Librationist cum classical theories of sets

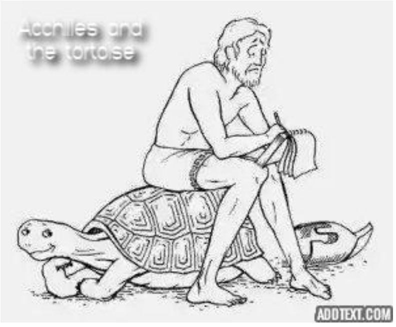
Draft - comments appropriate.

Frode Alfson Bjørdal

a

Copyright © 2023 Frode Alfson Bjørdal All rights reserved.

b



c

for Miriam

d

Preface

The investigations, which led to *librationist set theory*, began thirty years ago, in the spring of 1993, during an afternoon with coffe and a notebook, at Café Ni Muser, in Trondheim, Norway. I had just been employed as professor of Philosophy at *Universitetet i Trondheim*, as was then the endonym of *Norwegian University of Science and Technology*.

It was a struggle to build upon imprecise thoughts to express beliefs precise enough to be useful. But I pressed on, and began publishing unfinished ideas already in 1997, with [3].

There were two factors which gave me impetus to continue publishing essays, which I did not consider reasonably professional, according to desirable standards for mathematics and logic.

Firstly, the word *"essay"* signifies *trial* or *attempt*. Mathematical reality is still without well known resolutions, of the paradoxes, which have gained full support. As I, from the outset, have sincerely believed that I stumbled upon a distinct, and advantageous, approach, which might be developed, and brought to bear on the problems, my essays continued.

Secondly, I thought it was important to mark priority, as I thought the ideas could become useful for philosophically and mathematically satisfactory type free foundations for reasoning.

Acknowledgement

Several *essays* on librationism were published over the years, and I often thought that I had let too many see the light of day. The first was *Towards a Foundation for Type-Free Reasoning*, which was published in [3], fifteen years before I, eleven years ago, had decided upon using the neologism *librationism*, in [4]. [5] and [6] were printed in the first interval.

Most publications were lectures delivered at some of the annual LOGICA congresses, under the auspices of the Czech Academy of Sciences: in the Czech republic, in 1997, 2004, 2005 and 2010; at some of the annual Logic Colloquia, under the auspices of the Association of Symbolic Logic: in Barcelona in 2011, Vienna in 2014, Helsinki in 2014, Stockholm in 2015, and Prague in 2019; and for Sociedade Brasileira de Lógica in Petrópolis, 2014, and Salvador, 2022.

Lectures on librationism were as well delivered for The Steklov Mathematical Institute, in Moscow, at the Russian Academy of Sciences, in 2014; for the Euler International Mathematical Institute at the Steklov Mathematical Institute's division in Saint Petersburgh, in 2015; and for logic and mathematics events in Melbourne, 2008, Lisbon, 2010, Guangzhou, 2011, Rio de Janeiro, 2013 and Kolkata, 2014.

Most importantly, I delivered many lectures for the Seminary in Logic at the University of Oslo, where I have had the opportunity to talk about these matters for more then twenty five years.

All these encounters were important for my mathematical development. Resistance I met was usually useful. I cannot share responsibility for errors, and will not risk the names of others by thanking them especially.

Contents

Introduction

"Nur wenn man nicht auf den Nutzen nach aussen sieht, sondern in der Mathematik selbst auf das Verhältnis der unbenutzten Teile, bemerkt man das andere und eigentliche Gesicht dieser Wissenschaft. Es ist nicht zweckbedacht, sondern unökonomisch und leidenschaftlich. [. . .] Die Mathematik ist Tapferkeitsluxus der reinen Ratio, einer der wenigen die es heute gibt."

Robert Musil, *Der mathematische Mensch*, `Mitteilungen der Deutschen Mathematiker-Vereinigung`, No°20, page 50, 1912.

This text is as long as it is, and it is a shortening of, and improvement upon, a much longer one, which was at hand some few days ago. The length of the text which was at hand was one of the reasons I decided to use the documentclass for books. Another reason is that the book class makes it possible to number definitions, theorems, lemmas and equations with the same counter, as can be seen in Section 11.4, and a third one is that I experience better control over the manuscript by using the book class. But none of these reasons are written in stone, and the text may be rewrittee to submit as an article.

There will not be a focus upon the librationist treatments of paradoxes here, but rather upon how to achieve a librationist interpretation of classical set theory.

So I may address some of the more philosophical issues in this introduction, and some others may be addressed incidentally *en passant*, at various other places.

We regard a theory as inconsistent just if it has theses of the form $A \wedge \neg A$, and take it to be a contradictory just if it has contradictions as theses. So we hold that a theory is inconsistent just if it is contradictory. Given this, £ is consistent and not contradictory. As a consequence, librationism is not a dialetheist point of view, for *dialetheism* is canonically characterized, e.g. in [25], as a view which takes some contradictions to be true. Moreover, £ is not a paraconsistent point of view, as the latter are not conservative in the sense of Definition 4.0.1. Librationism, per Definition 4.0.4, may instead be taken to offer an extra-ordinary point of view. To distinguish, take librationism as well to offer a *bialethic* point of view, and not a dialetheic one.

Librationism meets a challenge which it is difficult to see can be met if one holds that contradictions are true. The challenge, for dialetheism and as well for librationism, is to explain what a true paradoxical sentence p says, which its true negation p $\neg p$contradicts. According to librationism, as set out here, the true sentence p and the true sentence $\neg p$are indeed contradictory. For the track of p opposes the track of $\neg p$, thus, by Definition 14.0.4, what p says opposes what $\neg p$ says, and so p contradicts $\neg p$.

A remark on designator is appropriate. One might hold that a theory is not a set theory if it presupposes more linguistic resources than *the language of set theory*, understood as first order logic plus the symbol $\in$. This tenet is not followed here, and it is instead presupposed that set theoretic reality should be investigated with such rescources which best reveal it. As will beceome clear, we make use of set abstracts, and these are not eliminable, due to the fact that ℔ and £ are highly non-extensional theories. The symbol $\in$, however, *is* eliminable.

As ℔ interprets classical set theory, it would at least seem misleading to hold that ℔ is not itself a set theory.

Nevertheless, a postulation of, and theory on *names* of terms, and *sentences*, is developed in Section 5. (For the use of small upper case letters, see Definition 1.4.4.) Names are taken as *Urelemente* in £ and ℔. Introduced predicates $\mathcal{T}$, for *is true*, $\mathcal{F}$, for *is false*, and $\mathcal{D}$, for *is provable in classical logic*, hold for some names. A motivation is to obtain a recovery of many of the identity losses caused by the prevalent non-extensionality of £ and ℔, and it also serves to connect the Liar type paradoxes with set theoretic paradoxes. So it might be said that librationism is more than a set theory, but that does not entail that it is not a set theory.

Chapter 1

Formal language

1.1 The number standpoint

The standpoint, which shall be presupposed here, is stronger than that adopted by Kurt Gödel in his classical 1931 work [14]. For it is here assumed that the formal expressions, as the variables, quantifiers, connectives, formulas, and so on, in £ and $\mathfrak{T}$, *are* natural numbers. Said natural numbers are denoted by numerals, of a bijective numeral system, which, as per chapter 5, are associated with *names* of formal expressions. Apposition of expressions, and other manipulations, are accounted for arithmetically.

1.2 The inclusion of abstracts

The pristine primitive formal language of £ and $\mathfrak{T}$ is Polish, and without identity symbol or membership symbol. The inclusion of abstracts is a trait shared with (Gandy, 1959), and with many contributions to the literature on non-classical set theories, including some which were at the time called *property theories*[1], as e.g. (Gilmore, 1974), and theories by (Cantini, 1996), and others, where abstracts were used because the principle of extensionality fails.

Set theoretic principles beyond number theory are presupposed, to provide the semantics.

1.3 The exclusion of "=" and "$\in$" from the primitive language

The membership relation will be defined by means of apposition of terms, and the identity relation is defined so that a and b are identical just if they are members of the same sets.

[1] I think the term "property theory", despite, I believe, its origin with Kurt Gödel, became unfortunate. The opening sentence of Roger Myhill's article *Paradoxes*, in Synthese 60 (1984), 129-143, is: "Gödel said to me more than once "There never were any set-theoretic paradoxes, but the property-theoretic paradoxes are still unresolved"; and he may well have said the same thing in print."

This remark, and cognates, must, from my experience, have had such influence that some later authors used the term "property-theory", for non-extensional set theories, which seek to account for more type-free accounts that approximate naive abstraction in dealing with the paradoxes.

Nevertheless, there are so many non-extensional set theories in the set theory literature, unrelated to theories which attempt to deal with the paradoxes, that it is not reasonable to consider them *property theoretic* and not *set theoretic*. I do not know that Gödel was aware of Dana Scott's contribution in [28], where the author proves the consistency of ZF including extensionality given the consistency of ZF without the extensionality axiom. Also, I do not know that Gödel was aware of Gilmore's contribution [13], and succeeding ones; moreover, Gödel did probably not study [12].

1.4 Metalinguistic conventions

Definition 1.4.1

(1) $==$ is metamathematical identification.[2]

(2) $\alpha, \beta, \gamma, \delta, \ldots$ are arbitrary ordinals.

(3) $m, n, o, p, \ldots$ are arbitrary finite ordinals.

(4) $\prec, \preceq, \succeq,$ and $\succ$ are the orderings on ordinals.

(5) Σ is the existential quantifier.

(6) Π is the universal quantifier.

(7) $\sim$ is negation.

(8) & is conjunction.

(9) ⓡ is disjunction.

(10) $\Rightarrow$ is for implication.

(11) $\Leftrightarrow$ is for bi-implication.

(12) $[x\colon \ \ldots]$ is for sets in the metalanguage.

(13) Ω is the term for the set of natural numbers in the underlying theory.

(14) ε stands for membership.

(15) μ is for the least operator.

(16) $a, b, c, d, e, f, a', \ldots$ are arbitrary terms.

(17) $i, j, k, l, m, n, i' \ldots$ are for numerical variables.

(18) $o, p, q, r, s, t, o', \ldots$ are for distinguished constants.

(19) $u, v, w, x, y, z, u', \ldots$ are arbitrary variables.

(20) $A, B, C, D, E, F, A', \ldots$ are arbitrary formulas.

(21) Other letters, or letter-like symbols, may be used as names of distinguished constants.

Definition 1.4.2 (The primitive signs, and their natural number denotata)

(a) •

(b) v̈

(c) $\downarrow$

[2]Occasionally $\stackrel{def}{==}$ will be used.

(d) $\forall$

(e) ς

(f) c̈

are the primitive signs, which denote numbers 1, 2, 3, 4, 5 and 6, in Ω, respectively.

Definition 1.4.3 (Strings of symbols-denotata)

(1) $\ell(n) == \lfloor log_6((n+1)\cdot(6-1))\rfloor$ invokes the floor function $\lfloor\ \rfloor$, and defines the length of the numeral needed to express the positive natural number n in the bijective base-6 numeral system.

(2) Concatenation $\frown$ is the function given by $m \frown n == m \cdot 6^{\ell(n)} + n$.

(3) We know that $\frown$, so defined, is associative.

(4) $m \frown n$ may be thought of as the apposition mn, when one thinks of m and n in terms of bijective base-6 numerals, which is what we do for strings of symbols of the formal language, via Definitions 1.4.2.

Definition 1.4.4 (The underlines)

To emphasize, and remind, that the expressions are used denote numbers, we in this definition underline, and write <u>variable</u>, <u>term</u>, <u>formula</u>, <u>sentence</u>, <u>constant</u>, and so on. To lighten the text, the underlines will not be used as from the next chapter.

Definition 1.4.5 (Primitive versus numerical forms)

An expression is in *primitive form* if a string of primitive signs, as per Definition 1.4.2, and in *numerical form* if it is the corresponding bijective base-6 numeral.

Definition 1.4.6 (<u>Variables</u>)

(1) v̈ is a <u>variable</u>.

(2) A <u>variable</u> succeeded by • is a <u>variable</u>.

(3) Nothing else is a <u>variable</u>.

(4) <u>Variables</u> are <u>terms</u>.

Definition 1.4.7 (<u>Handles</u>)

(1) c̈ is a <u>handle</u>.

(2) A <u>handle</u> succeeded by • is a <u>handle</u>.

(3) Nothing else is a <u>handle</u>.

(4) <u>Handles</u> are <u>terms</u>, without free <u>variables</u>, and so also, as per 1.4.11, <u>constants</u>.

Definition 1.4.8 (Terms and formulas)

(1) If a and b are terms, ba is a formula.

(2) If A and B are formulas, $\downarrow AB$ is a formula.

(3) If A is a formula and v is a variable, $\forall vA$ is a formula.

(4) If A is a formula and v is a variable, ςvA is a term.

(5) Nothing else is a term or a formula.

Definition 1.4.9 (Binders, binds and ties)

(1) In $\forall vA$, $\forall$ is the *binder*, and v is the *bind* of A as well as the *tie* of $\forall$. A is the *scope* of $\forall$.

(2) In ςvA, ς is the *binder*, and v is the *bind* of A as well as the *tie* of ς. A is the *scope* of ς.

Definition 1.4.10 (Free and bound variables)

(1) A variable occurrence in a formula, or term, is bound, just if it is a bind, or it is in the scope of a binder with another occurrence as tie.

(2) Variable occurrences in a formula, or term, are *free* if not bound.

(3) A variable is free in a formula, or term, just if an occurrence is.

(4) A variable is bound in a formula, or term, just if an occurrence is.

Definition 1.4.11 (Sentences and constants)

(1) A term without free variables is a *constant*.

(2) A formula without free variables is a sentence.

Definition 1.4.12 (Substitution)
If A is a formula (a is a term), b is a term and v is a variable, A^b_v is the formula (a^b_v is the term) obtained by substituting all free occurrences of v in A (a) with the term b.

Definition 1.4.13 (Substitutability)
Term b is substitutable for variable v in formula A (term a) just if no free occurrence of v in formula A (term a) is in the scope of a binder, with bind y, where y is a variable of b.

Definition 1.4.14 (Postfixed variable vector notation)
We may write $A(x, y, z)$ to signify that variables x, y and z are free in A, and, e.g., $a(x, y)$ to mean that variables x and y are free in a. $A(x, y, b)$ is short for $A(x, y, z)^b_z$.

Definition 1.4.15 (Prefixed variable vector notation)
Occasionally $\forall\vec{v}A$ is written. It conveys the idea that $\forall\vec{v}A$ is a sentence, and for some $n > 0$, and variables $v_1 \dots v_n$, $\forall\vec{v}A$ is $\forall v_1 \dots \forall v_n A$, or $n = 0$ and $\forall\vec{v}A$ is A.

Definition 1.4.16 (Parentheses, and defined operators for the object language)

(1) Delimiters for punctuation

(2) $\neg A == \downarrow AA$

(3) $(A \wedge B) ==\downarrow \neg A \neg B$

(4) $(A \vee B) == \neg \downarrow AB$

(5) $(A \rightarrow B) == (\neg A \vee B)$

(6) $(A \leftrightarrow B) == (A \rightarrow B) \wedge (B \rightarrow A)$

(7) $\exists v A == \neg \forall v \neg A$

(8) $a \in b == ba$

(9) $\{v|A\} == \varsigma v A$

Definition 1.4.17
Predicates $\mathcal{T}$, $\mathcal{F}$ and $\mathcal{D}$ are introduced in Section 5, and predicate $\mathcal{H}$ in Section **??**. These predicates are sets, but only have *propositions*, built up like those which were defined in Section 5, as members. The fact that these predicates do not have members which have members, answers for the choice of a distinct font.

Chapter 2

Semantics

The underlying theory in the metalanguage is $\Sigma_3\mathrm{KP}\Omega$ – Kripke-Platek set theory, with Σ_3-collection & -separation – where Ω is the least infinite von Neumann ordinal of the meta language.

2.1 The truth operator

Definition 2.1.1

To facilitate reading and accord with (Bjørdal, 2012) we posit:

$$\mathfrak{T}A == (\exists v)(v \in \{w|A\}), \quad \text{for } w == \mu x(\text{variable}(x)\ \&\ \text{x not free in } A)\ \&$$
$$v == \mu x \succ w(\text{variable}(x)\ \&\ \text{x not free in } A)$$

The conditions on v, w, x are to obtain uniformity, so that $\mathfrak{T}A$ has a unique definition.

2.2 Fair functions

Definition 2.2.1

For Γ a real number, i.e. a set of natural numbers, and A a formula, we write $\Gamma \Vdash A$ for $A\varepsilon\Gamma$.

A function Ξ from ordinal numbers to real numbers, i.e. sets of natural numbers, is *fair*, just if for any ordinal α:

Definition 2.2.2 (Fair functions)

(1) $\Xi(\alpha) \Vdash\downarrow AB$ just if neither $\Xi(\alpha) \Vdash A$ nor $\Xi(\alpha) \Vdash B$

(2) $\Xi(\alpha) \Vdash \forall vA(v)$ just if $\Xi(\alpha) \Vdash A^b_v$ for all b substitutable for v in A

(3) $\alpha \succ 0 \Rightarrow (\Xi(\alpha) \Vdash \mathfrak{T}A \Leftrightarrow \Sigma\gamma(\gamma \prec \alpha\ \&\ \Pi\delta(\gamma \preceq \delta \prec \alpha \Rightarrow \Xi(\delta) \Vdash A)))$

Theorem 2.2.3 (The Omega standard)

$$\Xi(\alpha) \Vdash \exists v A \Leftrightarrow \Xi(\alpha) \Vdash A^b_v \text{ for some } b \text{ substitutable for } v \text{ in } A.$$

Proof. $\Xi(\alpha) \Vdash \exists v A \Leftrightarrow \Xi(\alpha) \Vdash \neg\forall v \neg A \overset{1}{\Leftrightarrow} \Xi(\alpha) \nVdash \forall v \neg A \overset{2}{\Leftrightarrow}$ it is not the case that $\Xi(\alpha) \Vdash \neg A^b_v$ for all $b's$ substitutable for v in $A \overset{1}{\Leftrightarrow}$ for some b substitutable for v in A, $\Xi(\alpha) \nVdash \neg A^b_v \overset{1}{\Leftrightarrow} \Xi(\alpha) \Vdash A^b_v$ for some b substitutable for v in A. □

2.3 Closure

Definition 2.3.1 (Cover, stabilization and closure)

(1) $\mathrm{IN}(\alpha, A, \Xi)$ just if $\Pi\beta(\alpha \preceq \beta \Rightarrow \ \Xi(\beta) \Vdash \mathfrak{T}A)$

(2) $\mathrm{OUT}(\alpha, A, \Xi)$ just if $\Pi\beta(\alpha \preceq \beta \Rightarrow \ \Xi(\beta) \nVdash \mathfrak{T}A)$

(3) $\mathrm{IN}(A, \Xi)$ just if $\Sigma\alpha \mathrm{IN}(\alpha, A, \Xi)$

(4) $\mathrm{OUT}(A, \Xi)$ just if $\Sigma\alpha \mathrm{OUT}(\alpha, A, \Xi)$

(5) $\mathrm{STAB}(A, \Xi)$ just if $\mathrm{IN}(A, \Xi)$ ⓡ $\mathrm{OUT}(A, \Xi)$

(6) $\mathrm{UNSTAB}(A, \Xi)$ just if $\sim\mathrm{STAB}(A, \Xi)$

(7) α *covers* Ξ just if $\mathrm{IN}(A, \Xi) \Rightarrow \mathrm{IN}(\alpha, A, \Xi)$

(8) α *stabilizes* Ξ just if α covers Ξ, and $\Xi(\alpha) \Vdash \mathfrak{T}A \Rightarrow \mathrm{IN}(A, \Xi)$

(9) The *closure* ordinal ♀ is the least stabilizing ordinal

Theorem 2.3.2 (Herzberger)

There is a closure ordinal:

Proof. It is enough to show that there is a stabilizing ordinal:

Definition 2.3.3

$\mathsf{h}(A) = \mu\alpha(\mathrm{IN}(\alpha, A, \Xi))$

Fact 2.3.4

$\beta = \mathsf{h}(A)$ is Π_2 in the Levy hierarchy, as it is is equivalent with

$$\Pi\gamma(\beta \preceq \gamma \Rightarrow \Xi(\gamma) \Vdash \mathfrak{T}A) \ \& \ \Pi\delta(\Pi\gamma(\delta \preceq \gamma \Rightarrow \Xi(\gamma) \Vdash \mathfrak{T}A) \Rightarrow \beta \preceq \delta)$$

1. We first show that there is a covering ordinal:

   We have

$$\Pi A(\mathrm{IN}(A,\Xi) \Rightarrow \Sigma\beta(\beta = \mathsf{h}(A))) \tag{2.3.5}$$

   So

$$\Pi A\Sigma\beta(\mathrm{IN}(A,\Xi) \Rightarrow \beta = \mathsf{h}(A)) \tag{2.3.6}$$

   Π_2–collection and quantifier rules give us

$$\Pi B\Sigma Y\Pi A(A\varepsilon B \Rightarrow \Sigma\beta(\beta\varepsilon Y \& (\beta = \mathsf{h}(A)))) \tag{2.3.7}$$

   Instantiate with $B = [A\colon \mathrm{IN}(A,\Xi)]$ to obtain

$$\Sigma Y\Pi A(\mathrm{IN}(A,\Xi) \Rightarrow \Sigma\beta(\beta\varepsilon Y \& (\beta = \mathsf{h}(A))) \tag{2.3.8}$$

   Let Z be a witness for (2.3.8), and define the least covering ordinal by means of Π_2–separation,

$$\varkappa = [\nu\colon \nu\varepsilon Z \ \&\ Ordinal(\nu) \ \&\ \Sigma A(\mathrm{IN}(A,\Xi) \ \&\ \nu = \mathsf{h}(A))] \tag{2.3.9}$$

   Fact 2.3.4 entails that we invoked Π_2-collection in the step from (2.3.6) to (2.3.7), and, as Π_n-collection implies Σ_{n+1} collection in the context of Kripke–Platek set theory, this justifies our choice of $\Sigma_3 KP\Omega$ as the underlying set theory in the meta language.[1]

   It is worth stressing that $\mathfrak{T}$ above all gets its strength from its closures principles for *heritors*, as introduced in Definition 11.2.3, and not from the strength of the set theory needed to show that there is a closure ordinal. It is for that reason $\mathfrak{T}$ obtains sufficient strength to interpret ZF.

