HIGHER-ORDER VAGUENESS AND BORDERLINE NESTINGS –
A PERSISTENT CONFUSION[*]

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This paper shows that authors who have recently argued that higher-order vagueness is incoherent, paradoxical, illusory or non-existent[1] invariably confound elements of higher-order vagueness (of the kind relevant to the Sorites paradox) with elements of a different paradigm of borderline borderline cases; and that, once the elements of that other paradigm are removed from the description of higher-order vagueness, the basis for the claims of paradoxicality, etc., disappears.[2]

The paper sets out in detail the two paradigms (higher-order vagueness and borderline nestings) and their logics (iterated modalities vs. mixed-order non-empty predicates), illustrates how the prevalent notion of *hierarchical* higher-order vagueness gains its persuasiveness largely from a conflation of these paradigms, and shows how the alternative of *columnar* higher-order vagueness not only preserves coherence, but also is the sort of higher-order vagueness that is relevant to the Sorites paradox. As a corollary, the paper provides support for the increasing number of vagueness theorists who renounce clear borderline cases (such as Sainsbury, Wright, Williamson, Shapiro, Fara, Raffman, Cobreros and Smith).[3]

The paper is structured as follows: Section 1 presents the notion of hierarchical vagueness. Section 2 introduces and compares two ordinary language uses of "borderline case": epistemic and classificatory use. Section 3 lays out the basic logical requirements for higher-order

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[*] This paper is dedicated to the memory of Ruth Barcan Marcus.

[1] e.g. Sainsbury 1990, 1991; Wright 1992, 2010, Williamson 1994, Greenough 2003, Shapiro 2005, 2006, Raffman 2010.

[2] This paper is concerned only with the kind of higher-order vagueness that manifests itself in borderline borderline cases. For the question of the vagueness of "vague", whether there is a sharp border between the vague and the non-vague linguistic expressions, cf. e.g. Varzi (2003), Hyde (2003).

[3] Sainsbury 1990, 1991; Shapiro 2005, 2006, Williamson 1994, 1999; Smith 2008, Wright for example 2010, Raffman 2010, for example 513; Fara 2003; Cobreros 2011, 219-20. In this paper we discuss neither supervaluationism nor degree theories of vagueness. Even so, the logic of columnar radical higher-order vagueness (as set out in Section 3.3) can be utilized by such theories to explain the transition from the super-true or degree-1-true cases, respectively, to the borderline cases.

vagueness and shows that higher-order vagueness can be columnar rather than hierarchical. Section 4 presents the paradigm of borderline nestings and its logic. Section 5 lists possible sources for the conflation of the classificatory use of "borderline" plus borderline /2/ nestings with the epistemic use plus higher-order vagueness. Section 6 shows how such conflation undermines the arguments against higher-order vagueness by Sainsbury, Shapiro, Raffman, Wright, and Williamson. Section 7 concludes with a note on columnar higher-order vagueness and the Sorites.

1. Hierarchical Vagueness

First, the most-discussed type of – presumed – higher-order vagueness, which Mark Sainsbury has fittingly dubbed "hierarchical vagueness" (Sainsbury 1991, 1990). Sainsbury's 1991 is also the *locus classicus* for hierarchical vagueness. Because it is so apt (and we return to it later) we quote his description in full:

> "Here is one way in which the notion of higher-order vagueness might arise. The first thought is to represent the sense of a predicate like "green" or "child" by its effecting a division of categorially appropriate objects into three sets. This is supposed to do justice to the actuality or possibility of borderline cases: surfaces intermediate between blue and green, people intermediate between childhood and adulthood. It thus does justice to "tolerance" intuitions: a very small difference in shade cannot make the difference between something being green and being blue, so we need a class of borderlines; a very small difference in age cannot make the difference between childhood and adulthood, so we need a class of borderlines. On the classical conception I am investigating, this is an adequate representation of the lowest level of vagueness – vagueness$_1$.

> However, with most or even all vague predicates, it soon appears that the idea that there is a sharp division between the positive cases and the borderline ones, and between the borderline cases and the negative ones, can no more be sustained than can the idea that there is a sharp division between positive and negative cases. We find a new type of borderline case: for example, those things which seem intermediate between being definite cases of children, and being borderline cases of children. We decline to accept that there can be any sharp boundary here. If there were, it would remain true that there would be such a thing as the last heartbeat of my childhood, or at any rate the last heartbeat of my definite childhood, and that seems as crazy as the idea that the predicate "child" divides the universe into a set and its complement within the universal set. Hence we allow for the more than theoretical possibility of a higher order of vagueness – call it vagueness$_2$ – which consists in a five-fold division: into the definite positive cases for the predicate, the definite borderlines and the definite negative cases, together with the cases which are borderline between being definite positive cases and definite borderlines, and those borderline between being definite borderlines and definite negative cases. A predicate whose sense was correctly described by its effecting this fivefold division would count as vague$_2$. /3/

> The generalization is straightforward: a vague$_n$ predicate is one whose sense can be described by its drawing 2^n boundaries, thus dividing the categorially appropriate objects

into $2^n + 1$ sets. A vague predicate is vague$_n$ for some $n > 0$. A higher-order vague predicate is vague$_n$ for some $n > 1$. A radically higher-order vague predicate is vague$_n$ for all *n.*" (Sainsbury 1991, 168-9)

Radical hierarchical vagueness is commonly first presented as a possible way to escape the Sorites paradox; then criticized as itself leading to paradox or incoherence. Sometimes hierarchical vagueness is simply given as an illustration of higher-order vagueness. Frequently it is introduced by example. It comes in variations. But it always aims at characterizing or explaining the borderline cases of vague expressions. Typically, hierarchical vagueness and borderline cases are represented with the help of modal expressions ("clear", "definite", "determinate"). The standard way is to give an account of "*a* is a borderline case of F" in terms of clarity (definiteness, determinacy) as

1.1 $\qquad\qquad \neg CFa \wedge \neg C\neg Fa$

with C (or *D*, *Def*, DET, Δ, etc.) as a modal operator standing for "it is clear (definite, etc.) that". Some authors introduce a special operator for borderlineness, e.g. "it is indeterminate that". Some have the operator(s) take unanalyzed sentences. Others use first-order predicate notation, so they can discuss vague predicates. For now, we use BL for "is a borderline case of" and F and G for simple vague predicates.[4]

The hierarchy of radical hierarchical vagueness can be tentatively represented in modal terms as in Figure 1. (The point of the brackets in Figure 1 becomes clear in Section 4.6.)

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      (C)F                              BLF                          (C)¬F
    (C²)F         BL²F               (C)BLF           BL²F          (C²)¬F
  (C³)F    BL³F (C)BL²F    BL³F    (C²)BLF   BL³F  (C)BL²F   BL³F   (C³)¬F
            ...

(Cⁿ)F  BLⁿF (C)BLⁿ⁻¹F    ...     BLⁿF (Cⁿ⁻¹)BLF BLⁿF      ...    (C)BLⁿ⁻¹F  BLⁿF  (Cⁿ)¬F
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Figure 1: *provisional representation in modal terms of the hierarchy of radical hierarchical vagueness*

In Sainsbury's terms, the first line describes a vague$_1$ predicate that draws 2^1 boundaries, dividing the relevant objects into $2^1 + 1$ sets. The nth line has 2^n boundaries and $2^n + 1$ sets.[5] Some authors restrict their description of the hierarchy to the first pair of formulae on each line.

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[4] We disregard the question whether *a* is a borderline case of (being) F or whether *a* is a borderline case of the predicate "F", since for present purposes this is irrelevant. For some remarks why it is not irrelevant altogether for theories of vagueness see Bobzien 2011.1, 324-6.

[5] We do not find Williamson's criticism of Sainsbury's hierarchical vagueness (Williamson 1999, 135 footnote 6) convincing. Williamson's first objection, that hierarchical vagueness only accounts for one-dimensional domains can be countered with the fact that modifications for two or more dimensional domains are possible. Williamson's second objection, that the partition in hierarchical vagueness is too coarse grained in that it does not account for the difference between predicates for which *all* objects are borderline and those for which *no* objects are borderline is (also) irrelevant for standard Sorites paradoxes.

Some give only the first formula and, instead of the second, the negation of the first. In either case the context indicates that a hierarchy similar to that in Figure 1 is intended.[6]   /4/

2. Ordinary language uses of "borderline" and "borderline case"
In this section we introduce and discuss two ways in which the expressions "borderline" (in its adjectival use) and "borderline case" are used in non-technical language. We use "is (a) borderline …" and "is a borderline case of (being) …" interchangeably.[7]

2.1. Some general features of the non-technical use of "borderline"
We start with some features characteristic of all relevant non-technical uses of "borderline" and "borderline case":
- If something is borderline, or a borderline case, it is so regarding some predicate Φ, or some pair of predicates Φ and Ψ, etc.
- If something is borderline, it is so with respect to some parameter or dimension D or parameters or dimensions $D_1 … D_n$ (e.g. height).
- If something is borderline regarding some Φ and with respect to some D, then there are some things of the same sort that are Φ and not borderline Φ or that are ¬Φ and not borderline Φ (usually both). Moreover, such cases include the normal or standard ones.[8]

We add some restrictions as to what cases we consider, since our present interest is in borderline cases only insofar as they may play a rôle in standard Sorites paradoxes. From a perspective of language use, these restrictions are *arbitrary*.
- We do not consider cases which are borderline with respect to three or more predicates ("This is borderline yellow/orange/brown").
- We do not consider cases which are borderline with respect to more than one dimension ("This is borderline big with respect to height-cum-circumference"). /5/
- We do not consider cases which are borderline with respect to evaluative expressions ("good", "nice") or expressions of personal taste ("tasty", "fun") or absolute gradable adjectives ("pure", "straight").
- We only consider objects that can be denoted by designators of first-order logic. We do not consider types of objects or generics or properties (as in "Magenta is borderline red," "Pudding is borderline solid").[9]

All seven points are satisfied in the standard Sorites paradoxes discussed in the literature on higher-order vagueness understood as hierarchical, as introduced in Section 1.

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[6] Descriptions and discussions of hierarchical vagueness, of varying detail and various kinds, can be found in Fara 2003, 196-2004; Greenough 2005, 180-4; Keefe/Smith 1997, 15; Keefe 2000, 31-1, Raffman 2010, 509-10; Shapiro 2005, 147-9, 2006, 125-6; Soames 2003, 136-49; Williamson 1994, 157-8, 1999, 127, 136-7; Wright 1992, 2010 and Bobzien 2012, 190-4, to mention just some.

[7] Examples: "a borderline heap" / "a borderline case of a heap"; "borderline green" / "a borderline case of being green" (or "a borderline case of "green" ", see note 4).

[8] The paradigmatic cases which lack either non-borderline cases that are Φ or non-borderline cases that are not Φ are the so-called absolute gradable adjectives. For these see e.g. Kennedy 2007.

[9] Such cases have been discussed in Sorensen 2001, and are not always distinguished from those relevant to the Sorites paradox (e.g. Dummett, 1978, 182, Sorensen 2010, Wright, 2010, 526, 544 put them together). This lack of distinction goes back to antiquity. The Greek used "*Sorites*" to refer to arguments with series of both types. See e.g. Williamson 1994, 21, 25, Bobzien, 2002, 227.

Next, we present two distinct ordinary language uses of the expression "borderline" as applied to vague predicates: epistemic and classificatory. For each use, we provide an illustration and spell out some distinctive logical features. Each use corresponds to a conceptual paradigm which finds application in different basic types of everyday situations. Moreover, each use seems to be ingrained in our structuring and interacting with the world from an early age. We introduce these *pre-theoretical* notions of borderlineness, as they have an impact on how humans generally, and philosophers in particular, conceive of borderline and borderline borderline cases.[10] We are *not*, at this point, concerned with philosophical theories of vagueness. We simply zoom in on those more specific uses of "borderline" that may appear to be relevant to Sorites vagueness.

2.2 The classificatory use of "borderline"
This ordinary language use has its paradigm in situations of sorting. Someone sorting colour patches on the blue-to-green spectrum into boxes may utter this soliloquy: "The blue patches go into the box labeled 'blue'. The green patches go into the box labeled 'green'. Hm. This patch is *on the border between* the blue patches and the green patches. It is borderline blue/green. So I"ll need to get a third box for borderline blue/green patches." In the classificatory use, something is a borderline Φ/Ψ case only if it can be (since it has been or would be) identified as neither Φ nor Ψ. It is characteristic of the classificatory use that the speaker classifies an object *a* as belonging to a class of objects that, with respect to some dimension *D*, falls between two other classes which are denoted by (or to which otherwise correspond) two predicates Φ and Ψ. We abbreviate this as $BL(\Phi/\Psi)_D a$. A logical feature of the classificatory use is this: For there to be /6/ $BL(\Phi/\Psi)_D$ cases, the two predicates Φ and Ψ have to be contraries; that is, with regard to *D*

2.2.1 $\exists x\, BL(\Phi/\Psi)_D x \quad \rightarrow \quad \forall x\, [\Phi x \rightarrow \neg\Psi x],$ (CONTRARIETY$_{class}$)

Sorting items into categories is a very basic activity which eighteen months old babies tend to have mastered (as long as the objects are sufficiently distinct). It seems to be a critical skill for the early mental development of humans.[11] As a consequence, the paradigm of sorting has been internalized by every human being long before they encounter a Sorites paradox. Long before they encounter a Sorites paradox, if somewhat later than age eighteen months, they also have acquired sorting strategies for dealing with cases that do not fit pre-given categories. One of these strategies is adding a new category, if circumstances permit. Another is leaving items unsorted, if circumstances permit. Our exposition of the classificatory use is meant to represent simple everyday occurrences of sorting items into categories and of creating new categories if required. The purpose of the exposition is *not* (and we repeat: is *not*) to represent any philosophical theories of vagueness.

We can add further observations about the non-technical classificatory use of "borderline", if we add two restrictions motivated by our interest in cases that are potentially relevant to Sorites paradoxes. Thus we restrict the predicates under consideration to atomic vague predicates F, G

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[10] There are two further non-technical uses of "borderline": *factive* use, which understands "x is borderline Φ" as "*x* is Φ, but only just", and has as characteristic logical feature [$BL\Phi a \rightarrow \Phi a$]; and *anti-factive* use, which understands "*x* is borderline Φ" as "*x* is not quite Φ", and has as characteristic logical feature [$BL\Phi a \rightarrow \neg\Phi a$]. Raffman 2005 develops a theory of borderlineness that combines elements of the anti-factive use (see e.g. p.2) and of the classificatory use (see e.g. p.9).

