{\mathsf{M}} \forall y(y \in \upsilon^\omega \to \upsilon_\omega \neq y)\}$.

Proof: (i) If a were an element of υ_ω, $\vdash^{\mathsf{M}} a = \upsilon_\omega \land \exists y(y \in \upsilon^\omega \land a = y)$. Given Theorem 21.4, all members of υ^ω are empty sets. Consequently, if a were an element of υ_ω then υ_ω would be an empty set. So υ_ω is an empty set. (ii) A rendition of Theorem 22.2 is $\vdash^{\mathsf{M}} \forall x(x \notin \upsilon_\omega \leftrightarrow (x = \upsilon_\omega \to \forall y(y \in \upsilon^\omega \to x \neq y)))$, so, as a consequence,

$$\vdash^{\mathsf{M}} \upsilon_\omega \notin \upsilon_\omega \to (\upsilon_\omega = \upsilon_\omega \to \forall y(y \in \upsilon^\omega \to \upsilon_\omega \neq y)).$$

The proof finishes by invoking the maxim mode 9.2.5, as $\vdash^{\mathsf{M}} \upsilon_\omega \notin \upsilon_\omega$ on account of (i). □

22.5. Definition:

(1) c is an urset just if $c = \ulcorner n \urcorner$, for some $n \in \Omega_+$.

(2) In accordance with Definition 4.5.4, $\ulcorner 2222221 \urcorner$, $\ulcorner 222221 \urcorner$, $\ulcorner 22221 \urcorner$, $\ulcorner 2221 \urcorner$, $\ulcorner 221 \urcorner$, $\ulcorner 21 \urcorner$ and $\ulcorner 1 \urcorner$ are the *symbolic* ursets: $\ulcorner \# \urcorner$, $\ulcorner \ddot{c} \urcorner$, $\ulcorner \varsigma \urcorner$, $\ulcorner \forall \urcorner$, $\ulcorner \downarrow \urcorner$, $\ulcorner \ddot{v} \urcorner$, $\ulcorner \bullet \urcorner$.

(3) The symbolic ursets are the *atomic* names, which denote the primitive symbols.

(4) Recall Definition 4.5.5.2 of $\ell(n_0) \Longleftarrow \lfloor log_2(n_0 + 1) \rfloor$, which uses log_2 and the floor function $\lfloor \ \rfloor$, to define the length $\ell(n_0)$ of the bijective base-2 cipher needed to express a given chiffer n_0.

(5) So

$$\begin{aligned}\ell(\bullet) &= \ell(1) = 1\\ \ell(\ddot{v}) &= \ell(21) = 2\\ \ell(\downarrow) &= \ell(221) = 3\\ \ell(\forall) &= \ell(2221) = 4\\ \ell(\varsigma) &= \ell(22221) = 5\\ \ell(\ddot{c}) &= \ell(222221) = 6\\ \ell(\#) &= \ell(2222221) = 7.\end{aligned}$$

(6) Given Definition 4.5.5, the *joining of names* is defined by positing

$$\ulcorner n_0 \urcorner \bowtie \ulcorner n_1 \urcorner =\!= \ulcorner n_0 n_1 \urcorner =\!= \ulcorner n_0 \frown n_1 \urcorner =\!= \ulcorner n_0 \cdot 2^{\ell(n_1)} + n_1 \urcorner .$$

(7) Given Definition 22.5.5, we may use Definition 22.5.6 to construe composite names gramatically correct by joining *names* whilst obeying the formation rules of §4.5.

(8) For good $\in$ {symbol, symbol string, variable, formula, constant, term, sentence}, $\ulcorner \mathcal{N} \urcorner$ is a good name just if $\mathcal{N}$ is a good.

22.6. Caveat: In formula $\forall v \mathcal{T} \ulcorner A \urcorner$, $\ulcorner \; \urcorner$ is a term operating formula forming operator, so the evaluation of $\mathcal{T} \ulcorner A \urcorner{}^b_v$ is comparable with $\Box A^b_v$, where $\Box$ is any formula operating formula forming operator. So, for example, $\mathcal{T} \ulcorner v = v \urcorner{}^b_v$ is $\mathcal{T} \ulcorner b = b \urcorner$. A subtle substitution function, e.g. as with (Smorynski 1977, 837 et passim) in the proof of Gödel's incompleteness theorem, is not needed, for there is no use of quantification into an opaque, or otherwise "intensional", context.

Notice that at this point Theorem 21.4 may straightforwardly be extended:

22.7. Theorem (Sets of Urelemente to any order)

Given Definition 22.1.2, υ^ω in Definition 22.1.3 serves as the set of the expression names defined in §21. Given Theorems 22.2 and 22.4 (i), υ_ω is another empty set distinct from all members of υ^ω But we may now define a new omega ordered set of Ursets

$$\upsilon^{\omega \cdot 2} =\!= \{x | \forall y (\upsilon_\omega \in y \wedge \forall z (\upsilon_z \in y \rightarrow \upsilon_{z+1} \in y) \rightarrow x \in y)\}.$$

$\upsilon^{\omega \cdot 2}$, and indeed υ^β for any ordinal β, may serve as sets of Ursets, or *Urelemente*, for whatever purpose one may have in mind, including that of naming extramathematical things to equip £ with domains useful for applied mathematics, including logic.

23 Heritors and regulars

A man is like a fraction whose numerator is what he is and whose denominator is what he thinks of himself. The larger the denominator, the smaller the fraction.

Leo Tolstoy

Heritors and regulars are defined, and their behavior regulated so as to support the development of the interpretation of NBG set theory of §25.

23.1. Definition: The *Heritor* is $\mathcal{H} \Longleftarrow \{x|x = \{y|y \in x\}\}$.

23.2. Definition: a is an *heritor* just if $\vdash^{\mathsf{M}} a \in \mathcal{H}$.

23.3. Definition: $\mathcal{H}(a) \Longleftarrow a = \{x|x \in a\}$

23.4. Theorem: The Heritor and heritors are orthodox.

Proof: The Heritor is orthodox by identity theory, and heritors by Postulate 6.5.2.6. □

23.5. Definition: a is an *hyposet* of set b just if $a = \{x|x \in a \wedge x \in b\}$.

23.6. Axiom:

$$\vdash^{\mathsf{M}} \mathcal{H}(a) \wedge \mathcal{H}(b) \wedge a \subset b \rightarrow a = \{x|x \in a \wedge x \in b\}.$$

23.7. Theorem:

$$\vdash^{\mathsf{M}} \mathcal{H}(a) \wedge \mathcal{H}(b) \wedge a \overset{\mathrm{E}}{=} b \rightarrow a = b.$$

Proof: An instance of Axiom 23.6 is $\vdash^{\mathsf{M}} \mathcal{H}(b) \wedge \mathcal{H}(a) \wedge b \subset a \rightarrow b = \{x|x \in b \wedge x \in a\}$. $\{x|x \in a \wedge x \in b\} = \{x|x \in b \wedge x \in a\}$, given §12, so just wed with the statement instance $\mathcal{H}(a) \wedge \mathcal{H}(b) \wedge a \subset b \rightarrow a = \{x|x \in a \wedge x \in b\}$ of Axiom 23.6. □

23.8. Axiom:

$$\vdash^{\mathsf{M}} \mathcal{H}(a) \wedge \mathcal{H}(b) \wedge a \subset b \leftarrow a = \{x|x \in a \wedge x \in b\},$$

so if a is a hyposet of b, then a and b are heritors, and a is a *subheritor* of b.

23.9. Theorem:

$$\vdash^{\mathsf{M}} \mathcal{H}(a) \wedge \mathcal{H}(b) \rightarrow (a \subset b \leftrightarrow a = \{x|x \in a \wedge x \in b\}).$$

Proof: Invoke Axioms 23.6 and 23.8. □

23.10. Axiom (Heritors are hereditarily heritors)

$$\vdash^{\mathsf{M}} \mathcal{H}(y) \rightarrow \forall x(x \in y \rightarrow \mathcal{H}(x)).$$

23.11. Observation: This section's axioms do not commit to the existence of heritors.

23.12. Definition (Regular sets)

$$\mathcal{R}(x) \Longleftarrow \exists y(y \in x) \to \exists y(y \in x \wedge \forall z(z \notin x \vee z \notin y))$$

23.13. Exercise Regular hereditarily orthodox sets are hereditarily regular.

Our attention below will be upon regular heritors.