2. We next prove that there is a stabilizing ordinal:

Let $[f(n)\colon n\varepsilon\Omega]$, by some adaptation of Cantor's pairing function, be an enumeration of all elements of $\mathrm{UNSTAB}(\Xi)$ where each element recurs infinitely often so that if B=$f(m)$ and $m \prec n\varepsilon\Omega$, then there is a natural number o, $n \prec o\varepsilon\Omega$, such that $f(o) = B$. Let $g(0) = \varkappa$ and $g(n+1) =$ the least $\nu > g(n)$ such that

$$\Xi(\nu) \Vdash f(n) \Leftrightarrow \Xi(g(n)) \not\Vdash f(n)$$

Let $\ = [\gamma\colon \Sigma m\Sigma\nu(m\varepsilon\Omega \ \&\ \nu = g(m) \ \&\ \gamma\varepsilon\nu)]$. It is obvious that is a limit ordinal which covers Ξ. It is also clear that if $m \prec n\varepsilon\Omega$ then $g(m) \prec g(n)$. Since covers Ξ, it suffices to show that $\Xi() \Vdash \mathfrak{T}B$ entails that B in $\mathrm{STAB}(\Xi)$, to establish that stabilizes Ξ.

[1] [32] shows that $KP + \Sigma_3$-Determinacy suffices for the semantic account of a commensurate system AQI (*Arithmetical Quasi Induction*) introduced in [7], and [16] shows this equivalent to $KP + \Pi^1_2$-Monotone Induction. So a Σ_3-admissible ordinal may not be necessary, but it may be needed for the kind of proof we present which connects with the coding of the formal language with natural numbers of the meta theory. Welch has pointed out in private communication that a Σ_2-admissible ordinal, without further assumptions, can be proven to be insufficient.

Suppose $\Xi() \Vdash \mathfrak{T}B$. It follows that

a) $\Sigma\nu\Pi\xi(\nu \preceq \xi \prec \ \Rightarrow \Xi(\xi) \Vdash B)$

Since g is increasing with as its range, we will for some natural number $m\varepsilon\Omega$ have that $\nu \preceq g(m) \prec$, so that

b) $\Pi\xi(g(m) \preceq \xi \prec \ \Rightarrow \Xi(\xi) \Vdash B)$

Suppose $B \notin \mathrm{STAB}(\Xi)$. By our enumeration of unstable elements where each term recurs infinitely often, we have that $B = f(n)$ for some natural number n, $m \prec n \in \Omega$. It follows that $g(m) \prec g(n) \prec$. From a) and b) we can infer that $\Xi(g(n)) \Vdash B$, since we supposed that $\Vdash \mathfrak{T}B$. From the construction of the function g it follows that $\Xi(g(n+1)) \nVdash \neg B$, contradicting b). It follows that $\Xi() \Vdash \mathfrak{T}B$ only if $B \in \mathrm{STAB}(\Xi)$, so stabilizes Ξ. □

2.4 Validity

Definition 2.4.1

A is *valid* just if for all fair functions Ξ, $\Xi(♀) \Vdash \mathfrak{T}A$

r

for all fair functions Ξ, $\Xi(♀) \Vdash \neg\mathfrak{T}\neg A \wedge \neg\mathfrak{T}A$

Definition 2.4.2 (Varieties of tautology)

(1) $\vDash A$ for all fair functions Ξ, $\Xi(♀) \Vdash \neg\mathfrak{T}\neg A$

(2) $\vDash_{\mathsf{m}} A$ for all fair functions Ξ, $\Xi(♀) \Vdash \neg\mathfrak{T}\neg A \wedge \neg\mathfrak{T}A$

(3) $\overset{\mathsf{M}}{\vDash} A$ for all fair functions Ξ, $\Xi(♀) \Vdash \mathfrak{T}A$

$\vDash A$ signifies that A is valid, or a tautology, and the symbol is occasionally used when it is left open whether A is a maximal or a minor validity. $\vDash_{\mathsf{m}} A$ signifies that A is a *minor* tautology, while $\overset{\mathsf{M}}{\vDash} A$ means that A is a *maximal* validity.

Theorem 2.4.3 (Relations between varieties of tautology)

$$
\begin{array}{lllllll}
(1) & \vDash_{\mathsf{m}} A & \Leftrightarrow & \vDash A & \& & \vDash \neg A \\
(2) & \vDash A & \Leftrightarrow & \overset{\mathsf{M}}{\vDash} A & \text{r} & \vDash_{\mathsf{m}} A \\
(3) & \overset{\mathsf{M}}{\vDash} A & \Leftrightarrow & \vDash A & \& & \nvDash \neg A
\end{array}
$$

Proof. (1) is obvious.

The entailment from the right side to the left side of (2): this holds on account of (I) the fact that if A holds on all ordinals below the closure ordinal ♀, as from an ordinal $\gamma \prec$ ♀, so that $\Xi(♀) \Vdash \mathfrak{T}A$, and therefore $\overset{\mathsf{M}}{\vDash} A$, then A is unbounded under the closure ordinal, so that also $\Xi(♀) \Vdash \neg\mathfrak{T}\neg A$ and $\vDash A$; and (II) the fact that $\vDash_{\mathsf{m}} A \Rightarrow \vDash A$.

The entailment from the left side to the right side of (2): this holds as a consequence of *tertium non datur*. If A is unbounded under the closure ordinal, so that $\Xi(\female) \Vdash \neg\mathfrak{T}\neg A$, either also $\neg A$ is unbounded under the closure ordinal, so that $\models_{\mathsf{m}} A$, or $\neg A$ is not unbounded under the closure ordinal, i.e. $\Xi(\female) \nVdash \neg\mathfrak{T}\neg\neg A$. But in the latter case it follows, from Definition 2.2.2.1., that $\Xi(\female) \Vdash \mathfrak{T}A$, and we are done.

The entailment from the left side to the right side in (3): It follows from the first sentence of the proof of (2) that $\models^{\mathsf{M}} A \Rightarrow \models A$. Clearly, if $\models^{\mathsf{M}} A$, so that for all fair functions Ξ, $\Xi(\female) \Vdash \mathfrak{T}A$, we cannnot have $\models \neg A$, which holds just if for all fair functions Ξ, $\Xi(\female) \Vdash \neg\mathfrak{T}A$. So we are done.

The entailment from the right side to he left side of (3): The content in terms of Definition 2.4.2 is that

$$\Pi\Xi(\Xi(\female) \Vdash \neg\mathfrak{T}\neg A) \Rightarrow (\Sigma\Xi'(\Xi'(\female) \Vdash \mathfrak{T}A) \Rightarrow \Pi\Xi''(\Xi''(\female) \Vdash \mathfrak{T}A)).$$

In Prenex normal form this is:

$$\Sigma\Xi\Pi\Xi'\Pi\Xi''(\Xi(\female) \Vdash \neg\mathfrak{T}\neg A \Rightarrow (\Xi'(\female) \Vdash \mathfrak{T}A \Rightarrow \Xi''(\female) \Vdash \mathfrak{T}A)),$$

which is clearly true. So we are done. □

2.5 Varieties of orthodoxy

Definition 2.5.1

A sentence A is paradoxical just if A is a minor, i.e. $\models_{\mathsf{m}} A$.

Definition 2.5.2

A set a is paradoxical just if $b \in a$ is a minor for some b.

Definition 2.5.3

(1) Formula A is *orthodox* just if $\models^{\mathsf{M}} \forall\vec{v}(\mathfrak{T}A \vee \mathfrak{T}\neg A)$.

(2) Term a is orthodox just if $x \in a$ is orthodox.

(3) Formula A is *apocryphal* just if orthodox and $\not\models^{\mathsf{M}} A$ as well as $\not\models^{\mathsf{M}} \neg A$.

(4) Term a is apocryphal just if $b \in a$ is apocryphal for a term b.

(5) Formula A is *canonical* just if orthodox and not apocryphal.

(6) Term a is *canonical* just if orthodox and not apocryphal.

An apocryphal set: Let $s == \{v|v \in v\}$. s is apocryphal, for $s \in s$ is apocryphal. $s \in s$ is apocryphal as it is orthodox and as we neither have $\models^{\mathsf{M}} s \in s$ nor $\models^{\mathsf{M}} s \notin s$. We do not even have $\vdash s \in s$ or $\vdash s \notin s$, given soundness, and the fact that neither $\models s \in s$ nor $\models s \notin s$.

Canonical formulas and sets abound. $x = \{y|y \in y \vee y \notin y\}$ and $\{x|\exists y(x \in y)\}$ are examples.

2.6 Non-triviality assumptions

The assumption that there *are* fair functions for £, and $\mathfrak{T}$ below, amounts to the non-triviality of the systems. There are fair functions for $\mathfrak{T}$ if ZFC is consistent, and for £ under much weaker assumptions.

Chapter 3

Maxims, theses and modes

Section 2.4 gave a semantic distinction between *maximal* valdities, or *maxims*, and *minor* validities, or minors. In the following we invoke the syntactic notions *maxims*, *minors*, *theses*, *modes* and *alethic comprehension*.

On account of the important Theorem 2.4.3, we may develop £ and $\mathfrak{T}$ as logical systems, but with the peculiarty that we will have two turnstiles defined in terms om the one $\vdash$, in the same vein as Theorem 2.4.3.

Presentation resolve 3.0.1

We write $\vdash^{\mathsf{M}} A$ just if A is a maxim schema, and $\vdash A$ if A is thesis schema which may have minor instance. $\vdash_{\mathsf{m}} A$ is written just if is established that $\vdash A$ and $\vdash \neg A$, i.e. that A is a *minor*. The relations between the theses correspond with the relations between the varieties of tautology, as set forth in Theorem 2.4.3:

Theorem 3.0.2 (Relations between the varieties of theses)

$$
\begin{array}{llllll}
(1) & \vdash_{\mathsf{m}} A & \Leftrightarrow & \vdash A & \& & \vdash \neg A \\
(2) & \vdash A & \Leftrightarrow & \vdash^{\mathsf{M}} A & \textcircled{\textbf{r}} & \vdash_{\mathsf{m}} A \\
(3) & \vdash^{\mathsf{M}} A & \Leftrightarrow & \vdash A & \& & \nvdash \neg A
\end{array}
$$

Presentation resolve 3.0.3

We assume that £ and $\mathfrak{T}$ are sound, so that when $\vdash^{\mathsf{M}} A$ we also have that $\vDash^{\mathsf{M}} A$, and so that $\vDash B$ holds if $\vdash A$. Given this, we will mostly only need to present the syntactic version of central postulates, as it is understood that the semantical versions hold as well. Notice, however, that the statement that $\Xi(\alpha) \vDash A$ for all ordinals α is stronger than the statement that $\vDash^{\mathsf{M}} \alpha$, and in some sections, as Section 12, the posited postulates make use of the stronger statement.

3.1 Truth maxims

Postulate 3.1.1 (Classical logic maxims)

(L_1) $\vdash^{\mathsf{M}} A \rightarrow (B \rightarrow A)$

(L_2) $\overset{\mathsf{M}}{\vdash} (A \to (B \to C)) \to ((A \to B) \to (A \to C))$

(L_3) $\overset{\mathsf{M}}{\vdash} (\neg B \to \neg A) \to (A \to B)$

(L_4) $\overset{\mathsf{M}}{\vdash} \forall x(A \to B) \to (\forall xA \to \forall xB)$

(L_5) $\overset{\mathsf{M}}{\vdash} A \to \forall vA$, provided v is not free in A

(L_6) $\overset{\mathsf{M}}{\vdash} \forall vA \to A^b_v$, provided b is substitutable for v in A

(L_7) If $\overset{\mathsf{M}}{\vdash} \Gamma$ belongs to $(L_1 - L_6)$, then $\overset{\mathsf{M}}{\vdash} \forall v\Gamma$ belongs to $(L_1 - L_6)$.

Remark 3.1.2

Postulate 3.4.3.Od_3 plays, inter alia, the role of *modus ponens*.

Remark 3.1.3

(L_7) together with Postulate 3.4.3.Od_3 makes it possible to show, by adapting the proof of Metatheorem 45.4 of [17](174–175). that generalization holds as a derived inference.

Definition 3.1.4 (Russell's set)

$$\mathrm{r} == \{x|x \notin x\}$$

Postulate 3.1.5 (Truth maxims)

(M_1) $\overset{\mathsf{M}}{\vdash} \mathfrak{T}(A \to B) \to (\mathfrak{T}A \to \mathfrak{T}B)$

(M_2) $\overset{\mathsf{M}}{\vdash} \mathfrak{T}A \to \neg\mathfrak{T}\neg A$

(M_3) $\overset{\mathsf{M}}{\vdash} (\mathfrak{T}\mathrm{r} \in \mathrm{r} \vee \mathfrak{T}\mathrm{r} \notin \mathrm{r}) \to (\mathfrak{T}A \vee \mathfrak{T}\neg A)$

(M_4) $\overset{\mathsf{M}}{\vdash} \mathfrak{T}A \vee \mathfrak{T}\neg A \vee (\mathfrak{T}\neg\mathfrak{T}\neg B \to \mathfrak{T}B)$

(M_5) $\overset{\mathsf{M}}{\vdash} \mathfrak{T}A \vee \mathfrak{T}\neg A \vee (\mathfrak{T}B \to \mathfrak{T}\mathfrak{T}B)$

(M_6) $\overset{\mathsf{M}}{\vdash} \mathfrak{T}(\mathfrak{T}A \to A) \to (\mathfrak{T}A \vee \mathfrak{T}\neg A)$

(M_7) $\overset{\mathsf{M}}{\vdash} \mathfrak{T}(\mathfrak{T}A \to \mathfrak{T}\mathfrak{T}A) \to (\mathfrak{T}A \vee \mathfrak{T}\neg A)$

(M_8) $\overset{\mathsf{M}}{\vdash} (\neg\mathfrak{T}\neg A \to \mathfrak{T}\neg\mathfrak{T}\neg A) \to (\mathfrak{T}A \vee \mathfrak{T}\neg A)$

(M_9) $\overset{\mathsf{M}}{\vdash} \exists v\mathfrak{T}A \to \mathfrak{T}\exists vA$

(M_{10}) $\overset{\mathsf{M}}{\vdash} \mathfrak{T}\forall vA \to \forall v\mathfrak{T}A$

(M_{11}) $\overset{\mathsf{M}}{\vdash} \mathfrak{T}(TA \leftrightarrow A) \to \mathfrak{T}(T\neg A \leftrightarrow \neg A)$.

(M_{12}) $\overset{\mathsf{M}}{\vdash} \forall u(a \in u \to b \in u) \to (A^a_v \to A^b_v)$, for a and b substitutable for v in A.

(M_{13}) $a, b \in \prod \to (\forall x\mathfrak{T}(x \in a \leftrightarrow x \in b) \to \mathfrak{T}\forall x(x \in a \leftrightarrow x \in b)$ – $\prod$ as in Sections 3.2 and 11.3.

Proof. (Postulate 3.1.5.M_3) This somewhat extends the system presupposed in [4]. Let an ordinal β be *monogamous* just if a successor ordinal, so that we for any B have $\Xi(\delta) \Vdash \mathfrak{T}B$ just if $\Xi(\delta) \Vdash \neg\mathfrak{T}\neg B$. The postulate is justified by observing that the antecedent holds just at the monogamous successor ordinals, and all instances of the consequent as well hold at monogamous successor ordinals. □

Proof. (Postulate 3.1.5.M_4) Let an ordinal γ be reflected, just if $\Xi(\gamma) \Vdash \mathfrak{T}B$, provided $\Xi(\gamma) \Vdash \mathfrak{T}\neg\mathfrak{T}\neg B$. Any limit ordinal γ is reflected, for if B holds at all ordinals as from some ordinal below γ according to Ξ, then also $\neg\mathfrak{T}\neg B$ holds at all ordinals as from some ordinal below γ according to Ξ: $\neg\mathfrak{T}\neg B$ holds at $\eta + 1$ according to Ξ if B holds at η according to Ξ, and if B holds all under a limit λ, as from some ordinal below, according to Ξ, then $\Xi(\lambda) \Vdash \neg\mathfrak{T}\neg B$ expresses that B never stops recurring below λ according to Ξ.

Limit ordinals are reflected, and successors ordinals are monogamous, in the sense of the proof of Postulate Postulate 3.1.5.M_3.

The content of Postulate Postulate 3.1.5.M_4 is that all ordinals are reflected or monogamous, as for a monogamous ordinal δ, $\Xi(\delta) \Vdash \neg\mathfrak{T}\neg A \to \mathfrak{T}A$, and if δ is reflected, $\Xi(\delta) \Vdash \mathfrak{T}\neg\mathfrak{T}\neg B \to \mathfrak{T}B$. □

Proof. (Postulate 3.1.5.M_5) Let an ordinal γ be *transitive* just if for any A,

$$\exists\theta(\theta \prec \gamma \;\&\; \Pi\xi(\theta \preceq \xi \Rightarrow \Xi(\xi) \Vdash A))$$

only if

$$\exists\theta(\theta \prec \gamma \;\&\; \Pi\xi(\theta \preceq \xi \Rightarrow \Xi(\xi) \Vdash \mathfrak{T}A))$$

Limit ordinals are transitive, and successors ordinals are monogamous.

The content of M5 is that the ordinals either are transitive, or monogamous, in the sense of the proof of Postulate 3.1.5.M_3. Any given ordinal γ is monogamous just if $\Xi(\gamma) \Vdash \neg\mathfrak{T}\neg A \to \mathfrak{T}A$, and it is is transitive just if $\Xi(\delta) \Vdash \mathfrak{T}A \to \mathfrak{T}\mathfrak{T}A$. □

Proof. (Postulate 3.1.5.M_6) At successor ordinals this holds, because there the consequent is true. Let λ be a limit ordinal, and let ρ be such that

$$\Pi\xi(\rho \preceq \xi \prec \lambda) \Rightarrow \Xi(\xi) \Vdash \mathfrak{T}A \to A.$$

Suppose there is some ordinal $\varrho \prec \lambda$ and $\rho \prec \varrho$ such that $\Xi(\varrho) \Vdash A$. If so, $\Xi(\lambda) \Vdash \mathfrak{T}A$. If there, on the other hand, is no ordinal $\varrho \prec \lambda$ and $\rho \prec \varrho$ such that $\Xi(\varrho) \Vdash A$, then $\Xi(\lambda) \Vdash \mathfrak{T}\neg A$. □

Proof. (Postulate 3.1.5.M_7)

Suppose

a) $\Vdash^{\mathsf{M}} \mathfrak{T}(\mathfrak{T}A \to \neg\mathfrak{T}\neg A)$.