[11] Cf. e.g. the recent work by Barbara A Younger.

and the objects under consideration to a set $U_{ordered}$ of ordered objects a_1 to a_s on D, with a_s being (one of) the last object(s) and the set being finite or dense. We call a_1 and a_s polar cases. The dimensions are thus effectly restricted to *directional dimensions* (e.g. *from* green *to* blue, *from* short *to* tall).[12] If the series at issue displays tolerance, there will be subsets of $U_{ordered}$ that are Sorites series[13] with regard to F and D. The two restrictions allow us to add the following principles. Polar cases can be represented as governed by

2.2.2 $Fa_1 \wedge Ga_s.$ (POLAR$_{class}$)

Furthermore, there are two continuity principles. First, any a_i to the left of an a that is F is itself F and any a_i to the right of an a that is G is itself G, or

2.2.3 $[Fa_p \rightarrow Fa_{p-i}] \wedge [Ga_p \rightarrow Ga_{p+i}].$ (CONTINUITY$_{F,G}$)

Second, any object between two objects that are borderline Φ/Ψ is itself borderline Φ/Ψ, i.e.

2.2.4   If $BL(\Phi/\Psi)_D a_p$ and $BL(\Phi/\Psi)_D a_q$, then any a_i, with $p \leq i \leq q$, is also $BL(\Phi/\Psi)_D.$
                                                            (CONTINUITY$_{BL(\Phi/\Psi)}$)

/7/

In light of these two continuity relations, it is appropriate to describe the classificatory use with regard to an ordered series of objects $U_{ordered}$ on D in terms of *segments*. We call a segment S_Φ regarding D and $U_{ordered}$ the non-empty set of objects from $U_{ordered}$ that are categorized or identified as Φ regarding D. (In particular, the identification of borderline Φ/Ψ cases is a necessary condition for there to be a borderline segment $S_{BL(\Phi/\Psi)}$.[14]) Basically, the segment S_Φ are the things that are put in the box with the label "Φ". If there are no such things, there is no such segment on D. Thus we have by definition:

2.2.5   For each segment S_Φ there is an a such that $\Phi a.$ (NO-EMPTY-SEGMENT)

With the notion of a segment in place, *2.2.2*, *2.2.3* and *2.2.4* can be combined into the following general continuity principle:

2.2.6   Every segment S_Φ is continuous regarding D in the sense that for any a_p, a_q of S_Φ, any a_i with $p \leq i \leq q$ will also be in S_Φ.   (CONTINUITY$_S$)

And borderline Φ/Ψ cases can – *provisionally*[15] – be defined in terms of segments thus:

2.2.7 $BL(\Phi/\Psi)_D a$ $=_{df}$ a is located on D between two segments S_Φ and S_Ψ.

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[12] These restrictions do not make the classificatory use technical. Compare everyday sorting of objects into categories by height or weight.

[13] A Sorites series or Sorites sequence is a series of ordered objects that, regarding a predicate F, is prone to a Sorites paradox.

[14] A segment is a *borderline segment* iff it is of the form $S_{BL(\Phi/\Psi)}$. It is a *non-borderline segment* iff it is not of that form.

[15] Section 4.2 below explains why this is provisional.

A further restriction that is motivated by our interest in cases that are potentially relevant to Sorites paradoxes is this:

2.2.8   F and G are predicates that are at least *prima facie* thought together to cover the entire dimension at issue,   (PRIMA-FACIE-EXHAUSTIVE)

for example, "red" and "pink", "tall" and "of average height", as opposed to "red" and "white", "tall" and "short" – context will be pertinent here, of course. The rationale behind (PRIMA-FACIE-EXHAUSTIVE) is this: if, *when starting out sorting*, we already anticipated that there are cases between the red and the pink ones that go into a box labeled "red-pink", the Sorites relevance would from the start concern the pair of atomic predicates "red", "red-pink" (or the pair "red-pink", "pink").

2.3. The epistemic use of "borderline"
The second non-technical use of "borderline" is what we call its *epistemic use*. A fictitious monologue characteristic for this use is: "This patch is borderline blue. One really can"t tell (say, know, identify) whether it's blue. It's *on the border of* being blue and not being blue." It is characteristic of this use that the speaker expresses epistemic inaccessibility with regard to whether an object a is Φ; /8/ or alternatively, with regard to whether a corresponding sentence Φa is true. In the epistemic use, something is a borderline case of Φ only if it can be identified neither as Φ nor as $\neg\Phi$. A distinctive logical feature of the epistemic use can be expressed thus

2.3.1 $[BL\Phi a \leftrightarrow BL\neg\Phi a]$.

2.3.1 reveals the similarity of the epistemic use with contingency, which has a corresponding principle.[16]

This non-technical epistemic use of "borderline" is on two counts appropriate for *describing* the borderline zone of a Sorites paradox in a way the classificatory use is not.[17] First, the epistemic use is suitable to express the basic difficulties one encounters with a Sorites series: that there is a region on D that is, or seems, epistemically inaccessible with regard to F. Second, a Sorites paradox occurs only with a series of objects insofar as it runs from objects that satisfy F to objects that don't satisfy F (which in classical logic is equivalent to a series that runs from objects that satisfy F to objects that satisfy $\neg$F). It does not occur with a series insofar as it runs from objects that satisfy F to objects that satisfy G.

For the epistemic use of "borderline", the restriction to directional dimensions, or $U_{ordered}$, makes possible the following two specifications. One regards polar cases: the first object in the series is F and not borderline so, the last is not F and not borderline so, or

2.3.2 $[Fa_1 \wedge \neg BLFa_1] \wedge [\neg Fa_s \wedge \neg BL\neg Fa_s]$.   (POLAR_BL)

This is equivalent to stating that both Fa_1 and $\neg Fa_s$ are clear (cf. *1.1*)

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[16] "It is contingent that *A* iff it is contingent that $\neg A$". For the logic of contingency, see e.g. Montgomery & Routley 1966, Kuhn 1995.
[17] So also Greenough 2003.

2.3.3 $CFa_1 \wedge C\neg Fa_s.$ (POLAR$_C$)

The other specification is a continuity relation for non-borderline cases: Any a_i to the left of an a that is F is itself F and any a_i to the right of an a that is not F is itself not F

2.3.4 $[CFa_p \rightarrow CFa_{p-1}] \wedge [C\neg Fa_p \rightarrow [C\neg Fa_{p+1}].$ (CONTINUITY$_C$)

From *2.3.4* and the standard modal account of BL (*1.1*) it follows that

2.3.5   If $BLFa_p$ and $BLFa_q$, then any a_i, with $p \leq i \leq q$ is also BLF. (CONTINUITY$_{BL}$)

For, if $BLFa_p$ and $BLFa_q$ and $p<q$, then by application of *2.3.4* (using the contraposition of each conjunct) any a_j, with $p \leq j \leq q$ is both $\neg CF$ and $\neg C\neg F$, and hence also BL. The case with $p=q$ is trivial. /9/

2.4. The difference between the epistemic and the classificatory use of "borderline"

The most significant difference between the epistemic and the classificatory use is not immediately obvious. It may seem that it is the fact that one use concerns only one predicate, the other two. But this is not so. It is possible to introduce a variation of the epistemic use with two contrary predicates which retains its epistemic element. This results in epistemic inaccessibility with regard to a pair of predicates. Someone may say: "It just seems impossible to tell whether this is blue or green. It's borderline." Call this the *contrarietous* epistemic use of "borderline", as contrasted with the *contradictorious* epistemic use from Section 2.3. In this use, immediate relevance to the Sorites paradox – which requires contradictories – would be lost.[18] On the other hand, (CONTRARIETY$_{class}$) would not automatically hold, since nothing precludes that some borderline cases are both blue and green.

The crucial distinction between the epistemic and classificatory uses is both more profound and more subtle. It concerns the scopes of negation, possibility and conjunction that are part of the respective construal of their accounts. This comes out when one juxtaposes the two partial descriptions from Sections 2.2. and 2.3 above. (We use the contrarietous epistemic use to bring out the relevant difference more clearly. The contradictorious epistemic use can be recovered by substitution of "$\neg\Phi$" for "Ψ".)

2.4.1" "   In the epistemic use, something is a borderline case only if it can be identified neither as Φ nor as Ψ.

2.4.2" "   In the classificatory use, something is a borderline case only if it can be (since has been) identified as neither Φ nor Ψ.

We here use "identify" as an umbrella term. That is, we understand the expressions "can tell", "can know", "can say"(that a is F), and "a can be classified as F", "a can be sorted as F" in such a way that they entail "a can be identified as F". We also use "identifiable", "classifiable",

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[18] By the same token, one could substitute $\neg\Phi$ for Ψ in the classificatory use. For this special case, i.e. the contradictorious classificatory use, we refer the reader to Section 5.1.

"knowable", etc. as factive: something is identifiable, …, as F only if it is F.[19] The significant difference in logical structure can be brought out with the use of standard paraphrases from predicate and modal logics:

2.4.1"   Epistemically speaking, *a* is borderline Φ/Ψ only if *a* is such that both it is not possible to identify it as Φ and it is not possible to identify it as Ψ.

2.4.2"   Classificatorily speaking, *a* is borderline Φ/Ψ only if *a* is such that it is possible to identify it as being both not Φ and not Ψ.   /10/

Or, in a somewhat lax semi-formalized manner:

2.4.1   Epistemically speaking, $BL(Φ/Ψ)_D a$ only if [¬◊ to identify that Φ*a*] ∧ [¬◊ to identify that Ψ*a*].

2.4.2   Classificatorily speaking, $BL(Φ/Ψ)_D a$ only if ◊ to identify that [¬Φ*a* ∧ ¬Ψ*a*].

From these formulations one can read off the following three substantial differences between borderline cases as marked out by classificatory use and epistemic use respectively. (i) In epistemic use, it is possible for *a* to be both BL(Φ/Ψ) and Φ; and to be both BL(Φ/Ψ) and Ψ. In classificatory use, neither is possible. Rather, its account has the following implication:

2.4.3 $[BL(Φ/Ψ)_D a → ¬Φa] ∧ [BL(Φ/Ψ)_D a → ¬Ψa]$ (THIRD-KIND)

(ii) Epistemic use is compatible with a sharp border between the Φ and the Ψ on *D*. Classificatory use is not.[20] (iii) Epistemic use expresses a limitation of human epistemic ability; classificatory use does not. If anything, it marks progress in our ability to discriminate between things. Thus, although they can look similar, logically, the non-technical epistemic and classificatory uses of "borderline" are quite distinct. (For how they may be and have been combined or confounded in theories of vagueness cf. Sections 5 and 6 below.)

3. Higher-order vagueness

In ordinary language, the expression "borderline" is sometimes, if rarely, iterated. In philosophical discussions of hierarchical vagueness we regularly encounter iterations of the expression, often two or more. In Sections 3 and 4 we discuss two very different models of iterating "borderline". One is higher-order (borderline) vagueness, in its strict sense, couched in terms of modal logic. It harmonizes best with – but is not restricted to – the epistemic use of "borderline". The other type of iteration we call, for lack of a better term, *borderline nestings*. It is a natural expansion of our representation of the classificatory use of "borderline".

3.1. Syntax and terminology for higher-order vagueness

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[19] Of course it is possible for people to make mistakes both when sorting things and when stating or believing that they can tell or know or assert that F*a*. We are not concerned with those cases, since we maintain that any identifiability (tellability, sortability) that is germane to the Sorites is independent of the errors or handicaps of individual judgers.

[20] By *a sharp border between the Φ and the Ψ on D* we mean the following: There are on *D* no a_i, a_n, a_m, with $n<i<m$, such that Φa_n is true, Φa_m is false, and Φa_i is neither (or both) - with the continuity principles presupposed.

In Section 1 we gave the standard modal account of "*a* is a borderline case of F" in terms of clarity (definiteness, determinacy) as ¬CF*a* ∧¬C¬F*a*. We now generalize this, saying that (for any arbitrary formula *A*)

3.1.1   BL*A*   ↔   ¬C*A* ∧¬C¬*A*.   /11/

The modal operator C (or *D*, *Def*, DET, Δ, etc.) is modeled on the logic that governs the necessity operator □ and takes open or closed sentences in its scope. The syntax of C and of the connectives and quantifiers used is that of normal modal systems with □. CBLF*a* states that *a* is a clear borderline case of F; BLCF*a* that *a* is a borderline clear case of F.

Next we introduce the terminology of orders and of ranks. Thus *a* is a *first-order* borderline case of F, written BL^{1}F*a*, iff BLF*a*. And *a* is an *(n+1)th-order* borderline case (for *n*≥1), written BL^{n+1}F*a*, iff BLBL^{n}F*a*.[21] For example, using *3.1.1*, we get

3.1.2   BL^{2}F*a* ↔ BL[¬Cf*a*∧¬C¬F*a*].

Using *3.1.1* again, we get

3.1.3   BL^{2}F*a* ↔ ¬C[¬Cf*a*∧¬C¬F*a*] ∧¬C¬[¬Cf*a*∧¬C¬F*a*].

Generally, *a* is an *n*th-order borderline case of F iff *a* is a borderline case of (being) *n*−1th-order borderline F. A formula is of *rank 0* if it is without any occurrences of C or BL. A formula is of *rank n+1* if it has a formula of rank *n* within the scope of an occurrence of C or BL and has no formula of a rank larger than *n* within the scope of any occurrence of C or BL. Thus, e.g., BLF*a* is of rank 1, BL^{2}F*a* and CBLF*a* are of rank 2, etc.[22]

3.2. Minimal constitutive elements of higher-order vagueness
The following are the minimal constitutive elements of higher-order vagueness, or higher-order borderlineness, as we understand it:

Elements of modal logic:
3.2.1   Closed modalized sentences are bivalent.[23]

3.2.2   If *A* and *B* are theorems, then *A*∧*B* is a theorem.   (∧-INTRODUCTION)

3.2.3   C*A*→*A*.   (T$_c$)

3.2.4   [C*A*∧C*B*]→C[*A*∧*B*]   (K2$_c$) [24]

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[21] Cf. also Shapiro 2005, 148, 2006, ch.5.

[22] Our notion of a modal rank corresponds to that of a modal degree in Carnielli & Pizzi 2009, p.28, where also a formal definition is given.