24 Choice, power, potency and countability

The axiom of choice is obviously true, the well-ordering principle obviously false, and who can tell about Zorn's lemma?

Jerry Bona

Not all things worth counting are countable, and not all things that count are worth counting.

Albert Einstein

We show that the librationist universe is countable. Theorem 20.3 is one of the important reasons why that is so. Theorem 24.4.5 establishes that there is an orthodox bijection from the set of natural numbers ω to the full universe $\mathbf{U}$. §24.6 spells out how it is that Cantor's arguments, linked to power sets, are circumvented in £, with recourse to the bijection euro from ω to the universe, and the choice-function $\wp`x$ upon which it is based.

24.1 The denumerable wellordering

24.1.1. Definition:

$$\Pi a \Pi b \Big[\big(\mathsf{Constant}(a) \wedge \mathsf{Constant}(b) \wedge (\Xi, \alpha) \Vdash a \trianglelefteq b\big)$$

$$\Updownarrow$$

$$\big(\mu x(x \eta \Omega \ \& \ x \leq a \ \& \ (\Xi, \alpha) \Vdash x = a) \leq \mu y(y \eta \Omega \ \& \ y \leq b \ \& \ (\Xi, \alpha) \Vdash y = b)\big)\Big]$$

24.1.2. Corollary:

$$(\Xi, \alpha) \Vdash a = b \Leftrightarrow (\Xi, \alpha) \Vdash a \trianglelefteq b \ \& \ (\Xi, \alpha) \Vdash a \trianglerighteq b$$

24.1.3. Definition:

$$(\Xi, \alpha) \Vdash a \triangleleft b \Leftrightarrow (\Xi, \alpha) \Vdash a \trianglelefteq b \ \& \ (\Xi, \alpha) \Vdash a \neq b$$

24.1.4. Axiom The wellordering:

$$(\Xi, \alpha) \Vdash \forall x, y(x \triangleleft y \vee x = y \vee x \triangleright y)$$

24.1.5. Axiom The orthodoxy of the wellordering:

$\triangleleft$, and its cognate relations, are orthodox.

24.2 Function application notation

24.2.1. Definition:

$$f`a \risingdotseq b := \forall x \forall y \forall z\big(\big((x,y) \in f \wedge (x,z) \in f\big) \rightarrow y = z\big) \wedge (a,b) \in f.$$

24.2.2. Definition:

$$b \risingdotseq f`a := f`a \risingdotseq b$$

24.2.3. Definition:

$$x \in f`a := \exists y(f`a \risingdotseq y \wedge x \in y)$$

24.2.4. Definition:

$$f`a \in x := \exists y(f`a \risingdotseq y \wedge y \in x)$$

24.2.5. Remark: The notation $\risingdotseq$ is used instead of $=$, for there are paradoxical functions as e.g.

$$g = \{(x,y)|x = \{\emptyset\} \wedge ((r \in r \rightarrow y = \emptyset) \wedge (r \notin r \rightarrow y = \{\emptyset\}))\},$$

for $r = \{x|x \notin x\}$. For g we do have that $\vdash^{\mathsf{M}} \forall x \forall y \forall z\big(\big((x,y) \in g \wedge (x,z) \in g\big) \rightarrow y = z\big)$. But, notice that $(\Xi, \alpha) \Vdash (\{\emptyset\}, \{\emptyset\}) \in g$ just if $(\Xi, \alpha + 1) \Vdash (\{\emptyset\}, \emptyset) \in g$. So we cannnot write $g`\{\emptyset\} = \{\emptyset\}$ in the former case, and $g`\{\emptyset\} = \emptyset$ in the latter case. For identity is an orthodox equivalence relations. So we use $\risingdotseq$ to avoid problems with the theory of identity in exotic cases.

24.2.6. Remark: There certainly are sets of more orthodox functions so that a function h is an element in one of them only if $\stackrel{\mathsf{M}}{\vDash} \forall x \forall y(h`x \risingdotseq y \rightarrow \mathcal{T}\ulcorner h`x \risingdotseq y \urcorner)$.

24.2.7. Remark: The author introduced and discussed the notation $\risingdotseq$ in the article (Bjørdal 2008, pp. 55–66), whose English translation is «“2+2=4” is misleading», for such reasons which are adduced here.

24.3 The choice function

On account of ancient Greek διάλεξε, for *was selected*, we define $\wp`w$, the *atled* of w:

24.3.1. Definition The choice function:

$$\wp`w \risingdotseq \{x|(x \in w \wedge \forall y(y \in w \rightarrow x \trianglelefteq y)\}.$$

24.3.2. Definition Iterated choices from b:

$$a \risingdotseq \wp^m b \Leftrightarrow ((m = 0 \wedge a \risingdotseq \wp`b) \vee \exists n(n \in \omega \wedge m = n + 1 \ \wedge a \risingdotseq \wp`(b \setminus \bigcup_{i=0}^{i=n} \wp^i b))).$$

24.4 The enumerator

Given Axiom 24.1.5, the orthodoxy of ω and $\mathbf{U}$, and the fact that $\wp^n w$ is orthodox if w is orthodox, we posit

24.4.1. Definition of the Enumerator:

$$€ \Longleftarrow \{(n,x)|n \in \omega \wedge x \in \wp^n \mathbf{U}\}$$

24.4.2. Theorem (€ is orthdox)

Proof: As $\wp^n\mathbf{U}$, for $n \in \omega$ is orthodox. □

24.4.3. Theorem (The functionality of €)

$$\vdash^{\mathsf{M}} \forall x \forall y \forall z\big(\big((x,y) \in € \wedge (x,z) \in €\big) \rightarrow y = z\big)$$

Proof: Obvious, given Definitions 24.3.2 and 24.4.1 and Theorem 24.4.2. □

24.4.4. Theorem

$$\vdash^{\mathsf{M}} €`n \risingdotseq x \leftrightarrow (n,x) \in € \leftrightarrow n \in \omega \wedge x \in \wp^n(\mathbf{U})$$

Proof: On account of Definitions 24.2.1 and 24.4.1, and Theorem 24.4.3. □

24.4.5. Theorem

€ is a bijection from ω to $\mathbf{U}$.

Proof: Given §24.1, as the orders of ω and Ω match, and for any constant a, $a\eta\Omega$, as all sets are finite positive von Neumann ordinals according to the meta language. □

24.5 The enumeration postulates

For any ordinal α:

24.5.1. Postulate:

$$\Pi a \Pi b \Big(\text{constant}(a) \ \&\ \text{constant}(b) \Rightarrow (\Xi,\alpha) \Vdash \forall n(n \in \omega \rightarrow \big(\exists^{=n} x(x \lhd a) \wedge \exists^{=n} y(y \lhd b) \rightarrow a = b\big))\Big)$$

24.5.2. Postulate:

$$\Pi a,b,c,((\Xi,\alpha) \Vdash \forall n\Big(n \in \omega \rightarrow (\Big(\langle a,b\rangle \in € \ \&\ \exists^{=n} x(x \lhd b)\Big) \leftrightarrow \Big(\langle\{v|v \in a \vee v = a\}, c\rangle \in € \ \&\ \exists^{=(n+1)} x(x \lhd c)\Big))\Big)$$

24.5.3. Postulate:

$$(\Xi, \alpha) \Vdash \forall n(n \in \omega \rightarrow \exists y(\langle n, y\rangle \in €))$$

24.5.4. Postulate:

$$(\Xi, \alpha) \Vdash \forall y \exists n(n \in \omega \wedge \langle n, y\rangle \in €)$$

24.5.5. Postulate:

$$(\Xi, \alpha) \Vdash \forall n \forall n' \forall y(\langle n, y\rangle \in € \wedge \langle n', y\rangle \in € \rightarrow n = n')$$

24.5.6. Postulate:

$$(\Xi, \alpha) \Vdash \forall n \forall y \forall z(\langle n, y\rangle \in € \wedge \langle n, y\rangle \in € \rightarrow y = z)$$