So, by Postulate 3.1.5.M_1,

b) $\Vdash^{\mathsf{M}} \mathfrak{T}\mathfrak{T}A \to \mathfrak{T}\neg\mathfrak{T}\neg A$.

From Postulate 3.1.5.M_4 we have

c) $\Vdash^{\mathsf{M}} (\neg\mathfrak{T}\mathrm{r} \in \mathrm{r} \wedge \neg\mathfrak{T}\mathrm{r} \notin \mathrm{r}) \to (\mathfrak{T}\neg\mathfrak{T}\neg A \to \mathfrak{T}A)$.

So, by truth functional logic from **b)** and **c)**,

d) $\Vdash^{\mathsf{M}} (\neg\mathfrak{T}\mathrm{r} \in \mathrm{r} \wedge \neg\mathfrak{T}\mathrm{r} \notin \mathrm{r}) \to (\mathfrak{T}\mathfrak{T}A \to \mathfrak{T}A)$.

A weakening of **a)** gives us

e) $\Vdash^{\mathsf{M}} \mathfrak{T}(\mathfrak{T}A \to \mathfrak{T}\mathfrak{T}A) \to \mathfrak{T}(\mathfrak{T}A \to \neg\mathfrak{T}\neg\mathfrak{T}A)$,

so that, by contraposition and double negation,

f) $\Vdash^{\mathsf{M}} \mathfrak{T}(\mathfrak{T}A \to \mathfrak{T}\mathfrak{T}A) \to \mathfrak{T}(\mathfrak{T}\neg\mathfrak{T}A \to \neg\mathfrak{T}A)$.

It follows, bu using Postulate 3.1.5.M_6, that

g) $\Vdash^{\mathsf{M}} \mathfrak{T}(\mathfrak{T}A \to \mathfrak{T}\mathfrak{T}A) \to (\mathfrak{T}\mathfrak{T}A \vee \mathfrak{T}\neg\mathfrak{T}A)$.

c), **d)**, **g)** and logic give us

h) $\Vdash^{\mathsf{M}} (\neg\mathfrak{T}\mathrm{r} \in \mathrm{r} \wedge \neg\mathfrak{T}\mathrm{r} \notin \mathrm{r}) \to (\mathfrak{T}(\mathfrak{T}A \to \mathfrak{T}\mathfrak{T}A) \to (\mathfrak{T}A \vee \mathfrak{T}\neg A))$.

From a weakening of Postulate 3.1.5.M_3 we have

i) $\Vdash^{\mathsf{M}} (\mathfrak{T}\mathrm{r} \in \mathrm{r} \vee \mathfrak{T}\mathrm{r} \notin \mathrm{r}) \to (\mathfrak{T}(\mathfrak{T}A \to \mathfrak{T}\mathfrak{T}A) \to (\mathfrak{T}A \vee \mathfrak{T}\neg A))$.

A disjunctive syllogism with **h)** and **i)** finishes the proof. □

Proof. Postulate 3.1.5.M_8 This holds at successors as the consequent holds there. Suppose $\Xi(\lambda) \Vdash \neg\mathfrak{T}A \wedge \neg\mathfrak{T}\neg A$, for limit λ. It follows by Postulate 3.1.5.M_6 that $\Xi(\lambda) \Vdash \neg\mathfrak{T}\neg(\mathfrak{T}\neg A \wedge A)$. But then, $\Xi(\lambda) \Vdash \neg\mathfrak{T}\neg\mathfrak{T}\neg A \wedge \neg\mathfrak{T}\neg A$, and we are done. □

Proof. Postulate 3.1.5.M_9 Obvious .... □

Proof. Postulate 3.1.5.M_{10} Obvious .... □

Observation 3.1.6
Postulate 3.1.5.M_{11}
$\Vdash^{\mathsf{M}} \mathfrak{T}(TA \leftrightarrow A) \to \mathfrak{T}(T\neg A \leftrightarrow \neg A)$?

Suppose $\Xi(\lambda+1) \Vdash \neg(\mathfrak{T}(\mathfrak{T}A \leftrightarrow A) \to (\mathfrak{T}\neg A \leftrightarrow \neg A))$. As a consequence, $\Xi(\lambda+1) \Vdash \mathfrak{T}(\mathfrak{T}A \leftrightarrow A) \wedge \neg\mathfrak{T}(\mathfrak{T}\neg A \leftrightarrow \neg A)$, so $\Xi(\lambda+1) \Vdash \mathfrak{T}(\mathfrak{T}A \leftrightarrow A)$ as well as $\Xi(\lambda+1) \Vdash \neg\mathfrak{T}(\mathfrak{T}\neg A \leftrightarrow \neg A)$. So $\Xi(\lambda+1) \Vdash \mathfrak{T}(\mathfrak{T}A \leftrightarrow A)$ as well as $\Xi(\lambda+1) \Vdash \neg\mathfrak{T}\neg(\mathfrak{T}\neg A \leftrightarrow A)$, hence $\Xi(\lambda) \Vdash \mathfrak{T}A \leftrightarrow A$ as well as $\Xi(\lambda) \Vdash \mathfrak{T}\neg A \leftrightarrow A$. So $\Xi(\lambda) \Vdash \neg A$, and $\nVdash^{\mathsf{M}} \mathfrak{T}(TA \leftrightarrow A) \to \mathfrak{T}(T\neg A \leftrightarrow \neg A)$.

Proof. (Postulate 3.1.5.M_{12}) The validity of the maxim in Postulate 3.1.5.M_{12} is shown in the proof of Theorem 7.1.7.5. □

Remark 3.1.7

The semantic justification for some of the maxims of Postulate 3.1.5.M_1 – Postulate 3.1.5.M_{12} can be lifted from [4](340–341).

Remark 3.1.8

[8](396), indicates that the maxim schemas of Postulate 3.1.5.M_6, Postulate 3.1.5.M_7 and Postulate 3.1.5.M_8, originate with [30].

Remark 3.1.9

The maxims of Postulate 3.1.5.M_7 and Postulate 3.1.5.M_8 were not included in [4], as the author thought they were both derivable. The proof of Postulate 3.1.5.M_8 shows that this was correct for its maxim schema, but the proof of Postulate 3.1.5.M_7 suggests that Postulate 3.1.5.M_3 is needed for its semantical justification.

Remark 3.1.10

Although the converses of Postulate 3.1.5.M_5 and Postulate 3.1.5.M_6 hold at limit ordinals, they are not maxims, for we may at a sucessor σ have that

$$\Xi(\lambda) \Vdash (\mathfrak{T}\neg A \vee \mathfrak{T}A) \wedge \neg\mathfrak{T}(\mathfrak{T}A \to A),$$

and it happen for $A == \{x|x \notin x\} \in \{x|x \notin x\}$ at σ or $\sigma+1$. This contrasts with Remark 69.3.1.(ii) in [8](396).

3.2 Barcan for hereditarily orthodox co-extensionalities

Let $\prod$ be as in Theorem11.3.3, so that

$$\overset{\mathsf{M}}{\vDash}\ a \in \prod \leftrightarrow a \in \mathbb{h} \wedge \prod \in \mathbb{h} \wedge a \subset \prod).$$

Theorem 3.2.1

$$\overset{\mathsf{M}}{\vDash}\ a, b \in \prod \rightarrow (\forall x \mathfrak{T}(x \in a \leftrightarrow x \in b) \rightarrow \mathfrak{T}\forall x(x \in a \leftrightarrow x \in b)).$$

Proof. Assume

$$\overset{\mathsf{M}}{\nvDash}\ a, b \in \prod \rightarrow (\forall x \mathfrak{T}(x \in a \leftrightarrow x \in b) \rightarrow \mathfrak{T}\forall x(x \in a \leftrightarrow x \in b).$$

It follows, by Definitions 2.2.2 and 2.4.2, that for some fair function Ξ':

$$\Updownarrow$$

$$\Xi'(\text{♀}) \Vdash \neg\mathfrak{T}(a, b \in \prod \rightarrow (\forall x \mathfrak{T}(x \in a \leftrightarrow x \in b) \rightarrow \mathfrak{T}\forall x(x \in a \leftrightarrow x \in b))) \tag{3.2.2}$$

Using Definition 2.2.2.3,

$$\Updownarrow$$

$$\Pi\gamma(\gamma \prec \text{♀} \ \Rightarrow\ \Sigma\delta(\gamma \preceq \delta \prec \text{♀}\ \ \&\ \ \Xi'(\delta) \Vdash a, b \in \prod \wedge\ \forall x \mathfrak{T}(x \in a \leftrightarrow x \in b) \wedge \neg\mathfrak{T}\forall x(x \in a \leftrightarrow x \in b))) \tag{3.2.3}$$

Case 1/2 - δ is a limit: Suppose $\Xi'(\delta) \Vdash a, b \in \prod \wedge\ \forall x \mathfrak{T}(x \in a \leftrightarrow x \in b) \wedge \neg\mathfrak{T}\forall x(x \in a \leftrightarrow x \in b)$. Then, for all constants $c : \gamma$,

$$\Xi'(\psi) \Vdash a, b \in \prod \wedge c \in a \leftrightarrow c \in b \wedge$$

so as well

$$\Xi'(\psi) \Vdash a, b \in \prod \wedge \forall x(x \in a \leftrightarrow x \in b).$$

Also, however,

$$\Xi'(\delta) \Vdash \neg\mathfrak{T}\forall x(x \in a \leftrightarrow x \in b),$$

so that for some $\psi \preceq \phi \preceq \delta$,

$$\Xi'(\phi) \Vdash c \in a \leftrightarrow c \notin a.$$

So we cannot have

$$\Xi'(\delta) \Vdash a, b \in \prod \wedge\ \forall x \mathfrak{T}(x \in a \leftrightarrow x \in b) \wedge \neg\mathfrak{T}\forall x(x \in a \leftrightarrow x \in b)$$

at a limit δ.

Case 2/2 - $\delta = \gamma + 1$ is a successor. Suppose

$$\Xi'(\delta) \Vdash a, b \in \prod \wedge\ \forall x \mathfrak{T}(x \in a \leftrightarrow x \in b) \wedge \neg\mathfrak{T}\forall x(x \in a \leftrightarrow x \in b).$$

Then for some constant d,

$$\Xi'(\delta) \Vdash a, b \in \prod \wedge (d \in a \leftrightarrow d \in b) d \in a \not\leftrightarrow d \in b).$$

So

$$\overset{\mathsf{M}}{\vDash}\ a, b \in \prod \rightarrow (\forall x \mathfrak{T}(x \in a \leftrightarrow x \in b) \rightarrow \mathfrak{T}\forall x(x \in a \leftrightarrow x \in b).$$

□

3.3 Truth theses

Postulate 3.3.1 (Truth theses)

(T_1) $\vdash \mathfrak{T}A \to A$

(T_2) $\vdash A \to \mathfrak{T}A$

(T_3) $\vdash \mathfrak{T}\exists vA \to \exists v\mathfrak{T}A$

(T_4) $\vdash \forall v\mathfrak{T}A \to \mathfrak{T}\forall vA$

Remark 3.3.2

The theses in 3.3.1 cannot be strengthened to maxims. In the case of Postulate 3.3.1.T_1 and Postulate 3.3.1.T_2, this follows on account of well known paradoxicalities, as with Russell's r from Definition 3.1.4. In the case of the *attestor* schema of Postulate 3.3.1.T_3, we show the failure of general maximality in Section 9.2. The failure of the maximality for Postulate 3.3.1.T_4, the *Barcan thesis*, is shown in Section 9.3.

Remark 3.3.3

The following are amongst the more surprising paradoxical theses of £ and $\mathscr{Tb}$; we define the autocombatant $\mathring{a}$, and justify the following theses semantically in Theorem 10.3.1:

$$\vdash \forall x(x \in \mathring{a}) \ \&\ \vdash \forall x(x \notin \mathring{a}).$$

The autocombatant theses are instrumental in showing, in Section 10.4, that the power set $\mathcal{P}(a)$ is paradoxical, unless a is universal, and orthodox in the sense of Definition 7.1.6. This is of importance for the librationist account of infinity, as it helps the avoidance of the Cantorian conclusion that there are *nondenumerable* infinities.

The need to assume the autocombatant theses separately brings to the fore that we have not shown that the formal librationist system is complete.

3.4 Inference modes

We distinguish between simple and complex inference modes. The former are dyadic, and only $\neg$, $\mathfrak{T}$ and one occurence of a formula variable are allowed in the formulas in the antecedent and the consequent. Moreover, $\mathfrak{T}$ can only occur once in the antecedent and the consequent, and $\neg$ cannot be preceded by itself. A simple inference mode may also contain one quantifier in the antecedent and in the consequent.

We list many simple thetical modes to display the connections with the subsequent simple maximal modes.

Postulate 3.4.1 (Simple thetical modes)

(Tm_1) $\vdash A \Rightarrow \vdash \mathfrak{T}A$ Minor *certification*

(Tm_2) $\vdash A \Rightarrow \vdash \neg\mathfrak{T}\neg A$

(Tm$_3$) $\vdash \neg A \Rightarrow \vdash \mathfrak{T}\neg A$

(Tm$_4$) $\vdash \neg A \Rightarrow \vdash \neg\mathfrak{T}A$

(Tm$_5$) $\vdash \mathfrak{T}A \Rightarrow \vdash A$

(Tm$_6$) $\vdash \mathfrak{T}A \Rightarrow \vdash \neg\mathfrak{T}\neg A$

(Tm$_7$) $\vdash \mathfrak{T}\neg A \Rightarrow \vdash \neg A$

(Tm$_8$) $\vdash \mathfrak{T}\neg A \Rightarrow \vdash \neg\mathfrak{T}A$

(Tm$_9$) $\vdash \neg\mathfrak{T}A \Rightarrow \vdash \neg A$

(Tm$_{10}$) $\vdash \neg\mathfrak{T}A \Rightarrow \vdash \mathfrak{T}\neg A$

(Tm$_{11}$) $\vdash \neg\mathfrak{T}\neg A \Rightarrow \vdash A$

(Tm$_{12}$) $\vdash \neg\mathfrak{T}\neg A \Rightarrow \vdash \mathfrak{T}A$

(Tm$_{13}$) $\vdash \forall v\mathfrak{T}A \Rightarrow \vdash \mathfrak{T}\forall vA$

(Tm$_{14}$) $\vdash \neg\mathfrak{T}\forall vA \Rightarrow \vdash \neg\forall v\mathfrak{T}A$

(Tm$_{15}$) $\vdash \mathfrak{T}\exists vA \Rightarrow \vdash \exists v\mathfrak{T}A$

The following correlated maximal modes, up to Postulate 3.4.2.Mm$_{13}$, can be justified by the simple thetical modes on account of the syntactical correlate of Theorem 2.4.3.3. which says that $\stackrel{\mathsf{M}}{\vdash} A$ just if $\vdash A \,\&\, \nvdash \neg A$. Mode Postulate 3.4.2.Mm$_1$ is for example a consequence of the conjunction of the modes provided by Postulate 3.4.1.Tm$_1$ and Postulate 3.4.1.Tm$_9$. The other quantifier free dependencies are straightforward to establish.

Notice that Postulate 3.4.1.Tm$_{13}$ and Postulate 3.4.1.Tm$_{14}$ combine to consitute Postulate 3.4.2.Mm$_{13}$. Postulate 3.4.1.Tm$_{15}$ does not enter such a constitution.

Postulate 3.4.2 (Simple maximal modes)

(Mm$_1$) $\stackrel{\mathsf{M}}{\vdash} A \Rightarrow \stackrel{\mathsf{M}}{\vdash} \mathfrak{T}A$

(Mm$_2$) $\stackrel{\mathsf{M}}{\vdash} A \Rightarrow \stackrel{\mathsf{M}}{\vdash} \neg\mathfrak{T}\neg A$

(Mm$_3$) $\stackrel{\mathsf{M}}{\vdash} \neg A \Rightarrow \stackrel{\mathsf{M}}{\vdash} \mathfrak{T}\neg A$

(Mm$_4$) $\stackrel{\mathsf{M}}{\vdash} \neg A \Rightarrow \stackrel{\mathsf{M}}{\vdash} \neg\mathfrak{T}A$

(Mm$_5$) $\stackrel{\mathsf{M}}{\vdash} \mathfrak{T}A \Rightarrow \stackrel{\mathsf{M}}{\vdash} A$

(Mm$_6$) $\stackrel{\mathsf{M}}{\vdash} \mathfrak{T}A \Rightarrow \stackrel{\mathsf{M}}{\vdash} \neg\mathfrak{T}\neg A$

(Mm$_7$) $\stackrel{\mathsf{M}}{\vdash} \mathfrak{T}\neg A \Rightarrow \stackrel{\mathsf{M}}{\vdash} \neg A$

(Mm$_8$) $\stackrel{\mathsf{M}}{\vdash} \mathfrak{T}\neg A \Rightarrow \stackrel{\mathsf{M}}{\vdash} \neg\mathfrak{T}A$

(Mm$_9$) $\stackrel{\mathsf{M}}{\vdash} \neg\mathfrak{T}A \Rightarrow \stackrel{\mathsf{M}}{\vdash} \neg A$

(Mm$_{10}$) $\stackrel{\mathsf{M}}{\vdash} \neg\mathfrak{T}A \Rightarrow \stackrel{\mathsf{M}}{\vdash} \mathfrak{T}\neg A$

(Mm$_{11}$) $\stackrel{\mathsf{M}}{\vdash} \neg\mathfrak{T}\neg A \Rightarrow \stackrel{\mathsf{M}}{\vdash} A$

(Mm_{12}) $\vDash^{\mathsf{M}} \neg\mathfrak{T}\neg A \Rightarrow \vDash^{\mathsf{M}} \mathfrak{T}A$

(Mm_{13}) $\vDash^{\mathsf{M}} \forall v\mathfrak{T}A \Rightarrow \vDash^{\mathsf{M}} \mathfrak{T}\forall vA$

Postulate 3.4.3 (Ordinary distributive modes)

(Od_1) $\vDash^{\mathsf{M}} (A \to B) \Rightarrow (\vdash A \Rightarrow \vdash B)$

(Od_2) $\vDash^{\mathsf{M}} (A \to B) \Rightarrow (\vdash \neg B \Rightarrow \vdash \neg A)$

(Od_3) $\vDash^{\mathsf{M}} (A \to B) \Rightarrow (\vDash^{\mathsf{M}} A \Rightarrow \vDash^{\mathsf{M}} B$

Remark 3.4.4 (On Postulate 3.4.3)

The mode of Postulate 3.4.3.Od_3 is a logical consequence of the modes given by Postulate 3.4.3.Od_1 and Postulate 3.4.3.Od_2.

Remark 3.4.5 (On Postulate 3.4.3)

(Ld_1) and (Ld_2) are equivalent.

Postulate 3.4.6 (Librationist distributive modes)

(Ld_1) $\vdash (\neg A \vee B) \ \& \ \vdash A \Rightarrow (\vdash \neg A \ \textcircled{r} \vdash B)$

(Ld_2) $\vdash (A \to B) \Rightarrow (\vDash^{\mathsf{M}} A \Rightarrow \vdash B)$

Postulate 3.4.7 (Complex modes)

Re Postulate 3.4.7.Cm_5, see Definition 7.1.6.3.

(Cm_1) $\vDash^{\mathsf{M}} \mathfrak{T}A \vee \mathfrak{T}\neg A \Rightarrow \vDash^{\mathsf{M}} \mathfrak{T}A \to A$

(Cm_2) $\vDash^{\mathsf{M}} \mathfrak{T}\neg\mathfrak{T}\neg A \Rightarrow \vDash^{\mathsf{M}} \mathfrak{T}A$

(Cm_3) $\vDash^{\mathsf{M}} \mathfrak{T}(\mathfrak{T}A \to \mathfrak{T}B) \Rightarrow \vDash^{\mathsf{M}} \mathfrak{T}(A \to B)$

(Cm_4) $\vdash A \ \& \ \vdash B \Rightarrow \vdash \neg\mathfrak{T}\neg A \wedge \neg\mathfrak{T}\neg B$

(Cm_5) $\vDash^{\mathsf{M}} \mathfrak{D}(A(x)) \Rightarrow \vDash^{\mathsf{M}} \forall x\mathfrak{T}A(x) \to \mathfrak{T}\forall xA(x)$ (The Barcan mode)

(Cm_6) $\vDash^{\mathsf{M}} \mathfrak{D}(A(x)) \Rightarrow (\vDash^{\mathsf{M}} \exists xA \Rightarrow \vDash^{\mathsf{M}} A^a_x$ for some a substitutable for x in A).