[23] This semantic requirement is not a necessity; see e.g. Keefe 2000, ch.8. Still, if this condition is waved, the point of representing higher-order vagueness in modal terms *in order to prevent sharp borderline-region borders* seems to become moot.

The existence of borderline cases is not logically precluded, i.e.

3.2.5 $\neg CA \rightarrow C\neg A$ is not a theorem.   /12/

Lastly, "borderline" and "borderline case" are understood modally as expressed in *3.1.1*. The introduction of a modal *operator* BL, or similar, is optional.

Elements regarding series of ordered objects:
We have a series $U_{ordered}$ of objects a_1 to a_s ordered on some dimension D with regard to a (vague) predicate F. The ordering is strict.[25]   It can be finite or dense. The principles *2.3.2* (POLAR$_C$) and *2.3.4* (CONTINUITY$_C$) from Section 2.3 hold; and hence so does *2.3.5* (CONTINUITY$_{BL}$). In addition, the existence of borderline cases *within the ordered series* (i.e. on D between cases that satisfy CF and cases that satisfy C¬F) is not precluded:

3.2.6   It cannot be ruled out that there are a_p, a_i, a_q with $p \leq i \leq q$ and $[CFa_p \wedge [\neg CFa_i \wedge$
            $\neg C\neg Fa_i]] \wedge C\neg Fa_q$.

Moreover, it should be possible to supplement a theory of higher-order vagueness with some kind of "forward continuity principle" that links the elements of a Sorites series, such as the Sorites premise $Fa_n \rightarrow Fa_{n+1}$ or, more likely, some weakened form of it.[26] Otherwise it will be useless as a theory that advances the solution of the Sorites paradox.[27]

Compositionality:
For there to be genuine higher orders, the compositionality of the C operator needs to be reflected in the operator"s account. Here is a simple example (one which we do not condone). Interpret C as "it is knowable that". S*econd-order clarity* of Fa then requires that it is knowable that it is knowable that Fa. If there is a BL operator, the same holds. For example, if you interpret BL as "it is unknowable whether", s*econd-order borderlineness* of Fa requires that it is unknowable whether it is unknowable whether Fa. So far this is trivial. Less trivial – in the sense that it has been repeatedly discounted – is the fact that, if there is no borderline operator in the modal system, the account of higher-order borderline cases must in the same way correspond to the equivalent of BL^nFa expressed in terms of the C operator.[28] Finally, for something to be a

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[24] The agglomeration (or collection) axiom K2 is weaker than axiom K. Even if bivalence is assumed, systems of genuine higher-order vagueness need not be normal modal systems, i.e. extensions of system K.
[25] The requirement that the ordering is strict is not a necessity either. It provides the simplest case, and also the case that is most directly relevant to the Sorites. Non-strict orderings can be mapped on strict ones via equivalence classes.
[26] Examples for weakened forms discussed in the literature are Williamson's $KFa_n \rightarrow \neg Fa_{n+1}$ and versions of $D^k(D^mFa_n \rightarrow \neg D \neg D^{m-1}Fa_{n+1})$ with $m \geq 1$ and $k \geq 0$, for which see e.g. Wright 1992, Asher, Dever & Pappas 2009.
[27] Other such desiderata are (ii) that a theory of higher-order vagueness help explain the intuition that there appears to be a *seamless transition* in a Sorites series with regard to Φ from the cases where Φ applies to those where it doesn't (cf. Fara 2003, 197); and (iii) that it tally with the *boundarylessness intuition* that there is no determinable boundary that marks the non-borderline cases from the (totality of the) borderline cases in a Sorites series (cf. Sainsbury 1990).
[28] Such compositionality is absent e.g. in the formal systems provided by Williamson 1999 and in the account of higher-order vagueness discussed in Shapiro 2006.

clear /13/ *borderline case*, the accounts of clarity and of borderlineness both need to be compositional in the way described.

Levels and level generation:

For an understanding of the rôle different axioms of modal iteration play in different theories of higher-order vagueness, it is expedient to distinguish higher orders by levels. *Levels* are the result of sorting basic formulae[29] by rank and applying the result to an arbitrary series of ordered objects U_{ordered}. Thus *each level n has only basic formulae of rank n*. Take the pair of predicates F, ¬F of some series U_{ordered} as base level or level 0 (since there are zero modal operators). Higher levels *n* are generated by prefixing C or ¬C to each predicate of level *n*–1. The resulting basic formulae are those of level *n*. Generally, there are 2^{n+1} possible basic formulae at level *n*. They can be paired up, so that there are 2^n possible (and not necessarily extensionally distinct) ways of dividing *D* and a maximum of 2^n+1 possible distinct regions on *D* thus partitioned.

Level 1 has CF and C¬F and their negations ¬CF and ¬C¬F. The conjunction of the latter two describes *first-level first-order borderline cases*. Evidently, such conjunctions have the same rank as their conjuncts. Level 2 has the four pairs obtained by prefixing either C or ¬C to the formulae of Level 1, i.e. C^2F, ¬C^2F, C^2¬F, ¬C^2¬F, C¬CF, ¬C¬CF, C¬C¬F, ¬C¬C¬F. Using conjunction, we get two types of *second-level first-order borderline cases*, cases which can be described as ¬C^2Fa∧¬C¬CFa and ¬C^2¬Fa∧¬C¬C¬Fa respectively. At Level 2, there *may*, but need not, be *clear* borderline cases. Such cases could be described as C¬CFa∧C¬C¬Fa.[30]

There are various coherent modal theories of higher-order vagueness. They are in the main determined by what additional axioms, rules and further constraints are introduced. These may, but need not, include any of the following: classical logic for the non-modalized sentences; *modus ponens*; and for the C operator axioms K, B, S4, S5 and the Rule of Necessitation (RN). Some combinations of these restrict the number of logically distinct (i.e. non-equivalent) orders, preclude the existence of clear borderline cases or have some other significant consequences. The semantics of higher-order vagueness depends on what additional rules and axioms are chosen.

3.3. Columnar higher-order vagueness

We now explain why higher-order vagueness as such does not entail a hierarchy of orders as introduced in Section 1 – not even the beginning of such a hierarchy, i.e. cases on *D* that satisfy C[¬CFx∧¬C¬Fx] (clear borderline cases), "buffered" on one side by cases that satisfy neither C[¬CFx∧¬C¬Fx] nor C^2Fx, on the other by cases that satisfy neither C[¬CFx∧¬C¬Fx] nor C^2¬Fx. /14/

The following deliberations show this. With higher-order vagueness, it is possible for there to be first-order borderline cases at each level. They all would be describable by the formula ¬CΦa∧¬C¬Φa with C^nF for Φ, and with *n* increasing with each level (*n*=1 at level 1, *n*=2 at level 2, etc.).[31] However, the only (non-negotiable) relation *between levels* of borderline cases and/or

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[29] By a basic formula we here mean a formula that contains no operators other than the C operator and the negation operator.

[30] Given (K2$_c$), this formula entails C[¬CFa∧¬C¬Fa] and thus CBLFa.

[31] See also Wright 2010, 529.

clear cases is provided by (T_C). And (T_C) only minimally regiments the extensional relations between first-order borderline cases of different levels.[32]

In particular, if we consider the case of *maximal* possible intersection of the extensions of the basic modal formulae with regard to D, we may have no more than four extensions that cut out no more than three areas on D − similar to the classical four modal notions with contingency added. [33] With four extensions and three areas, the description in terms of first-level first-order modalities would be as in Figure 2.

Figure 2: *Representation of maximal possible intersection of the extensions of the basic modal formulae described in terms of first-level first-order modalities*

All other descriptions of the four extensions in terms of higher-order clarity and higher-order borderlineness would follow the co-extension rules:

- Formulae starting with C and containing an even number of ¬ are co-extensional with CFx.
- Formulae starting with C and containing an odd number of ¬ are co-extensional with C¬Fx.
- Formulae starting with ¬C and having an even number of ¬ are co-extensional with ¬C¬Fx.
- Formulae starting with ¬C and containing an odd number of ¬ are co-extensional with ¬CFx.

This may look arbitrary. But it is not, as some reflection makes clear. (If you substitute G for ¬F you get perfect left-right symmetry.) We call this maximal overlap scenario *columnar* (as opposed to hierarchical) higher-order vagueness. In it, the columnar borderline region in the middle of D could be described as

3.3.1 $\neg C^n F \wedge \neg C^n \neg F$ for any *n*.

This shows that higher-order vagueness of any order is compatible with there being (no section for clear borderline cases and hence) no clear borderline cases. /15/

If to the assumption of maximal overlap we add the assumption that *n* in the above description is indefinite (i.e. there is no limit on modal iterations), we obtain columnar radical higher-order vagueness. Columnar radical higher-order vagueness is a theory of *radical unclarity*. Borderlineness of Fa is characterized not only by the fact that it is unclear whether Fa, but also by the fact that it is unclear whether BLFa, BL^{2}Fa, and so forth.

If to columnar radical higher-order vagueness we add the further assumption that C is defined in terms of epistemic accessibility (i.e. that the use of "borderline" is epistemic),[34] then *it cannot be*

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[32] Cf. also Asher, Dever & Pappas 2009, 921.

[33] Of course the overlap is not strictly maximal – that would be the case in which CFx↔¬C¬Fx, resulting in only two areas on D.

[34] This latter assumption also preserves or warrants direct relevance to the description of the borderline region in a Sorites paradox.

ruled out that there are no borderline cases at all (i.e. of no borderline case is it epistemically accessible that it is a borderline case). For, at any level n and for any a in the columnar borderline region, it cannot be ruled out that a does not satisfy the description of the borderline region, but rather the description of one of the regions bordering it at the left or right. This is so, since $\neg C \neg C^n F \wedge \neg C \neg C^n \neg F$ also holds of the same region. For, by the above co-extension rules, $\neg C^n F$ and $\neg C \neg C^n \neg F$ have the same extension, and so do $\neg C^n \neg F$ and $\neg C \neg C^n F$.[35]

Thus, if we combine columnar radical higher-order vagueness with an epistemic reading of the C-operator, we have a theory in which it is epistemically inaccessible (at least for humans) whether, with respect to D, there is any a at all that satisfies the – co-extensive – predicates BLF, BL^2F, …, BL^nF. In other words, in this theory it is epistemically inaccessible whether, on D there are any borderline cases of any order. Thus radical higher-order vagueness is compatible with there being no borderline cases at all.[36]

With an epistemic reading of the C-operator, columnar radical higher-order vagueness is naturally complemented by the "forward continuity principle" for Sorites series that one cannot ever rule out that it is not the case that a clear case of F is followed by a clear case of ¬F. This can be expressed more formally as: for any a_i of a Sorites series, it cannot be ruled out that $\neg[CFa_i \wedge C \neg Fa_{i+1}]$. This should provide a satisfactory logical underpinning for the persuasiveness of the conditional Sorites premise.[37] /16/

Columnar radical higher-order vagueness can be formally described with the help of two principles, one for clarity, one for borderlineness (in addition to the modal-logical minimal constitutive elements from 3.2 and *modus ponens*). The first principle is the C-equivalent to the S4 axiom:

3.3.2 $CA \rightarrow CCA$ (MAXIMAL-OVERLAP$_C$)

The second essentially states that if something is borderline, it is borderline borderline:[38]

3.3.3 $[\neg CA \wedge \neg C \neg A] \rightarrow [\neg C \neg CA \wedge \neg C \neg C \neg A]$ (MAXIMAL-OVERLAP$_{BL}$)

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[35] Cf. Asher, Dever & Pappas 2009, 923-5, for a suggestion of a similar minimal model of higher-order vagueness (couched in the terminology of gap principles) that is columnar in our sense. They conclude that "[t]he insistence that gap principles be arbitrarily determinately true is in effect an insistence that vagueness is an ineliminable feature of language. But there seems to be no good reason to accept this." If one interprets the modal operator epistemically (describing the epistemic inaccessibility of the borderline zone) rather than semantically, one is not bound by the conclusion that vagueness is an ineliminable feature of natural languages (although we believe it may well be).

[36] Those who consider the existence of borderline cases as constitutive of vague predicates would have to add the existence of borderline cases as an independent postulate. It would then follow that one can tell or know of none of these borderline cases that it is a borderline case.

[37] Epistemically interpreted columnar radical higher-order vagueness also fulfills the other two desiderata for theories of higher-order vagueness (see note 27). (ii) All borderline cases are (epistemically) indistinguishable from the cases adjacent to the borderline zone on the left and on the right, insofar as for each of them it cannot be ruled out that they, too, are $C^n F$ or $C^n \neg F$ respectively. (iii) Since it cannot be ruled out that there are no borderline cases at all, there is no determinable boundary between the non-borderline cases and the (totality of the) borderline cases.

[38] Bobzien 2010 sets out columnar radical higher-order vagueness in terms of an unclarity or BL operator rather than a clarity operator, which gives the characteristic axiom the form $BLA \rightarrow BL^2A$ rather than (MAXIMAL-OVERLAP$_{BL}$). This brings out the nature of columnar higher-order vagueness more directly.

In the normal modal system S4 (for which one needs to further add (RN) and axiom K and eliminate uncountably infinite series), *3.3.3* is co-extensional with the McKinsey Axiom $C\neg C\neg A \rightarrow \neg C\neg CA$. Put differently, in normal modal logic the modal system that underlies columnar radical higher-order vagueness is equivalent to the system S4M. Generally, if columnar radical higher-order vagueness is combined with bivalence-preserving and agnostic theories of vagueness, Axiom B (i.e. $A \rightarrow C\neg C\neg A$) fails to hold for the C operator.[39]

To sum up the main results of this section:
* Higher-order vagueness can be columnar rather than hierarchical
* Higher-order vagueness of any order is compatible with there being no clear borderline cases on *D.*
* *Radical* higher-order vagueness is compatible with there being no borderline cases of any order.