Some consequences of the enumeration postulates:

24.5.7. Theorem:

$$\vdash^{\mathsf{M}} \exists^{=0} x(x \lhd \mathrm{L})$$

24.5.8. Theorem:

$$\Pi a \Pi b \Big(\text{constant}(a) \ \& \ \text{constant}(b) \Rightarrow \vdash^{\mathsf{M}} \forall n(n \in \omega \rightarrow \big(\exists^{=n} x(x \lhd a) \wedge \exists^{=n} y(y \lhd b) \rightarrow a = b\big))\Big)$$

24.5.9. Theorem:

$$\Pi b\Big[\text{constant}(b) \Rightarrow \vdash^{\mathsf{M}} \big(\langle \varnothing, b\rangle \in € \leftrightarrow \exists^{=0} x(x \lhd b)\big)\Big]$$

24.5.10. Theorem:

$$\Pi a, b, c, (\vdash^{\mathsf{M}} \forall n\Big(n \in \omega \rightarrow (\big(\langle n, b\rangle \in € \ \& \ \exists^{=n} x(x \lhd b)\big) \leftrightarrow \big(\langle \{v | v \in n \vee v = n\}, c\rangle \in € \ \& \ \exists^{=(n+1)} x(x \lhd c)\big))\Big))$$

24.5.11. Theorem:

$$\vdash^{\mathsf{M}} \forall y \exists n(n \in \omega \wedge \langle n, y\rangle \in €)$$

Proof: As all sets are finite von Neumann ordinals of the meta language, and ω has the same order as Ω. □

24.6 Absolutely all sets are countable

If some set is uncountable, some set of subsets of ω is uncountable. We have earlier introduced the power set $\mathcal{P}(\mathrm{a}) = \{x|x \subset \mathrm{a}\}$, and will first consider its import on the question. Thereon we consider the *potency set* of a set a as given by

24.6.1. Definition: $\mathcal{P}(\mathrm{a}) = \{x|x = \{y|y \in x \wedge y \in \mathrm{a}\}\}$.

The potency set construction is very important in §25. Here the preoccupation is with showing that neither power sets nor potency sets generate uncountable sets.

24.6.1 € restricted to $\mathcal{P}(\{x|x \in \omega\})$

€ restricted to the power set of $\{x|x \in \omega\}$ is

$$€|_{\mathcal{P}(\{x|x\in\omega\})} = \{(x,y)|(x,y) \in € \wedge y \in \mathcal{P}(\{x|x \in \omega\})\}, \qquad (24.6.2)$$

which has ω as domain and $\mathcal{P}(\{x|x \in \omega\})$ as range. Given Definitions 24.3.2 and 24.4.1, equation 24.6.2 may be equivalently stated as

$$€|_{\mathcal{P}(\{x|x\in\omega\})} = \{(x,y)|x \in \omega \wedge y \in \wp^{x}\mathbf{U} \wedge y \in \mathcal{P}(\{x|x \in \omega\})\}. \qquad (24.6.3)$$

24.6.4. Theorem

$$\vdash^{\mathsf{M}} \forall u \forall v \forall w((u,v) \in €|_{\mathcal{P}(\{x|x\in\omega\})} \wedge (u,w) \in €|_{\mathcal{P}(\{x|x\in\omega\})} \rightarrow v = w).$$

Proof: Obvious, from the built up of $€|_{\mathcal{P}(\{x|x\in\omega\})}$ with orthodox function $\wp^{x}$. □

To attempt Cantor's proof by contradiction for uncountability, assume that $€|_{\mathcal{P}(\{x|x\in\omega\})}$ surjects from ω to $\mathcal{P}(\{x|x \in \omega\})$ and posit

24.6.5. Definition:

$$\mathrm{S} = \{x|x \in \omega \wedge x \notin €|_{\mathcal{P}(\{x|x\in\omega\})}`x\}.$$

24.6.6. Theorem For an $m \in \omega$,

$$\vdash^{\mathsf{M}} (m, \mathrm{S}) \in €|_{\mathcal{P}(\{x|x\in\omega\})}.$$

Proof: A consequence of Equation 24.6.3 and alethic comprehension is

$$\vdash^{\mathsf{M}} (m, \mathrm{S}) \in €|_{\mathcal{P}(\{x|x\in\omega\})} \leftrightarrow \mathcal{T}\ulcorner m \in \omega \wedge \mathrm{S} \in \wp^{m}\mathbf{U} \wedge \mathrm{S} \in \mathcal{P}(\{x|x \in \omega\})\urcorner.$$

Let $m \in \omega$ be the natural number such that $\vdash^{\mathsf{M}} \mathrm{S} \in \wp^{m}\mathbf{U}$, so $\vdash^{\mathsf{M}} m \in \omega \wedge \mathrm{S} \in \wp^{m}\mathbf{U}$. But besides, $\vdash^{\mathsf{M}} \mathrm{S} \in \mathcal{P}(\{x|x \in \omega\})$, as $\vdash^{\mathsf{M}} \mathrm{S} \subset \{x|x \in \omega\}$. So $\vdash^{\mathsf{M}} m \in \omega \wedge \mathrm{S} \in \wp^{m}\mathbf{U} \wedge \mathrm{S} \in \mathcal{P}(\{x|x \in \omega\})$. Thus, on account of inference mode 9.1.2.1, $\vdash^{\mathsf{M}} \mathcal{T}\ulcorner m \in \omega \wedge \mathrm{S} \in \wp^{m}\mathbf{U} \wedge \mathrm{S} \in \mathcal{P}(\{x|x \in \omega\})\urcorner$. Finish by using the maxim mode 9.2.5. □

24.6.7. Theorem There is an $m \in \omega$ such that $\vdash^{\mathsf{M}} €|_{\mathcal{P}(\{x|x\in\omega\})}`m \risingdotseq \mathrm{S}$.

Proof: Invoke Theorems 24.6.4 and 24.6.6, and Definition 24.2.1. □

From Definition 24.6.5 and alethic comprehension,

$$\vdash^{\mathsf{M}} m \in \mathrm{S} \leftrightarrow \mathcal{T}\ulcorner m \in \omega \wedge m \notin \in\!\!\!|_{\mathcal{P}(\{x|x\in\omega\})}\text{‘}m\urcorner. \tag{24.6.8}$$

Given Definition 24.2.3,

$$\vdash^{\mathsf{M}} m \in \mathrm{S} \leftrightarrow \mathcal{T}\ulcorner m \in \omega \wedge \forall y(\in\!\!\!|_{\mathcal{P}(\{x|x\in\omega\})}\text{‘}m \doteqdot y \to m \notin y)\urcorner. \tag{24.6.9}$$

Given Theorems 24.6.4 and 24.6.7, and the fact that there is only one $m \in \omega$ such that $\mathrm{S} \in \wp^m\mathbf{U}$, for the appropriate m, $\forall y(\in\!\!\!|_{\mathcal{P}(\{x|x\in\omega\})}\text{‘}m \doteqdot y \leftrightarrow y = \mathrm{S})$. So that

$$\vdash^{\mathsf{M}} m \in \mathrm{S} \leftrightarrow \mathcal{T}\ulcorner m \in \omega \wedge m \notin \mathrm{S})\urcorner. \tag{24.6.10}$$

But it was assumed that $m \in \omega$, which is an orthodox statement, so that

$$\vdash^{\mathsf{M}} m \in \mathrm{S} \leftrightarrow \mathcal{T}\ulcorner m \notin \mathrm{S})\urcorner. \tag{24.6.11}$$

As

$$\vdash \mathcal{T}\ulcorner m \notin \mathrm{S})\urcorner \to m \notin \mathrm{S} \tag{24.6.12}$$

and

$$\vdash m \in \mathrm{S} \to \mathcal{T}\ulcorner m \in \mathrm{S})\urcorner, \tag{24.6.13}$$

it follows that

$$\vdash m \notin \mathrm{S} \tag{24.6.14}$$

and

$$\vdash \mathcal{T}\ulcorner m \notin \mathrm{S}\urcorner \to \mathcal{T}\ulcorner m \in \mathrm{S}\urcorner. \tag{24.6.15}$$