(Cm_7) $\vDash^{\mathsf{M}} A^a_v$ for any constant $a \Rightarrow \vDash^{\mathsf{M}} \forall vA$

Proof. (Postulate 3.4.7.Cm_1) By elementary logic, $\vDash^{\mathsf{M}} \mathfrak{T}A \to (\mathfrak{T}A \to A)$. From Postulate 3.1.5.$\mathrm{M}_2$, i.e. $\vDash^{\mathsf{M}} \mathfrak{T}\neg A \to \neg\mathfrak{T}A$, and elementary logic, it again follows that $\vDash^{\mathsf{M}} \neg\mathfrak{T}A \to (\mathfrak{T}A \to A)$. As $\vDash^{\mathsf{M}} \mathfrak{T}A \vee \mathfrak{T}\neg A$, the proof is finished by a disjunctive syllogism with Postulate 3.4.7.Cm_1. $\square$

Proof. (Postulate 3.4.7.Cm_3) Suppose $\Xi(\female) \Vdash \mathfrak{T}(\mathfrak{T}A \to \mathfrak{T}B)$. (i) Let ρ be be a ordinal as from which $\mathfrak{T}A \to \mathfrak{T}B$ holds, so that

$$\Pi\xi(\rho \preceq \xi \prec \female \Rightarrow \Xi(\xi) \Vdash (\mathfrak{T}A \to \mathfrak{T}B).$$

Thus $\Xi(\rho+1) \Vdash (\mathfrak{T}A \to \mathfrak{T}B)$, and therefore $\Xi(\rho) \Vdash (A \to B)$. Consequently, succeeding successors will have $\mathfrak{T}A \to \mathfrak{T}B$ and $A \to B$. (ii) Let limit ordinal $\lambda \prec \female$, above ρ, have $\mathfrak{T}A \to \mathfrak{T}B$, and $A \to B$ below, as from ρ. As $\lambda \prec \female$, from the assumption on ρ, $\Xi(\lambda) \Vdash (\mathfrak{T}A \to \mathfrak{T}B)$. As $\Xi(\lambda+1) \Vdash (\mathfrak{T}A \to \mathfrak{T}B)$, also $\Xi(\lambda) \Vdash (A \to B)$. (iii) By a repetition of (i) and (ii) it follows that $A \to B$ holds as from ρ below $\female$, so that $\Xi(\female) \Vdash \mathfrak{T}(A \to B)$. □

Proof. (Postulate 3.4.7.Cm_5) Suppose $\overset{\mathsf{M}}{\models} \mathfrak{D}(A)$ and that $\overset{\mathsf{M}}{\not\models} \forall x\mathfrak{T}A(x) \to \mathfrak{T}\forall xA(x)$. By Definition 2.4.2.3, for at least one fair function Ξ, $\Xi(\female) \nVdash \mathfrak{T}(\forall x\mathfrak{T}A(x) \to \mathfrak{T}\forall xA(x))$. So on account of Definition 2.2.2.1 we have that $\Xi(\female) \Vdash \neg\mathfrak{T}(\forall x\mathfrak{T}A(x) \to \mathfrak{T}\forall xA(x))$ so that $\Xi(\female) \Vdash \neg\mathfrak{T}\neg(\forall x\mathfrak{T}A(x) \wedge \neg\mathfrak{T}\forall xA(x))$. This means that $\forall x\mathfrak{T}A(x) \wedge \neg\mathfrak{T}\forall xA(x))$ is unbounded under $\female$. As $\overset{\mathsf{M}}{\models} \mathfrak{D}(A)$, i.e. $\forall x(\mathfrak{T}A(x) \vee \mathfrak{T}\neg A(x))$, for all terms a, either $\mathrm{IN}(,A(a),\Xi)$ or $\mathrm{IN}(,\neg A(a),\Xi)$. □

Proof. (Postulate 3.4.7.Cm_6) We prove tha validity of this on page 39, in the proof of Theorem 9.1.3. □

3.5 Alethic comprehension

Postulate 3.5.1 (Alethic comprehension without parameters)

$$\overset{\mathsf{M}}{\models} \forall x(x \in \{y|A\} \leftrightarrow \mathfrak{T}A^x_y), \text{where } x \text{ is substitutable for } y \text{ in } A.$$

Given Postulate 3.4.7.Cm_7, also:

Postulate 3.5.2 (Alethic comprehension with parameters)

$$\overset{\mathsf{M}}{\models} \forall\vec{v}\forall x(x \in \{y|A\} \leftrightarrow \mathfrak{T}A^x_y), \text{where } x \text{ is substitutable for } y \text{ in } A.$$

Chapter 4

Ordinary and extraordinary theories

Let L be *classical* logic:

Definition 4.0.1

(1)	T is conservative	just if	$\models_{\mathrm{L}} A \Rightarrow \models_{\mathrm{T}} A$
(2)	T is moderate	just if	$\models_{\mathrm{T}} A \Rightarrow \not\models_{\mathrm{T}} \neg A$
(3)	T is ordinary	just if	T is conservative and moderate

Corollary 4.0.2
Classical logic is ordinary, as it is conservative and moderate.

Definition 4.0.3
Theory T is *polarized*, just if for some sentence B,

$$\models_{\mathrm{T}} B \ \& \ \models_{\mathrm{T}} \neg B.$$

Definition 4.0.4
Theory T is *extra-ordinary* just if it is ordinary and polarized.

So-called paraconsistent systems are not ordinary, or extra-ordinary, as they are not conservative. As a rule, paraconsistent systems are not even moderate, if polarized, as $p \wedge \neg p$ is taken to be a thesis if p is paradoxical. Some non-adjunctive paraconsistent logics, as Jaskowski's, in [18] (translated in [19]), may be moderate, even if polarized, but are not conservative.

$\mathfrak{T}$ and £ are polarized on account of their comprehension principles. They are, moreover, ordinary and extra-ordinary, so also moderate and conservative systems. All classical logical theses are preserved by $\mathfrak{T}$ and £, in their language, and no classical logical theses are contradicted by $\mathfrak{T}$ or £.

Chapter 5

Names, expressions and Urelemente

This chapter is devoted to the introduction of *codes* of expressions, and the role the codes play in the theories of sets put forth here. The codes are to be understood as names of the expressions they are codes of.

5.1 Codes as names and Urelemente

For expression $\mathcal{E}$, $\ulcorner\mathcal{E}\urcorner$ is the name of $\mathcal{E}$. The name $\ulcorner a\urcorner$ refers to term a, and the name $\ulcorner A\urcorner$ refers to formula A, and we will here mostly be preoccupied with terms which are constants and formulas which are sentences. $\ulcorner A\urcorner$ and $\ulcorner a\urcorner$ are *Urelemente*.

We introduce codes for formulas and terms in Definitions 5.2.3 and 5.2.10–5.2.12. The codes are introduced primitively, and not by means of an arithmetization, as with Gödel, and followers.

It was noted in the introduction, towards the end:

> "Nevertheless, a postulation of, and theory on *names* of terms, and *sentences*, is developed in Section 5. (...) Names are taken as *Urelemente* in £ and Ꝥ. Introduced predicates $\mathcal{T}$, for *is true*, $\mathcal{F}$, for *is false*, and $\mathcal{D}$, for *is provable in classical logic*, hold for some names. A motivation is to obtain a recovery of many of the identity losses caused by the prevalent non-extensionality of £ and Ꝥ, and it also serves to connect the Liar type paradoxes with set theoretic paradoxes ... ."

One might include codes of other expressions, as variables, quantifiers, connectives, formulas, and so on, and maintain the same distinction between the *Urelement* and the expression it refers to. We have not done that yet in this text.

Incidentally, although W.V. Quine used corner quotes for another purpose earlier, it seems to have been Georg Kreisel who first introduced the notation $\ulcorner A\urcorner$ for Gödel numbers, in [21] and [22] and [23]. We use a slightly different notation than $\ulcorner\ \urcorner$, viz. $^\ulcorner\ ^\urcorner$, as per Definition 5.2.12, to distinguish: one distinct feature is that codes as $\ulcorner A\urcorner$ and $\ulcorner a\urcorner$ are not elements of ω, the set of finite von Neumann ordinals, which we use to represent natural numbers, according to £ or Ꝥ.

Notice, e.g., that $\varsigma\ddot{v}\ddot{v}\ddot{v}$ *is* the number 5222, as expressed in the binary base-6 numeral system, according to the meta theory. However, according to £ and Ꝥ, it is, via Definition 1.4.16, the term $\{u|u \in u\}$, which *we* think of as a set, though £ and Ꝥ do not think that something is a set.

Similarly, for a natural number m, $\ulcorner\exists u(u \in u)\urcorner$ *is* the number $6m$ according to the meta theory. £ and Ꝥ do think of $\ulcorner\exists u(u \in u)\urcorner$ as a *term*, but we have no way to think of $\ulcorner\exists u(u \in u)\urcorner$ as a set, or number, provided by £, and suggest that $\ulcorner\exists u(u \in u)\urcorner$, and similar terms, should instead be regarded as linguistic entities.

5.2 The codes

Definition 5.2.1

1. If a has free variables, $\ulcorner a \urcorner$ is an *Urelemente* which is an incomplete *name* of a.
2. If a is a constant, $\ulcorner a \urcorner$ is an *Urelemente* which is a name of a.

Definition 5.2.2

1. If A has free variables, $\ulcorner A \urcorner$ is an *Urelemente* which is an incomplete *name* of A.
2. If A is a sentence, $\ulcorner A \urcorner$ is an *Urelemente* which is a *name* of A.

Definition 5.2.4 introduces the truth predicate $\mathcal{T}$, whose meaning is regulated by Postulate 5.2.14, and Definition 5.2.5 introduces the falsity predicate $\mathcal{F}$, whose meaning is regulated by Postulate 5.2.15.

The derivability predicate $\mathcal{D}$ is taken as primitive, and is not, as with [14], introduced via arithmetization. This helps us avoid the identification of $\mathcal{D}$ with a natural number. $\mathcal{T}$ is, similarly, taken as an *Urelemente*, which is not a number, or a set.

Definition 5.2.3

$$\# == \ddot{c}$$

Definition 5.2.4

$$\mathcal{T} == \ddot{c}\bullet$$

Definition 5.2.5

$$\mathcal{F} == \ddot{c}\bullet\bullet$$

Definition 5.2.6

$$\mathcal{D} == \ddot{c}\bullet\bullet\bullet$$

Definition 5.2.7

If a is a term, $\mathcal{T}a$ is a formula.

Definition 5.2.8

If a is a term, $\mathcal{F}a$ is a formula.

Definition 5.2.9

If a is a term, $\mathcal{D}a$ is a formula.

Definition 5.2.10

If a is a term, $\#a$ is a term.

Definition 5.2.11

If A is a formula, $\#A$ is a term.

Definition 5.2.12

$$\ulcorner A \urcorner == \#A$$

Definition 5.2.13

$$\ulcorner a \urcorner == \#a.$$

Postulate 5.2.14

$$\stackrel{\mathsf{M}}{\vDash} \forall \vec{x}(\mathcal{T}\ulcorner A(\vec{x})\urcorner \leftrightarrow \mathfrak{T}A(\vec{x}))$$

Postulate 5.2.15

$$\stackrel{\mathsf{M}}{\vDash} \forall \vec{x}(\mathcal{F}\ulcorner A(\vec{x})\urcorner \leftrightarrow \mathcal{T}\ulcorner \neg A(\vec{x})\urcorner)$$

Postulate 5.2.16

$\stackrel{\mathsf{M}}{\vDash} \mathcal{D}\ulcorner A \urcorner \Leftrightarrow$ Classical logic has A as a thesis.

Definition 5.2.17
We take $\ulcorner A \urcorner$ and $\ulcorner B \urcorner$ to be *Tarski-Lindenbaum*-congruent, or *alphabetological variants* of each other, just if $\stackrel{\mathsf{M}}{\vDash} \mathcal{D}\ulcorner \forall x(A \leftrightarrow B)\urcorner$.

Remark 5.2.18
5.2.14 correlates the truth *operator* $\mathfrak{T}$, which operates upon a sentence A to create a new sentence $\mathfrak{T}A$, interpreted as *it is true* that A, with the truth *predicate* $\mathcal{T}$, and $\mathcal{T}\ulcorner A \urcorner$ is interpreted as proposition $\ulcorner A \urcorner$ *is true*.

Remark 5.2.19
The primitive derivability predicate $\mathcal{D}$ only complies with the last of the Hilbert–Bernays provability conditions, presupposed for provability predicates to express Gödel's incompleteness proof, viz.

$$\stackrel{\mathsf{M}}{\vDash} \mathcal{D}\ulcorner A \to B \urcorner \to (\mathcal{D}\ulcorner A \urcorner \to \ulcorner B \urcorner).$$

Remark 5.2.20
In the statement $\mathit{Classical\ logic\ has\ A\ as\ a\ thesis}$ in Postulate 5.2.16, we take the language of the classical logic to be understood such that $\in$ is the only relational predicate, $\mathcal{T}$, $\mathcal{F}$ and $\mathcal{D}$ are monadic predicates, $\mathfrak{T}A$ is interpreted via Definition 2.1.1, while sets, names and propositions are constants. Parentheses and defined connectives and terms are used in confomity with Definition 1.4.16.

Remark 5.2.21
Propositions and characters are elements, but are not sets: It was stressed above that codes of expressions are not sets. But the code of a *constant*, as $\ulcorner a \urcorner$, is a *name*, and the code of a

sentence, as $\ulcorner a = a \urcorner$, is a *proposition*. Notice that theorems are sentences, and not propositions. We may say that the names *name* the corresponding term, and that the sentence *names* the matching proposition.

We take names and propositions to be *Urelemente*. $\mathcal{T}$, $\mathcal{F}$ and $\mathcal{D}$ are sets which happen to have only $Urelemente$, and, in this case, more precisely, propositions, as members. This follows from the fact that $\mathcal{T}$, $\mathcal{F}$ and $\mathcal{D}$ have their members given by Definitions 5.2.4–5.2.6 and Definition 1.4.16.8. The fact that $\mathcal{T}$, $\mathcal{F}$ and $\mathcal{D}$ only have propositions as members follows from the presupposed fact that only Postulates 5.2.14–5.2.16 govern their meaning.

We have not postulated sets which *only* have names as members. But we have allowed names into our ontology, so, if a is constant, $\ulcorner a \urcorner$ is a name which names a. As $\ulcorner a \urcorner = \ulcorner a \urcorner$, $\ulcorner a \urcorner \in \mathrm{V}$.

Chapter 6

The Liar Russellized

In 1925 Frank Ramsey argued, in [26](20), that there is an essential difference between *syntactical paradoxes* which "involve only logical or mathematical terms such as class and number", and *semantic paradoxes*, which "... are not purely logical, and cannot be stated in logical terms alone; for they all contain some reference to thought, language, or symbolism".

Ramsey considered Russell's paradox a canonical representative of syntactic paradoxes, and the Liar he considered an archetypical semantic paradox.

Abraham A. Fraenkel and Yehoshua Bar-Hillel, in [10](1958,5), adjudged:

> Since Ramsey [26] it has become customary to distinguish between logical and semantic (sometimes also called syntactic or epistemological) antinomies.

Despite this, it will be argued below, that one should take semantic paradoxes, as the Liar, to be so inextricably intertwined with syntactic paradoxes, as Russell's paradox, that one should not consider them to be different kinds of paradoxes.

Others reached the same conclusion, but on the basis of considerations different from those adduced below:

Dana Scott argued, in [27](1967), that the Zermelo axioms were justified by type theoretic reasoning:

> "The truth is that there is only one way of avoiding the paradoxes: namely, the use of some form of the theory of types. That was at the basis of both Russell's and Zermelo's intuitions. Indeed the best way to regard Zermelo's theory is as a simplification and extension of Russell's. (We mean Russell's *simple* theory of types, of course.) The simplification was to make the types *cumulative*." [27](208)

Alonzo Church, in [9], virtually equated Russell's theory of types and Alfred Tarski's resolution of the Liar paradox, as he stated:

> "In the light of this it seems justified to say that Russell's resolution of the semantical antinomies is not a different one than Tarski's but is a special case of it."[9](756)

The interest of Scott's and Tarski's points of view, for our purposes here, are that they, jointly, take Tarski's resolution of the alleged *semantic* paradoxes to be the same as Russell's, and Zermelo's, resolution of the, allegedly syntactical, set theoretic paradoxes.

So one may, I shall assume, postulate bridge principles, as below, between given, supposedly syntactical paradoxes, and supposedly semantical paradoxes.

Theorem 6.0.1
There is a semantical *liar sentence* L equivalent with $\neg\mathcal{T}\ulcorner\mathrm{L}\urcorner$.

Proof. An instance of 5.2.14 is

$$\models^{\mathsf{M}} \mathcal{T}\ulcorner \mathrm{r} \notin \mathrm{r})\urcorner \leftrightarrow \mathfrak{T}\mathrm{r} \notin \mathrm{r}. \tag{6.0.2}$$

By a use of alethic comprehension we arrive at

$$\models^{\mathsf{M}} \mathcal{T}\ulcorner \mathrm{r} \notin \mathrm{r})\urcorner \leftrightarrow \mathrm{r} \in \mathrm{r}. \tag{6.0.3}$$

By negating both sides of the biconditional in 6.0.3, we get

$$\models^{\mathsf{M}} \neg\mathcal{T}\ulcorner \mathrm{r} \notin \mathrm{r})\urcorner \leftrightarrow \mathrm{r} \notin \mathrm{r}. \tag{6.0.4}$$

Observation 6.0.5 (Genealogies of Liar like paradoxes)

One may identify classical liar sentences, and their variants, whose provenances stem from classical Greek philosophy, with the maxims of Theorems as 6.0.4.

Given

Definition 6.0.6

$$\mathrm{L} \Longleftarrow \mathrm{r} \notin \mathrm{r},$$

and a substitution with L for $\mathrm{r} \notin \mathrm{r}$, in 6.0.4, we obtain the more canonical form

$$\models^{\mathsf{M}} \mathrm{L} \leftrightarrow \neg\mathcal{T}\ulcorner\mathrm{L}\urcorner. \tag{6.0.7}$$

□

Semantical paradoxes, as 6.0.7, are resolved as their corresponding set theoretical paradoxes.

Theorem 6.0.8
$\vdash \mathrm{L}$, $\vdash \neg\mathrm{L}$, $\vdash \mathcal{T}\ulcorner\mathrm{L}\urcorner$, $\vdash \neg\mathcal{T}\ulcorner\mathrm{L}\urcorner$, $\vdash \mathcal{T}\ulcorner\neg\mathrm{L}\urcorner$ and $\vdash \neg\mathcal{T}\ulcorner\neg\mathrm{L}\urcorner$.

Proof. We know that $\vdash \mathrm{r} \in \mathrm{r}$ and $\vdash \mathrm{r} \notin \mathrm{r}$, so from Definition 6.0.6, $\vdash \mathrm{L}$ and $\vdash \neg\mathrm{L}$. Finish with Postulate 3.4.3.Od_1 and Postulate 3.4.3.Od_2. □

Chapter 7

The theory of identity

We improve upon sections 4 and 5 of [4](342–345). An important streamlining is the use of the additional inference modes provided by Postulate 3.4.7.Cm_1–Postulate 3.4.7.Cm_3, which are for that reason shown to be important ingredients in the librationist theory of sets. A valuable consequence of the inclusion of the needed inference modes is that we do not, as e.g. seen in the systems of [8], need additional axiomatic principles for having well behaved notions of identity and natural number.

7.1 Membership uniformity

We define the identity relation by means of a notion of *membership uniformity*, which is similar to the relation named *membership congruency* introduced by Abraham A. Fraenkel and Yehoshua Bar-Hillel, and discussed in [11](27), though not in the previous edition [10](1958).

Membership uniformity, as defined in Definition 7.1.1, does not require that $\forall u(u \in a \leftrightarrow u \in b)$, as that can be shown to be superfluous, by an elementary argument.

Notice also that the *definiens* in Definition 7.1.1 is a conditional, and not a biconditional. The observations which justified the analogous definition $*13 \cdot 01$ in Principia Mathematica, will most probably not justify the former definition. The sufficiency of Definition 7.1.1, in £ and $\mathfrak{F}$, is proven by Theorem 7.1.7.4, and without appealing to principles of *predicativity*, as in the proof of $*13 \cdot 01$ by Alfred N. Whitehead and Bertrand Russell.