3.4 Intersective Unclarity

Genuine higher-order borderline cases are sometimes confounded with what we call *intersective unclarity* ($U_\cap$). For any order n, a is intersectively$_n$ unclear iff it is not n^{th}-order clear that F and it is not n^{th}-order clear that ¬F: /17/

3.4.1 $U_{\cap n}Fa$ $=_{df}$ $\neg C^n Fa \wedge \neg C^n \neg Fa$

Intersective$_n$ unclarity is named after the fact that it is the result of *intersecting* the cases that are not n^{th}-order clearly F with the cases that are not n^{th}-order clearly ¬F. We call intersective$_n$ borderline cases the cases that are intersectively$_n$ unclear. They are in the intersection of the cases that are not n^{th}-order clearly F with those that are not n^{th}-order clearly ¬F. The account of the clarity operator used in the definition of intersective unclarity may well preserve compositionality for higher orders of clarity. In that case, the higher-order clarity is genuine. However, the "higher-order" intersective$_n$ borderline cases (such as $U_{\cap 2}Fa$ $U_{\cap 3}Fa$, etc.) are not genuine higher-order borderline cases. The account of $U_{\cap n}Fa$ is not compositional in the required sense. Nor does intersective unclarity provide a hierarchy of borderline regions the way hierarchical vagueness does. This is so since there is always only one borderline region per level.[40]

4. The classificatory use of "borderline" and borderline nestings

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[39] The reason is the following. It is a constitutive element of genuine higher-order vagueness that, for any vague predicate F, one cannot rule out that there are borderline cases of F. Now, if bivalence holds, it hence cannot be ruled out that there is an a_i such that $BLFa_i$ and Fa_i are both true. But if $BLFa_i$ is true, so is $\neg C\neg C\neg Fa_i$ (by (maximal-overlap$_{bl}$)). Hence both Fa_i and $\neg C\neg C\neg Fa_i$ are true. Hence Axiom B does not hold. The same result ensues if instead of a theory that preserves bivalence we have a theory that is agnostic in that it neither postulates nor precludes bivalence for borderline cases. For a detailed explanation and the basics of a philosophical justification of an agnostic theory that combines S4 and (MAXIMAL-OVERLAP$_{BL}$) and hence forsakes Axiom B, see Bobzien 2010.

[40] The borderline cases in Williamson's Logic of Clarity (Williamson 1994, Appendix; Williamson 1999) are intersective borderline cases, and the relevance of the Logic of Clarity to the Sorites paradox can be questioned. See Bobzien 2012, Section 8.3 for details.

Descriptions of hierarchical vagueness (as introduced in Section 1) rarely fit the model of higher-order vagueness as set out in Sections 3.1 and 3.2. Rather, they tend to include various assumptions and descriptive elements that instead match the classificatory use of "borderline" and situations of sorting objects into categories. In this section, we expand the sorting paradigm so as to include borderline borderline (etc.) cases, and provide a formal representation of this expansion (as borderline nestings). This will facilitate the identification of elements of the sorting paradigm that have found their way, perhaps subliminally, into theories of – presumed – higher-order vagueness. We emphasize that *in this section we do not intend in any way whatsoever to represent any theories of borderlineness from the literature on higher-order vagueness*.

4.1. Borderline borderline cases in the classificatory use of "borderline"

Back to the classificatory use of "borderline" in everyday situations of sorting objects. What happens if in such situations it is not just possible to identify borderline blue/green cases, but also, say, cases that are on the border between the green cases and the borderline blue/green cases? That is, what happens, if there are borderline (borderline blue/green)/green cases? No doubt, such classifications of borderline borderline cases can and do occur. Here is a possible scenario. At some point, sorters can identify cases that are between the blue and the green cases and add for them a box labeled "borderline blue//18/green" to their initial two boxes. After some further sorting, they learn to identify cases that are between the borderline blue/green and the green cases and add an additional box, labeled "borderline (borderline blue/green) / green" for those cases. The objects going into this box would reasonably be called borderline borderline cases.[41] It is conceivable that there also turn out to be identifiable cases between the green and the borderline (borderline blue/green)/green cases. These borderline (borderline (borderline blue/green)/green)/green cases would reasonably be called borderline borderline borderline cases. The sorters" interests, circumstances, discriminatory abilities and practice will determine how far this can go and what additional categories are introduced.

There are several distinctive features of this kind of sorting, i.e. sorting which includes the addition of new labeled boxes (or labeled categories) for cases identified as borderline, borderline borderline, etc.

- There are no empty boxes.
- The introduction of new boxes is stepwise. A box for borderline (borderline blue/green)/green cases can only be introduced *if* a box for borderline blue/green cases was previously introduced. Consequently, it is not possible to have a box for non-borderline cases and one for borderline borderline cases, but none for simple borderline cases.
- The labels on the boxes are mutually distinct.
- Box labels of existing boxes do not change with the introduction of new boxes. Consequently, at any stage of a sorting process that contains a box or boxes for borderline

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[41] Google "*borderline borderline, borderline* lepromatous", "*borderline-borderline/borderline*-tuberculoid (BB/BT) patients" for possible examples of borderline borderline cases.

cases, there will be boxes with base labels (like "green"), with simple borderline labels, and possibly with borderline borderline labels and so on.[42]

- No item is in more than one box – both regarding any one time and regarding the entire sorting process (i.e. no item changes boxes).[43] In particular, no item is classified both as borderline and as borderline borderline; and no item is classified both as non-borderline (e.g. green) and as borderline (e.g. borderline (borderline blue/green)/green).

If the objects-to-be-sorted are ordered, we additionally have the following concatenation feature when there is at least one box for borderline cases: /19/

- With the boxes lined up in accordance with the order-numbers of their objects, for every two labels of adjacent boxes, the label of one of the boxes forms part of the label of the other.

The semantics for objects a_1 to a_s that can be ordered with regard to some dimension D and have been sorted into labeled boxes, as in our blue/green example, is straightforward. It is the semantics of exclusive and exhaustive disjunction with n for the number of boxes labeled "Φ_1", …, "Φ_n", and with the predicates $\Phi_1, \ldots, \Phi_n$ bound by a universal quantifier. Using "$\oplus$" for exlusive and exhaustive disjunction, this is

4.1.1 $\forall x \, [\Phi_1 x \oplus \Phi_2 x \oplus \ldots \oplus \Phi_n x]$

which is the same as

4.1.2 $\forall x \, [\, [\Phi_1 x \vee \Phi_2 x \vee \ldots \vee \Phi_n x] \; \wedge \; \bigwedge_{1 \leq i < j \leq n} \neg[\Phi_i x \wedge \Phi_j x]\,].$[44]

The latter is logically equivalent to

4.1.3 $\forall x \, [\Phi_1 x \vee \Phi_2 x \vee \ldots \vee \Phi_n x] \quad \wedge \quad \forall x \bigwedge_{1 \leq i < j \leq n} \neg[\Phi_i x, \wedge \Phi_j x].$

The first conjunct of *4.1.3* expresses exhaustiveness, the second exclusiveness. If we include unsorted objects in our consideration, naturally, exhaustiveness for the box labels $\Phi_1 \ldots \Phi_n$ no longer holds.

4.2. Domain, syntax, elements and generation of borderline nestings

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[42] Sorters may of course resort to relabeling boxes, especially when labels become too long: "blue-green" or "teal" for "borderline blue/green" are possible examples. We disregard relabeling, since the results do away with borderline borderline, etc., cases, and are not relevant to our present endeavor.

[43] We acknowledge that in some real-life sorting processes the introduction of borderline cases and borderline segments may lead to a reassessment of cases already sorted. (Cf. also Wright 2010, 545.) However, our interest *here* is solely in the subset of sorting processes that are factive/veridical and with the context of sorting fixed. Does this include an element of idealization? Perhaps. In any case, our choice is based on the assumptions made in the works on higher-order vagueness we criticize in Section 6.

[44] Cf. e.g. Pelletier & Hartline 2008, p.78.

We suggest that everyday sorting situations like those laid out in Section 4.1 are best represented by what we call nestings of borderline cases, or borderline nestings. In this section we introduce domain, syntax, basic elements, some terminology and generation rules for borderline nestings.

The domain for borderline nestings is a set of objects U. The syntax is as follows:
- a, a_1, …, a_s are designators.
- F, G are atomic vague predicates.
- $BL(F/G)_D$ is a well-formed predicate.
- The brackets indicate the scope of a "BL" in the usual way. We call the bracket that indicates the scope of a "BL" the *scope bracket* of that "BL".
- If $BL(\Phi/\Psi)_D$ is a well-formed predicate, then $BL(BL(\Phi/\Psi)/\Psi)_D$ is a well-formed predicate.
- If $BL(\Phi/\Psi)_D$ is a well-formed predicate, then $BL(\Phi/BL(\Phi/\Psi))_D$ is a well-formed predicate. /20/
- For connectives and quantifiers, the syntax of classical predicate logic applies.[45]

For every predicate Φ, the segment S_Φ is the non-empty set of things that satisfy Φ (i.e. *2.2.5* or (NO-EMPTY-SEGMENT) holds). The letters a, a_1, …, a_s represent the things to be sorted. The segments represent the boxes. The predicates (devoid of any connectives or quantifiers) Φ_1, …, Φ_n represent the labels for the boxes.

We sort types of borderline cases and borderline segments by their *nesting ordinals*. A borderline nesting (or nesting of "BL") is the occurrence of "BL" within the scope brackets of a "BL".
- A *zeroth-nesting* borderline case is one which satisfies a BL formula that contains no borderline nesting.
- A *first-nesting* borderline case is one which satisfies a BL formula that contains a single nesting as greatest nesting of "BL".
- A *second-nesting* borderline case is one which satisfies a BL formula that contains a double nesting (i.e. a nesting of "BL" in the scope of a singly nested "BL") as greatest nesting of "BL".
- An nth-*nesting* borderline case is one which satisfies a BL formula that contains an n-tuple nesting (i.e. a nesting of "BL" in the scope of an $n-1$-tuple nested "BL") as greatest nesting of "BL".

Given the two base predicates F and G and the nesting syntax, there is one possibility for zeroth-nesting borderline cases; and there are 2^n possibilities for n^{th}-nesting borderline cases, with $n \geq 1$. With infinite n, we have *radical nestings*. The nesting terminology is applied to segments thus: A segment $S_{BL(\Phi/\Psi)}$ is an nth-nesting borderline segment iff its objects are n^{th}-nesting borderline cases. A nesting formula Φ is of *rank n* iff it contains an $n\text{-}1^{th}$ nesting of "BL" as greatest nesting.

Since borderline nestings are designed to represent the classificatory use of "borderline", we adopt from Section 2.2, as minimal constitutive elements of borderline nestings, the following principles introduced there to represent the classificatory use.

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[45] This does not make formulae with quantifiers or connectives within the BL scope brackets well-formed.

Elements based on the account of "BL":

2.2.1 $\qquad \exists x\, \mathrm{BL}(\Phi/\Psi)_D x\ \rightarrow\ \forall x\, [\Phi x \rightarrow \neg\Psi x],$ $\qquad\qquad$ (CONTRARIETY$_{class}$)

2.4.3 $\qquad [\mathrm{BL}(\Phi/\Psi)_D a \rightarrow \neg\Phi a] \wedge [\mathrm{BL}(\Phi/\Psi)_D a \rightarrow \neg\Psi a]$ $\qquad$ (THIRD-KIND)

Elements regarding series of ordered objects:

The domain is restricted to $U_{ordered}$, i.e. to a series of objects a_1 to a_s ordered on some dimension D with regard to a (vague) atomic predicate F, where the ordering is strict and can be finite or dense. The principles *2.2.2* (POLAR$_{class}$), /21/ *2.2.6* (CONTINUITY$_S$), and *2.2.8* (PRIMA-FACIE-EXHAUSTIVE) hold. Two segments S_Φ and S_Ψ are said to *share a border*, or to be *bordering segments*, regarding a dimension D and a domain $U_{ordered}$ iff on D for any $a_p \in S_\Phi$, $a_q \in S_\Psi$ there is no a_i with $p<i<q$ such that $a_i \in S_X$ and $X\neq\Psi\neq\Phi$.

Levels, generation rules, and admissible nesting combinations:

Once it is possible to identify $\mathrm{BL}(\Phi/\Psi)_D$ cases, we have on D a new segment $S_{\mathrm{BL}(\Phi/\Psi)}$. Such new segments correspond to newly added labeled boxes in a sorting process. We represent the process of adding new boxes by three rules that govern the generation of borderline segments.

We start with a pair of predicates F, G and a dimension D on which (a set $U_{ordered}$ of) objects of the relevant kind are ordered. To these correspond two non-borderline segments: the left-most segment S_F and the right-most segment S_G (by CONTRARIETY$_{class}$, POLAR$_{class}$ and CONTINUITY$_S$). Somewhere on D the two segments *share a border* (by PRIMA-FACIE-EXHAUSTIVE). This set-up is *level 0*, since there are zero borderline segments. The following rules govern the generation of borderline segments with nestings of any number at progressively higher levels.

Rule 1: Take any pair of segments at level n that share a border on D. The level $n+1$ borderline cases between these two segments are the objects that satisfy the predicate one obtains by substituting the subscript of the left-hand level-n segment for Φ and the subscript of the right-hand level-n segment for Ψ in $\mathrm{BL}(\Phi/\Psi)_D$. If there are such objects, the resulting predicate provides the subscript of a new borderline segment at level $n+1$. The borderline cases of this new segment are n^{th}-nesting borderline cases. (SEGMENT INTRODUCTION)

Rule 2: Take any level n segment. For each such segment there is a corresponding level $n+1$ segment identical with it. There is no change of nesting ordinals of the segments. (SEGMENT ASCENT)

Rule 3: Order the segments of level $n+1$ on dimension D in accordance with the location of their objects on D.[46] (SEGMENT ORDERING)

An *admissible nesting combination of predicates* regarding some dimension D and set $U_{ordered}$ is any possible combination of the predicates that are segment subscripts of the segments one obtains as a result of the exhaustive application of *Rules* 1 to 3 to the segments of some level n

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[46] There is an alternative way of ordering the segments on D, without reference to the location of the objects. Take the borderline segment(s) with the highest nesting number. Each has the form $S_{\mathrm{BL}(\Phi/\Psi)}$. For each such segment identify which of Φ and Ψ has the highest nesting number. If it is Φ, there will be a segment S_Φ. Put this segment to the left of $S_{\mathrm{BL}(\Phi/\Psi)}$. If it is Ψ, there will be a segment S_Ψ. Put this segment to the right of $S_{\mathrm{BL}(\Phi/\Psi)}$. Repeat with all segments that have at least one "open" or as yet "unconnected" side until you have them all lined up and concatenated in this way. Cf. also (CONCATENATION) below in 4.3.

with regard to D and U_{ordered}. Each admissible nesting combination corresponds to a possible situation of sorting. /22/

Idealization of borderline nestings:
Relative to U_{ordered}, at any level the border shared by two segments S_Φ, S_Ψ may be sharp or non-sharp. The border is *non-sharp*, iff there are p, i, q with $p<i<q$ such that $a_p \in S_\Phi$, $a_q \in S_\Psi$ and a_i belongs to (is in) neither S_Φ nor S_Ψ. It is *sharp* iff it is not non-sharp. At any level, if there is a sharp border, there are no further applications of *Rule* 1 possible. For any non-sharp borders, the possibility of further applications of *Rule* 1 depends on whether further cases are identifiable as located between the relevant two segments.[47] Once *Rule* 1 can no longer be applied at a level, higher levels are identical with that level with regard to their segments (by *Rule* 2).