But

$$\vdash^{\mathsf{M}} \mathcal{T}\ulcorner m \in \mathrm{S}\urcorner \to \neg\mathcal{T}\ulcorner m \notin \mathrm{S}\urcorner, \tag{24.6.16}$$

so that

$$\vdash \mathcal{T}\ulcorner m \notin \mathrm{S}\urcorner \rightarrow \neg\mathcal{T}\ulcorner m \notin \mathrm{S}\urcorner, \tag{24.6.17}$$

and consequently

$$\vdash \neg\mathcal{T}\ulcorner m \notin \mathrm{S}\urcorner. \tag{24.6.18}$$

But an instance of the inference mode 9.1.2.11 is

$$\vdash \neg\mathcal{T}\ulcorner m \notin \mathrm{S}\urcorner \Rightarrow \vdash m \in \mathrm{S}, \tag{24.6.19}$$

so that

$$\vdash m \in \mathrm{S}. \tag{24.6.20}$$

A joining of equations 24.6.14 and 24.6.20 results in

$$\vdash m \in \mathrm{S} \;\&\; \vdash m \notin \mathrm{S}. \tag{24.6.21}$$

But this merely amounts to an argumentum ad paradoxo, and it has not been proven that $€|_{\mathcal{P}(\{x|x\in\omega\})}$ is not a function with domain ω which is onto its range $\mathcal{P}(\{x|x \in \omega\})$.

24.6.2 € restricted to $\mathcal{P}(\omega)$

The potency set of ω is

$$\mathcal{P}(\omega) = \{x|x = \{y|y \in x \wedge y \in \omega\}\}. \tag{24.6.22}$$

€ restricted to the potency set of ω is

$$€|_{\mathcal{P}(\omega)} = \{(x, y)|(x, y) \in € \wedge y \in \mathcal{P}(\omega)\}, \tag{24.6.23}$$

which has ω as domain and $\mathcal{P}(\omega)$ as its range. Given Definition 24.4.1, equation 24.6.23 may be equivalently stated as

$$€|_{\mathcal{P}(\omega)} = \{(x, y)|x \in \omega \wedge y \in \wp^{x}(\mathbf{U}) \wedge y \in \mathcal{P}(\omega)\}, \tag{24.6.24}$$

24.6.25. Fact €, $\mathcal{P}(\omega)$ and $€|_{\mathcal{P}(\omega)}$ are orthodox.

Proof: € is orthodox given Theorem 24.4.2, $\mathcal{P}(\omega)$ on account of the theory of identity, and $€|_{\mathcal{P}(\omega)}$ is orthodox because € and $\mathcal{P}(\omega)$ are orthodox. □

24.6.26. Fact

$$\vdash^{\mathsf{M}} \forall x\forall y\forall z((x, y) \in €|_{\mathcal{P}(\omega)} \wedge (x, z) \in €|_{\mathcal{P}(\omega)} \rightarrow y = z).$$

Proof: As $\Subset$ is functional. □

24.6.27. Assumption Orthodox function $\Subset|_{\mathcal{P}(\omega)}$ surjects from ω to $\mathcal{P}(\omega)$:

$$\forall w(w \in \mathcal{P}(\omega) \to \exists v(v \in \omega \wedge \Subset|_{\mathcal{P}(\omega)}`v \risingdotseq w)).$$

24.6.28. Definition:

$$\mathrm{S} = \{x|x \in \omega \wedge x \notin \Subset|_{\mathcal{P}(\omega)}`x\}.$$

24.6.29. Assumption $\mathrm{S} = \{y|y \in \omega \wedge y \in \mathrm{S}\}$: $\mathrm{S} \in \mathcal{P}(\omega)$.

24.6.30. Assumption S is orthodox.

Proof: From Assumption 24.6.29, Axiom 23.8 and Theorem 23.4. □

24.6.31. Assumption An $m \in \omega$ is such that $\Subset|_{\mathcal{P}(\omega)}`m \risingdotseq \mathrm{S}$.

Proof: From Assumption 24.6.27. □

24.6.32. Assumption $\vdash^{\mathsf{M}} \forall x(x \in \mathrm{S} \leftrightarrow x \in \omega \wedge x \notin \Subset|_{\mathcal{P}(\omega)}`x)$.

Proof: Given Definition 24.6.28 and the fact that S is orthodox. □

24.6.33. Assumption $\vdash^{\mathsf{M}} \forall x(x \in \mathrm{S} \leftrightarrow x \in \omega \wedge \forall y(\Subset|_{\mathcal{P}(\omega)}`x \risingdotseq y \to x \notin y)$.

Proof: On account of Definition 24.2.3 and Assumption 24.6.32. □

24.6.34. Assumption $\vdash^{\mathsf{M}} (m \in \mathrm{S} \to m \notin \mathrm{S})$.

Proof: It was agreed in Assumption 24.6.31 that for an $m \in \omega$, $\Subset|_{\mathcal{P}(\omega)}`m \risingdotseq \mathrm{S}$. □

24.6.35. Assumption $m \notin S \to \exists y(\Subset|_{\mathcal{P}(\omega)}`m \risingdotseq y \wedge m \in y)$.

Proof: From Assumption 24.6.33, the agreement of Assumption 24.6.31. □

24.6.36. Theorem For functional f:

$$\text{if } \vdash^{\mathsf{M}} \exists y(f`a \risingdotseq y \wedge a \in y) \text{ and } \vdash^{\mathsf{M}} f`a \risingdotseq c, \text{then } \vdash^{\mathsf{M}} a \in c.$$

Proof: Because $\vdash^{\mathsf{M}} [(a, y) \in f \wedge (a, c) \in f] \to y = c$, as f is functional, and because $\vdash^{\mathsf{M}} ((d, e) \in f \leftrightarrow f`d \risingdotseq e)$ if $\vdash^{\mathsf{M}}$ (f is functional). □

24.6.37. Assumption $\vdash^{\mathsf{M}} (m \notin \mathrm{S} \to m \in \mathrm{S})$.

Proof: Appeal to Assumption 24.6.35 and Theorem 24.6.36. □

24.6.38. Assumption $\vdash^{\mathsf{M}} m \in \mathrm{S} \wedge m \notin \mathrm{S}$

Proof: From Assumptions 24.6.34 and 24.6.37. □

The contradiction in the maximal context of Assumption 24.6.38 is false, so it follows that a previous assumption is to be discarded. We do that by stating the following

24.6.39. Theorem Assumption 24.6.29 is false, and so $\vdash^{\mathsf{M}} \mathrm{S} \neq \{x|x \in \mathrm{S} \ \wedge \ x \in \omega\}$.

Proof: The discussion in §24.6.2. □

25 $\mathfrak{T}$ and the theories of vonsets

If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is.

John von Neumann

Recall Definitions 23.3 and 23.12.

25.1. Definition Set theory $\mathfrak{T}$ is £ plus Axioms 23.6, 23.8, 23.10.

25.2. Definition $\mathfrak{T}\mathcal{H}\&\mathcal{R}(\mathbf{D})$ is $\mathfrak{T}$ plus $\mathcal{H}(\mathbf{D})$ plus $\mathcal{R}(\mathbf{D})$, with $\mathbf{D}$ as in Definition 25.4.6.

Let $\mathbf{NBGC}+\mathbf{TA}$ be Neumann-Bernays-Gödel set theory with Global Choice and Tarski's Axiom. An interpretation of $\mathbf{NBGC}+\mathbf{TA}$ is developed in $\mathfrak{T}\mathcal{H}\&\mathcal{R}(\mathbf{D})$ below.

Natural weakenings and extensions of $\mathbf{NBGC}+\mathbf{TA}$ are as well taken to be theories of *vonsets*. Needless to say, but all vonsets are sets, though some sets are not vonsets.

The term "natural" in the previous paragraph is left undefined, as investigations should not be restrained. So we here disregard philosophical quandaries related to the fact that the term "vonset" may have different *meanings*, whatever that is, in natural extensions of $\mathbf{NBG}$ which are not consistent with each other, such as **NBG** + the *Axiom of choice*, and **NBG** + the *Axiom of determinacy*.

25.1 The potency vonset

We saw in §20 that power sets as classically defined are mathematically useless, as they are paradoxical lest of a non-paradoxical universal set.

