FIRST CAUSE EXISTENCE FROM PROPOSITIONAL COMPACTNESS

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Abstract. We show that, assuming impossibility of absolute nothingness, a necessary being does exist. Our argument is an elaboration of the "Subtraction argument" known in philosophical circles, and it makes use of the compactness theorem of propositional logic and an interpretation of the notion of "possible worlds" through propositional valuations.

1. Introduction

Many authors have inquired about the possibility of an empty world (see e.g. [2]). Some of them [?] have proposed a "subtraction argument" to show that absolute nothingness is actually possible. Others [4] have given arguments to show that the concept of an empty world necessarily leads to a contradiction.

In this paper, we propose a rigorous mathematical argument showing that, granting the validity of subtraction argument for finite worlds, one is necessarily led to one out of two possible outcomes:

1. Empty world is a possibility, or
2. A necessary being does exist.

However, it was argued in [4] that an empty world necessarily leads to a contradiction. Their argument is based on a causal account of possible worlds that is somewhat at odds with our mathematical argument. We leave the investigation of the possibility of an empty world to a future paper, and we concentrate here on the issue of subtraction principle. Recall that the subtraction argument runs as follows:

1. Assume that there is a possible world w with n possible beings, or n contingent objects.
2. Assume also that the existence or nonexistence of either n objects does not necessitate the existence of any other object among the inhabitants of the world w.
3. By assumption (2), we can subtract (mentally and not physically) one of the objects. Let us call it a_n.
4. The new world w' will consist of $n-1$ objects.
5. We may repeat the same steps 1,2 and 3 to the world w' thus obtaining a world w''.
6. This process can be repeated until we get a world w_0 with only one object left.
7. Since the object in w_0 is a possible being (recall that we assume that no necessary being exists), it can also be eliminated from w_0 leaving out an empty world.

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So, starting with a possible world with a finite number of inhabitants (all of them are contingent and they do not depend on each other) we arrive a a world with nothing in it at all, i.e. empty world. So the empty world is possible.
Several authors have criticized this argument. One clear objection is that it assumes that a possible world might have a finite number of inhabitants. In this paper, we show that, putting aside the philosophical complaints against this argument, we can in fact show that even if the actual world has infinitely many objects, the argument can be made through essentially the same steps and using a special interpretation of possible worlds and an application of the compactness theorem of propositional calculus.

The plan of the paper is as follows: in section 2 we introduce elementary facts from propositional logic and valuations. In section 3 we argue for a special interpretation of the notion of possible worlds using valuations as introduced in section 2. In section 4 we introduce the compactness theorem of propositional logic and in section 5 we apply it to the problem at hand. In section 6 we lay down our conclusions.

2. Propositional calculus

A standard reference on mathematical logic is, e.g. [3].
In propositional logic, propositions are statements which can be true or false. They are denoted by $p, q, \ldots$. An example of a proposition is $p : 5 > 6$. Another example is $7 > 5$. In particular we do not allow formulas like $x > 5$ whose veracity depends on a variable, neither we allow quantifiers like $\exists x(x > 5)$, these are the business of predicate calculus. Propositions are facts about real or fictional individuals, and they do not apply to classes of individuals. By a propositional domain we mean a set (finite or infinite) of statements about individuals of some world of discourse.
Recall that every proposition can be assigned a truth value in the set $\{T, F\}$ where the value T is assigned to a true proposition and the value F is assigned to a false proposition. In the examples above, the proposition $5 > 6$ is assigned the value F since it is clearly false. On the other hand, the proposition $7 > 5$ is assigned the value T since it is true. We may formally replace the symbols T, F by $1, 0$ respectively.

Let $\mathcal{P}$ be a propositional domain. Then a map $v : \mathcal{P} \to \{0, 1\}$ is an assignment that associates to each propositional variable (i.e. atomic proposition) a value $\in \{0, 1\}$. We denote by $\mathcal{V}$ the set of maps from $\mathcal{P}$ into the set $\{0, 1\}$. So, if $v \in \mathcal{V}$, $v(p) = 1$ if and only if p is true (according to v) and similarly $v(p) = 0$ if and only if p is false (according to v).
It is a standard observation that the cardinality of $\mathcal{V}$ is 2^α where α is the cardinality (finite or infinite) of $\mathcal{P}$.
By a propositional formula we mean a formula built out from the propositional symbols $p, q, \ldots$ by successive applications of the connectives $\wedge$ (and) $\vee$ (or) and $\neg$ (not). So for instance $\neg$ (My car is black) means that my car is not black. Let $\mathcal{F}$ be the set of propositional formulas which can be defined inductively as follows:
1. Every propositional variable p is a formula.