If we disregard any limitations on available objects (say U_{ordered} is dense on D) and on identifiability (i.e. we allow for beings with the capacity for developing indefinitely finer powers of discrimination), repeated application of *Rules* 1 to 3 produces the following idealization of borderline nestings:

- At each level n, with $n\geq1$, we have n^{th}-nesting borderline segments alternating with less-than-n^{th}-nesting segments.
- There is an increase of segments at each progressively higher level, with the following sequence pattern of segments. The number of segments at each level n is twice that of the previous level minus one, or, what is the same, the number of segments at each level n is 2^n+1 (2, 3, 5, 9, 17, 33 …).
- The sequence pattern of nestings (with 1 for non-borderline segments, 2 for zeroth-nesting segments, 3 for first-nesting segments, etc.) on D, with dense U_{ordered}, is the series starting 11, 121, 13231, 143424341, 15453545254535451 … . (The generation of the series, with a_n for the nth number of the series and $a_1 = 11$, works thus: a_n is obtained by filling the space between any neighboring entries of a_{n-1} by n − where the two digits of a_1 and each inserted numeral count as an entry.)

Each level of the idealized nesting pattern provides an admissible nesting combination.

Definition and compositionality:
The introduction of a new segment $S_{BL(\Phi/\Psi)}$ has the effect that, strictly, S_Φ and S_Ψ no longer share a border. Consequently, the *definiens* of the tentative definition of $BL(\Phi/\Psi)_D$ cases in *2.2.7* needs to be relativized to a point before the introduction of those cases. The three rules take care of this. The adjusted definition for borderline nestings is

4.2 $BL(\Phi/\Psi)_D a$ $=_{df}$ at the level from which a"s segment is generated, a is located on D between the two bordering segments S_Φ and S_Ψ. /23/

Example: "$BL(BL(BL(F/G)/G)/G)_D a$ iff (at level 2) a is located on D between the two bordering segments $S_{BL(BL(F/G)/G)}$ and S_G." This can be unraveled in terms of the accounts for each of the borderline-segment generations roughly as follows: "(at level 2) a is located on D between [the

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[47] If we have density of objects on D, the limit on higher nestings is not set by the available number of objects, but by the ability to discriminate ever more finely. (Sainsbury's 1991 version of the higher-order vagueness paradox accounts for the possibility of a dense ordering; so does Shapiro's 2006 version (Shapiro 2006, 128).)

segment whose members are (at level 1) located on D between [[the segment whose members are (at level 0) located on D between S_F and S_G]] and S_G] and S_G."[48] Borderline nestings thus display a compositionality of sorts.

4.3. Borderline nestings as representation of sorting processes of ordered objects *at a level*

In Sections 4.3 and 4.4 we show that borderline nestings accurately represent borderline borderline cases, etc., as they may occur in everyday sorting processes of ordered objects into categories, as presented in Section 4.1.

First, it was required that there are no empty boxes. To this feature corresponds the (NO-EMPTY-SEGMENT) principle *2.2.5*. Second, the introduction of new boxes (categories) could only be done stepwise. A corresponding principle holds for borderline nestings:

4.3.1 n^{th}-nesting borderline cases can only be introduced stepwise, that is, after (i.e., a level up from the level at which) the bordering $n-1^{th}$-nesting borderline cases have been introduced. (STEPWISE GENERATION)

This follows directly from generation *Rule* 1. Third, it was a required feature for successful sorting that no two boxes (or categories) carry the same label (or name). A corresponding result holds for borderline nestings:

4.3.2 For each admissible combination, no two predicates are the same. (UNIQUENESS)

Or, in terms of segments, for each admissible nesting combination it holds that, no two segments have the same subscript. *4.3.2* follows from generation *Rules 1* and *2*.[49] Fourth, corresponding to the fact that existing box labels do not change when new labeled boxes are added, it holds that
/24/

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[48] The sorter's identification of a as between S_Φ and S_Ψ always works the same way, regardless of level and nesting numbers: a is identified as located on D between S_Φ and S_Ψ by whatever method the first borderline cases were identified as between S_F and S_G – e.g. by looking at those cases and comparing them with the sorted members of S_F and S_G (plus application of the continuity principles).

[49] Here is an informal proof:
(1) The two segments at level 0 have different names.
(2) *Rule 1*
(3) *Rule 2*
(4) From (2) it follows that if no two segments at level n have the same name, then no segments generated by *Rule* 1 for level $n+1$ have the same name.
(5) From (3) it follows that if no two segments at level n have the same name, then no two segments generated by *Rule* 2 for level $n+1$ have the same name.
(6) From (2) it follows that all segments generated by *Rule* 1 are n^{th}-nesting borderline segments.
(7) From (3) it follows that all segments generated by *Rule* 2 are either less than n^{th}-nesting borderline segments or non-borderline segments.
(8) From (1), (4), (5), (6), and (7) it follows that (with exhaustive application of *Rules 1* and *2*) no two segments at level $n+1$, with $n \geq 0$, have the same name.
(9) Since at each level, the admissible combinations are a subset of the segment names, it follows from (8) that in each admissible combination no two predicates are the same.

4.3.3   Segments that ascend from lower to higher levels do not change their subscript in the
process.   (NO-LABEL-CHANGE)

This follows directly from *Rule* 2. Fifth, the concatenation feature is also represented. For any
admissible combination of predicates at levels 1 and up, any two bordering segments S_Φ, S_Ψ are
concatenated in the following way:

4.3.4   For any two bordering segments S_Φ, S_Ψ, the predicate (= subscript) of one is in the
immediate scope of the largest-scope BL of the predicate of the other.

(CONCATENATION)

This follows from generation *Rules* 1 to 3. Moreover, it is characteristic of borderline nestings
that each admissible combination (except the one at level 0) contains predicates of unequal rank:

4.3.5   All (non-basic) admissible combinations of predicates include predicates of different rank
(i.e. with different nesting ordinals).   (UNEQUAL-RANKS-AT-A-LEVEL)

This follows from (STEPWISE GENERATION) and generation *Rule* 2.

4.4. The semantics of admissible nesting combinations

In this section we demonstrate that the semantics of borderline nestings corresponds to the
semantics for ordered objects a_1 to a_s that have been sorted, or are sortable, into labeled boxes, as
set out in Section 4.1. Technically, one can conceive of $BL(F/G)_D x$ as a three-place mixed-order
predicate that takes two first-order one-place predicates and one object as arguments; and of
$BL(F/BL(F/G))_D x$ as a four-place mixed-order predicate that takes three first-order one-place
predicates and one object as arguments; and so forth.[50] The predicates of the admissible nesting
combinations would thus be of ever-increasing complexity. What we show in this section is that,
despite such complexity of the *predicates* in admissible nesting combinations, the semantics for
admissible nesting combinations is nothing but the semantics of exclusive and exhaustive
disjunction (*4.1.1*).
/25/
In our representation of the sorting process in terms of borderline nestings, the counterpart to the
ordered objects sorted into labeled boxes at some particular sorting stage is a sub-domain of
$U_{ordered}$ associated with a particular admissible nesting combination. We call such a sub-domain
U_{sorted}. U_{sorted} can be described as the members of the union of segments of an admissible nesting
combination with regard to some *D*. There will be a one-to-one correspondence between box
labels of that sorting stage and the subscripts of the segments. Our goal is to show that *4.1.1*
(exclusive exhaustive disjunction) is true for any admissible nesting combination of *n* predicates
over U_{sorted} (i.e. for any *a* in U_{sorted}). We make use of the logical equivalence of *4.1.1* with *4.1.2*
and *4.1.3* and show separately for

4.4.1 $\forall x \ [\Phi_1 x \lor \Phi_2 x \lor \ldots \lor \Phi_n x]$ (EXHAUSTIVENESS)

and

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[50] This is not the only way of construing nesting predicates, but this is irrelevant for present purposes.

4.4.2 $\qquad \forall x \bigwedge_{1 \leq i < j \leq n} \neg[\Phi_i x, \wedge \ \Phi_j x]$ $\qquad\qquad$ (EXCLUSIVENESS)

that they hold for any particular admissible nesting combination of *n* predicates.

(EXHAUSTIVENESS) holds trivially: Since U_{sorted} is nothing but the members of the union of the segments of the nesting combination, for each admissible combination the domain under consideration (U_{sorted}) does not surpass the set of objects which satisfy *at least one* of the predicates of the combination.

We show (EXCLUSIVENESS) as follows. Of any admissible combination of predicates it holds that each of its pairs of predicates corresponds either to (i) S_F, S_G; or to (ii) two bordering segments S_Φ, S_Ψ of which at least one is a borderline segment; or to (iii) two non-bordering segments S_Φ, S_Ψ – all with regard to some *D*. (This follows from the generation of segments by means of *Rules* 1 to 3 from the base segments, together with the definitions of "bordering segment" and "non-bordering segment" from Section 4.2.) We show that $\forall x\ [\Phi x \rightarrow \neg\Psi x]$ holds for any pair of predicates Φ, Ψ of an admissible combination that corresponds to (i) or (ii) or (iii). Thereby we will have shown that (EXCLUSIVENESS) holds for any admissible combination of *n* predicates.

Take an arbitrary admissible nesting combination of predicates. It consists of two vague predicates F, G, together with any BL-predicates for which segments have been generated as part of the generation of borderline segments. Then

- Regarding (i), for the pair of predicates F, G, (CONTRARIETY$_{\text{class}}$) warrants $\forall x[\Phi x \rightarrow \neg\Psi x]$.
- Regarding (ii), for all pairs of predicates that correspond to two bordering segments of which one is a borderline segment (CONCATENATION) in tandem with (THIRD-KIND) warrant $\forall x[\Phi x \rightarrow \neg\Psi x]$
- Regarding (iii), i.e. for the case of the non-bordering segments we sketch an informal proof. Suppose $\exists x[\Phi x \wedge \Psi x]$. This requires S_Φ and S_Ψ to intersect. By definition of non-bordering segments, for any pair of non-bordering segments S_Φ and S_Ψ there exists a segment S_X between /26/ them on *D*. By (NO-EMPTY-SEGMENT) we have $\exists x X x$. Since the objects of U_{sorted} are ordered on *D*, and we have (CONTINUITY$_S$), there is an object in S_X with an order-number larger than those of the objects that satisfy Φ and smaller than those of the objects that satisfy Ψ (or vice versa).[51] Hence, regarding *D* and U_{sorted}, S_Φ and S_Ψ cannot intersect. Hence $\forall x[\Phi x \rightarrow \neg\Psi x]$ holds for any pair of predicates Φ, Ψ corresponding to two non-bordering segments.

Thus both *4.4.1* and *4.4.2* hold for any admissible nesting combination of *n* predicates. Hence so does *4.1.1*, i.e. exclusive and exhaustive disjunction.

This is the semantics for admissible nesting combinations; more precisely, for combinations of borderline nestings for the domain U_{sorted} at a level *n*, with regard to *D* and the predicates $\Phi_1 \ldots \Phi_m$ that correspond to the *m* segments on *n*, as generated by *Rules* 1 to 3 from the atomic vague predicates F and G at level 0. *At* any level, borderline nestings concern sorted objects only. This is why the semantics can be handled in classical first-order logic. No surprise here. The sorting

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[51] This holds also if S_Φ, S_Ψ and S_X each correspond to a dense ordering of objects: there still will be a first and a last object in each segment.

of (ordered) things into boxes, and the representation of the sorted things at any stage of the sorting process is in no way paradoxical, even if the labels on the boxes are vague predicates.

4.5. The relations of borderline nestings across levels

For statements regarding the relation between borderline cases *from different levels*, unsorted cases need to be taken into account. This is so, because the ascent to higher levels represents the sorting of *formerly unsorted* items into added boxes. Thus the domain needs to be enlarged from U_{sorted} back to U_{ordered}. As a result, (EXHAUSTIVENESS) is lost, while (EXCLUSIVENESS) still holds. (EXCLUSIVENESS) together with the rules that govern nestings suffices to show that borderline nestings accurately represent the characteristic feature of veridical sorting processes that *during a sorting process* an item is only ever in one and the same box. This can be done by showing the corresponding feature of borderline nestings that even across levels no object can ever be in two distinct segments. We here prove the weaker claim that even across levels no n^{th}-nesting borderline case can ever be a non-borderline case or an mth-nesting borderline case (with $m \neq n$ and $m, n \geq 0$), since this is all we need for Section 6. We proceed in two parts. We start with

4.5.1   An object a that is an nth-nesting borderline case at some level l cannot be an mth-nesting borderline case (with $m \neq n$) at any level.   (NO-BORDERLINE-OVERLAP)

The proof is as follows. First, a direct consequence of *4.4.2* (EXCLUSIVENESS) is

4.5.2   *At* any level, no n^{th}-nesting borderline case can also be an mth-nesting borderline case, with $n \neq m$.   (NO-BORDERLINE-OVERLAP-AT-A-LEVEL)

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Second, it follows immediately from *Rule* 2 (SEMANTIC ASCENT) that, given a is an nth-nesting borderline case at level l, a is an nth-nesting borderline case at all levels $\geq l$, and subsequently from *4.5.2* that a is not an mth-nesting borderline case at any level $\geq l$. Third, suppose a was an mth-nesting borderline case at some level $< l$. Then, by *Rule* 2, a would be an mth-nesting borderline case at all levels higher than that level, including level l. But by assumption, a is an nth-nesting borderline case at level l, and by *4.5.2* not an mth-nesting borderline case at level l. Hence the supposition is false. Consequently *4.5.1* holds. *Mutatis mutandis* the same proof can be given for

4.5.3   An object a that is an nth-nesting borderline case at some level l cannot be a non-borderline case at any level.[52]   (NO-NON-BORDERLINE-OVERLAP)

4.5.1 and *4.5.3* are the counterpart to the axioms that govern modal iteration in theories of higher-order vagueness. The proof that across levels no object can ever be in two distinct segments works along the same lines of that for *4.5.1* and *4.5.3*, with individual segments rather

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[52] There are two possibilities for the levels $< l$ here: (i) that a is sorted at *some* level as F (or G); (ii) that a is unsorted at *any* level, or unsortable. The sub-proof for (i) runs exactly parallel to the proof in the main text, with "F (or G)" instead of "an m-th nesting borderline case". The sub-proof for (ii) is: If a is unsorted at any level, or unsortable, then it is unsorted at level l. Yet, by assumption it is sorted at l. For the possibility of integrating unsortable objects into the paradigm of borderline nestings by combining it with the epistemic use of "borderline" see below, section 5.4.

than n^{th}-nestings. Thus all the features of the sorting paradigm based on the classificatory use of "borderline" are adequately represented by the model of borderline nestings.