Potency vonsets are potency sets, as all vonsets are sets.

The notion of potency set was introduced in Definition 24.6.1:

$$\mathcal{P}(a) =\!= \{x|x = \{y|y \in x \land y \in a\}\}.$$

25.1.1. Theorem: The *potency vonset* of a vonset a contains precisely a's hypovonsets, in the sense of Definition 23.5.

Proof: Use Axioms 23.10 and 25.4.10 and Theorem 23.9.entail that vonsets are heritors, and from Axioms 23.6 and 23.8. □

25.1.2. Theorem:

$\mathcal{P}(a)$ is orthodox, and all of its members are hereditarily heritors.

Proof: $\mathcal{P}(a)$ is orthodox by the logic of identity. Its members, if any, are heritors on account of Axiom 23.8, and are hereditarily heritors given Axiom 23.10. □

25.1.3. Theorem:

$\mathcal{P}(a)$ is empty if a is not an heritor.

Proof: Appeal to Axiom 23.8. □

25.1.4. Theorem:

$$\vdash^{\mathsf{M}} \forall x(x \in \mathcal{P}(a) \leftrightarrow \mathcal{H}(x) \wedge \mathcal{H}(a) \wedge x \subset a).$$

Proof: Appeal to Theorem 23.9 and Definition 24.6.1. □

25.2 The Grothendieck vonset of w relative to v

25.2.1. Definition: Let

$$\begin{aligned} \mathrm{G}(v, w, v_0, v_1,) \Longleftrightarrow & \forall y\Big(w \in y \wedge \forall z\big[z \in y \rightarrow (z \in \mathcal{P}(v_1) \wedge \mathcal{P}(z) \in \mathcal{P}(v_1) \wedge \mathcal{P}(z) \in y)\big] \wedge \\ & \forall z\big(z \in \mathcal{P}(y) \wedge z \notin y \rightarrow \exists f[f \in v \wedge \text{Bijection}(f) \wedge \\ & (\forall x_0)(x_0 \in y \rightarrow \exists x_1(x_1 \in z \wedge (x_0, x_1) \in f)) \\ & (\forall x_1)(x_1 \in z \rightarrow \exists x_0(x_0 \in y \wedge (x_0, x_1) \in f))]\big) \rightarrow v_0 \in y\Big) \end{aligned}$$

Use Theorem 19.1.5 to obtain the manifestation set with parameters $\mathcal{G}(v, w)$,

25.2.2. Theorem The *Grothendieck* of w relative to v:

$$\begin{aligned} \vdash^{\mathsf{M}} \forall u \forall w\big(u \in \mathcal{G}(\mathrm{v}, \mathrm{w}) \leftrightarrow \mathcal{T}\mathcal{T} \forall y\Big(& w \in y \wedge \forall z\big[z \in y \rightarrow \\ & (z \in \mathcal{P}(\mathcal{G}(\mathrm{v}, \mathrm{w})) \wedge \mathcal{P}(z) \in \mathcal{P}(\mathcal{G}(\mathrm{v}, \mathrm{w})) \wedge \mathcal{P}(z) \in y)\big] \wedge \\ & \forall z\big(z \in \mathcal{P}(y) \wedge z \notin y \rightarrow \exists f[f \in v \wedge \text{Bijection}(f) \wedge \\ & (\forall x_0)(x_0 \in y \rightarrow \exists x_1(x_1 \in z \wedge (x_0, x_1) \in f)) \\ & (\forall x_1)(x_1 \in z \rightarrow \exists x_0(x_0 \in y \wedge (x_0, x_1) \in f))]\big) \rightarrow u \in y\Big)\big) \end{aligned}$$

25.2.3. Theorem: $\mathcal{G}(v, w)$ is orthodox for orthodox v and w, so that

$$\begin{aligned} \vdash^{\mathsf{M}} \forall u \forall w\big(u \in \mathcal{G}(\mathrm{v}, \mathrm{w}) \leftrightarrow \forall y\Big(& w \in y \wedge \forall z\big[z \in y \rightarrow \\ & (z \in \mathcal{P}(\mathcal{G}(\mathrm{v}, \mathrm{w})) \wedge \mathcal{P}(z) \in \mathcal{P}(\mathcal{G}(\mathrm{v}, \mathrm{w})) \wedge \mathcal{P}(z) \in y)\big] \wedge \\ & \forall z\big(z \in \mathcal{P}(y) \wedge z \notin y \rightarrow \exists f[f \in v \wedge \text{Bijection}(f) \wedge \\ & (\forall x_0)(x_0 \in y \rightarrow \exists x_1(x_1 \in z \wedge (x_0, x_1) \in f)) \\ & (\forall x_1)(x_1 \in z \rightarrow \exists x_0(x_0 \in y \wedge (x_0, x_1) \in f))]\big) \rightarrow u \in y\Big)\big) \end{aligned}$$

Proof: As in the proof that ω is orthodox, of Theorem 13.2.3 on page 43, noting that $\mathcal{P}(\mathcal{G}(\mathrm{v}, \mathrm{w}))$ is an orthodox heritor by cause of Theorem 25.1.2. □

25.2.4. Remark: For appropriate v and w, Theorem 25.2.3 amounts to *Tarski's axiom*, which states that all sets are members of a Grothendieck-universe. Tarski-Grothendieck set theory is usually presented as **ZFC** + Tarski's axiom.

25.3 Capture

In this section we presuppose that the sets and conditions invoked are orthodox.

25.3.1. Definition Capture with B from w:

$$\mathcal{C}(\mathrm{B}, w) \Longleftarrow \{x|\exists y(y \in w \wedge \forall z((x,y)^z_y \in \mathrm{B} \leftrightarrow y = z))\}$$

25.3.2. Theorem: Capture is equivalent with replacement.

Proof: i) If a vonset is obtained from capture with B from w, it can be obtained from replacement by using the functional condition $\forall z((x,y)^z_y \in \mathrm{B} \leftrightarrow y = z)$. ii) If a vonset is obtained from replacement by functional B so that $\forall x \forall y \forall z((x,y) \in \mathrm{B} \wedge (x,z) \in \mathrm{B} \rightarrow y = z)$, it can be obtained from capture by using the condition as in Definition 25.3.1. □

25.3.3. Theorem: Capture, as replacement, entails specification.

Proof: Use the functional $\mathrm{B}' \Longleftarrow \{(x,y)|x \in \mathrm{B} \wedge x = y)\}$ as capture vonset relative to a vonset a, and observe that the existence of the vonset $\{x|x \in a \wedge \mathrm{B}(x)\}$ is justified by capture and extensionality, which holds for $\mathbf{V}$ and $\mathbf{D}$ below, as per Theorem 25.4.13. □

25.4 V and D

25.4.1. Definition of the *drift* of u:

$$\begin{aligned}\mathfrak{D}(u) = \{w|w \in u \vee \forall v\big(\big[&u \in v \wedge \mathrm{E} = \{(x_i,x_j)|(x_i,x_j) \in u^2 \wedge x_i \in x_j\} \in v \wedge \\ &\forall x_i(x_i \in v \rightarrow \{y|y \in u \wedge y \notin x_i\} \in v) \wedge \forall x_i \forall x_j(x_i \in v \wedge x_j \in v \rightarrow x_i \cap x_j \in v) \wedge \\ &\forall x_i(x_i \in v \rightarrow \mathrm{dom}(x_i) = \{y|\exists x((y,x) \in x_i)\} \in v) \wedge \\ &\forall x_i(x_i \in v \rightarrow \{y|\exists x_j, x_k(y = (x_j,x_k) \wedge x_j \in x_i \wedge v_k \in v_1)\} \in v) \wedge \\ &\forall x_i(x_i \in v \rightarrow \{y|\exists x_j \exists x_k \exists x_l(y = (x_j,x_k,x_l) \wedge (x_k,x_l,x_j) \in x_i\} \in v) \wedge \\ &\forall x_i(x_i \in v \rightarrow \{y|\exists x_j \exists x_k \exists x_l(y = (x_j,x_k,x_l) \wedge (x_j,x_l,x_k) \in x_i\} \in v)\big] \rightarrow w \in v\big)\}\end{aligned}$$