Definition 7.1.1

$$a = b \;==\; \forall u(a \in u \to b \in u)$$

We say that a and b are membership congruent just if $\overset{\mathsf{M}}{\vDash} \forall u(a \in u \to b \in u)$. Consult **??**; membership congruencies are descendents:

Lemma 7.1.2

$$\overset{\mathsf{M}}{\vDash} \mathfrak{T}(\forall u(a \in u \to b \in u) \to \mathfrak{T}\forall u(a \in u \to b \in u))$$

Proof. Suppose $\overset{\mathsf{M}}{\vDash} \forall u(a \in u \to b \in u)$. By instantiation we have:

$$\overset{\mathsf{M}}{\vDash} \mathfrak{T}(\forall u \mathfrak{T}((a \in u \to b \in u) \to$$
$$(a \in \{v | \forall u(a \in u \to v \in u)\} \to b \in \{v | \forall u(a \in u \to v \in u)\}).$$

But $\overset{\mathsf{M}}{\vdash} a \in \{v|\forall u(a \in u \to v \in u)\}$, so that

$$\overset{\mathsf{M}}{\vdash} \forall u(a \in u \to b \in u) \to b \in \{v|\forall u(a \in u \to v \in u)\}.$$

Finish with **alethic comprehension and Postulate 3.1.5.M_1**. □

Lemma 7.1.3
$\overset{\mathsf{M}}{\vdash} \mathfrak{T}(a = b \to \mathfrak{T}a = b)$

Proof. From Definition 7.1.1 and Lemma 7.1. □

Lemma 7.1.4

$$\overset{\mathsf{M}}{\vdash} \mathfrak{T}(\mathfrak{T}\neg\forall u(a \in u \to b \in u) \to \neg\forall u(a \in u \to b \in u))$$

Proof. Use Lemma 7.1, Postulate 3.1.5.M_2 and Postulate 3.4.2.Mm_1. □

Lemma 7.1.5

$$\overset{\mathsf{M}}{\vdash} \mathfrak{T}\forall u(a \in u \to b \in u) \to \forall u(a \in u \to b \in u)$$

Proof. From Lemma 7.1.4 and Postulate 3.1.5.M_6 we obtain

$$\overset{\mathsf{M}}{\vdash} \mathfrak{T}\forall u(a \in u \to b \in u) \vee \mathfrak{T}\neg\forall u(a \in u \to b \in u),$$

and we finish by using Postulate 3.4.7.Cm_1. □

Definition 7.1.6

(1) A is *orthodox* just if $\overset{\mathsf{M}}{\vdash} \forall\vec{v}(\mathfrak{T}A \vee \mathfrak{T}\neg A)$.

(2) a is orthodox just if $\overset{\mathsf{M}}{\vdash} \forall\vec{v}\forall x(\mathfrak{T}x \in a \vee \mathfrak{T}x \notin a)$.

(3) $\mathfrak{O}(A)$ for A is orthodox, and $\mathfrak{O}(a)$ for a is orthodox.

Theorem 7.1.7 (Orthodoxy, equivalence and subsitutability of identicals)

(1) $\overset{\mathsf{M}}{\vdash} \mathfrak{T}a = b \vee \mathfrak{T}a \neq b$ Orthodoxy

(2) $\overset{\mathsf{M}}{\vdash} a = a$ Reflexivity

(3) $\overset{\mathsf{M}}{\vdash} a = b \wedge b = c \to a = c$ Transitivity

(4) $\overset{\mathsf{M}}{\vdash} a = b \to b = a$ Symmetry

(5) $\overset{\mathsf{M}}{\vdash} a = b \to (A^a_v \to A^b_v)$, where a and b are substitutable for v in A.

Proof.

1. Use Lemma 7.1.4 and Postulate 3.1.5.M_6.
2. Trivial
3. Trivial, given Definition 7.1.1

4. $\vdash^{\mathsf{M}} \forall u(a \in u \to b \in u) \to$

$$a \in \{v|\forall u(v \in u \to a \in u)\} \to b \in \{v|\forall u(v \in u \to a \in u)\}.$$

But

$$\vdash^{\mathsf{M}} a \in \{v|\forall u(v \in u \to a \in u)\}.$$

So

$$\vdash^{\mathsf{M}} \forall u(a \in u \to b \in u) \to b \in \{v|\forall u(v \in u \to a \in u)\}.$$

By alethic comprehension,

$$\vdash^{\mathsf{M}} \forall u(a \in u \to b \in u) \to \mathfrak{T}\forall u(b \in u \to a \in u).$$

A permutation of a and b in Lemma 7.1.5 gives us

$$\vdash^{\mathsf{M}} \mathfrak{T}\forall u(b \in u \to a \in u) \to \forall u(b \in u \to a \in u).$$

So a hypothetical syllogism gives us

$$\vdash^{\mathsf{M}} \forall u(a \in u \to b \in u) \to \forall u(b \in u \to a \in u).$$

Invoking Definition 7.1.1 suffices to finish.

5. We satisfy the promissory note issued in Remark 3.1 on Postulate 3.1.5.M_{12}. Suppose for a and b substitutable for v in A, and fair function Ξ,

$$\Xi(Ϙ) \nVdash \mathfrak{T}(\forall u(a \in u \to b \in u) \to (A^a_v \to A^b_v)).$$

On account of the validity of the mode of Postulate 3.4.7.Cm_2 we get

$$\Xi(Ϙ) \nVdash \mathfrak{T}\neg\mathfrak{T}(\forall u(a \in u \to b \in u) \wedge A^a_v \wedge \neg A^b_v).$$

It follows from Definition 2.2.2.1 that

$$\Xi(Ϙ) \Vdash \neg\mathfrak{T}\neg\mathfrak{T}(\forall u(a \in u \to b \in u) \wedge A^a_v \wedge \neg A^b_v).$$

On account of Postulate 3.1.5.M_1,

$$\Xi(Ϙ) \Vdash \neg\mathfrak{T}\neg(\mathfrak{T}\forall u(a \in u \to b \in u) \wedge \mathfrak{T}A^a_v \wedge \mathfrak{T}\neg A^b_v).$$

From Postulate 3.1.5.M_2 we get

$$\Xi(Ϙ) \Vdash \neg\mathfrak{T}\neg(\mathfrak{T}\forall u(a \in u \to b \in u) \wedge \mathfrak{T}A^a_v \wedge \neg\mathfrak{T}A^b_v).$$

On account of the validity of Lemma 7.1.4, we get

$$\Xi(Ϙ) \Vdash \neg\mathfrak{T}\neg(\forall u(a \in u \to b \in u) \wedge \mathfrak{T}A^a_v \wedge \neg\mathfrak{T}A^b_v).$$

From alethic comprehension and existential generalization we obtain

$$\Xi(Ϙ) \Vdash \neg\mathfrak{T}\neg(\forall u(a \in u \to b \in u) \wedge \exists u(a \in u \wedge b \notin u)),$$

which is impossible. So Postulate 3.1.5.M_{12} and Theorem 7.1.7.5 are valid, given Definition 2.4.2.3, and we are done. □

7.2 Lindenbaum-Tarski congruent terms, and alphabetological variants

Recall Definitions 5.2.6 and 5.2.9, Postulate 5.2.16 and Definition 5.2.17:

Postulates 7.2.1 and 7.2.2 say that identity is an equivalence relation which is neutral with respect to Lindenbaum-Tarski congruent terms, and alphabetological variants.

Postulate 7.2.1
(The Lindenbaum-Tarski closure for identity)

$$\vdash^{\mathsf{M}} \mathcal{D} \ulcorner \forall x(A(x) \leftrightarrow B(x)) \urcorner \to \{x|A(x)\} = \{x|B(x)\}$$

Postulate 7.2.2
(Alphabetical indifference)

$$\{x|A(x)\} = \{x|B(x)\} \to \{x|A(x)\} = \{y|B(x)^{y}_{x}\},$$

where y is substitutable for x in B.

Postulates 7.2.1 and 7.2.2 compensate somewhat for the loss of extensionality in £ and $\mathfrak{T}$, as per Section 10.5, and secure such theorems as:

$$\vdash^{\mathsf{M}} \{x|A(x)\} = \{y|A(y) \wedge \exists z(B(z) \vee \neg B(z))\}.$$

Chapter 8

Arithmetic

First order number theory justified in £

The exposition here takes place in ω, i.e. the least von Neumann ordinal which contains all finite von Neumann ordinals.

Definition 8.0.1

(1) $\varnothing = \{x | x \neq x\}$

(2) $a' = \{x | x = a \vee x \in a\}$

(3) $\omega = \{x | \forall y(\varnothing \in y \wedge \forall z(z \in y \rightarrow z' \in y) \rightarrow x \in y)\}$

Theorem 8.0.2

(1) $\overset{\mathsf{M}}{\vdash} \varnothing \in \omega$

(2) $\overset{\mathsf{M}}{\vdash} \forall x(x \in \omega \rightarrow x' \in \omega)$

(3) ω is orthodox

(4) $\overset{\mathsf{M}}{\vdash} \forall y(\varnothing \in y \wedge \forall z(z \in y \rightarrow z' \in y) \rightarrow \forall x(x \in \omega \rightarrow x \in y))$

(5) $\overset{\mathsf{M}}{\vdash} A(\varnothing) \wedge \forall x(A(x) \rightarrow A(x')) \rightarrow \forall y(y \in \omega \rightarrow A(y))$

Proof.

1. Combine alethic comprehension and the fact that

$$\overset{\mathsf{M}}{\vdash} \mathfrak{T}\forall y(\varnothing \in y \wedge \forall z(z \in y \rightarrow z' \in y) \rightarrow \varnothing \in y)$$

2. This follows from alethic comprehension and

$$\overset{\mathsf{M}}{\vdash} \forall x(\mathfrak{T}(\forall y(\varnothing \in y \wedge \forall z(z \in y \rightarrow z' \in y) \rightarrow x \in y)) \rightarrow \\ \mathfrak{T}(\forall y(\varnothing \in y \wedge \forall z(z \in y \rightarrow z' \in y) \rightarrow x' \in y)))$$

3. From logic

$$\overset{\mathsf{M}}{\vdash} \varnothing \in \omega \wedge \forall x(x \in \omega \to x \in \omega) \to (\forall y(\varnothing \in y \wedge \forall x(x \in y \to x' \in y) \to a \in y) \to a \in \omega)$$

By combining with 1 and 2 we have

$$\overset{\mathsf{M}}{\vdash} \forall y(\varnothing \in y \wedge \forall x(x \in y \to x' \in y) \to a \in y) \to a \in \omega)$$

Postulate 3.4.2.Mm_1, Postulate 3.1.5.M_1 and alethic comprehension give us

$$\overset{\mathsf{M}}{\vdash} a \in \omega \to \mathfrak{T}a \in \omega$$

Postulate 3.4.3.Od_3 along with Postulate 3.1.5.M_1, Postulate 3.1.5.M_2 and Postulate 3.1.5.M_6 give us

$$\overset{\mathsf{M}}{\vdash} \mathfrak{T}a \in \omega \vee \mathfrak{T}a \notin \omega$$

As a was arbitrary, we have

$$\overset{\mathsf{M}}{\vdash} \forall x(\mathfrak{T}x \in \omega \vee \mathfrak{T}x \notin \omega),$$

and we are done.

4. Immediate, as it is equivalent with

$$\overset{\mathsf{M}}{\vdash} \forall x(x \in \omega \to \forall y(\varnothing \in y \wedge \forall z(z \in y \to z' \in y) \to x \in y))$$

5. Confer [8](356): Let, for arbitrary sentence $A(x)$,

$$A'(x) \Longleftarrow A(\varnothing) \wedge \forall y(A(y) \to A(z') \to A(x))$$

By logic,

$$\overset{\mathsf{M}}{\vdash} A'(\varnothing) \;\&\; \overset{\mathsf{M}}{\vdash} \forall x(A'(x) \to A'(x'))$$

Inference mode Postulate 3.4.2.Mm_1, and Postulate 3.1.5.M_{12}, give us

$$\overset{\mathsf{M}}{\vdash} \mathfrak{T}A'(\varnothing) \;\&\; \overset{\mathsf{M}}{\vdash} \forall x\mathfrak{T}(A'(x) \to A'(x'))$$

By quantifier distribution and Postulate 3.1.5.M_1 we get

$$\overset{\mathsf{M}}{\vdash} \mathfrak{T}A'(\varnothing) \;\&\; \overset{\mathsf{M}}{\vdash} \forall x(\mathfrak{T}A'(x) \to \mathfrak{T}A'(x'))$$

Alethic comprehension gives us

$$\overset{\mathsf{M}}{\vdash} \varnothing \in \{y|A'(y)\} \;\&\; \overset{\mathsf{M}}{\vdash} \forall x(x \in \{y|A'(y)\} \to x' \in \{y|A'(y)\})$$

Adjunction gives us

$$\overset{\mathsf{M}}{\vdash} \varnothing \in \{y|A'(y)\} \wedge \forall x(x \in \{y|A'(y)\} \to x' \in \{y|A'(y)\})$$

4 and the inference of mode Postulate 3.4.3.Od_3 give us

$$\overset{\mathsf{M}}{\vdash} \forall x(x \in \omega \to x \in \{y|A'(y)\})$$

From 3 and Postulate 3.4.7.Cm_1 we have

$$\stackrel{\mathsf{M}}{\vdash} \forall x(\mathfrak{T}x \in \omega \to x \in \omega),$$

so that

$$\stackrel{\mathsf{M}}{\vdash} \forall x(\mathfrak{T}x \in \omega \to x \in \{y|A'(y)\})$$

Alethic comprehension gives us

$$\stackrel{\mathsf{M}}{\vdash} \forall x(\mathfrak{T}x \in \omega \to \mathfrak{T}A'(x)),$$

which, combined with Postulate 3.4.7.Cm_3 establish

$$\stackrel{\mathsf{M}}{\vdash} \forall x(x \in \omega \to A'(x))$$

A use of Theorem 8.0.1.2, and rearrangement, finishes the proof. □

Chapter 9

Shortcomings and redresses

We supplement Section 3 with negative results whose justification depend upon Section 7.

9.1 Shortcoming of existential instantiation

Theorem 9.1.1 (Maximal deficit)
There are cases such that $\vdash^{\mathsf{M}} \exists x A$ and for no term a, $\vdash^{\mathsf{M}} A^a_x$.

Proof. $\vdash^{\mathsf{M}} \exists x(x = \varnothing \leftrightarrow \mathrm{r} \in \mathrm{r})$, but for no $a \vdash^{\mathsf{M}} (a = \varnothing \leftrightarrow \mathrm{r} \in \mathrm{r})$. □

Corollary 9.1.2
Maximal existential instantiation in the form

$$\vdash^{\mathsf{M}} \exists x A \Rightarrow \text{for some term } a, \vdash^{\mathsf{M}} A^a_x$$

cannot be adopted as an inference rule, as it is not always valid.

Theorem 9.1.3 (Orthodox redress)
We show the validity of Postulate 3.4.7.Cm_6, as announced on page 20:

$$\vdash^{\mathsf{M}} \mathfrak{O}(A(x)) \Rightarrow \ (\vdash^{\mathsf{M}} \exists x A \Rightarrow \vdash^{\mathsf{M}} A^a_x \text{ for some } a \text{ substitutable for } x \text{ in A}).$$

Proof.

$$\text{Assume that } A(x) \text{ is orthodox, i.e. } \vdash^{\mathsf{M}} \mathfrak{T}A(x) \vee \mathfrak{T}\neg A(x). \tag{9.1.4}$$

By soundness,

$$\vdash^{\mathsf{M}} \exists x A \Rightarrow \vDash^{\mathsf{M}} \exists x A, \text{so for all fair functions } \Xi, \Xi(\text{♀}) \Vdash \mathfrak{T}\exists x A. \tag{9.1.5}$$

$$\text{As ♀ is a stabilising ordinal, } \Xi(\text{♀}) \Vdash \exists x A. \tag{9.1.6}$$

$$\text{Given Definition 2.2.2 and Theorem 2.2.3, for a } a, \Xi(\text{♀}) \Vdash A^a_x. \tag{9.1.7}$$

$$\text{As } A(x) \text{ is orthodox, } \Xi(\text{♀}) \Vdash \mathfrak{T}A^a_x. \tag{9.1.8}$$

$$\text{So } \vDash^{\mathsf{M}} A^a_x. \tag{9.1.9}$$

$$\text{So Postulate 3.4.7.}\mathrm{Cm}_6 \text{ is valid.} \tag{9.1.10}$$

□

9.2 Shortcoming of attestor

As stated in Remark 3.3.2, it will here be shown that the attestor schema of Postulate 3.3.1.T_3 does not in general hold as a maxim, i.e. some instances of the schema are minors.

Theorem 9.2.1
For some A,

$$\overset{\mathsf{M}}{\vDash} \mathfrak{T}\exists x A \;\&\; \overset{\mathsf{M}}{\nvDash} \exists x \mathfrak{T} A.$$

Proof. Let A be as in Theorem 9.1.1. Obviously, $\overset{\mathsf{M}}{\vDash} \mathfrak{T}\exists x(x = \varnothing \leftrightarrow \mathrm{r} \in \mathrm{r})$ holds. Suppose $\overset{\mathsf{M}}{\vDash} \exists x\mathfrak{T}A$, so that $\Xi(\text{♀}) \Vdash \mathfrak{T}\exists x\mathfrak{T}A$. It follows that there is an ordinal γ such that $\Xi(\beta) \Vdash \exists x\mathfrak{T}A$ holds whenever $\gamma \prec \beta \prec \text{♀}$. Let limit ordinal λ satisfy $\gamma \prec \lambda \prec \text{♀}$, so that $\Xi(\lambda) \Vdash \exists x\mathfrak{T}A$. On account of Definition 2.2.2.1 and Definition 2.2.2.2, there is a term a and an ordinal δ such that $a = \varnothing \leftrightarrow \mathrm{r} \in \mathrm{r}$ holds at all ordinals θ which satisfy $\delta \prec \theta \prec \lambda$. But this is impossible, as $r \in r$ holds at some of those ordinals, and $\mathrm{r} \notin \mathrm{r}$ holds at others, and identity is orthodox. □

9.3 Shortcoming of the Barcan thesis

As mentioned in Remark 3.3.2, it will be shown that the axiomatic Barcan schema in Postulate 3.1.5.M_3 does not hold as a maxim, but only as a thesis.

The precursor to this result, in a truth theoretic context, is *McGee's paradox*, first published upon in [24], which we adapt to our context. Compare [8](380–382) and [4](357).

First we decide upon some notions:

Definition 9.3.1
For r in 9.3.1.5, recall Definition 3.1.4:

(1) $a' \Longleftarrow \{x|x \in a \vee x = a\}$.

(2) $\{a, b\} \Longleftarrow \{x|x \in a \vee x \in b\}$.

(3) $\{a\} \Longleftarrow \{a, a\}$.

(4) $a_\omega \Longleftarrow \{u|\forall x(\langle\varnothing, a\rangle \in x \wedge \forall y, z(\langle y, z\rangle \in x \rightarrow \langle y', \{v|v \in z\}\rangle) \rightarrow u \in x)\}$.

(5) $\mathrm{t} \Longleftarrow \{x|x = r \wedge x \notin x \wedge \neg\mathfrak{T}x \in x\}$.

(6) Use $\overline{0}$, $\overline{1}$, $\overline{2}$, . . . for the members of ω.

(7) Let $\mathrm{t}_{\overline{0}} \Longleftarrow \mathrm{t}$ and $\mathrm{t}_{\overline{n+1}} \Longleftarrow \{v|v \in \mathrm{t}_{\overline{n}}\}$.

(8) $\mathrm{B}(\mathrm{t}_{\overline{i}}) \Longleftarrow \exists w(\langle w, \mathrm{t}_{\overline{i}}\rangle \in \mathrm{t}_\omega) \rightarrow r \notin \mathrm{t}_{\overline{i}}$

(9) $\mathrm{B}(x) \Longleftarrow \exists w(\langle w, x\rangle \in \mathrm{t}_\omega) \rightarrow r \notin x$

Lemma 9.3.2
For any a, a_ω is orthodox.

Proof. Adapt the proof of Theorem 8.0.2.3. □

Lemma 9.3.3

$$\Xi(\lambda) \Vdash r = r \wedge \mathrm{r} \notin \mathrm{r} \wedge \neg\mathfrak{T}\mathrm{r} \in \mathrm{r},$$

just if λ is a limit.