4.6 Sorter versus observer perspective in classificatory use and borderline nestings

Situations of sorting like the ones characteristic for the classificatory use of "borderline" and for borderline nestings can be considered and described from two points of view. One is the perspective of the sorter. The other is the perspective of someone who observes the sorting, i.e. an observer. The sorter normally describes the sorted objects in direct correspondence with the labels on the boxes: a_1 is blue, a_{40} is borderline blue/green, a_{100} is green. The observer would standardly describe the sorted objects qua sorted into the labeled boxes (qua sorted into some category or other): a_1 is *sorted/categorized/classified as* blue, a_{40} as borderline blue/green, a_{100} as green.

In everyday life, the sorter perspective is the norm. Ordinarily, people asked what is in the box labeled "F","BL(F/G)", etc., will respond "the blue things", not "the things sorted/categorized/classified as blue", etc. (Instructions to sorters will be of the kind "put the Φ things in the box labeled "Φ"", so for the sorters themselves it is natural to describe the things in the box labeled "Φ" as Φ rather than as the items sorted as Φ.) So far, in Sections 2 and 4, we have described classificatory use and borderline nestings entirely from the sorter /28/ perspective. However, the observer perspective is a potential source for the conflation of higher-order vagueness with borderline nestings, and for this reason we here say something more about it. In particular, someone may conceive of "sorted (as)" as akin in function to the modal expressions "clear" or "determinate" as they are used in the vagueness literature, with "unsorted" paralleling unclarity or indeterminacy. Such an assumption of similarity in function is however misguided. To highlight the dissimilarity, we introduce a "sorted (as)" operator for the representation of the classificatory use from the observer perspective. (This is the only purpose of the operator.)

We express the sorting situation from the observer perspective using an operator ("$_{SORTED}$") that expresses that an object a has been categorized as (or put by the sorter into the box labeled) Φ, or in any case would be categorized (or boxed) thus if it came up for sorting.[53] In analogy with formulations like "a is definitely/clearly F" and "It is definite/clear that Fa", we could use "a is sortedly F" or "It is sorted that Fa" for "a is sorted as F".[54] Formally, where Fa, BL(F/G)$_D a$, etc. express the sorter perspective, $_{SORTED}$Fa, $_{SORTED}$BL(F/G)$_D a$, etc., express the observer perspective. Note that from the observer perspective, borderline cases are to be described as $_{SORTED}$BL(Φ/Ψ)$_D a$, not as $_{SORTED}$BL($_{SORTED}\Phi$/$_{SORTED}\Psi$)$_D a$. The observer observes them as being sorted by the sorter as BL(Φ/Ψ)$_D a$, not as being sorted as BL($_{SORTED}\Phi$/$_{SORTED}\Psi$)$_D a$.

In Section 2.2, we presented the principles that represent the classificatory use of "borderline" from the sorter perspective. We here rephrase the principles to make them represent the observer perspective by prefixing "$_{SORTED}$" to the relevant sub-formulae. We add "O" to the names of the principles as indicator of the observer perspective :

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[53] This added disjunct is intended to provide for *all and only* the cases covered by the continuity principles and by (POLAR$_{class}$), which each represent commonly accepted idealized features of the classificatory use that are independent of whether an item has actually been sorted.

[54] The awkwardness of such formulations is but an indicator of how forced such an analogy is.

4.6.1 $\exists x \, _{\text{SORTED}}BL(\Phi/\Psi)x \to \forall x \, [\Phi x \to \neg \Psi x],$ (O-CONTRARIETY$_{\text{class}}$)

4.6.2 $_{\text{SORTED}}Fa_1 \wedge _{\text{SORTED}}Ga_s$ (O-POLAR$_{\text{class}}$)

4.6.3 $[_{\text{SORTED}}Fa_n \to _{\text{SORTED}}Fa_{n-i}] \wedge [_{\text{SORTED}}Ga_n \to _{\text{SORTED}}Ga_{n+i}].$ (O-CONTINUITY$_{F,G}$)

4.6.4   If $_{\text{SORTED}}BL(\Phi/\Psi)_Da_n$ and $_{\text{SORTED}}BL(\Phi/\Psi)_Da_m$, then any a_i, with $n \leq i \leq m$, will also be
$_{\text{SORTED}}BL(\Phi/\Psi)_D.$ (O-CONTINUITY$_{BL(\Phi/\Psi)}$)

4.6.5   For each segment S_Φ there is an a such that $_{\text{SORTED}}\Phi a$.[55]

(O-NO-EMPTY-SEGMENT)

4.6.6 $[_{\text{SORTED}}BL(\Phi/\Psi)_Da \to \neg \Phi a] \wedge [_{\text{SORTED}}BL(\Phi/\Psi)_Da \to \neg \Psi a]$[56]   (O-THIRD-KIND)

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The O-counterpart to the definition of "borderline" is

4.6.7 $_{\text{SORTED}}BL(\Phi/\Psi)_Da$ $=_{df}$ a is such that, at the level from which a"s segment is
generated, a has been sorted as located on D between two bordering segments
S_Φ and S_Ψ (or in any case would be thus sorted if it came up for sorting).

The modal principle *2.4.2* (i.e. "BL$(\Phi/\Psi)_Da$ only if it is possible to identify that $[\neg \Phi a \wedge \neg \Psi a]$") is not impacted by the perspective chosen. It is not an explicit part of the definition from either perspective. Rather, it is as a result of having been identified and sorted as BL(Φ/Ψ) that a is in the box or category labeled BL(Φ/Ψ); moreover, that there even is a box or category labeled BL(Φ/Ψ). From the fact that a has been identified and sorted as BL(Φ/Ψ) (or in any case would be thus sorted if it came up for sorting) we, as well as the sorter and any observers, can infer that a is identifiable and classifiable as BL(Φ/Ψ). The truth of *2.4.2* is based on the fact that veridical classifying of something as Φ presupposes identifying that thing as Φ. Perspective aside, Φ and $_{\text{SORTED}}\Phi$ do not differ in any significant way. They have the same extension. Accordingly, each principle, as well as the definition, taken from the observer perspective is materially equivalent to its counterpart of the sorter perspective.[57]

For borderline nestings, too, the switch to the observer perspective is straightforward. We adjust the syntactic rules by adding the rule "$_{\text{SORTED}}F$, $_{\text{SORTED}}G$ are well formed predicates" and prefixing "$_{\text{SORTED}}$" to the "BL" with the largest scope in each BL formula. The generation of nestings starts with a pair of predicates $_{\text{SORTED}}F$, $_{\text{SORTED}}G$, instead of the pair F, G; the principles used are as given above; finally, Generation *Rule 1* is rewritten thus (with changes in italics)

Rule 1' : Take any pair of segments at level n that share a border on D. The level $n+1$ borderline cases between these two segments are the objects that satisfy the predicate one obtains by substituting the subscript of the left-hand level-n segment for Φ and the subscript of the

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[55] Segment indices do not have the $_{\text{SORTED}}$ operator, since they correspond to the labels on the boxes, and those labels give the sorter perspective. So *2.2.6* remains unaltered. (Alternatively, one could have such an operator and think of $S_{\text{SORTED}\Phi}$ as consisting of "S_{SORTED}" as "the segment, from the observer perspective" and "$_\Phi$" as "the label from the observer perspective" (which is the same as the label from the sorter perspective.)

[56] Should it not be $[_{\text{SORTED}}BL(\Phi/\Psi)_Da \to \neg_{\text{SORTED}}\Phi a] \wedge [_{\text{SORTED}}BL(\Phi/\Psi)_Da \to \neg_{\text{SORTED}}\Psi a]$? No. In order to be sorted as BL(Φ/Ψ), a needs to be identified as $\neg\Phi$ and as $\neg\Psi$, not as $\neg_{\text{SORTED}}\Phi$ and $\neg_{\text{SORTED}}\Psi$.

[57] We disregard questions regarding whether the existence of an observer is required for the truth of the observer sentences, since this issue is irrelevant here. In any event, one could argue that the sorter may double as observer.

right-hand level-n segment for Ψ in $_{SORTED}BL(\Phi/\Psi)_D$. If there are such objects, the resulting predicate *minus its prefix "$_{SORTED}$"* provides the subscript of a new borderline segment at level $n+1$. The borderline cases of this new segment are n^{th}-nesting borderline cases. (O-SEGMENT INTRODUCTION)

Just as the observer version of zeroth-nesting borderline cases is $_{SORTED}BL(F/G)$ and not $_{SORTED}BL(_{SORTED}F/_{SORTED}G)$, so the observer version of $BL(BL(F/G)/G)$, say, is $_{SORTED}BL(BL(F/G)/G)$, and not $_{SORTED}BL(_{SORTED}BL(_{SORTED}F/_{SORTED}G)/_{SORTED}G)$. The sorter and observer do not interact. The observer observes the sorter sort objects into boxes labeled without a sorting operator and introduce new boxes labeled without sorting operators. /30/

There is also no substantial difference in the ranks of the formulae at a level. *If* one were to think of the iterated "BL" and the "$_{SORTED}$" as each counting equally towards the rank of a formula, one could say this: The sequence pattern of nestings displayed by the idealization of borderline nestings (11, 121, 13231, …, see Section 4.3) describes the nesting ranks at a level from the observer perspective (with 11 for level 0). From the sorter perspective, in the idealized case the nesting ranks at a level are those of the observer perspective minus one (00, 010, 02120, 032313230, …).[58] Thus *4.3.5* (UNEQUAL-RANKS-AT-A-LEVEL) would hold regardless of whether one takes the sorter or the observer perspective.[59] Overall a switch to the observer perspective does not change anything of substance.

5. Merging classificatory and epistemic use of "borderline"

In this section we briefly present five ways in which the classificatory and epistemic uses of "borderline" may be, and quite possible have been, merged.[60] We do so, because conflations of the two uses often go hand-in-hand with the confusion of higher-order vagueness and borderline nestings.

5.1. *First way of merging the two uses:* The most basic amalgamation is the replacement of the contrary-but-not-contradictory vague base predicates F and G in the classificatory use of "borderline" with a vague base predicate F and its negation ¬F, thus incorporating an element from the epistemic use. The result is what in Section 2.3 we called the contradictorious classificatory use. Here are two formulations that each bring out the elements of both types of use:

5.1.1   *a* is borderline F/¬F iff *a* is identifiable (at the base level) as located on *D* between the things that are F and the things of which it is not the case that they are F.

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[58] It is of course arbitrary that the sequence pattern of nesting coincides with the ranks of the *observer perspective*. If we had chosen to start with zero rather than one for non-borderline segments, the sequence patterns would coincide with the ranks of the sorter perspective. (Our choice is motivated by non-philosophical reasons which we will not to go into here.)

[59] This would be true even if someone insisted that the observer perspective is one in which the observers add their knowledge that the cases at issue are all sorted cases into the description, so that we have $_{SORTED}BL(_{SORTED}F/_{SORTED}G)$, $_{SORTED}BL(_{SORTED}BL(_{SORTED}F/_{SORTED}G)/_{SORTED}G)$, etc. In this case, the nesting rank pattern would be obtained by adding $n+1$ to each rank number n of the sorter perspective.

[60] We provide an in-depth discussion of these possibilities of combining the two uses elsewhere.

5.1.2　　　　*a* satisfies "is borderline F/¬F" iff *a* is identifiable (at the base level) as located on D between the a_1-a_q that satisfy "F" and the a_r-a_s that do not satisfy "F".　/31/

Those who suggest that something along these lines is the kind of borderlineness relevant to the Sorites paradox face the tasks of explaining what its relation is to the paradox, and how they would deal with the higher-order vagueness paradoxes. (The proposal of some non-classical logic is no substitute for answering either question.)

5.2. *Second way of merging the two uses:* Another simple way to mix the classificatory with epistemic use is the characterization of borderline and non-borderline cases in terms of the following categorization scheme: There is a category in which the F things belong, a category in which the G things belong, and a category for the cases where one can"t tell whether they are F or G. Or, combined with the amalgamation described in 5.1: There is a category for the things that are F, a category for the things that are not F, and a category for those cases where one can"t tell whether they are F or not F.

5.3. *Third way of merging the two uses:* Still another way of merging the two uses is in answer to this question: Is it possible to have objects *a* that are both in the epistemic and in the classificatory use borderline cases with respect to the same predicate F (or pair of predicates F, ¬F)? If we take an epistemic verb of identifiability for epistemic use and a sorting verb for classificatory use and then form the conjunction of the two descriptions of borderline uses (from 2.4), we get something of the kind: *a* is borderline F/¬F iff

5.3.1　　　　[¬◊ to tell whether or not F*a*] ∧ [◊ to classify *a* as not being F ∧ as not being ¬F].
5.3.2　　　　[¬◊ to tell whether or not F*a*] ∧ [◊ to classify both F*a* and ¬F*a* as not true].

This kind of fusion combines in the rendering of BLF (or BL(F/¬F) (i) a *description* of the epistemic inaccessibility that is a defining element of the phenomenon of the borderline zone in a Sorites series with (ii) a philosophical (metaphysical, semantic, pragmatic) *explanation* of this very phenomenon, where the explanation is of the kind Crispin Wright has dubbed "Third Possibility".[61]　If such a conjunction is intended as an account of borderlineness, it is doubtful that compositionality can be preserved for second-and-higher-order borderline cases. Hence there would be no genuine higher-order vagueness. In any event, it is difficult to see how set-ups like this one can avoid the objection that they entail sharp borders between the non-borderline cases of F and the borderline cases of F.　/32/

5.4. *Fourth way of merging the two uses:* A subtler way of combining the classificatory and epistemic uses of "borderline" is in response to the question: Is it possible to have F*a* appear in the same coherent formula both in the scope of BL epistemically used, and in the scope of BL classificatorily used? This seems possible:

5.4.1　　　　[[¬◊**I**F*a*]∧[¬◊**I**¬F*a*]] ∧ ◊**I**[[¬◊**I**F*a*]∧[¬◊**I**¬F*a*]]

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[61] E.g. Wright 2003, 2010.