25.4.2. Definition of $\mathrm{V}(v_0, v_1)$:

$$\begin{aligned}\mathrm{V}(v_0,v_1) = \forall v\big((&\omega \in v \wedge \forall w \in v \forall x \in v : \{w,x\} \in v \wedge \forall w \in v : \bigcup w \in v \wedge \\ &\forall w \in v : \mathcal{P}(w) = \{x|x = \{y|y \in x \wedge y \in w\}\} \in v \wedge \\ &\forall w \in v : \wp(w) = \{x|(x \in w \wedge \forall y(y \in w \rightarrow x \trianglelefteq y))\} \in v \wedge \\ &\forall w \in v : \mathcal{G}(\mathfrak{D}(v_1), w) \in v \wedge \\ &\forall w \in v \forall \mathrm{B} \in \mathfrak{D}(v_1) : \mathcal{C}(\mathrm{B}, w) = \{x|\exists y(y \in w \wedge \forall z((x,y)^z_y \in \mathrm{B} \leftrightarrow y = z))\} \in v) \\ &\rightarrow v_0 \in v\big)\end{aligned}$$

25.4.3. Definition of **V** via manifestation from Definition 25.4.2:

$$\vdash^{\mathsf{M}} \forall u\Big[u \in \mathbf{V} \leftrightarrow \mathcal{T}\mathcal{T}\forall v\big[[\omega \in v \wedge \forall w \in v \forall x \in v : \{w,x\} \in v \wedge \forall w \in v : \bigcup w \in v \wedge \forall w \in v : \mathcal{P}(w) = \{x | x = \{y | y \in x \wedge y \in w\}\} \in v \wedge \forall w \in v : \wp(w) = \{x | (x \in w \wedge \forall y(y \in w \to x \trianglelefteq y)\} \in v \wedge \forall w \in v : \mathcal{G}(\mathfrak{D}(\mathbf{V}), w) \in v \wedge \forall w \in v \forall \mathrm{B} \in \mathfrak{D}(\mathbf{V}) : \mathcal{C}(\mathrm{B}, w) = \{x | \exists y(y \in w \wedge \forall z((x,y)^z_y \in \mathrm{B} \leftrightarrow y = z))\} \in v] \to u \in v\big]\Big]$$

As **V** is orthodox on account of Theorems 23.4 and 25.4.11,

25.4.4. Theorem:

$$\vdash^{\mathsf{M}} \forall u\Big[u \in \mathbf{V} \leftrightarrow \forall v\big[[\omega \in v \wedge \forall w \in v \forall x \in v : \{w,x\} \in v \wedge \forall w \in v : \bigcup w \in v \wedge \forall w \in v : \mathcal{P}(w) = \{x | x = \{y | y \in x \wedge y \in w\}\} \in v \wedge \forall w \in v : \wp(w) = \{x | (x \in w \wedge \forall y(y \in w \to x \trianglelefteq y)\} \in v \wedge \forall w \in v : \mathcal{G}(\mathfrak{D}(\mathbf{V}), w) \in v \wedge \forall w \in v \forall \mathrm{B} \in \mathfrak{D}(\mathbf{V}) : \mathcal{C}(\mathrm{B}, w) = \{x | \exists y(y \in w \wedge \forall z((x,y)^z_y \in \mathrm{B} \leftrightarrow y = z))\} \in v] \to u \in v\big]\Big]$$

25.4.5. The drift equation: $\mathbf{D} = \mathfrak{D}(\mathbf{V})$.

25.4.6. Definition of the drift of all classes:

$$\mathbf{D} = \{w | w \in \mathbf{V} \vee \forall v\big(\big[\mathbf{V} \in v \wedge \mathrm{E} = \{(x_i,x_j) | (x_i,x_j) \in \mathbf{V}^2 \wedge x_i \in x_j\} \in v \wedge \forall x_i(x_i \in v \to \{u | u \in \mathbf{V} \wedge u \notin x_i\} \in v) \wedge \forall x_i \forall x_j(x_i \in v \wedge x_j \in v \to x_i \cap x_j \in v) \wedge \forall x_i(x_i \in v \to \mathrm{dom}(x_i) = \{w | \exists x((w,x) \in x_i)\} \in v) \wedge \forall x_i(x_i \in v \to \{u | \exists x_j, x_k(u = (x_j,x_k) \wedge x_j \in x_i \wedge v_k \in \mathbf{V})\} \in v) \wedge \forall x_i(x_i \in v \to \{u | \exists x_j \exists x_k \exists x_l(u = (x_j,x_k,x_l) \wedge (x_k,x_l,x_j) \in x_i\} \in v) \wedge \forall x_i(x_i \in v \to \{u | \exists x_j \exists x_k \exists x_l(u = (x_j,x_k,x_l) \wedge (x_j,x_l,x_k) \in x_i\} \in v)\big] \to w \in v\big)\}$$

25.4.7. Fact: $\vdash^{\mathsf{M}} \mathbf{V} \subset \mathbf{D}$.

25.4.8. Theorem The definition of **V**, with recourse to **D**:

$$\vdash^{\mathsf{M}} \forall u\Big[u \in \mathbf{V} \leftrightarrow \forall v\big[[\omega \in v \wedge \forall w \in v \forall x \in v : \{w,x\} \in v \wedge \forall w \in v : \bigcup w \in v \wedge \forall w \in v : \mathcal{P}(w) = \{x | x = \{y | y \in x \wedge y \in w\}\} \in v \wedge \forall w \in v : \wp(w) = \{x | (x \in w \wedge \forall y(y \in w \to x \trianglelefteq y)\} \in v \wedge \forall w \in v : \mathcal{G}(\mathbf{D}, w) \in v \wedge \forall w \in v \forall \mathrm{B} \in \mathbf{D} : \mathcal{C}(\mathrm{B}, w) = \{x | \exists y(y \in w \wedge \forall z((x,y)^z_y \in \mathrm{B} \leftrightarrow y = z))\} \in v] \to u \in v\big]\Big]$$

25.4.9. Definition: **V** is the class of all vonsets.

25.4.10. Axiom: $\mathcal{H}(\mathbf{D})$.

25.4.11. Theorem: $\mathcal{H}(\mathbf{V})$.

Proof: On account of Axioms 23.10 and 25.4.10. □

25.4.12. Corollary: **V** and **D** are orthodox.

Proof: Use Axiom 25.4.10, Theorem 25.4.11 and Theorem 23.4. □

25.4.13. Theorem: Co-extensional members of $\mathbf{V} \cup \mathbf{D}$ are identical.

Proof: Use Axioms 23.10 and 25.4.10, Theorem 25.4.11 and, finally, Theorem 23.7. □

25.4.14. Theorem: $\mathbf{V} = \{x | x \in \mathbf{V} \wedge x \in \mathbf{D}\}$.

Proof: As $\mathcal{H}(\mathbf{V})$ and $\mathcal{H}(\mathbf{D})$ on account of Axiom 25.4.10 and Theorem 25.4.11, appeals to Theorems 23.9 and Fact 25.4.7 suffice to finish the proof. □

25.4.15. Axiom The drift is wellfounded $\mathcal{R}(\mathbf{D})$.

25.4.16. Theorem All classes are wellfounded.

Proof: Invoke the result of Exercise 23.13. □

25.4.17. Theorem All vonsets are wellfounded.

Proof: Given Fact 25.4.7, a vonset in **V** is as well a class member of **D**. So the vonset is wellfounded on account of Theorem 25.4.16. □

25.4.18. Theorem **D** is not a class.

Proof: If **D** were a class, it would on account of Definition 25.4.6 follow that $\mathbf{D} \in \mathbf{D}$, which contradicts Axiom 25.4.15. □

25.4.19. Remark: Instead of postulating Axiom 25.4.15, one may obtain a suitable regular class $\mathbf{V}^*$ of all regular vonsets by taking it to be the class of all elements of a potency set of an ordinal in **V**. That invokes the consistency proof of **ZFC** with regularity given the consistency of $\mathbf{ZFC}^- = \mathbf{ZFC}$ without regularity, by (Kunen 1980, chapter 3), or a similar relative consistency proof. Given Kunen's result, however, and the relative consistency results obtained earlier by (Skolem 1923) and (Neumann 1929), we know that we can safely posit Axiom 25.4.15.