2. If φ and ψ are formulas then $\neg\varphi$, $\varphi \wedge \psi$ and $\varphi \vee \psi$ are also formulas.
3. Formulas are only given by one of the two procedures above.

Consider a valuation $v \in \mathcal{V}$. It is a natural question to ask if we can extend v to $\mathcal{F}$, that is: is there an application $v' : \mathcal{F} \to \{0, 1\}$ which assigns truth values to formulas such that, for all $p \in \mathcal{P}$, $v(p) = v'(p)$ and the following conditions hold:
1. $v'(\varphi \wedge \psi) = 1$ iff $v'(\varphi) = 1$ and $v'(\psi) = 1$.
2. $v'(\varphi \vee \psi) = 1$ iff $v'(\varphi) = 1$ or $v'(\psi) = 1$.
3. $v'(\neg\varphi) = 1$ iff $v'(\varphi) = 0$.

It turns out that the answer to the above question is affirmative and from now on, we will denote the valuation v' extending v for $v \in \mathcal{V}$ simply by v.

3. Possible worlds

The notion of possible worlds is essential in ontology. A possible world is , roughly, a possible state of affairs. The use of possible worlds concept is implicit in our everyday's discourse. For example, " If it had not rained this morning I would have got to work on time", or "If my father and mother had not met I would not have been born" etc. Some philosophers believe that possible worlds are actually existing, all on the same footing. We do not believe in that. Some nuance exists about the relation of the concept of possible worlds in metaphysics and the concept of "many-worlds" in some interpretations to quantum mechanics. We will refrain from discussing this issue further since we believe that it complicates greatly the exposition and is not our main goal.

Consider all the inhabitants of this world, and of all possible worlds. We agree that the use of "all" in (all inhabitants) and in (all possible worlds) might be problematic, and might rise set theoretic issues. We will ignore these issues for the time being, leaving their discussion to possible future publications. So let $\mathcal{S}$ be the set of all objects and events in the actual world, in the past and the future, as well as all objects and events in all possible worlds. Let $\mathcal{P}$ be the set of all existential facts about objects in $\mathcal{S}$, i.e. every $p \in \mathcal{P}$ is of the form "a exists" for some $a \in \mathcal{S}$.

A causality statement about the world is a statement of the form: "Necessarily if a exists then b exists" for $a, b \in \mathcal{S}$. This last statement indicates that a has a cause, which is b in this case. Of course such a statement does not belong to $\mathcal{P}$ since we use the word "Necessarily". However, we can enforce the "constraint" $p \to q$ (which means, by the way, $\neg p \vee q$). Let us call $\mathcal{S}$ the "Superworld". So $\mathcal{S}$ is subject to many constraints, all of the form "if a exists, then b exists" for $a, b \in \mathcal{S}$. In fact, we have most probably an infinite set of such constraints denoted Φ_i (where i is some index), corresponding to the multiple and various causality relationships among the objects in the superworld.

Finally we reach our conception of possible worlds: a possible world is simply a valuation $v \in \mathcal{V}$ (where $\mathcal{V}$ is the set of all maps $\mathcal{P} \to \{0, 1\}$ as above) such that the extension of v to $\mathcal{F}$ (where $\mathcal{F}$ is the set of propositional formulas constructed out of $\mathcal{V}$) always satisfies all the constraints Φ_i, i.e. $v(\Phi_i) = 1$ for all i.