(with $\Diamond\mathbf{I}$ for "it is possible to identify that", and "identify" as a term used indiscriminately for epistemic and classificatory purposes). The first conjunct of the largest-scope conjunction applies the (partial) account of BL in the epistemic use to Fa. The second conjunct applies the (partial) account of BL in the classificatory use to $\Diamond\mathbf{I}$Fa and $\Diamond\mathbf{I}\neg$Fa, which are, as required, contraries. In effect, the result is one way of expressing *genuine clear borderlineness* of Fa:

5.4.2 $[[\neg CFa]\wedge[\neg C\neg Fa]] \wedge C[[\neg CFa]\wedge[\neg C\neg Fa]]$.

There is nothing incoherent *per se* with such genuinely clear borderline cases.[62] However, coherent as they may be, they raise various problems, if they are employed as part of a Sorites solution. Most markedly, with the introduction of clear borderline cases the whole caboodle of higher-order vagueness paradoxes returns through the back door.[63]

5.5. *Fifth way of merging the two uses:* A fifth way of linking classificatory and epistemic use is by applying the (epistemically interpreted) modalities of higher-orders on the results of the generation of nestings with the BL(/)$_D$ predicates, counting all sorted cases as clear cases, all unsorted ones as borderline.[64] If the epistemic use of BL is supplemented with a logic of columnar radical higher-order vagueness (as set out in 3.3), one obtains, with regard to *D*, a multi-column result that avoids sharp boundaries between all adjacent segments with non-sharp borders.

These five possibilities of merging the epistemic and classificatory uses of "borderline" illustrate how easily elements of both can be part of one and the /33/ same view or theory of vagueness. Not surprisingly, often some amalgamation of the two is combined with the conflation of higher-order vagueness and borderline nestings.

6. Evidence for the conflation of higher-order vagueness with the classificatory use and borderline nestings
This section provides evidence for our thesis that the conflation of higher-order vagueness with borderline nestings is common and is what led philosophers to the erroneous conclusions that higher-order vagueness is incoherent, paradoxical, illusory or non-existent and deserves to be demoted. We consider arguments by Sainsbury, Shapiro, Williamson, Raffman and Wright.

We use the presence of features characteristic for the nesting paradigm as evidence for a likely confusion of nestings with orders and as indicator that the authors may have been, perhaps

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[62] This combination of classificatory and epistemic use calls for an S4 axiom for the clarity or definiteness operator. Otherwise a non-circular, intelligible account of clear borderline cases and higher-order borderline cases is hard to provide. See Shapiro 2006, ch.5, for some of the problems one encounters; Bobzien 2011, Sections 6-8, for how they can be bypassed.

[63] Moreover, genuine clear borderline cases are not compatible with a sharp border between the *a* that are F and those that are ¬F. Epistemicists of the Williamson variety thus have to do without them. And they usually do (though they may not shout it out): see Williamson 1994, Appendix; Williamson 1999. The same holds for contextualist epistemicists, see e.g. Fara 2003.

[64] Soames introduction of superindefinedness in addition to indeterminacy (Soames 2003, 147-9) displays some elements of this kind of combination, if in our view, unsuccessfully executed.

unwittingly, in the grip of the nesting paradigm when presenting their considerations against higher-order vagueness. Here is a list of these features:

- With *nestings,* if something is borderline F/G, it is neither F nor G (THIRD KIND). With *orders*, something can be both borderline F and F.
- With *nestings*, it is assumed that each predicate that expresses borderlineness relative to some domain $U_{ordered}$, dimension D and level n, is satisfied by some objects of $U_{ordered}$ (NO-EMPTY-SEGMENT). With *orders*, this is not so.
- With *nestings*, it is impossible for there to be an nth-nesting borderline case without there being an $n-1^{th}$-nesting borderline case (STEPWISE GENERATION). With *orders*, this is not so.
- With *nestings*, there can be no overlap of borderline and non-borderline segments across levels (NO-NON-BORDERLINE-OVERLAP). With *orders*, there can be overlap of borderline and non-borderline regions across levels. For example, something can be clearly F and borderline clearly F.
- With *nestings*, there can be no overlap of borderline segments of different ranks across levels (NO-BORDERLINE-OVERLAP). With *orders*, there can be overlap of borderline regions of different ranks across levels. Something can be borderline F and borderline borderline F.
- With *nestings*, all segments (at and across levels) for non-borderline cases are of the same rank, i.e. rank zero (SEGMENT ASCENT). With *orders*, there can be non-borderline cases with different rank numbers, with no upper limit to that number: something can be clear, or clearly clear, etc.
- With *nestings*, at any level ≥ 1, the predicate rank numbers are unequal. In particular, at level 1, the rank numbers are $0-1-0$ (UNEQUAL-RANKS-AT-A-LEVEL). With *orders*, at any level, the predicate rank numbers are equal. In particular, at level 1, the rank numbers are $1-1-1$.

Of course, the presence of these features is no uncontestable proof that an author confounds higher orders and nestings. Nor does this matter. What matters is that by removing the elements of the authors" description of /34/ higher-order vagueness that are characteristic of borderline nestings, their arguments against higher-order vagueness collapse.[65]

6.1. Sainsbury 1990 and 1991
We start with the *locus classicus* for hierarchical vagueness, Sainsbury 1991, 167-170, with a parallel in Sainsbury 1990, 252-55. Sainsbury sets out to show that "vague predicates are incoherent, if the classical conception <that the extensions of vague predicates are sets> is correct" (1991, 167). In order to do so, he (i) adds hierarchical vagueness (which he refers to as higher-order vagueness) to the classical conception and (ii) attempts to prove that a paradox results.

(i) *Sainsbury's description of hierarchical vagueness*: (Remember, the position Sainsbury sketches is not one he condones.) The main telltale sign of conflation is that Sainsbury oscillates between equal and unequal rank numbers. He repeatedly describes the first level as that with

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[65] For reasons of simplicity, we mostly keep to the vagueness-related terminologies of the authors we discuss.

positive cases, negative cases and borderline cases (1991, 167, 168). Ordered on *D*, this gives the modal ranks $0 – 1 – 0$, a pattern of unequal ranking. This pattern is characteristic of zeroth nestings (THIRD-KIND), not first orders. Sainsbury introduces the second level, saying "We find a new type of borderline case: for example, those things which seem intermediate between being definite cases of children, and being borderline cases of children" (1991, 168, bottom). This suggests DF – <BLBLF> – BLF, with a pattern of modal ranks $1 – <2> – 1$ for the left-hand side of *D*. The move from F, BLF to DF, BLF suggests a move from nestings to orders.[66] Such a move is not trivial, but philosophically significant. Sainsbury's introduction of borderline cases as a third set, besides the F and the non-F cases, was incompatible with bivalence-plus-transparent-truth. The prefixing of "definite" in the discussion of second-order vagueness makes bivalence-plus-transparent-truth possible again: The sequence DF – BLF – Dnon-F (a pattern of equal ranking) is compatible with just two base sets and with a sharp boundary between the F and the non-F cases. At the same time the prefixing of "definite" removes the necessity of three non-intersecting base sets. Sainsbury's next sentence corroborates our assertion that he is wavering between nestings and orders, where he writes: "the last heartbeat of my childhood, or at any rate the last heartbeat of my definite childhood" (1991, 168). Sainsbury fleshes out second-order vagueness thus: it "consists in a five-fold division: into the definite positive cases for the predicate, the definite borderlines and the definite negative cases, together with the cases which are borderline between definite positive cases and definite borderlines, and those borderline between being definite borderlines and definite negative cases." (1991, 168-169) This suggests DF, BL(DF/DBLF), DBLF, BL(DBL/Dnon-F), Dnon-F. The modal ranks are again uneven. We seem to have either $1 – 2 – 2 – 2 – 1$ or $1 – 3 – 2 – 3 – 1$; the latter is the /35/ sequence pattern of nestings (UNEQUAL-RANKS-AT-A-LEVEL). For second-*order* vagueness, we expect $2 – 2 – 2 – 2 – 2$. Contrasting with the uneven modal ranks, later on, Sainsbury assumes that the leftmost and the rightmost segments are $D^n F$ and $D^n non-F$ for all *n* (1991, 169). (We here read his "unimpunibly definite" as "definite of order *n* for all *n*", as the context suggests.) This later assumption tallies with the rank numbers of higher-order vagueness but not of nestings.

There are several further indicators that Sainsbury's hierarchical vagueness is influenced by the nesting paradigm.[67] He describes the number of segments resulting from the boundaries which an nth-order vague predicates draws by means of a generation process of borderline borderline, etc., cases which parallels the generation *Rule 1* for nestings (SEGMENT INTRODUCTION). Sainsbury's depictions of the second-order borderline cases read like an application of (SEGMENT INTRODUCTION). Moreover, Sainsbury's formulations indicate that he assumes throughout that for each vague predicate the 2^n+1 sets do not intersect (NO-BORDERLINE-OVERLAP). Additionally, Sainsbury's formulations imply that the non-overlapping sets are all non-empty. First, it is a division of objects into sets that brings the sets about (NO-EMPTY-SET). Second, for a predicate to be presumed n^{th}-order vague, it needs to effect the division of object into 2^n+1 sets. Thus when no such division of objects into sets is possible, there are no further presumed higher orders for that predicate. This corresponds to (STEPWISE).[68] The *locus classicus* for hierarchical vagueness

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[66] The "definite" is not a renaming of "positive" from pp. 167-8 as "definite" – see p.169 "definite positive".

[67] We invite the reader to consult the quote from Sainsbury (1991) in Section 1 above for confirmation of our claims.

[68] Further indicators that the nesting paradigm impacts Sainsbury's account of hierarchical vagueness is his description of what he calls "an adequate representation of" first-order vagueness in terms of contrary rather than contradictory expressions: "surfaces intermediate between blue and green, people intermediate between childhood

is thus also the *locus classicus* for the conflation of higher-order vagueness with borderline nestings.

(ii) *Sainsbury's argument for the incoherence of higher-order vagueness:* Sainsbury concludes that higher-order vagueness, including radical higher-order vagueness, is incompatible with the extension of predicates conceived as sets (1991, 170). He argues that higher-order vagueness of any order leads to a division of the categorially appropriate objects into exactly, and hence at least, three non-overlapping non-empty classes – the outer ones, of which the vague predicate is true, or false, "beyond the shadow of vagueness"; and the middle one containing "the things touched by vagueness" (1991, 170). This argument presupposes that there must be *at least* three non-overlapping non-empty classes. However, it is only the nesting paradigm that requires this. Genuine higher-order vagueness is *compatible with* there being only two non-overlapping non-empty classes of objects at the base level, and with a sharp boundary between them;[69] the only caveat is that it is epistemically inaccessible /36/ whether there are two or three such classes. Thus Sainsbury's argument is successful in showing that the addition of non-overlapping non-empty sets for borderline borderline cases, etc., does nothing to remove sharp boundaries. However, the argument does not show that higher-order vagueness is incoherent or incompatible with the conception of the extensions of vague predicates as sets.

6.2. Shapiro 2005 and 2006
Stewart Shapiro presents a different version of a higher-order vagueness paradox in his 2005 (at 147-51) and 2006 (at 125-31). Evidence of the conflation of the epistemic use of "borderline" and higher-order vagueness with the nesting paradigm can be found here, too. Shapiro's basic assumption of *open texture* provides an explanation of the epistemic inaccessibility of the borderline zone by introducing a theory that requires a non-overlapping, non-empty area on *D* that contains the borderline cases (THIRD-KIND).

Like Sainsbury, Shapiro vacillates regarding the modal rank numbers, between those corresponding to nestings (UNEQUAL-RANKS-AT-A-LEVEL) and those corresponding to orders. He starts with the tripartite distinction "bald", "borderline bald", "not bald" (2005, 147);[70] in the next paragraph he uses "(determinately) bald", "borderline bald", "(determinately) non-bald"; at the end of the paragraph, the "determinately" has shed its brackets ("a second-order borderline case … determinately non-bald"). Here Shapiro moves from an uneven rank pattern 0 – 1 – 0 via an intermediate one with brackets (1) – 1 – (1) to the even rank pattern 1 – 1 – 1. This mirrors a move from the classificatory use of "borderline" via some dithering to the modal pattern for orders of vagueness. Shortly after, the brackets are back: "A man is determinately-determinately bald if he is (determinately) bald and (determinately) not borderline-borderline bald … a man is determinately-determinately not bald if he is not bald and (determinately) not borderline-

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[69] and adulthood" (1991, 168); and that he never mentions any requirement of compositionality for the expressions "borderline" or "definite".

[69] Above we saw that by definitizing the predicates describing the left and right sets at level 1 (1991, 168) Sainsbury himself removed the requirement of three non-overlapping non-empty sets.

[70] See also 2005, 150, citing Raffman, "there is tolerance between "baldness" and any other category – even a "borderline" category"; later in the same paragraph Shapiro contrasts borderline cases with determinate cases (Shapiro 2006, 130).

borderline bald" (2005, 148). For genuine higher-order vagueness, rank-wise, we would expect the accounts

6.2.1   DDFa = DFa∧¬BLBLFa and   DD¬Fa =   D¬F∧¬BLBLFa.[71]

Instead we get the equivalent of

6.2.2   DDFa = (D)Fa∧(D)¬BLBLFa and   DD¬Fa =   ¬Fa∧(D)¬BLBLFa

The lack of "determinately" before ¬F (and the bracketed "determinately" before F) suggest influence of the nesting paradigm. What the bracketed /37/ "determinately" before "¬BLBL" is for, we cannot tell.[72] Bracketed "determinately" occur also at (2005, 149) and (2006, 126).