Proof. For any successor ordinal $\chi + 1$, $\Xi(\chi + 1) \Vdash \neg\mathfrak{T}\mathrm{r} \in \mathrm{r} \leftrightarrow \mathrm{r} \in \mathrm{r}$. Precisely at any limit ordinal λ, $\Xi(\lambda) \Vdash \mathrm{r} \notin \mathrm{r} \wedge \neg\mathfrak{T}\mathrm{r} \in \mathrm{r}$. □

Theorem 9.3.4

Let $\alpha \prec \Omega$ be a limit ordinal, and β be $\alpha + \omega$:

1. $\Xi(\beta) \Vdash \forall x \mathfrak{T}\mathrm{B}(x)$
2. $\Xi(\beta) \Vdash \neg\mathfrak{T}\forall x \mathrm{B}(x)$.

Proof.

1. If $\Xi(\beta) \Vdash \neg\exists w(\langle w, x\rangle \in \mathrm{t}_\omega)$, it follows that $\Xi(\beta) \Vdash \mathfrak{T}\mathrm{B}(x)$ on account of Lemma 9.3.2. If, on the other hand, $\Xi(\beta) \Vdash \exists w(\langle w, \mathrm{t}_{\bar{i}}\rangle \in \mathrm{t}_\omega)$ we have that $\Xi(\beta) \Vdash \mathfrak{T}\mathrm{B}(\mathrm{t}_{\bar{i}})$, as there is a $\gamma \succeq \alpha + i$ such that
$$\forall \delta(\alpha \prec \gamma \preceq \delta \prec \beta \Rightarrow \Xi(\delta) \Vdash \mathrm{B}(\mathrm{t}_{\bar{i}})).$$
So for any term y, $\Xi(\beta) \Vdash \mathfrak{T}\mathrm{B}(y)$, and so $\Xi(\beta) \vdash \forall x \mathfrak{T}\mathrm{B}(x)$.
2. Otherwise, $\Xi(\beta) \Vdash \mathfrak{T}\forall x \mathrm{B}(x)$, and we would have $\Xi(\delta) \Vdash \forall x \mathrm{B}(x)$ as from some ordinal δ below β and above α. Let $\delta == \alpha + (n + 1)$, for finite ordinal $n \succeq 0$, be such an ordinal. A $\Xi(\delta) \Vdash \mathrm{B}(\mathrm{t}_{\bar{n}})$, by instantiation, this entails that $\Xi(\alpha + (n+1)) \Vdash \mathrm{B}(t_{\bar{n}})$. As $\vDash \exists w(\langle w, \mathrm{t}_{\bar{n}}\rangle \in \mathrm{t}_\omega)$, it follows that $\Xi(\alpha + (n + 1)) \Vdash r \notin \mathrm{t}_{\bar{n}}$. As a consequence, $\Xi(\alpha + 1) \Vdash r \notin \mathrm{t}_{\bar{0}}$. But the latter entails $\Xi(\alpha) \Vdash (r \neq r \vee \mathrm{r} \in \mathrm{r} \vee \mathfrak{T}\mathrm{r} \in \mathrm{r})$ which contradicts Lemma 9.3.3, as α is presupposed to be a limit ordinal. □

Theorem 9.3.5

$$\nVdash^{\mathsf{M}} \forall x \mathfrak{T}\mathrm{B}(x) \rightarrow \mathfrak{T}\forall x \mathrm{B}(x)$$

Proof. Theorem 9.3.4 with Definition 2.2.2 entail that

$$\Xi(\beta) \nVdash \forall x \mathfrak{T}\mathrm{B}(x) \rightarrow \mathfrak{T}\forall x \mathrm{B}(x).$$

It follows that

$$\Xi(\Omega) \nVdash \mathfrak{T}(\forall x \mathfrak{T}\mathrm{B}(x) \rightarrow \mathfrak{T}\forall x \mathrm{B}(x)),$$

and an appeal to Definition 2.4.2.3 finishes the proof. □

Corollary 9.3.6

$$\nVdash^{\mathsf{M}} \forall x \mathfrak{T}\mathrm{B}(x) \rightarrow \mathfrak{T}\forall x \mathrm{B}(x)$$

Corollary 9.3.7

$$\vdash \forall x \mathfrak{T}\mathrm{B}(x) \rightarrow \mathfrak{T}\forall x \mathrm{B}(x) \;\&\; \vdash \forall x \mathfrak{T}\mathrm{B}(x) \wedge \neg\mathfrak{T}\forall x \mathrm{B}(x)$$

9.4 The orthodox redresses

Theorem 9.1.3 (Orthodox existential instantiation)

$$\models^{\mathsf{M}} \mathfrak{D}(A(x)) \Rightarrow (\models^{\mathsf{M}} \exists x A \Rightarrow \models^{\mathsf{M}} A^a_x \text{ for some } a \text{ substitutable for } x \text{ in A}).$$

Proof. As on page 39. □

Theorem 9.4.1 (Orthodox attestor)
If $A(x)$ is orthodox, then

$$\models^{\mathsf{M}} \mathfrak{T}\exists x A(x) \Rightarrow \models^{\mathsf{M}} \exists x \mathfrak{T} A(x).$$

Proof. Appeal to Theorem 9.1.3, and existential generalization. □

Theorem 9.4.2 (The Barcan formula holds for orthodox formulas)

$$\models^{\mathsf{M}} \mathfrak{D}(B(x)) \Rightarrow \models^{\mathsf{M}} (\forall x \mathfrak{T}\mathrm{B}(x) \rightarrow \mathfrak{T}\forall x \mathrm{B}(x)).$$

Proof. As on page 21. □

Chapter 10

Manifestations

10.1 Introduction

The manifestation set construction is very strong. We will see below that it has important *positive* consequences for $\mathfrak{B}$'s ability to account for strong set theoretic principles.

The foci in this section will be upon *negative* results. We account for the *autocombatant*, as was promised in Remark 3.3.3. Next we elucidate the quase universal paradoxicality of power sets, and the prevalent failure of extensionality. We put an emphasis upon relating the constructions needed for the results.

10.2 Orthodox manifestation

For the following construction, cfr. [4](345–46), [8](76), [31](695–96) and earlier literature referred to there. One may, plausibly, find that Roger's theorem and Kleene's second recursion theorem are related, but the proof that there are manifestation sets does not rely upon any presuppositions of computability.

Definition 10.2.1 (Kuratowskian ordered pairs)

$$\langle a, b\rangle \Longleftarrow \{\{a\}, \{a, b\}\}$$

Definition 10.2.2 (The manifestation set $\mathbb{a}$ of formula $A(x, y)$)

(1) $v\eta b \Longleftarrow \exists w(w = \langle v, b\rangle \wedge w \in b)$

(2) $\mathfrak{a} \Longleftarrow \{z|\exists x, y(z = \langle x, y\rangle \wedge A(x, y)_y^{\{v|v\eta y\}})\}$

(3) $\mathbb{a} \Longleftarrow \{v|v\eta\mathfrak{a}\}$

Theorem 10.2.3
For formula $A(x, y)$

$$\stackrel{\mathsf{M}}{\vdash} \forall x(x \in \mathbb{a} \leftrightarrow \mathfrak{T}\mathfrak{T}A(x, \mathbb{a}))$$

Proof. From Definition 10.2.2.3 and alethic comprehension,,

$$\stackrel{\mathsf{M}}{\vdash} c \in \mathbb{a} \leftrightarrow \mathfrak{T}c\eta\mathfrak{a}$$

Combining with Definition 10.2.2.1 we get

$$\overset{\mathsf{M}}{\vdash} \mathfrak{T}c\eta\mathfrak{a} \leftrightarrow \mathfrak{T}\exists w(w = \langle c, \mathfrak{a}\rangle \wedge w \in \mathfrak{a})$$

Combinining the two previous steps, Definition 10.2.2.2, alethic comprehension and Postulate 3.4.3.Od_3 we have

$$\overset{\mathsf{M}}{\vdash} c \in \mathbb{a} \leftrightarrow \mathfrak{T}\exists w(w = \langle c, \mathfrak{a}\rangle \wedge \mathfrak{T}\exists x, y(w = \langle x, y\rangle \wedge A(x, y)_y^{\{v|v\eta y\}}))$$

By means of the theory of identity we infer

$$\overset{\mathsf{M}}{\vdash} c \in \mathbb{a} \leftrightarrow \mathfrak{T}\mathfrak{T}A(c, y)_y^{\{v|v\eta\mathfrak{a}\}},$$

so that we on account of Definition 10.2.2.3 and Definition 1.4.14 have

$$\overset{\mathsf{M}}{\vdash} c \in \mathbb{a} \leftrightarrow \mathfrak{T}\mathfrak{T}A(c, \mathbb{a})$$

Finish with universal generalization, i.e. Postulate 3.4.7.Cm_7. □

Corollary 10.2.4 (Orthodox manifestation)

If $A(x, y)$ is orthodox, $\overset{\mathsf{M}}{\vdash} \forall x(x \in \mathbb{a} \leftrightarrow A(x, \mathbb{a}))$

10.3 The heretical autocombatant

In contrast to orthodox manifestation sets, there are many paradoxical ones, for example the following quite heretical one, which generates contradictory theses.

Theorem 10.3.1 (The *autocombatant*)
As indicated in Remark 3.3.3, we, for formula $\mathring{\mathrm{A}}(x, y) \Longleftarrow x \notin y$, and associated manifestation set $\mathring{\mathbb{a}}$, have the kindred theses for the autocombatant:

$$\vdash \forall x(x \in \mathring{\mathbb{a}}) \ \& \ \vdash \forall x(x \notin \mathring{\mathbb{a}}).$$

Proof. We reason semantically on the basis of Theorem 10.2.3, as we have

$$\overset{\mathsf{M}}{\vDash} \forall x(x \in \mathring{\mathbb{a}} \leftrightarrow \mathfrak{T}\mathfrak{T}x \notin \mathring{\mathbb{a}}).$$

If λ is any limit below the closure ordinal ♀, we will, for any term a, and any fair function Ξ, have that $\Xi(\lambda) \Vdash a \notin \mathring{\mathbb{a}}$; otherwise a contradiction would follow as $a \notin \mathring{\mathbb{a}}$ would hold at succeeding successor ordinals σ, $\sigma + 1$ and $\sigma + 2$ below λ. Consequently, we for such a limit λ as well have that $\Xi(\lambda + 2) \Vdash a \in \mathring{\mathbb{a}}$. From Definition 2.2.2.2 we have that $\Xi(\lambda) \Vdash \forall x(x \notin \mathring{\mathbb{a}})$ and $\Xi(\lambda + 2) \Vdash \forall x(x \in \mathring{\mathbb{a}})$. As a result, $\Xi(\text{♀}) \Vdash \neg\mathfrak{T}\neg\forall x(x \in \mathring{\mathbb{a}})$ and $\Xi(\text{♀}) \Vdash \neg\mathfrak{T}\neg\forall x(x \notin \mathring{\mathbb{a}})$. The proof finishes by invoking Definition 2.4.2.1. □

10.4 Powersets are paradoxical lest as $\mathcal{P}(\{x|x = x\})$

We use standard notation and write

Definition 10.4.1

$$\mathcal{P}(a) \Longleftarrow \{x|x \subset a\}.$$

As mentioned in Remark 3.3.3, we show that power sets are paradoxical unless they are the power set of a set b such that $\overset{\mathsf{M}}{\vDash} \forall x(x \in b)$.

We follow tradition:

Definition 10.4.2

$$\mathrm{V} \Longleftarrow \{x|x = x\}$$

Theorem 10.4.3

If it's a thesis that a is extensionally distinct from V, then the power set $\mathcal{P}(a)$ is paradoxical.

Proof.

(1) If $\overset{\mathsf{M}}{\vDash} \exists x(x \notin a)$, use the autocombatant $\mathring{a}$, of Postulate 3.3.3, for which

$$\vdash \forall x(x \in \mathring{a}) \;\&\; \vdash \forall x(x \notin \mathring{a})$$

It follows that $\vdash \mathring{a} \notin \mathcal{P}(a)$ and $\vdash \mathring{a} \in \mathcal{P}(a)$, so that $\mathcal{P}(a)$ is paradoxical.

(2) If $\vDash_{\mathsf{m}} \exists x(x \notin a)$, $\vdash \mathrm{V} \in \mathcal{P}(a)$ and $\vdash \mathrm{V} \notin \mathcal{P}(a)$, so $\mathcal{P}(a)$ is paradoxical. □

10.5 Non-extensionality

The phenomenon of non-extensionality in type free theories is well known, and several have contributed to the deposit of knowledge.

We presuppose the notation imposed by

Definition 10.5.1
$a \overset{\mathsf{e}}{=} b \Longleftarrow \forall x(x \in a \leftrightarrow x \in b)$.

A particularly simple proof of extensionality failure in £ and $\mathfrak{T}$ is obtained by making use of the fact that for any limit ordinal λ,

$$\Xi(\lambda + 1) \vDash \{x|x = x\} \overset{\mathsf{e}}{=} \{x|x \notin x\} \wedge \{x|x = x\} \neq \{x|x \notin x\}.$$

Consequently, there are sets a and b such that $\nvDash a \overset{\mathsf{e}}{=} b \to a = b$, and, a fortiori, $\overset{\mathsf{M}}{\nvDash} \; a \overset{\mathsf{e}}{=} b \to a = b$; but $\overset{\mathsf{M}}{\vDash} a \overset{\mathsf{e}}{=} b \to a = b \Rightarrow \overset{\mathsf{M}}{\vDash} \; a \overset{\mathsf{e}}{=} b \to a = b$ is a soundness requirement, so that $\overset{\mathsf{M}}{\nvDash} a \overset{\mathsf{e}}{=} b \to a = b$.

As related in [8](73), Gilmore [13] showed, for a partial set theory, that it proves the existence of an orthodox set a such that $a \overset{\mathsf{e}}{=} \varnothing$ and $a \neq \varnothing$. In conversation, Lev Gordeev related that he communicated a much simpler proof of the same result, based upon combinatoric logic, in the context of Explicit Mathematics, to Solomon Feferman, around 1981. This was published in 1985, with acknowledgement to Gordeev, by Michael Beeson, in [2]. Andrea Cantini, in [8](74), relates a proof, by Pierluigi Minari, to the effect that we for any orthodox set a may find a distinct orthox set b such that a and b are nevertheless co-extensional.

Theorem 5 (ii) in [4](346) relates the result that Minari's construction can be generalized, as in Theorem 10.5.2 below, and left it as an exercise to prove.

The content of Theorem 10.5.2, just after the next paragraph, is on a par with Theorem 5 (ii) in [4](346), but its proof is more precise than the proof of Theorem 5 in [4](346), and it is more informative than the latter.

The result expressed by Theorem 10.5.2 appears to be the most general result available, and we do not relate proofs of other non-extensionality results mentioned here, as they are corollaries.

Theorem 10.5.2
For *any* ortodox set a there are infinitely many co-extensional and pairwise distinct orthodox sets which, in their turn, are all co-extensional with a and distinct from a.

Proof. The induction hypothesis, for $n > 1$, and orthodox sets $a_1, \ldots a_n$, is that

$$\bigwedge_{i=1}^{i=n} \bigwedge_{j=1}^{j=n} (i \neq j \to (a_i \neq a_j \wedge a_i \stackrel{\mathsf{e}}{=} a_j)). \tag{10.5.3}$$

Let a_{n+1} be the manifestation set of

$$\Big(\bigvee_{i=1}^{i=n} y = a_i \wedge \bigwedge_{i=1}^{i=n} a_i \notin a_i\Big) \vee \Big(\bigwedge_{i=1}^{i=n} y \neq a_i \wedge x \in a_i\Big) \tag{10.5.4}$$

so that, by the logic of identity,

$$\begin{gathered}\forall x (x \in a_{n+1} \leftrightarrow \\ (\Big(\bigvee_{i=1}^{i=n} a_{n+1} = a_i \wedge \bigwedge_{i=1}^{i=n} a_i \notin a_i\Big) \vee \Big(\bigwedge_{i=1}^{i=n} (a_{n+1} \neq a_i \wedge x \in a_i)\Big))).\end{gathered} \tag{10.5.5}$$

If $\bigvee_{i=1}^{i=n}(a_{n+1} = a_i)$, it follows that $\vdash^{\mathsf{M}} a_{n+1} \in a_{n+1} \leftrightarrow a_{n+1} \notin a_{n+1}$, which is impossible. So $\bigwedge_{i=1}^{i=n}(a_{n+1} \neq a_i)$. It follows from the induction hypothesis that $\bigwedge_{i=1}^{i=n}(a_i \stackrel{\mathsf{e}}{=} a_{n+1})$. The process can be iterated ad libitum, so we are done. □

Chapter 11

Platforms

We isolate a variety of hereditarily orthodox sets which may be used to interpret classical set theories, via natural strengthenings, despite that $\mathfrak{W}$ proves that there are only denumerably many sets.

11.1 Auxiliaries

Definition 11.1.1 (Co-extensionality)

$$a \overset{\mathsf{e}}{=} b \Longleftarrow\!\!= \forall x(x \in a \leftrightarrow x \in b).$$

Definition 11.1.2 (Extent-functionality)

A is extent-functional on x and y just if $\forall \vec{v}, x, y(A \wedge A^y_x \rightarrow x \overset{\mathsf{e}}{=} y)$.

Definition 11.1.3 (Notation for binders restricted to set b)

(1) A^b and a^b signifiy all variables bound in A and a are restricted to b.

(2) If a is a variable, then a^b is a.

(3) $(c \in d)^b$ is $c^b \in d^b$.

(4) $\neg A^b$ is $\neg(A^b)$, $(A \wedge B)^b$ is $(A^b \wedge B^b)$, and so on for other connectives.

(5) $\{v|A\}^b \Longleftarrow\!\!= \{v|v \in b \wedge A^b\}$.

(6) $(\forall v)(A)^b \Longleftarrow\!\!= (\forall v)(v \in b \rightarrow A^b)$.

(7) $(\forall \vec{v})(A)^b \Longleftarrow\!\!=$ is the sentence given by the least $n \geq 0$ such that

$$\Big(n > 0 \;\&\; (\forall v_1 \ldots \forall v_n)(v_1 \in b \wedge \ldots \wedge v_n \in b \rightarrow A^b)\Big)$$

r

$$\Big(n = 0 \,\&\, A^b\Big).$$

Definition 11.1.4 (The *join* of a and b)

$$\mathcal{J}(a, b) == \{x | x \in a \vee x \in b\} == (a\mathcal{J}b)$$

Definition 11.1.5 (The union set of a)

$$\bigcup(a) == \{x | \exists y(x \in y \wedge y \in a)\}$$

Definition 11.1.6 (Capture)

$$\mathcal{C}(a, \ulcorner B \urcorner) == \{x | \exists y(y \in a \wedge B(x, y) \wedge \forall z(B(x, y)^z_x \to x \stackrel{\mathsf{e}}{=} z))\}$$

Definition 11.1.7 (Capture on codes)

$$\mathcal{C}(a, \ulcorner B \urcorner) == \{x | \exists y(y \in a \wedge \mathrm{T}\ulcorner B(x, y) \urcorner \wedge \forall z(\mathrm{T}\ulcorner B(x, y)^z_x \urcorner \to x \stackrel{\mathsf{e}}{=} z))\}$$

Recall Definition 10.4.1, of $\mathcal{P}(a)$ as $\{x | x \subset a\}$:

Definition 11.1.8 (Scott infinity)

$$\begin{aligned} \varpi == \{x | \forall v(\Big(\forall w\big(\forall x(x \notin w) \to w \in v\big) \wedge \\ \forall w\big(w \in v \to \mathcal{P}(w) \in v\big)\Big) \to x \in v)\} \end{aligned}$$

11.2 Hoard, Holder – heiresses and heritors

Definition 11.2.1

a is $hereditarily$ orthodox iff orthodox with just hereditarily orthodox members.

Definition 11.2.2

The Hoard is set

$$\hbar == \{x | \{w | w \notin x\} = \{w | w \in \{w | w \notin x\}\}\}.$$

Definition 11.2.3

The Holder is set

$$\mathbb{h} == \{x | \{w | w \in x\} = x\}.$$

Definition 11.2.4

a is an *heiress* just if $\overset{\mathsf{M}}{\vDash} a \in \hbar$.

Definition 11.2.5

a is an *heritor* just if $\overset{\mathsf{M}}{\vDash} a \in \mathbb{h}$

.

Theorem 11.2.6

The Hoard is orthodox.

Proof. By the theory of identity. □

Theorem 11.2.7

The *heiresses* are orthodox.