4. Compactness theorem

Let us forget about the superworld for a moment and consider a propositional domain $\mathcal{P}$ and the set $\mathcal{F}$ of propositional formulas constructed from it. Let also $\mathcal{V}$ be the set of extended valuations v, $v : \mathcal{F} \to \{0, 1\}$. Let $\mathcal{F}'$ be a subset of $\mathcal{F}$. We say that $\mathcal{F}'$ is consistent, or satisfiable if there exists a valuation $v \in \mathcal{V}$ such that for every $\varphi \in \mathcal{F}'$ $v(\varphi) = 1$, otherwise we say that $\mathcal{F}'$ is inconsistent. For a set of propositional formulas to be consistent it suffices that we cannot deduce (through usual processes of propositional logic, like *modus ponens*) a contradiction from it, a formula of the form $\neg\varphi \wedge \varphi$.
We can now state the compactness theorem of propostional logic (see [3]):

4.1. **Theorem.** *Using the above notation, let $\mathcal{F}'$ be an arbitrary set of propositional formulas, $\mathcal{F}' \subset \mathcal{F}$. Then $\mathcal{F}'$ is consistent if and only if every finite subset of it is consistent.*

A contrapositive to this theorem is the following: $\mathcal{F}'$ is inconsistent if and only if there exists $\mathcal{F}_0 \subset \mathcal{F}'$ finite inconsistent.

5. The infinitary subtraction argument

Let us return to the setting of section 3. We can represent the set of dependencies of all objects in the superworld by a huge oriented graph which has no oriented cycles. Let $\mathcal{F}'$ be the set of all Φ_i where Φ_i are as in section 3. If there exists a necessary being, so it exists in all possible worlds.

Let $\mathcal{F}''$ be the following set $\mathcal{F}'' = \mathcal{F}' \cup \{\neg p | p \in \mathcal{P}\}$. $\mathcal{F}''$ is the set of constraints union the set of negative statements about all objects in the superworld. So in fact, the empty world is possible if and only if there exists a valuation $v \in \mathcal{V}$ such that for every $\varphi \in \mathcal{F}''$ $v(\varphi) = 1$.

Let us assume that there does not exist a necessary being, otherwise there would be nothing to prove. Hence for every $p \in \mathcal{P}$, the two possibilities of p being true or false are available (because every p is of the form "a exists" for some $a \in \mathcal{S}$). So in particular, any finite susbet $\mathcal{F}_0$ of $\mathcal{F}''$ is consistent (this can be seen by an easy expansion). To show the last statement, consider a finite subset $\mathcal{F}_0$ of $\mathcal{F}''$. Since $\mathcal{F}_0$ is finite, there are finitely many individuals involved in all the statements of $\mathcal{F}''$. For instance, let $\mathcal{F}_0 = \{\Phi_1, \ldots, \Phi_n, \neg p_1, \ldots, \neg p_k\}$ with Φ_i: "If a_i exists then b_i exists". Recall that the proposition p_j is of the form: "b_j exists" where b_j is intended to be a name of a contingent being for $j = 1, \ldots, k$. If we enlarge $\mathcal{F}_0$ into $\mathcal{F}_0'$ we would obtain a finite set of statements of the form:

$$\mathcal{F}_0' := \{\text{"}a_i \text{ does not exist"} \mid i = 1, \ldots, N\} \\ \cup \{\text{"If } a_i \text{ exists then } a_j \text{ exists"} \mid \text{for some } i, j \in \{1, \ldots, N\}\},$$

where N is some natural number greater than n and k, and for some ℓ we have $a_\ell = b_1, \ldots, a_{\ell+k} = b_k$. It is clear that there exists a valuation $\mathcal{V}$ which assigns the value 1 to all the statements in $\mathcal{F}_0'$, namely by considering that none of the a_i (for $i = 1, \ldots, N$) exists. In this case the conditionals (i.e. statement Φ_i of the form "If

a_i exists then a_j exists") are vacuously satisfied since a conditional of the form $p \to q$ is true if p does not hold or q holds. It follows that since $\mathcal{F}'_0$ is consistent, $\mathcal{F}_0 \subset \mathcal{F}'_0$ is also consistent. Hence by applying theorem 4.1 we have that $\mathcal{F}''$ is consistent, so there is a valuation $v \in \mathcal{V}$ satisfying all formulas of $\mathcal{F}''$ which means that the empty world is a possible world.

6. Conclusion and future work

In this paper we showed that assuming the metaphysical validity of the subtraction argument, we can deduce that there exists one out of two possibilities:
1. That a necessary being (i.e. God) exists, or
2. An empty world is a possible world. We shall argue in a future work that in fact 2 is unavailable so one is left only with the fact that a necessary being exists.

References

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