Anselm's God in Isabelle/HOL

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Contents

1 Introduction

theory *AnselmGod*
imports *Main*
begin

This paper presents an automated verification of Anselm's ontological argument, as reconstructed by Paul Oppenheimer and Edward Zalta [5], in Isabelle/HOL, an interactive theorem prover for higher-order logic. Previously, the argument has being automated by Oppenheimer and Zalta in Prover9 [6], an automated theorem prover for first-order logic, and by John Rushby in PVS [8], an automated theorem prover for higher-order logic. Automations of other versions of the argument include [1], [9] and [3]. My purpose here is to present a basis for comparison in the spirit of [10], which compares automated proofs of the irrationality of $\sqrt{2}$.

Oppenheimer and Zalta's reconstruction is based on the idea of treating 'that than which nothing greater can be conceived' as a definite description,

1

and treating definite descriptions as singular terms. But in Isabelle/HOL all terms, including definite descriptions, are assumed to denote. So the main task is to embed a free logic for definite descriptions within Isabelle/HOL. (Previously, a free logic has been embedded into Isabelle/HOL by Christoph Benzmuller and Dana Scott [2]. But theirs differs from Zalta and Oppenheimer's in several ways). Once Isabelle/HOL is equipped with free definite descriptions, reconstructing the argument is straightforward.

2   Free Logic

Isabelle treats definite descriptions as singular terms of the form *THE x. φ x*. However, all terms in Isabelle are assumed to denote, and so from universal elimination we have the validity of the argument form:

lemma $\forall$ *x. ψ x $\Longrightarrow$ ψ (THE x. φ x)* **by** *(rule allE)*

In the presence of definite descriptions which do not denote, this argument form is invalid; for example, from 'everyone has hair' we should not infer 'the present King of France has hair', since the present King of France does not exist.

This problem can be avoided by introducing a null individual n to serve as the reference of non-denoting definite descriptions, as follows:

typedecl *i* — the type of individuals
consts *n:: i (n)* — the null individual

Then the universal and particular quantifiers can be restricted to individuals excluding the null-individual as follows, where the new free quantifiers are distinguished from the classical quantifiers by bold type:

abbreviation *universal-quantifier*:: $(i \Rightarrow bool) \Rightarrow bool$ ($\forall$)
   where $\forall$ $\varphi \equiv \forall x::i. (\neg x = n \longrightarrow \varphi x)$
abbreviation *universal-syntax*:: $(i \Rightarrow bool) \Rightarrow bool$ (**binder** $\forall$ *[8] 9*)
   where $\forall$ *x. φ x $\equiv$* $\forall$ φ

abbreviation *particular-quantifier*:: $(i \Rightarrow bool) \Rightarrow bool$ ($\exists$)
   where $\exists$ $\varphi \equiv \exists x::i. (x \neq n \wedge \varphi x)$
abbreviation *particular-syntax*:: $(i \Rightarrow bool) \Rightarrow bool$ (**binder** $\exists$ *[8] 9*)
   where $\exists$ *x. φ x $\equiv$* $\exists$ φ

Note that the quantifiers here range over both existent and non-existent individuals, whereas the quantifiers in [2] range only over existent individuals.

In the free logic employed by Oppenheimer and Zalta, statements of identity in which terms do not denote are always false [5], p. 511. So the domain of the identity relation should be restricted to exclude the null-individual:

abbreviation *identity*:: $i \Rightarrow i \Rightarrow bool$ (*is*)

where *is x y* ≡ *x* ≠ *n* ∧ *x* = *y*
abbreviation *identity-syntax*:: *i* ⇒ *i* ⇒ *bool* (**infix** = *50*)
  where *x* = *y* ≡ *is x y*

Once identity is introduced, the uniqueness quantifier can then be defined in the usual way:

abbreviation *uniqueness-quantifier*:: (*i* ⇒ *bool*) ⇒ *bool* (*unique*)
 where *unique φ* ≡ (∃ *x*::*i*. *φ x* ∧ (∀ *y*::*i*. *φ y* ⟶ *x* = *y*))
abbreviation *uniqueness-syntax*:: (*i* ⇒ *bool*) ⇒ *bool* (**binder** *unique* [*8*] *9*)
  where *unique x. φ x* ≡ *unique φ*

Finally, the logic employed by Oppenheimer and Zalta is a negative free logic, in that applications of atomic predicates to non-denoting terms are always false [5], p. 511. So it's necessary to introduce a higher-order predicate distinguishing between atomic and non-atomic predicates, and to introduce an axiom stipulating that no atomic predicate is true of the null individual:

consts *atomic-predicates*:: (*i* ⇒ *bool*) ⇒ *bool* (*atomic*)
axiomatization where *negativity-constraint*: *atomic φ* ⟹ ¬ *φ n*

In addition, it has to be stated that identity is atomic:

axiomatization where *identity-atomic*: ⋀ *x. atomic* (*is x*)

One of the most controversial premises of the ontological argument is that 'exists' is a genuine or atomic predicate. But surprisingly, we shall see below that the argument does not require this premise.