Proof. Assume a is an heiress. It follows that a is in the hoard, and therefore

$$\{x|x \notin a\} = \{x|x \in \{x|x \notin a\}\}. \tag{11.2.8}$$

As a consequence,

$$\overset{\mathsf{M}}{\vDash} \forall x(x \in \{x|x \notin a\} \leftrightarrow x \in \{x|x \in \{x|x \notin a\}\}). \tag{11.2.9}$$

So $\overset{\mathsf{M}}{\vDash} \forall x(\mathfrak{T}x \notin a \leftrightarrow \mathfrak{T}x \in \{x|x \notin a\})$, and it follows that

$$\overset{\mathsf{M}}{\vDash} \forall x(\mathfrak{T}x \notin a \rightarrow \mathfrak{T}\mathfrak{T}x \notin a). \tag{11.2.10}$$

By Postulate 3.1.5.M_7 and detachment, $\overset{\mathsf{M}}{\vDash} \forall x(\mathfrak{T}x \in a \vee \mathfrak{T}x \notin a)$, so a is orthodox. □

Theorem 11.2.11

The Holder is orthodox, so $\overset{\mathsf{M}}{\vDash} \mathfrak{O}(\mathbb{h})$

Proof. By the theory of identity. □

Theorem 11.2.12

The heritors are orthodox.

Proof. By the theory of identity. □

11.3 Minimal platforms for hereditarily orthodox sets

$\hbar$ and $\mathbb{h}$ may be very large sets, if one wants them to be, and empty, if one so decrees.

Below we use heritors $x \in \mathbb{h}$, which, as per Theorem 11.2.12, are orthodox elements of orthodox Holder $\mathbb{h}$.

Definition 11.3.1

$$A(x,y) == x \in \mathbb{h} \wedge y \in \mathbb{h} \wedge x \subset y$$

Definition 11.3.2 (Definition by manifestation)

$$\models^{\mathsf{M}} \forall x(x \in \prod \leftrightarrow \mathfrak{T}\mathfrak{T}x \in \mathbb{h} \wedge \prod \in \mathbb{h} \wedge x \subset \prod)$$

Theorem 11.3.3

We see in our heads that $\prod$ is orthodox, so

$$\models^{\mathsf{M}} \forall x(x \in \prod \leftrightarrow x \in \mathbb{h} \wedge \prod \in \mathbb{h} \wedge x \subset \prod)$$

Definition 11.3.4

$\prod$, in Theorem 11.3.3, is a *platform*.

Observation 11.3.5

Platforms are hereditarily orthodox sets.

Observation 11.3.6

As it is not declared that $\prod$ has members, we should take it to be empty, as the statement that it has members is not valid. So $\prod$ is a minimal platform, and there are distinct minimal platforms.

11.4 The invariant platform

Let

$$\mathfrak{i}(a) \leftrightarrow \forall x, y(x \in a \wedge x \stackrel{E}{=} y \rightarrow y \in a)$$

and $\prod$ be as in Section 11.3:

Definition 11.4.1

$$a \in \lceil \mathfrak{i} \rceil \leftrightarrow (a \in \prod \wedge \mathfrak{i}(a) \wedge \exists x(x \in \prod \wedge a \subset x \wedge \forall y(y \in \prod \wedge y \in x \rightarrow \mathfrak{i}(y) \wedge y \subset x)))$$

Theorem 11.4.2

$\lceil \mathfrak{i} \rceil$ is hereditarily orthodox.

Proof. Given Lemma 11.4.3, this is displayed to our intuitions by the internal forum. □

Lemma 11.4.3

$a \in \prod \wedge \mathfrak{i}(a)$ is orthodox.

Proof. Assume

$$a \in \prod \wedge \mathfrak{i}(a). \tag{11.4.4}$$

Using Theorem 11.3.3,

$$\Updownarrow$$

$$a \in \mathbb{h} \wedge \prod \in \mathbb{h} \wedge a \subset \prod \wedge \mathfrak{i}(a) \tag{11.4.5}$$

$$\Updownarrow$$

$$a \in \mathbb{h} \wedge \prod \in \mathbb{h} \wedge a \subset \prod \wedge \forall x, y(x \in a \wedge \forall u(u \in x \leftrightarrow u \in y) \rightarrow y \in a)) \tag{11.4.6}$$

$$\Updownarrow$$

$$a \in \mathbb{h} \wedge \prod \in \mathbb{h} \wedge a \subset \prod \wedge (\mathfrak{v}_1 \in a \wedge \forall u(u \in \mathfrak{v}_1 \leftrightarrow u \in \mathfrak{v}_2) \rightarrow \mathfrak{v}_2 \in a)) \tag{11.4.7}$$

$$\Updownarrow$$

$$a \in \mathbb{h} \wedge \prod \in \mathbb{h} \wedge a \subset \prod \wedge \mathfrak{v}_1 \in a \wedge (\mathfrak{v}_3 \in \mathfrak{v}_1 \leftrightarrow \mathfrak{v}_3 \in \mathfrak{v}_2) \rightarrow \mathfrak{v}_2 \in a)) \tag{11.4.8}$$

$$\Updownarrow$$

$$\begin{gathered} a \notin \mathbb{h} \vee \prod \notin \mathbb{h} \vee \ a \not\subset \prod \vee \mathfrak{v}_1 \notin a \vee ((\mathfrak{v}_3 \in \mathfrak{v}_1 \wedge \mathfrak{v}_3 \notin \mathfrak{v}_2) \vee \\ (\mathfrak{v}_3 \notin \mathfrak{v}_1 \wedge \mathfrak{v}_3 \in \mathfrak{v}_2)) \vee \mathfrak{v}_2 \in a \end{gathered} \tag{11.4.9}$$

$$\Downarrow$$

$$\begin{gathered} \mathfrak{T}a \notin \mathbb{h} \vee \mathfrak{T}\prod \notin \mathbb{h} \vee \ \mathfrak{T}a \not\subset \prod \vee \mathfrak{T}\mathfrak{v}_1 \notin a \vee \mathfrak{T}((\mathfrak{v}_3 \in \mathfrak{v}_1 \wedge \mathfrak{v}_3 \notin \mathfrak{v}_2) \vee \\ (\mathfrak{v}_3 \notin \mathfrak{v}_1 \wedge \mathfrak{v}_3 \in \mathfrak{v}_2)) \vee \mathfrak{T}\mathfrak{v}_2 \in a \end{gathered} \tag{11.4.10}$$

$$\Downarrow$$

$$\begin{gathered} \mathfrak{T}a \notin \mathbb{h} \vee \mathfrak{T}\prod \notin \mathbb{h} \vee \ \mathfrak{T}a \not\subset \prod \vee \mathfrak{T}\mathfrak{v}_1 \notin a \vee \mathfrak{T}((\mathfrak{v}_3 \in \mathfrak{v}_1 \wedge \mathfrak{v}_3 \notin \mathfrak{v}_2) \vee \\ (\mathfrak{v}_3 \notin \mathfrak{v}_1 \wedge \mathfrak{v}_3 \in \mathfrak{v}_2)) \vee \mathfrak{T}\mathfrak{v}_2 \in a \end{gathered} \tag{11.4.11}$$

$$\Downarrow$$

$$\begin{gathered} \mathfrak{T}a \notin \mathbb{h} \vee \mathfrak{T}\prod \notin \mathbb{h} \vee \ \mathfrak{T}a \not\subset \prod \vee \mathfrak{T}\mathfrak{v}_1 \notin a \vee \mathfrak{T}((\mathfrak{v}_3 \in \mathfrak{v}_1 \wedge \mathfrak{v}_3 \notin \mathfrak{v}_2) \vee \\ (\mathfrak{v}_3 \notin \mathfrak{v}_1 \wedge \mathfrak{v}_3 \in \mathfrak{v}_2)) \vee \mathfrak{T}\mathfrak{v}_2 \in a \end{gathered} \tag{11.4.12}$$

$$\Downarrow$$

$$\begin{gathered} \mathfrak{T}\Big(a \notin \mathbb{h} \vee \prod \notin \mathbb{h} \vee \ a \not\subset \prod \vee \mathfrak{v}_1 \notin a \vee ((\mathfrak{v}_3 \in \mathfrak{v}_1 \wedge \mathfrak{v}_3 \notin \mathfrak{v}_2) \vee \\ (\mathfrak{v}_3 \notin \mathfrak{v}_1 \wedge \mathfrak{v}_3 \in \mathfrak{v}_2)) \vee \mathfrak{v}_2 \in a\Big) \end{gathered} \tag{11.4.13}$$

$$\Downarrow$$

$$\mathfrak{T}\Big(a \notin \mathbb{h} \vee \prod \notin \mathbb{h} \vee \;\; a \not\subset \prod \vee \boldsymbol{\upsilon}_1 \notin a \vee \exists u((u \in \boldsymbol{\upsilon}_1 \wedge u \notin \boldsymbol{\upsilon}_2) \vee \\ (u \notin \boldsymbol{\upsilon}_1 \wedge u \in \boldsymbol{\upsilon}_2)) \vee \boldsymbol{\upsilon}_2 \in a\Big) \tag{11.4.14}$$

$$\Downarrow$$

$$\mathfrak{T}\Big(a \notin \mathbb{h} \vee \prod \notin \mathbb{h} \vee \;\; a \not\subset \prod \vee \exists x, y(x \notin a \vee \exists u((u \in x \wedge u \notin y) \vee \\ (u \notin x \wedge u \in y)) \vee y \in a)\Big) \tag{11.4.15}$$

$$\Updownarrow$$

$$\mathfrak{T}\Big(a \in \mathbb{h} \wedge \prod \in \mathbb{h} \wedge \;\; a \subset \prod \wedge \forall x, y(x \in a \wedge \forall u(u \in x \leftrightarrow u \in y) \to y \in a\Big) \tag{11.4.16}$$

$$\Updownarrow$$

$$\mathfrak{T}(a \in \mathbb{h} \wedge \prod \in \mathbb{h} \; \wedge a \subset \prod \wedge I(a)) \tag{11.4.17}$$

$$\Updownarrow$$

$$\mathfrak{T}(a \in \prod \wedge I(a)) \tag{11.4.18}$$

So $\vDash^{\mathsf{M}} \mathfrak{T}(a \in \prod \wedge I(a) \to \mathfrak{T}a \in \prod \wedge I(a))$, and $a \in \prod \wedge I(a)$ is orthodox on account of Postulate 3.1.5.M_2 and Postulate 3.1.5.M_6. □

Chapter 12

The set of everything is countable

We show that the results lead to the conclusion that the universe is countable, and demonstrate how the librationist set theory is able to evade Cantor's theorem.

12.1 The denumerable wellordering of the universe

Definition 12.1.1
Let $\preceq$ be the natural ordering of Ω :

$$\Xi(\alpha) \Vdash a \mathrel{\underline{\blacktriangleleft}} b$$
$$\Updownarrow$$
$$\mu x(x \in \Omega \,\&\, x \preceq a \,\&\, \Xi(\alpha) \Vdash x = a) \preceq \mu y(y \in \Omega \,\&\, y \preceq b \,\&\, \Xi(\alpha) \Vdash y = b)$$

Corollary 12.1.2

$$\Xi(\alpha) \Vdash a = b \Leftrightarrow \Xi(\alpha) \Vdash a \mathrel{\underline{\blacktriangleleft}} b \,\&\, \Xi(\alpha) \Vdash a \mathrel{\underline{\blacktriangleright}} b$$

Definition 12.1.3

$$\Xi(\alpha) \Vdash a \blacktriangleleft b \Leftrightarrow \Xi(\alpha) \Vdash a \mathrel{\underline{\blacktriangleleft}} b \,\&\, \Xi(\alpha) \Vdash a \neq b$$

Theorem 12.1.4 (The wellordering)

$$\vdash^{\mathsf{M}} \forall x, y(x \blacktriangleleft y \vee x = y \vee x \blacktriangleright y)$$

Proof. Assume $\neg a \blacktriangleleft b \wedge a \neq b \wedge \neg a \blacktriangleright b$. From Definition 12.1.3 it follows that $a = b$, so the assumption is impossible. □

Definition 12.1.5 (The counting quantifier)

For $n \in \mathbb{N}$, $\exists^{=n} xA$ just if there are exactly n objects which are A.

Definition 12.1.6 (The enumeration handle)

$$€ = \ddot{c} \bullet\bullet\bullet\bullet$$

Semantic enumeration postulates, for any ordinal α:

Postulate 12.1.7

$$\Xi(\alpha) \Vdash \mathfrak{O}(€)$$

Postulate 12.1.8

$$\Xi(\alpha) \Vdash \langle \varnothing, \ddot{c}\bullet \rangle \in €$$

Postulate 12.1.9

$$\Pi a \Pi b \Big(\mathsf{constant}(a) \;\&\; \mathsf{constant}(b) \Rightarrow \\ \Xi(\alpha) \Vdash \forall n (n \in \omega \to \Big(\exists^{=n} x (x \blacktriangleleft a) \wedge \exists^{=n} y (y \blacktriangleleft b) \to a = b \Big)) \Big)$$

Postulate 12.1.10

$$\Pi a, b, c, (\Xi(\alpha) \Vdash \forall n \Big(n \in \omega \to (\Big(\langle a, b \rangle \in € \;\&\; \exists^{=n} x (x \blacktriangleleft b) \Big) \leftrightarrow \\ \Big(\langle \{ v | v \in a \vee v = a \}, c \rangle \in € \;\&\; \exists^{=(n+1)} x (x \blacktriangleleft c) \Big)) \Big)$$

Postulate 12.1.11

$$\Xi(\alpha) \Vdash \forall y \exists n (n \in \omega \wedge \langle n, y \rangle \in €)$$

Maximal consequences of the Semantic enumeration postulates:

Postulate 12.1.12

$$\vdash^{\mathsf{M}} € \text{ is orthodox.}$$

Postulate 12.1.13

$$\vdash^{\mathsf{M}} \exists^{=0} x (x \blacktriangleleft \ddot{c}\bullet)$$

Postulate 12.1.14

$$\Pi a \Pi b \Big(\mathsf{constant}(a) \;\&\; \mathsf{constant}(b) \Rightarrow \\ \vdash^{\mathsf{M}} \forall n (n \in \omega \to \Big(\exists^{=n} x (x \blacktriangleleft a) \wedge \exists^{=n} y (y \blacktriangleleft b) \to a = b \Big)) \Big)$$

Postulate 12.1.15

$$\Pi b \Big[\mathsf{constant}(b) \Rightarrow \vdash^{\mathsf{M}} \Big(\langle \varnothing, b \rangle \in € \leftrightarrow \exists^{=0} x (x \blacktriangleleft b) \Big) \Big]$$

Postulate 12.1.16

$$\Pi a, b, c, (\Vdash^{\mathsf{M}} \forall n\Big(n \in \omega \to \big(\Big(\langle n, b\rangle \in \text{€} \;\&\; \exists^{=n} x(x \blacktriangleleft b)\Big) \leftrightarrow$$
$$\Big(\langle\{v|v \in n \vee v = n\}, c\rangle \in \text{€} \;\&\; \exists^{=(n+1)} x(x \blacktriangleleft c)\big)\Big)\Big)\Big))$$

Postulate 12.1.17

$$\Vdash^{\mathsf{M}} \forall y \exists n(n \in \omega \wedge \langle n, y\rangle \in \text{€})$$

Proof. As all sets in V are finite von Neumann ordinals of the meta language. ☐

12.2 The bijection from the natural numbers to the full universe

Theorem 12.2.1 (The bijection)

$\Vdash^{\mathsf{M}}$ € is a bijection from ω to V.

Proof. Suppose $n \in \omega$. Given 12.1.17, there is a b such that $\langle n, b\rangle \in$ €. ☐

12.3 The escape from ucountable cardinals

In order to reproduce Cantor's argument for the existence of uncountble sets in a relevant sense for our context, it should take place in a hereditarily orthodox set, such as $\lceil\mathfrak{i}\rceil$, or at least in a universe which is orthodox to some order larger than one. As a consequence, Cantor's argument is turned into a reductio which simply establishes that the assumed function cannot be a member of the set wherein the initial set and its power set are members.

To see this, suppose we impose no restrictions upon the power set and assume there is a function f from ω unto $\{x|x \subset \omega\}$. We here take ω to be orthodox, as per Theorem 8.0.2.3. If we try to follow the Cantorian argument attempt just below, we must already discare assumption 1, for $\{x|x \subset \omega\}$ is demonstrably not an element in $\lceil\mathfrak{i}\rceil$, and as a consequence there cannot be a function in $\lceil\mathfrak{i}\rceil$ from omega onto $\{x|x \subset \omega\}$. We may still assume, however, that there is a function $\mathfrak{f}$ from ω onto $\{x|x \subset \omega\}$. Let $\mathfrak{S} = \{x|x \in \omega \wedge x \notin \mathfrak{f}(x)\}$. Let $m = \mathfrak{f}^{-}1(\mathfrak{S})$. What we get, by means of alethic comprehension, is that

a) $m \in \mathfrak{S} \leftrightarrow \mathfrak{T}(m \in \omega \wedge m \notin \mathfrak{f}(m))$
b) $m \in \mathfrak{S} \leftrightarrow \mathfrak{T}(m \in \omega \wedge m \notin \mathfrak{f}(\mathfrak{f}^{-}1(\mathfrak{S})))$
c) $m \in \mathfrak{S} \leftrightarrow \mathfrak{T}(m \in \omega \wedge m \notin \mathfrak{S})$
d) $m \in \mathfrak{S} \leftrightarrow \mathfrak{T}(m \notin \mathfrak{S})$.

d) expresses an ordinary Liar sentence, so the attempted argument has no Cantorian force.

A proper Cantorian argument attempt:

1. Suppose $f \in \lceil\mathfrak{i}\rceil$ is a function from $\omega \in \lceil\mathfrak{i}\rceil$ onto $\mathcal{P}(\omega)^{\lceil\mathfrak{i}\rceil} \in \lceil\mathfrak{i}\rceil$.
2. Let $S = \{x|x \in \omega \wedge x \notin f(x)\}$.

3. Let $f^{-1}(S) = n$.
4. So $n \in S \leftrightarrow n \in \omega \wedge n \notin f(n)$.
5. On account of 3 we have $n \in S \leftrightarrow n \in \omega \wedge n \notin f(f^{-1}(S))$.
6. So $n \in S \leftrightarrow n \in \lceil \mathrm{i} \rceil \wedge n \notin S$.
7. As $n \in \lceil \mathrm{i} \rceil$, $n \in S \leftrightarrow n \notin S$.

What we can say about this is that $\lceil \mathrm{i} \rceil$ "believes" that the set $\mathcal{P}(\varpi)^{\lceil \mathrm{i} \rceil}$ is uncountable, for if we suppose a function $f \in \lceil \mathrm{i} \rceil$ surjects from $\varpi \in \lceil \mathrm{i} \rceil$ to $\mathcal{P}(\varpi)^{\lceil \mathrm{i} \rceil}$, a contradiction follows.

But *still and still*, the function € surjects from $\omega \in \lceil \mathrm{i} \rceil$ to $\mathcal{P}(\omega)^{\mathbb{h}} \in \lceil \mathrm{i} \rceil$, and, indeed, to $\{x|x = x\}$, so € is, a fortiori, a function from ϖ onto the full universe V. €, though, is not a member of $\lceil \mathrm{i} \rceil$.

The analysis which now forces itself upon us, is very much like Skolem's resolution of the imbroglio, in [29], and it in the librationist framework simply follows as a theorem that $f \notin \lceil \mathrm{i} \rceil$ if f is a function from $\omega \in \lceil \mathrm{i} \rceil$ onto $\mathcal{P}(\omega)^{\lceil \mathrm{i} \rceil} \in \lceil \mathrm{i} \rceil$. Moreover, it is, in £ and $\mathfrak{T}\!\!\mathfrak{b}$, abundabtly clear that the notion of uncountability may just be local, so the librationist attitude fully agrees with Skolem's in holding that uncountability is just a relative notion. But Skolem's conclusion that the notion of *set* is *relative*, is not supported, as € is not taken to be contained in a *classical* set theory.

Chapter 13

A librationist interpretation of ZFC

$\mathcal{J}(a,b) =\!= \{x|x \in a \vee x \in b\}$ is the *join* of a and b. Let ϖ, Scott–ω, be the least set that contains all empty sets in $\lceil \mathrm{i} \rceil$, and $\mathcal{P}(a)^{\lceil \mathrm{i} \rceil}$ if it contains a. $A(x,y)$ is *extent-functional* just if $A(x,y)$ and $A(x,z)$ only if y and z are co-extensional. (Scott, 1961) showed that Zermelo set theory minus the axiom of extensionality, with join instead of pairing, ϖ instead of ω, *plus replacement* for extent-functional first order conditions, interprets ZF minus the axiom of foundation. Given (von Neumann, 1929), Scott's set theory interprets ZF; given (Gödel, 1938), or our Theorem 13.8, Scott's set theory, and, as a consequence of what is related below, also $\mathfrak{T}$, interprets ZFC.