25.4.20. Theorem: **V** is not a vonset.

Proof: Appeal to Definition 25.4.9 and Theorem 25.4.15. □

25.5 Primitive theorems for classes

We leave is as an exercise to prove the following from Definition 25.4.6.

25.5.1. THEOREM **V** is a class:

$$\mathbf{V} \in \mathbf{D}.$$

25.5.2. THEOREM Membership class:

$$\mathrm{E} = \{(x, y) | x \in \mathbf{V} \wedge y \in \mathbf{V} \wedge x \in y\} \in \mathbf{D}.$$

25.5.3. THEOREM Intersection class:

$$\forall \mathrm{A} \in \mathbf{D} \forall \mathrm{B} \in \mathbf{D} \exists \mathrm{C} \in \mathbf{D} \forall x (x \in \mathrm{C} \leftrightarrow x \in \mathrm{A} \wedge x \in \mathrm{B}).$$

25.5.4. THEOREM Complement class:

$$\forall \mathrm{A} \in \mathbf{D} \exists \mathrm{B} \in \mathbf{D} \forall x (x \in \mathrm{B} \leftrightarrow x \notin \mathrm{A}).$$

25.5.5. THEOREM Domain class:

$$\forall \mathrm{A} \in \mathbf{D} \exists \mathrm{B} \in \mathbf{D} \forall x (x \in \mathrm{B} \leftrightarrow \exists y ((x, y) \in \mathrm{A})).$$

25.5.6. THEOREM Product by V class:

$$\forall \mathrm{A} \in \mathbf{D} \exists \mathrm{B} \in \mathbf{D} \forall x (x \in \mathrm{B} \leftrightarrow \exists y \exists z (x = (y, z) \wedge y \in \mathrm{A} \ \wedge z \in \mathrm{V})).$$

25.5.7. THEOREM Circular permutation class:

$$\forall \mathrm{A} \in \mathbf{D} \exists \mathrm{B} \in \mathbf{D} \forall x \forall y \forall z ((x, y, z) \in \mathrm{B} \leftrightarrow (y, z, x) \in \mathrm{A}).$$

25.5.8. THEOREM Transposition class:

$$\forall \mathrm{A} \in \mathbf{D} \exists \mathrm{B} \in \mathbf{D} \forall x \forall y \forall z ((x, y, z) \in \mathrm{B} \leftrightarrow (x, z, y) \in \mathrm{A}).$$

25.6 The Tuple-lemmas

25.6.1. LEMMA:

$$\forall \mathrm{A} \in \mathbf{D} \exists \mathrm{B}_1 \in \mathbf{D} \forall x \forall y \forall z ((x, y, z) \in \mathrm{B}_1 \leftrightarrow (x, y) \in \mathrm{A} \wedge z \in \mathrm{V}).$$

25.6.2. LEMMA:

$$\forall \mathrm{A} \in \mathbf{D} \exists \mathrm{B}_2 \in \mathbf{D} \forall x \forall y \forall z ((x, z, y) \in \mathrm{B}_2 \leftrightarrow (x, y) \in \mathrm{A} \wedge z \in \mathrm{V}).$$

25.6.3. LEMMA:

$$\forall \mathrm{A} \in \mathbf{D} \exists \mathrm{B}_3 \in \mathbf{D} \forall x \forall y \forall z ((z, x, y) \in \mathrm{B}_3 \leftrightarrow (x, y) \in \mathrm{A} \wedge z \in \mathrm{V}).$$

25.6.4. LEMMA:

$$\forall \mathrm{A} \in \mathbf{D} \exists \mathrm{B}_4 \in \mathbf{D} \forall x \forall y ((y, x) \in \mathrm{B}_4 \leftrightarrow (x, y) \in \mathrm{A}).$$

Proof: Use Theorem 25.5.6 to get B_1, Theorem 25.5.8 on B_1 to get B_2, Theorem 25.5.7 on B_1 to get B_3, and use Theorem 25.5.7 on B_2, plus Theorem 25.5.5, to get B_4. □

25.7 The class existence theorem

25.8 The expansion lemma

25.9 Proof that V is orthodox

25.10 Proof that all members of V are orthodox

As $\mathcal{H}$ by Axiom ... This is done already.

25.11 Global well ordering

Useful explanation of

Global well ordering given global choice.

26 Space for librationist category theory?

La filosofia è scritta in questo grandissimo libro, che continuamente ci sta aperto innanzi agli occhi (io dico l'Universo), ma non si può intendere, se prima non il sapere a intender la lingua, e conoscer i caratteri ne quali è scritto. Egli è scritto in lingua matematica, e i caratteri son triangoli, cerchi ed altre figure geometriche, senza i quali mezzi è impossibile intenderne umanamente parola; senza questi è un aggirarsi vanamente per un oscuro labirinto.

Galilei

The author has learned that set theories as $\mathbf{NBGC} + \mathbf{TA}$ are considered ideal for category theory, and wants to investigate whether that can be done in the librationist framework set up for mentioned set theories in §25.

27 The theory of vansets NF in $\mathfrak{B}\mathcal{H}(\mathbf{W})$

The analogy between the myth of mathematics and the myth of physics is, in some additional and perhaps fortuitous ways, strikingly close. Consider, for example, the crisis which was precipitated in the foundations of mathematics, at the turn of the century, by the discovery of Russell's paradox and other antinomies of set theory. These contradictions had to be obviated by unintuitive, ad hoc devices; our mathematical myth-making became deliberate and evident to all. But, what, of physics? An antinomy arose between the undular and the corpuscular accounts of light; and if this was not as out-and-out a contradiction as Russell's paradox, I suspect that the reason is that physics is not as out-and-out as mathematics.

Willard van Orman Quine, in (Quine 1961, pp. 18–19)

We give an account of Willard van Quine's set theory *New Foundations*, of (Quine 1937), via the axiomatization offered by (Hailperin 1944, p. 10), which is adapted here:

$P_0 : \exists\beta\forall x(x \in \beta \leftrightarrow \exists y(x \in y \wedge x \notin y)$

$P_1 : \forall u\forall v\exists\beta\forall x(x \in \beta \leftrightarrow (x \notin u \vee x \notin v))$

$P_2 : \forall\alpha\exists\beta\forall x\forall y((\{x\}, \{y\}) \in \beta \leftrightarrow (x, y) \in \alpha)$

$P_3 : \forall\alpha\exists\beta\forall x\forall y\forall z((x, y, z) \in \beta \leftrightarrow (x, y) \in \alpha)$

$P_4 : \forall\alpha\exists\beta\forall x\forall y\forall z((x, z, y) \in \beta \leftrightarrow (x, y) \in \alpha)$

$P_5 : \forall\alpha\exists\beta\forall x\forall y((y, x) \in \beta \leftrightarrow x \in \alpha)$

$P_6 : \forall\alpha\exists\beta(x \in \beta \leftrightarrow \forall u((u, \{x\}) \in \alpha))$

$P_7 : \forall\alpha\exists\beta\forall x\forall y((y, x) \in \beta \leftrightarrow (x, y) \in \alpha)$

$P_8 : \exists\beta\forall x(x \in \beta \leftrightarrow \exists y(x = \{y\}))$

$P_9 : \exists\beta\forall x\forall y((\{x\}, y) \in \beta \leftrightarrow x \in y)$

Notice that P_0 was not included in (Hailperin 1944, p. 10).

$\mathbf{U}$ was reserved for the full universal set $\{x|x = x\}$ of £. In the previous section $\mathbf{V}$ was reserved for the class of all vonsets, as defined via manifestation there.

$\mathbf{W}$, with associated mnemonic device *die Welt*, is reserved the Quinean vanset of all vansets, as defined via manifestation below in this section.