3   Definite Descriptions

The main idea of Oppenheimer and Zalta's reconstruction of the ontological argument is to treat definite descriptions as genuine singular terms, which leads to the following syntax in Isabelle/HOL:

consts *definite-description*:: (*i* ⇒ *bool*) ⇒ *i* (**τ**)
abbreviation *description-syntax*:: (*i* ⇒ *bool*) ⇒ *i* (**binder τ** [*8*] *9*)
where **τ** *x. φ x* ≡ **τ** *φ*

In Oppenheimer and Zalta's reconstruction of the argument, definite descriptions are governed by the Russellian axiom schema [5], p. 513:

axiomatization where *description-axiom*:
atomic ψ ⟹ *ψ* (**τ** *x. φ x*) ≡ (∃ *x. φ x* ∧ (∀ *y. φ y* ⟶ *x* = *y*) ∧ *ψ x*)

From this axiom schema, Oppenheimer and Zalta derive two intermediary theorems to be used in the reconstruction of their argument [5], pp. 513-4. According to the first:

theorem *description-theorem-1*: *unique x. φ x* ⟹ ∃ *y. y* = (**τ** *x. φ x*)
using *description-axiom identity-atomic* **by** *blast*

The second theorem follows directly from the following lemma:

lemma *lemma-1*: $a = (\boldsymbol{\tau} \ x. \ \varphi \ x) \Longrightarrow \varphi \ (\boldsymbol{\tau} \ x. \ \varphi \ x)$
using *description-axiom identity-atomic* **by** *blast*

theorem *description-theorem-2*: $\exists \ x. \ x = (\boldsymbol{\tau} \ x. \ \varphi \ x) \Longrightarrow \varphi \ (\boldsymbol{\tau} \ x. \ \varphi \ x)$
by (*simp add*: *lemma-1*)

In the course of verifying the argument using Prover9, Oppenheimer and Zalta discovered a simplified proof which uses instead [6], p. 345:

theorem *description-theorem-3*:
atomic $\psi \Longrightarrow \psi \ (\boldsymbol{\tau} \ x. \ \varphi \ x) \Longrightarrow \exists \ y. \ y = (\boldsymbol{\tau} \ x. \ \varphi \ x)$
using *negativity-constraint* **by** *fastforce*

Notice that it is only this last theorem which presupposes the negativity constraint, whereas the first two theorems depend only on the atomicity of identity.

4   Anselm's Argument

The argument proper employs the following non-logical vocabulary:

consts *existence*:: $i \Rightarrow bool$ (E) — exists in reality
consts *greater-than*:: $i \Rightarrow i \Rightarrow bool$ (G) — is greater than
consts *conceivable*:: $i \Rightarrow bool$ (C) — exists in the understanding

Note that $E \ a$ is not intended by Oppenheimer and Zalta to be equivalent to $\exists x. \ a = x$ since according to their reading of the argument, some things do not exist in reality [5], p. 514.

Finally, the presentation of the argument is simplified by introducing the following abbreviation for the predicate 'is a being greater than which none can be conceived':

abbreviation *none-greater-than* :: $i \Rightarrow bool$ (Φ)
where $\Phi \ x \equiv (C \ x \land \neg(\exists \ y. \ G \ y \ x \land C \ y))$

With this vocabulary in place, a name for God can be introduced as an abbreviation for the description 'the being greater than which none can be conceived':

definition g :: i **where** $g \equiv (\boldsymbol{\tau} \ x. \ \Phi \ x)$

In Oppenheimer and Zalta's presentation every name is assumed to denote, so a name for God cannot be introduced until it is proved that the description $\boldsymbol{\tau}x. \ \Phi \ x$ denotes [5], p, 520. But since it's not assumed in this presentation that every name denotes or, in other words, since it's not assumed that no names denote the null individual, it's not necessary to postpone this step.