It was related in the lecture in this seminary, October 13th, 2022, that a strengthening $\mathfrak{T}$, of £, interprets Scott's set theory, and so also ZFC via (von Neumann, 1929) and (Gödel, 1938).

Librationist interpretations of the axiomatic principles of ZF may now be made relative to a *platform* $\lceil \mathrm{i} \rceil$, as in Definition 11.4.1, so that

$$\circledast \quad a \in \lceil \mathrm{i} \rceil \leftrightarrow (a \in \sqcap \wedge \mathfrak{i}(a) \wedge \exists x(x \in \sqcap \wedge \mathfrak{i}(x) \wedge a \subset x \wedge \forall y(y \in \sqcap \wedge y \in x \rightarrow \mathfrak{i}(y) \wedge y \subset x)))$$

We know from Theorem 11.4.2 that set $\lceil \mathrm{i} \rceil$ is hereditarily orthodox. As $\sqcap$, $\lceil \mathrm{i} \rceil$ is empty, unless instructed otherwise. So we enjoin that $\lceil \mathrm{i} \rceil$ has $\varpi =\!= \omega$ with all binders restriced to $\lceil \mathrm{i} \rceil$ as an initial set, and that $\lceil \mathrm{i} \rceil$ is closed under the following four closures principles, restricted to $\lceil \mathrm{i} \rceil$: i) $power$, ii) $join$, iii) $union$ and iv) $capture$. The latter closure principle, capture, is the last also in the following list:

$$\vdash^{\mathsf{M}} \varpi \in \lceil \mathrm{i} \rceil; \; \vdash^{\mathsf{M}} a \in \lceil \mathrm{i} \rceil \rightarrow \mathcal{P}(a)^{\lceil \mathrm{i} \rceil} \in \lceil \mathrm{i} \rceil;$$

$$\vdash^{\mathsf{M}} a \in \lceil \mathrm{i} \rceil \wedge b \in \lceil \mathrm{i} \rceil \rightarrow \mathcal{J}(a,b)^{\lceil \mathrm{i} \rceil} \in \lceil \mathrm{i} \rceil; \; \vdash^{\mathsf{M}} a \in \lceil \mathrm{i} \rceil \rightarrow \bigcup(a)^{\lceil \mathrm{i} \rceil} \in \lceil \mathrm{i} \rceil; \text{ and}$$

$$\vdash^{\mathsf{M}} a \in \lceil \mathrm{i} \rceil \rightarrow \{x|\exists y(y \in a \wedge B(x,y) \wedge \forall z(B(x,y)^z_x \rightarrow x \overset{\mathsf{e}}{=} z))\}^{\lceil \mathrm{i} \rceil} \in \lceil \mathrm{i} \rceil.$$

$\mathcal{P}(a)^{\lceil \mathrm{i} \rceil}$ is $\{x|x \in \lceil \mathrm{i} \rceil \wedge x \subset a\}$. Similarly for $\mathcal{J}$ and $\bigcup$.

The last condition is a *schema*, for extent-fuctional *capture*, which is to hold for *any* 1st order condition B.

Theorem 13.0.1

Capture is equivalent with specification and replacement.

Proof. i) If a set is obtained by capture, it can be obtained by replacement by using the extent-functional condition $\exists y(y \in a \wedge B(x,y) \wedge \forall z(B(x,y)^z_x \to x \overset{\mathsf{e}}{=} z))$ used for the capture. ii) If a set is obtained by replacement, it can be obtained from capture by using the extent-functional replacement condition for B, in capture, along with the now redundant second clause for capture. iii) Capture and replacement both have specification as a consequence. Use the condition $B'(x,y) == B(x) \wedge x \overset{e}{=} y)$ as capture condition relative to a set a, and observe that the existence of the set $\{x|x \overset{e}{=} y \wedge y \in a\}$ is justified by capture. $\square$

We show that if A is a theorem of ZF, then $A^{\ulcorner i \urcorner}$ is a maxim of $\mathfrak{U}$. We in the following rely upon the axiomatization of ZF presupposed by [28](116–117).

13.1 Principle of extensionality

The interpretation relative to $\ulcorner i \urcorner$ of the extensionality principle

$$\forall x(x \in a \leftrightarrow x \in b) \to a = b$$

is

$$\begin{aligned}&\forall x(x \in \ulcorner i \urcorner \to (x \in a \leftrightarrow x \in b)) \to \\ &\forall x(x \in \ulcorner i \urcorner \to (a \in x \to b \in x)).\end{aligned}$$

$$\Big((a \in \ulcorner i \urcorner \wedge b \in \ulcorner i \urcorner \wedge \forall x\Big(x \in \ulcorner i \urcorner \to (x \in a \leftrightarrow x \in b)\Big)) \\ \to \forall x\Big(x \in \ulcorner i \urcorner \to (a \in x \to b \in x)\Big)\Big).$$

Notice that

$$(a \in \ulcorner i \urcorner \wedge b \in \ulcorner i \urcorner \to \forall x\Big(x \notin \ulcorner i \urcorner \to (x \in a \leftrightarrow x \in b)\Big)),$$

as $a \in \ulcorner i \urcorner \wedge b \in \ulcorner i \urcorner$ entails that $x \notin \ulcorner i \urcorner$ only if $x \notin a$ and $x \notin b$ are hereditary heritors.
Also, $a \overset{\mathsf{e}}{=} b \to \forall x(x \in \ulcorner i \urcorner \to (x \in a \leftrightarrow x \in b))$, so that

$$\big(a \in \ulcorner i \urcorner \wedge b \in \ulcorner i \urcorner \to \Big(\forall x\big(x \in \ulcorner i \urcorner \to (x \in a \leftrightarrow x \in b)\big) \leftrightarrow a \overset{\mathsf{e}}{=} b\Big)\big),$$

and the interpretation may be stated more succinctly as

$$a \in \ulcorner i \urcorner \wedge b \in \ulcorner i \urcorner \wedge a \overset{\mathsf{e}}{=} b \to \forall x(x \in \ulcorner i \urcorner \Leftrightarrow (a \in x \to b \in x)).$$

Suppose $a \in \ulcorner i \urcorner \wedge b \in \ulcorner i \urcorner \wedge a \overset{E}{=} b \wedge c \in \ulcorner i \urcorner \wedge a \in c$. As c is invariant, $b \in c$, and we are done.

13.2 Join

We interpret $\forall a, b(\exists w(\forall x(x \in w \leftrightarrow x \in a \vee x \in b)))$ via $\{x|x \in a \vee x \in b\}^{\ulcorner i \urcorner}$. As $\ulcorner i \urcorner$ is hereditarily orthodox, we have:

$$\forall a, b(a \in \ulcorner i \urcorner \wedge b \in \ulcorner i \urcorner \to \forall x(x \in \{x|x \in a \vee x \in b\} \leftrightarrow x \in a \vee x \in b)^{\ulcorner i \urcorner})$$

The consequent eintails $\exists w(\forall x(x \in w \leftrightarrow x \in a \vee x \in b))^{\ulcorner i \urcorner}$, so that we are done, as we have:

$$a \in \ulcorner i \urcorner \wedge b \in \ulcorner i \urcorner \to \exists w(w \in \ulcorner i \urcorner \wedge \forall x(x \in \ulcorner i \urcorner \to (x \in w \leftrightarrow x \in a \vee x \in b)))$$

13.3 Union

$\forall a \exists w(\forall x(x \in w \leftrightarrow \exists y(x \in y \wedge y \in a)))$ is an axiom of ZF, and we want to show that we for any $a \in \lceil \mathrm{i} \rceil$ have:

$$\exists w(w \in \lceil \mathrm{i} \rceil \wedge \forall x(x \in \lceil \mathrm{i} \rceil \to (x \in w \leftrightarrow \exists y(y \in \lceil \mathrm{i} \rceil \wedge x \in y \wedge y \in a))))$$

$\bigcup(a)^{\lceil \mathrm{i} \rceil} = \{x | \exists y(x \in y \wedge y \in a)\}^{\lceil \mathrm{i} \rceil}$, serves for w, so the interpretation succeeds.

13.4 Power set

We interpret $\forall a \exists w \forall x(x \in w \leftrightarrow x \subset a)$. For given $a \in \lceil \mathrm{i} \rceil$ we have $\{x | x \subset a\}^{\lceil \mathrm{i} \rceil} \in \lceil \mathrm{i} \rceil$, so that $\forall x(x \in \{x|x \subset a\}^{\lceil \mathrm{i} \rceil} \leftrightarrow x \subset a)^{\lceil \mathrm{i} \rceil}$ becomes

$$\forall x(x \in \lceil \mathrm{i} \rceil \to (x \in \{x|x \subset a\}^{\lceil \mathrm{i} \rceil} \leftrightarrow \forall y(y \in \lceil \mathrm{i} \rceil \to (y \in x \to y \in a))))$$

Obvious steps shows that this gives $\forall a \exists w \forall x(x \in w \leftrightarrow x \subset a)^{\lceil \mathrm{i} \rceil}$, so the interpretation succeeds.

13.5 Infinity

$\vdash^{\mathsf{M}} \varpi^{\lceil \mathrm{i} \rceil} \in \lceil \mathrm{i} \rceil$, so that we, as $\varpi^{\lceil \mathrm{i} \rceil}$ is hereditarily orthodox, have

$$\begin{aligned}\vdash^{\mathsf{M}} \exists u \forall x\Big(x \in u \leftrightarrow \forall v(\Big(&\forall w(\forall x(x \notin w) \to w \in v) \wedge \\ &\forall w(w \in v \to \mathcal{P}(w) \in v)\Big) \to x \in v)\Big)^{\lceil \mathrm{i} \rceil}.\end{aligned}$$

So we are done.

13.6 Replacement

We want to interpret replacement as given by

$$\begin{aligned}&\forall a(\forall x, y, z\Big(B(x,y) \wedge B(x,z) \to y = z\Big) \to \\ &\quad \exists w\Big(\forall y(y \in w \leftrightarrow \exists x\Big(x \in a \wedge B(x,y)\Big))\Big)).\end{aligned}$$

We have that

$$\begin{aligned}&\forall a(\forall x, y, z\Big(B(x,y) \wedge B(x,z) \to y = z\Big) \to \\ &\quad \exists w\Big(\forall y(y \in w \leftrightarrow \exists x\Big(x \in a \wedge B(x,y)\Big))\Big))^{\lceil \mathrm{i} \rceil}\end{aligned}$$

is given by

$$\forall a\Big(a \in \lceil\mathsf{i}\rceil \to \big(\forall x, y, z(B(x,y)^{\lceil\mathsf{i}\rceil} \wedge B(x,z)^{\lceil\mathsf{i}\rceil} \to y \overset{E}{=} z) \to$$
$$\exists w(w \in \lceil\mathsf{i}\rceil \wedge \forall y(y \in \lceil\mathsf{i}\rceil \to (y \in w \leftrightarrow \exists x(x \in \lceil\mathsf{i}\rceil \wedge x \in a \wedge B(x,y)^{\lceil\mathsf{i}\rceil})))))\Big).$$

Given $a \in \lceil\mathsf{i}\rceil$, $\lceil\mathsf{i}\rceil$ has set $\{y|\exists x(x \in a \wedge B(x,y))\}^{\lceil\mathsf{i}\rceil}$, which is as required, for first order extent functional condition $B(x,y)^{\lceil\mathsf{i}\rceil}$, as $\{y|\exists x(x \in a \wedge B(x,y))\}^{\lceil\mathsf{i}\rceil}$ is orthodox, and because for $a \in \lceil\mathsf{i}\rceil$, $\vdash^{\mathsf{M}} \forall x(x \in a \leftrightarrow x \in \lceil\mathsf{i}\rceil \wedge x \in a)$.

13.7 Specification

Specification was not interpreted, as it, given Theorem 13.0.1, is derivable from replacement.

13.8 Choice

As discussed in Section 12.1, the facts that terms of $\mathfrak{T}$ are natural numbers, according to the meta language, and that $\mathfrak{T}$ *proves that there are only denumerably many sets*, suffice to justify the following orthodox denumerable wellordering of constants, congruent with the one supported by ZF + V=HOD:

$$\mathrm{WOC}: \qquad \vdash^{\mathsf{M}} \ a \blacktriangleleft b \vee a = b \vee a \blacktriangleright b).$$

Given WOC, which supports global choice, it's easy to justify Russell's multiplicative axiom for hereditary orthodox sets that fulfill the conditions Russell stated. So choice holds in $\lceil\mathsf{i}\rceil$.

Chapter 14

$\ulcorner A \urcorner$ is true just if A hits the truth

Chapters 5 and 6 should be recalled at this point.

Definition 14.0.1
The *trace* of A is the emta-language set of ordinals $[\delta\colon \Xi(\delta) \Vdash \mathrm{A}]$.

Definition 14.0.2
The *truth* is the closure ordinal ♀.

Definition 14.0.3
The *track* of A is the intersection of its trace with the truth.

Definition 14.0.4
A *hits* its track.

Definition 14.0.5
$\ulcorner A \urcorner$ is true just if A hits the truth.

Definition 14.0.6
$\ulcorner A \urcorner$ is false just if $\ulcorner \neg A \urcorner$ is true.

Definition 14.0.7
The track of $A \wedge B$ is the track of A intersected with the track of B.

Remark 14.0.8
Let L be as in step d in the proof of Theorem 6.0.1, so that $\overset{\mathsf{M}}{\vdash} \neg\mathcal{T}\ulcorner L \urcorner \leftrightarrow L$.

By Theorem 6.0.8,

$$\vdash L \text{ and } \vdash \neg L$$

as well as

$$\vdash \mathcal{T}\ulcorner L \urcorner,\ \vdash \neg\mathcal{T}\ulcorner L \urcorner,\ \vdash \mathcal{T}\ulcorner \neg L \urcorner, \text{ and } \vdash \neg\mathcal{T}\ulcorner \neg L \urcorner.$$

May we square these facts about L with Definition 14.0.5? Yes, the appropriate reading of Definition 14.0.5 should be such that the sentence $\ulcorner A \urcorner$ *is true* should be interpreted as $\vdash \mathcal{T}\ulcorner A \urcorner$, and the sentence *A hits the truth* is, on account of the semantics, equivalent with $\vdash A$.

So Definition 14.0.5 may be interpeted in terms of the bidirectional inferential mode

$$\vdash \mathcal{T}\ulcorner A \urcorner \Leftrightarrow \vdash A.$$

It is, moreover, a fact that $\vdash\ \neg\mathcal{T}\ulcorner A\urcorner\ \Leftrightarrow\ \vdash\ \neg A$, as well as $\overset{\mathsf{M}}{\vdash}\ \mathcal{T}\ulcorner A\urcorner\ \Leftrightarrow\ \overset{\mathsf{M}}{\vdash}\ A$. The bidirectional inferential modes in this paragraph do not have instances which contradict the maxim $\overset{\mathsf{M}}{\vdash} \neg\mathcal{T}\ulcorner L\urcorner \leftrightarrow L$. ■

The connectives do not work truth-functionally in librationism, but they work *track-functionally* and by following classical interdefinability connections as in any Boolean algebra. The track of the negation $\neg A$ of A, is truth minus the track of A, and the track of the conjunction $A \wedge B$ is the intersection of the track of A and the track of B. The track of sentences built up from other connectives follow from their definitions in terms of negation and conjunction.

Bibliography

[1] Y. Bar-Hillel, E. I. J. Poznanski, M. O. Rabin, and A. Robinson, editors. *Essays on the foundations of mathematics*. North-Holland Publishing Company, Amsterdam, 1961.

[2] M. Beeson. *Foundations of Constructive Mathematics*. Springer, 1985.

[3] F. A. Bjørdal. Towards a Foundation for Type-free Reasoning. In Timothy Childers, editor, *The Logica Yearbook 1997*, pages 259–273. FILOSOFIA, by the Academy of Sciences of the Czech Republic, 1998.

[4] F. A. Bjørdal. Librationist closures of the paradoxes. *Logic and Logical Philosophy*, 21(4):323–361, 2012.

[5] Frode Bjørdal. There are only Countably Many Objects. In M. Bilkova and L. Behounek, editors, *The Logica Yearbook 2004*, pages 47–58. FILOSOFIA, Prague, 2005.

[6] Frode Bjørdal. Considerations Contra Cantorianism. In M. Pelis and V. Puncochar, editors, *The Logica Yearbook 2010*, pages 43–52. College Publications, London 2011, 2011a.

[7] J. P. Burgess. The truth is never simple. *Journal of Symbolic Logic*, 51(3):663–681, 1986.

[8] A. Cantini. *Logical Frameworks for Truth and Abstraction: An Axiomatic Study*. Elsevier, 1996. ISBN 0-444-82306-9.

[9] A. Church. Comparison of Russell's resolution of the semantical antinomies with that of Tarski. *Journal of Symbolic Logic*, 41(4):747–760, 1976.

[10] A. A. Fraenkel and Y. Bar-Hillel. *Foundations of Set Theory*. North-Holland Pub. Co, first edition, 1958.

[11] A. A. Fraenkel and Y. Bar-Hillel. *Foundations of Set Theory*. Atlantic Highlands, NJ, USA: Elsevier, second revised edition, 1973.

[12] H. Friedman. The consistency of classical set theory relative to a set theory with intuitionistic logic. *Journal of Symbolic Logic*, 38(2):315–319, 1973.

[13] P. C. Gilmore. *The consistency of partial set theory without extensionality*. In [20], 147–153.

[14] K. Gödel. Über formal unentscheidbare sätze der principia mathematica und verwandter systeme i. *Monatshefte für Mathematik*, 38(1):173–198, 1931.

[15] K. Gödel. The consistency of the axiom of choice and of the generalized continuum-hypothesis. *Proceedings of the U.S. National Academy of Sciences*, 24:556–557, 1938.

[16] S. Hachtman. Determinacy and monotone inductive definitions. *Israeli Journal of Mathematics*, 230:71–96, 2019.

[17] G. Hunter. *Metalogic: An Introduction to the Metatheory of Standard First Order Logic*. Berkeley: University of California Press, 1971.

[18] S. Jaśkowski. Rachunek zdań dla systemow dedukcyjnych sprzecznych. *Studia Societatis Scientiarum Torunensis*, Vol. I (No. 5):57–77, 1948.

[19] S. Jaśkowski. A propositional calculus for inconsistent deductive systems – (translation of [18]). *Logic and Logical Philosophy*, 7:35–56, 1999.

[20] T. Jech, editor. *Axiomatic Set Theory, Part 2*. American Mathematical Society, 1974.

[21] G. Kreisel. Reflection principle for Heyting's arithmetic. *Journal of Symbolic (Logic*, 28:306–307, 1963. Abstract.

[22] G. Kreisel. Reflection principle for Heyting's arithmetic. *Journal of Symbolic (Logic*, 28:307–308, 1963. Abstract.

[23] G. Kreisel. The subformula property and reflection principles. *Journal of Symbolic (Logic*, 28:305–306, 1963. Abstract.

[24] V. McGee. How truthlike can a predicate be? A negative result. *Journal of Philosophical Logic*, 14(4):399–410, 1985.

[25] G. Priest, F. Berto, and Z. Weber. Dialetheism. In *Stanford Encyclopedia of Philosophy*. 2022.

[26] F. Ramsey. The foundations of mathematics. *Proceedings of the London Mathematical Society*, 25:338–384, 1925.

[27] D. Scott. *Axiomatizing Set Theory*. In [20], 207–214.

[28] D. Scott. *More on the axiom of extensionality*. In [1], pages 115–131.

[29] T. Skolem. Einige bemerkungen zur axiomatischen begründung der mengenlehre. In *Proceedings of the 5th Scandinavian Mathematical Congress*, pages 137–152, Helsinki, 1922.

[30] R. Turner. *Truth and Modality for Knowledge Representation*. MIT Press, 1990.

[31] A. Visser. *Handbook of Philosophical Logic*, volume 4, chapter Semantics and the Liar Paradox, pages 617–706. Springer, first edition, 1989.

[32] P. D. Welch. Weak systems of determinacy and arithmetical quasi-inductive definitions. *Journal of Symbolic Logic*, 76:418–436, 2011.

Index