27.1. DEFINITION:

$$
\begin{aligned}
\mathrm{W}(v_0, v_1) \Longleftarrow \forall v\Big(\big[&\{x|\exists y(x \in y \wedge x \notin y)\} \in v \wedge \\
&\forall w \forall x(w \in v \wedge x \in v \rightarrow \{y \in v_1|\ (y \notin w \vee y \notin x)\} \in v) \wedge \\
&\forall w(w \in v \rightarrow \{(\{x\}, \{y\} \in v_1|(x, y) \in w\} \in v) \wedge \\
&\forall w(w \in v \rightarrow \{(x, y, z) \in v_1|(x, y) \in w\} \in v) \wedge \\
&\forall w(w \in v \rightarrow \{(x, z, y) \in v_1|(x, y) \in w\} \in v) \wedge \\
&\forall w(w \in v \rightarrow \{(y, x) \in v_1|(x, y) \in w\} \in v) \wedge \\
&\forall w(w \in v \rightarrow \{x \in v_1|\forall y(y \in v_1 \rightarrow (y, \{x\}) \in w)\} \in v) \wedge \\
&\forall w(w \in v \rightarrow \{(y, x) \in v_1|x \in w\} \in v) \wedge \\
&\forall w(w \in v \rightarrow \{x \in v_1|\exists y(y \in v_1 \wedge x = \{y\})\} \in v) \wedge \\
&\forall w(w \in v \rightarrow \{(\{x\}, y) \in v_1|x \in y\} \in v)\big] \\
&\rightarrow v_0 \in v\Big)
\end{aligned}
$$

Use Definitions 27.1 and 19.1.2 to obtain

27.2. THEOREM:

$$
\begin{aligned}
\forall u(u \in \mathbf{W} \leftrightarrow \mathcal{T}\mathcal{T}\forall v\Big(\big[&\{x|\exists y(x \in y \wedge x \notin y)\} \in v \wedge \\
&\forall w \forall x(w \in v \wedge x \in v \rightarrow \{y \in \mathbf{W}|(y \notin w \vee y \notin x)\} \in v) \wedge \\
&\forall w(w \in v \rightarrow \{(\{x\}, \{y\} \in \mathbf{W}|(x, y) \in w\} \in v) \wedge \\
&\forall w(w \in v \rightarrow \{(x, y, z) \in \mathbf{W}|(x, y) \in w\} \in v) \wedge \\
&\forall w(w \in v \rightarrow \{(x, z, y) \in \mathbf{W}|(x, y) \in w\} \in v) \wedge \\
&\forall w(w \in v \rightarrow \{(y, x) \in \mathbf{W}|(x, y) \in w\} \in v) \wedge \\
&\forall w(w \in v \rightarrow \{x \in \mathbf{W}|\forall y(y \in \mathbf{W} \rightarrow (y, \{x\}) \in w)\} \in v) \wedge \\
&\forall w(w \in v \rightarrow \{(y, x) \in \mathbf{W}|x \in w\} \in v) \wedge \\
&\forall w(w \in v \rightarrow \{x \in \mathbf{W}|\exists y(y \in \mathbf{W} \wedge x = \{y\})\} \in v) \wedge \\
&\forall w(w \in v \rightarrow \{(\{x\}, y) \in \mathbf{W}|x \in y\} \in v)\big] \\
&\rightarrow u \in v\Big))
\end{aligned}
$$

27.3. THEOREM: $\mathbf{W}$ is orthodox.

Proof: Adapt the the proof of Theorem 13.2.3. □

27.4. THEOREM:

$$\begin{aligned}\forall u(u \in \mathbf{W} \leftrightarrow \forall v\Big(\big[&\{x|\exists y(x \in y \wedge x \notin y)\} \in v \wedge\\ &\forall w\forall x(w \in v \wedge x \in v \rightarrow \{y \in \mathbf{W}|(y \notin w \vee y \notin x)\} \in v) \wedge\\ &\forall w(w \in v \rightarrow \{(\{x\},\{y\} \in \mathbf{W}|(x,y) \in w\} \in v) \wedge\\ &\forall w(w \in v \rightarrow \{(x,y,z) \in \mathbf{W}|(x,y) \in w\} \in v) \wedge\\ &\forall w(w \in v \rightarrow \{(x,z,y) \in \mathbf{W}|(x,y) \in w\} \in v) \wedge\\ &\forall w(w \in v \rightarrow \{(y,x) \in \mathbf{W}|(x,y) \in w\} \in v) \wedge\\ &\forall w(w \in v \rightarrow \{x \in \mathbf{W}|\forall y(y \in \mathbf{W} \rightarrow (y,\{x\}) \in w)\} \in v) \wedge\\ &\forall w(w \in v \rightarrow \{(y,x) \in \mathbf{W}|x \in w\} \in v) \wedge\\ &\forall w(w \in v \rightarrow \{x \in \mathbf{W}|\exists y(y \in \mathbf{W} \wedge x = \{y\})\} \in v) \wedge\\ &\forall w(w \in v \rightarrow \{(\{x\},y) \in \mathbf{W}|x \in y\} \in v)\big]\\ &\rightarrow u \in v\Big))\end{aligned}$$

Proof: A consequence of Theorem 27.2 as $\mathbf{W}$ is orthodox, given Theorem 27.3. □

27.5. AXIOM: $\mathcal{H}(\mathbf{W})$

27.6. THEOREM: Co-extensional sets in $\mathbf{W}$ are identical.

Proof: Use Axiom 23.10 and Theorem 23.7. □

The proper identity for $\mathbf{W}$ is of course given by

27.7. DEFINITION:

$$a \overset{\mathbf{W}}{=} b \Longleftrightarrow \forall v(v \in \mathbf{W} \rightarrow (a \in v \rightarrow b \in v).$$

By Axiom 27.5, Theorem 27.6 and Theorem 27.4 combined with the results of (Hailperin 1944), it follows that $\mathfrak{W}\mathbf{W}$ accounts for Quine's set theory **NF**.

28 ⌜A⌝ is true just if A states the truth

La logique est l'hygiène des mathématiques.
André Weil

The following perspective upon the semantics is useful for some purposes.

28.1. Definition: The closure ordinal ♀ is *the truth*.

28.2. Definition: The *way* of sentence A is $[\delta : \delta \leq ♀\ \&\ (\Xi, \delta) \Vdash \mathrm{A}]$.

28.3. Definition: A *states* the supremum of its way.

28.4. Definition: A *expresses* its way.

28.5. Definition: ⌜A⌝ is true just if A states the truth.

28.6. Definition: ⌜A⌝ is false just if ⌜$\neg A$⌝ is true.

28.7. Definition: The way of $A \wedge B$ is the way of intersected with the way of B.

28.8. Definition: The way of $\neg A$ is the truth minus the way of A.

Here the sentence ⌜A⌝ *is true* may be interpreted as $\vdash \mathcal{T}$⌜A⌝, and the sentence A *states the truth* as equivalent with $\vdash \mathrm{A}$.

Moreover, "just if" is here to be interpreted via the bidirectional entailment in

$$\vdash \mathcal{T}\ulcorner \mathrm{A} \urcorner \Leftrightarrow \vdash \mathrm{A}.$$

It is a fact that

$$\vdash \neg\mathcal{T}\ulcorner \mathrm{A} \urcorner \Leftrightarrow \vdash \neg\mathrm{A},$$

so, consequently,

$$\vdash^{\mathsf{M}} \mathcal{T}\ulcorner \mathrm{A} \urcorner \Leftrightarrow \vdash^{\mathsf{M}} \mathrm{A}.$$

The connectives are not truth-functional in librationism, but they are *way-functional*, and can be accounted for by following classical interdefinability connections as in any Boolean algebra: The way of the negation $\neg\mathrm{A}$ of A, is truth minus the way of A, and the way of the conjunction $\mathrm{A} \wedge \mathrm{B}$ is the intersection of the way of A and the way of B. The ways of sentences built up from other connectives follow from their definitions in terms $\neg$ and $\wedge$.

According to librationism, a true paradoxical sentence L and its true companion sentence $\neg\mathrm{L}$ complement each other. For the way of L, as defined in Definition 28.2, is in such a case a set of ordinals with ♀ as least upper bound, whereas as well the way of $\neg\mathrm{L}$ is a set of ordinals with ♀ as least upper bound; moreover, the ways of L and $\neg\mathrm{L}$ do not overlap. Thus, by the Definition 28.4, L does not express the same as what $\neg\mathrm{L}$ expresses, for L and $\neg\mathrm{L}$ have different ways.

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