The final quasi-logical premise in Oppenheimer and Zalta's reconstruction of the argument is the connectivity of 'is greater than', which is used in the proof of the following lemma [5], p. 518:

lemma *lemma-2*:
  assumes *connectivity*: $\forall\ x.\ \forall\ y.\ G\ x\ y \lor G\ y\ x \lor x = y$
  shows $\exists\ x.\ \Phi\ x \implies unique\ x.\ \Phi\ x$
  using *connectivity* **by** *blast*

Note that *connectivity* disallows any ties with respect to greatness. This is implausible, since you and I, for example, may be equally great, without being the same person. So *connectivity* should not be thought of as merely stipulative, and a weaker premise would be desirable.

With this vocabulary in place, Anselm's ontological argument, as reconstructed by Oppenheimer and Zalta, can be stated as follows:

theorem
  assumes *premise-1*: $\exists\ x.\ \Phi\ x$
  — there exists in the understanding a being greater than which none can be conceived
and *premise-2*: $\neg\ E\ (\boldsymbol{\tau}\ x.\ \Phi\ x) \longrightarrow (\exists\ y.\ G\ y\ (\boldsymbol{\tau}\ x.\ \Phi\ x) \land C\ y)$
  — if the being greater than which none can be conceived does not exist in reality, then a being exists in the understanding which is greater than the being greater than which none can be conceived
and *connectivity*: $\forall\ x.\ \forall\ y.\ G\ x\ y \lor G\ y\ x \lor x = y$
shows $E\ g$ — God exists.

Isabelle can verify the argument in one line with the command *using premise-1 premise-2 connectivity lemma-1 g-def description-theorem-1 by smt*. But since proofs in Isabelle using *smt* are currently considered impermanent, I instead give Zalta and Oppenheimer's handwritten proof [6], p. 337:

proof (*rule ccontr*)
  assume *atheism*: $\neg\ E\ g$
  from *premise-1* **and** *connectivity* **and** *lemma-2* **have** *unique x*. $\Phi\ x$ **by** *simp*
  with *description-theorem-1* **have** $\exists\ y.\ y = (\boldsymbol{\tau}\ x.\ \Phi\ x)$ **by** *simp*
  with *description-theorem-2* **have** $\Phi\ (\boldsymbol{\tau}\ x.\ \Phi\ x)$ **by** *simp*
  hence *god-is-greatest*: $\neg(\exists\ y.\ G\ y\ (\boldsymbol{\tau}\ x.\ \Phi\ x) \land C\ y)$ **by** (*rule conjE*)
  from *atheism* **and** *premise-2* **and** *g-def* **have** $(\exists\ y.\ G\ y\ (\boldsymbol{\tau}\ x.\ \Phi\ x) \land C\ y)$ **by** *simp*
  with *god-is-greatest* **show** *False*..
qed

Note that neither Oppenheimer and Zalta's proof nor the one line *smt* proof depend on the negativity constraint or whether any of the non-logical vocabulary is atomic (though they do depend indirectly on the atomicity of identity).

5 The Prover9 Argument

In the course of verifying the argument using Prover9, Oppenheimer and Zalta discovered a simplified version which employs only *premise-2*, but not *premise-1* or the connectivity of 'greater than' [6].

theorem
assumes *premise-2*: $\neg\ E\ (\boldsymbol{\tau}\ x.\ \Phi\ x) \longrightarrow (\exists\ y.\ G\ y\ (\boldsymbol{\tau}\ x.\ \Phi\ x) \wedge C\ y)$
shows $E\ g$ **nitpick** [*user-axioms*] **oops**

However, Isabelle not only fails to verify this argument, but finds a counterexample using *nitpick*. The reason is that it needs to be specified that 'greater than' is atomic, in order for *description-theorem-3* to be applicable:

theorem *Prover9Argument*:
assumes *premise-2*: $\neg\ E\ (\boldsymbol{\tau}\ x.\ \Phi\ x) \longrightarrow (\exists\ y.\ G\ y\ (\boldsymbol{\tau}\ x.\ \Phi\ x) \wedge C\ y)$
and *G-atomic*: $\bigwedge x.\ atomic\ (G\ x)$
shows $E\ g$

Once the atomicity of 'greater than' is added as a premise, a call to *sledgehammer* suggests the following two-step proof, which Isabelle verifies easily:

proof −
  have $C\ g \wedge (\forall i.\ i = n \vee \neg\ G\ i\ g \vee \neg\ C\ i) \vee n = g$
    by (*metis* (*lifting, full-types*) *g-def lemma-1*)
  then show *?thesis*
    by (*metis* (*lifting*) *G-atomic g-def negativity-constraint premise-2*)
qed

If provided with all premises, *sledgehammer* still suggests a proof using only *premise-2*:

theorem
assumes *connectivity*: $\forall\ x.\ \forall\ y.\ G\ x\ y \vee G\ y\ x \vee x = y$
and *premise-1*: $\exists\ x.\ \Phi\ x$
and *premise-2*: $\neg\ E\ (\boldsymbol{\tau}\ x.\ \Phi\ x) \longrightarrow (\exists\ y.\ G\ y\ (\boldsymbol{\tau}\ x.\ \Phi\ x) \wedge C\ y)$
and *G-atomic*: $\bigwedge x.\ atomic\ (G\ x)$
shows $E\ g$
proof −
  have $\Phi\ g \vee n = g$
    by (*metis* (*lifting, full-types*) *g-def lemma-1*)
  then show *?thesis*
    by (*metis* (*lifting*) *G-atomic g-def negativity-constraint premise-2*)
qed

Note that this version of the argument does employ the *negativity-constraint*, as well as the premise that identity is atomic via *lemma-1*. So although it has less non-logical premises than the original version of the argument, it has more, and more controversial, logical premises.

6 Soundness

Since *premise-1* and the connectivity of 'is greater than' are both dispensable, and the atomicity of 'is greater than' is not especially controversial, the main non-logical premise of the argument turns out to be *premise-2*. Note that *premise-2* is entailed by God's existence:

theorem
  assumes *theism*: E g
  shows $\neg$ E $(\tau$ $x.$ Φ $x) \longrightarrow (\exists$ $y.$ G y $(\tau$ $x.$ Φ $x) \wedge C$ $y)$
  using *g-def theism* **by** *auto*

So under the supposition that 'is greater than' is atomic, *premise-2* is equivalent to God's existence, suggesting an atheist might wish to reject it as question-begging (see [6], pp. 348-9 and [4] for more detailed discussion of this point).

However, Ted Parent has pointed out that *premise-2* need not stand on its own, but may be further supported by the following argument [7], p. 478:

lemma
assumes *premise-3*: $\forall$ $y.$ $\forall$ $z.$ $((E$ y $\wedge$ $\neg$ E $z) \longrightarrow ((y = (\tau$ $x.$ Φ $x) \vee z = (\tau$ $x.$ Φ $x)) \longrightarrow y = (\tau$ $x.$ Φ $x)))$ **and** *something-exists*: $\exists$ $x.$ E x **and** *god-is-conceivable*: C g **and** *C-atomic*: *atomic* C
shows $\neg$ E $(\tau$ $x.$ Φ $x) \longrightarrow (\exists$ $y.$ C y $\wedge$ G y $(\tau$ $x.$ Φ $x))$
by (*metis* (*no-types, lifting*) *C-atomic description-theorem-3 g-def god-is-conceivable premise-3 something-exists*)

But as Parent says, the premise that 'exists in the understanding' is atomic is particularly questionable. If 'exists in the understanding' is atomic, then it follows from *description-theorem-3* that, for example, if the largest positive integer exists in the understanding, then something is the largest positive integer. But since 'the largest positive integer' is a grammatical description, there is a case to be made that the largest positive integer does exist in the understanding, even though nothing is the largest positive integer [7], p. 480-1.

7 Conclusion

The main difference between Oppenheimer and Zalta's reconstruction of the argument in Prover9 and the reconstruction presented here in Isabelle/HOL is that whereas Prover9 employs first-order logic, Isabelle/HOL employs higher-order logic. That means that the Russellian *description-axiom* schema can be stated directly in Isabelle/HOL, whereas in Prover9 it has to be represented indirectly using first-order quantifiers ranging over predicates and relations [6], pp. 338-41.

Because of the way Oppenheimer and Zalta carry out this embedding, it is presupposed in their presentation that all the non-logical predicates which occur in their argument are atomic. In contrast, in the presentation in Isabelle/HOL, whenever the assumption that a certain predicate is atomic is needed, this has to be made explicit as a premise of the argument. This is not a merely practical matter since, as Parent points out, the question of whether 'exists in the understanding' is an atomic predicate turns out to be crucial.

Abstracting from the peculiarities of different software, a surprising result is that whereas every version of the argument requires the premise that identity is atomic, and some versions require the additional premises that 'is greater than' is atomic and 'exists in the understanding' is atomic, no version of the argument requires the premises that 'exists in reality', or in other words 'exists' simpliciter, is atomic. This is in spite of the fact that the question of whether 'exists' is a genuine predicate has historically being one of the most controversial questions raised by Anselm's argument.

end

8 Acknowledgements

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