Shapiro introduces borderline cases thus: "Intuitively, there is no (sharp) border between the men that are bald at the beginning and those that are not bald at the end. The fellows in the middle are the borderline cases." (2005, 147). "It also seems that there is no sharp boundary between the (determinately) bald men at the start and the borderline bald men in the middle, nor … between the borderline bald men in the middle and the (determinately) non-bald men at the end." (2005, 148). These quotes are indicators that Shapiro assumes (THIRD-KIND) and (NO-EMPTY-SEGMENT). Both are features of the nesting paradigm. Shapiro later provides an account of "borderline" in epistemic terms (e.g. 2005, 157). In fact, Shapiro is one of the very few philosophers who explicitly discuss the requirement of compositionality of "borderline" in its epistemic use. No doubt then, that in Shapiro's description of borderline cases features of the classificatory and epistemic uses are amalgamated.

The way Shapiro introduces his notions of higher-order determinacy and borderlineness, *n*-determinately and *n*-borderline, echoes the generation *Rule 1* of nestings (SEGMENT INTRODUCTION). Compare e.g. "say that a man is borderline-borderline-borderline bald if he is either a borderline case of "determinately-determinately bald" with "borderline-borderline bald" or if he is a borderline case of borderline-borderline bald with "determinately-determinately non-bald"" (2006, 126) and "We will not inquire about the borderline between the men that are determinately borderline bald and those that are (merely) borderline-borderline bald" (2006, 126).

In the context of presenting his version of a higher-order vagueness paradox, Shapiro writes: "… we will eventually run out of distinctions to make, provided that the categories do not overlap and there is at least one man in each …" (2005, 149). "… we will eventually run out of distinctions to make, provided we want at least on man in each category. A finite series can support only finitely many distinctions" (2006, 126). These quotes show first that Shapiro assumes (NO-BORDERLINE-OVERLAP), an assumption he never questions, despite the fact that his definitions of "borderline" and "determinately" allow for overlap. Second, Shapiro never questions his proviso that there are no empty segments (NO-EMPTY-SEGMENT). His formulations, e.g. the use of definite articles, in the description of categories such as "the fellows in the middle"

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[71] Of course, for genuine higher-order vagueness we would not expect any account of DDF in terms of BL at all. That we have one, is another indicator of a possible impact of the nesting paradigm.

[72] In Shapiro 2006, at 126, the first conjuncts have the desired form, with an unbracketed "determinately", but bracketed "determinately" are still prefixed to the second conjuncts.

(2005, 147) "the borderline men in the middle", "those that are borderline-borderline bald" (2005, 148) further confirm this. Again, Shapiro's definition of "borderline" does not require that the categories are non-empty. Third, the quotes, in particular the last sentence, suggest that Shapiro considers non-empty n-borderline sets or segments as a condition for the introduction of $n+1$-borderline sets or segments, which is (STEPWISE). This is a further assumption required for nestings, but not for orders. Thus three principal elements of the nesting paradigm seem to have slipped into Shapiro's presentation of his higher-order-vagueness paradox. /38/

As Sainsbury's, so Shapiro's version of the higher-order vagueness paradox loses its paradoxicality once the elements of the nesting paradigm are eliminated from the argument. Shapiro's conclusion is that either a_1 is not determinately$_n$F (which makes the determinately operator irrelevant to the Sorites) or there is a sharp boundary between the determinately$_n$ and the borderline$_n$ cases (2005, 149). This conclusion is based on his assumptions of (NO-EMPTY-SEGMENT), (NO-BORDERLINE-OVERLAP) and (STEPWISE). For details how without these assumptions Shapiro's argument loses its force we refer the reader to Bobzien 2010, 23-4.

6.3. Williamson 1994

In his book *Vagueness* Timothy Williamson presents a related consideration for the non-existence of higher-order vagueness (Williamson 1994, 160-1), possibly an attempt to formally capture the main idea behind Sainsbury's incoherence argument. Patrick Greenough presents an easily accessible version of Williamson's argument:

> Suppose we have an operator D in the object-language. Even if we suppose that D is vague, we can still define an operator using D which appears to be non-vague as follows: Let D^* be defined to mean the infinite conjunction "A & DA & DDA &...". (Think of D^* as capturing a notion of absolute determinacy.) Suppose that D^*A. Given the definition just given, and &-E (on the first conjunct), it follows that DA & DDA & .... Given an infinitary rule of collection for D, it follows that $D(A$ & DA & DDA &...), which, given our definition, is equivalent to DD^*A. Hence, if D^*A then DD^*A. Moreover, by a similar pattern of reasoning we can establish that $DD(A$ & DA & DDA & ...) and that $DDD(A$ & DA & DDA & ...) and so on. But then, given our definition, we can show that if D^*A then D^*D^*A. In other words, D^* obeys an S4-like principle and so, Williamson concludes, "higher-order vagueness disappears". (Greenough 2005, 182)

We have shown elsewhere why S4-like principles alone do not make higher-order vagueness disappear.[73] Greenough correctly adds that for that to happen, we need further assumptions and offers the following reconstruction of Williamson"s argument:

> Suppose that D^* is vague. Suppose also that this entails that there are determinate borderline cases between D^*A and $\neg D^*A$. In so far as one thinks that one can express the vagueness of D* using D itself, then it follows that there is an x such that $D(\neg DD^*A$ &$\neg D\neg D^*A)$. The left conjunct of this existential generalization is equivalent to $\neg D^*A$. Given the closure of D, we then conclude that $D\neg D^*A$ which contradicts the right conjunct. Conclusion: D^* is not vague. (Greenough 2005, 183) /39/

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[73] Bobzien 2011.2, 196-200.

This reconstruction makes use of the assumption that the vagueness of *D** entails that there are determinate borderline cases for *D**.[74] However, the assumption of definite (or identifiable) borderline cases is a feature of the classificatory use of "borderline". It marks the beginning of a hierarchy of borderline cases. It is not required for the epistemic use of "borderline", which as we have seen, can be columnar. Greenough acknowledges (183, footnote 15) that *D** may not entail definite borderline cases. He offers as alternative the assumption that *D** obeys the B axiom (or Brouwerian axiom) in addition to the S4 axiom, from which it can be inferred that *D** is not vague (*ibid*.). However, assuming that *D** is governed by the B axiom is just another way of precluding columnar higher-order vagueness (see above Section 3.3). Williamson himself states in a footnote that his "construction … does not validate the Brouwerian schema", and acknowledges that therefore "a kind of higher-order vagueness may remain" (1994, 297, footnote 33).

6.4. Raffman 2010

In her paper "Demoting higher-order vagueness", Diana Raffman argues that there exist no higher-order borderline cases (Raffman 2010, 509-15). She couches her arguments in the form of four *ruminations*. We point out how her ruminations make use of the classificatory use of "borderline" and of elements of higher-order nestings.

Rumination #1: Following Sainsbury 1990, whose reasoning she considers decisive, in this argument Raffman presupposes a theory of borderline cases that includes three non-overlapping non-empty classes (510), so our response to Sainsbury in Section 6.1 applies.

Rumination #2: This argument is directed at a theory of hierarchical vagueness which presupposes (NO-BORDERLINE-OVERLAP): "If there can be borderline cases between **Φ** and **not-Φ**, the thinking goes, then surely there can be (second-order) borderline cases between **Φ** and borderline **Φ**; and then surely there can be (third-order) borderline cases between **Φ** and the second-order borderlines; and so on." (510, cf. 511, 512). Moreover, at some point in this argument, Raffman uses a bracketed "definitely": "there is nothing more to being borderline than failing to (definitely) belong either in the category **Φ** or in the category **not-Φ**" (2010, 511). The unbracketed presence of this "definitely" would provide the standard modal definition of "borderline" and would be compatible with higher-order vagueness, bivalence and a sharp border between the **Φ** and the **not-Φ** things. The absence of the "definitely" would provide (THIRD-KIND) from the nesting paradigm and would not be compatible with bivalence and a sharp border between the **Φ** and the **not-Φ** things.

Rumination #3: This argument is directed at a theory of higher-order borderline cases which maintains (THIRD-KIND): "Borderline cases are supposed to /40/ be of indefinite or indeterminate or uncertain status with respect to being **Φ**. So borderline cases have a status other than being **Φ**" (512).

Rumination #4: This is an argument for the impossibility of higher-order borderline cases in which Raffman argues from "no item can be definitely borderline **Ψ**" to "no item can be borderline borderline **Ψ**" (513). In her defense of this step, Raffman relinquishes compositionality of higher-order vagueness (514).

Thus none of Raffman's arguments proves that genuine higher-order vagueness does not exist.

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[74] The reconstruction also makes use of the assumption that the vagueness of D* is to be expressed using D itself, a point Williamson is aware of, and that Rosanna Keefe has questioned, cf. Keefe (2000) ch.8.

6.5 Wright 1992, 2010

In his 2010 paper "The Illusion of Higher-order Vagueness" Crispin Wright provides a distinction between two phenomena, or theses, of higher-order borderline vagueness, (a) and (c) (527-8), then shows that (a) is not higher-order vagueness but an illusion, then strives to argue that (a) and (c) are equivalent and that it follows that (c), too, is merely an illusion of higher-order vagueness. Wright's (a) displays most of the characteristic features of the classificatory use of "borderline" and borderline nestings. We agree that (a) does not manifest higher-order vagueness.

For (c), Wright refers to Williamson (1999). However, (c) is neither Williamson's intersective unclarity (for which see above Section 3.4) nor genuine higher-order vagueness. Wright describes how (c) countenances higher-order vagueness thus: "in general, each successive nth order of vagueness, n > 1, is conceived as consisting in the vagueness of the boundary between the Def_{n-1} Fs – the things that are definitely … definitely (n-1 times) F – and the definite borderline cases of order n-1, that is, as consisting in the (potential) existence of cases satisfying the condition: $\sim Def_n F \& \sim Def(\text{Borderline}_{n-1}F)$"[75] (529) This quote shows that (c) invokes or implies the following elements of the nesting paradigm:

- First-order borderline cases exist. – This is not part of the account of "borderline case" Wright starts out from, i.e. things that are neither definitely F nor definitely G (524, the "Basic Formula").
- Definite borderline cases exist. – This is neither implied by the account of "borderline" nor part of Williamson's 1999 theory, and is the beginning of a hierarchy of borderline cases.
- The existence of definite Borderline_{n-1} cases is presupposed for the existence of Borderline_n cases. – Hence the existence of Borderline_{n-1} cases is presupposed for the existence of Borderline_n cases, which is (STEPWISE). /41/
- The Borderline_n cases are said to be *between* the Def_{n-1} cases and the $Def(\text{Borderline}_{n-1})$ cases. – Since the latter are also Borderline_{n-1} cases (Wright accepts the factivity of *Def*, 529), there is, for any n, at most partial overlap of Borderline_n and Borderline_{n-1} cases.
- Additionally, later in the discussion of (c), Wright assumes that the principle "There are no true instances of F & $\sim Def F$" holds for (c) (529, with footnote 13). – This principle is incompatible with Williamson's 1999 theory and goes counter to the epistemic use of "borderline".
- Related to this may be the fact that earlier, where Wright introduces his Basic Formula in his discussion of Dummett (524), he moves from the modal account of borderlineness directly to the classificatory use and (THIRD-KIND) (524). Wright never mentions the fact that, unlike (THIRD-KIND), his Basic Formula permits the epistemic use of "borderline" as well as bivalence and a sharp boundary (all features of Williamson 1999, Wright's prototype for (c)).

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[75] Wright's formalizations of the condition for second-order and n^{th}-order cases are faulty (a simple oversight, as is clear from the context and as Wright has confirmed in correspondence). The easiest way of formally expressing what Wright intended appears to be $\sim Def_2 Fa \& \sim Def \sim Def Fa$ for the second-level second-order borderline cases leftmost on a dimension *D* (the only ones Wright is concerned with) and $\sim Def_n Fa \& \sim Def \sim Def_{n-1} Fa$ for the nth-level n^{th}--order borderline cases leftmost on a dimension *D*. We wonder whether the formulations in terms of "Borderline_n" may be another indicator of the nesting paradigm's subliminal impact.

Thus there are many indications that assumptions from the nesting paradigm have crept into Wright's conception of higher-order borderline cases of type (c). But without these assumptions, Wright cannot show that (a) and (c) are equivalent, and hence also not that higher-order vagueness as set out in (c) is an illusion. We have shown in Bobzien (2010), 21-22, how the presence of some of these elements in Wright's formal argument for the incoherence of higher-order vagueness (as set out first in Wright 1992) are necessary for drawing the conclusion that higher-order vagueness is incoherent, and how their removal simultaneously removes the incoherence and illusionary character of higher-order vagueness.

7. Conclusion: columnar higher-order vagueness and the Sorites

None of the authors discussed in Section 6 has successfully demonstrated that higher-order vagueness *per se* is incoherent, paradoxical, non-existent, illusory or incompatible with sharp boundaries of vague predicates. We have shown that the arguments of all five philosophers conflate elements of the higher-order vagueness paradigm with elements of the paradigm of borderline nestings and the classificatory use of "borderline". In particular, all five philosophers appear to conceive of higher-order vagueness as necessarily hierarchical and as necessarily including clear borderline cases.

In Section 3 we argued that it is not necessary for higher-order vagueness to be hierarchical or to include clear borderline cases. We introduced columnar higher-order vagueness as an example of a coherent notion of higher-order vagueness. We showed that with an epistemic interpretation of the modal operators, columnar radical higher-order vagueness fulfills all requirements a theory of higher-order vagueness needs to meet to further a solution of the Sorites. Given the arguments against hierarchical higher-order vagueness in Section 6, columnar radical higher-order /42/ vagueness appears to be the only theory of higher-order vagueness that has this virtue.

At first encounter, theories of columnar higher-order vagueness are often met with skepticism. We believe that some of this skepticism has its roots in the very conflation of the paradigms of higher orders and nestings that we found in the arguments against higher-order vagueness *tout court*. However, there is nothing in the Sorites paradox that requires a hierarchical notion of higher-order vagueness. The *phenomenon* of a borderline region of vague predicates (with regard to some D and $U_{ordered}$) that is characteristic for Sorites-prone predicates is one of epistemic inaccessibility and – apparent – seamless transition. In discussions of vagueness, higher-order vagueness is invoked in the first instance to provide an adequate *description* of this combination of phenomena. There is prima facie no element of a hierarchy present.[76] Neither epistemic inaccessibility nor seamless transitions are intrinsically hierarchical. Accordingly, nothing suggests that higher-order vagueness is to reflect and hence display a hierarchical pattern. Once one lets go of the enticement of the nesting paradigm, the apparent requirement of a hierarchical structure, including the apparent requirement of clear or definite borderline cases, disappears, or at least loses major appeal.[77]

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[76] Nothing in descriptions like "it's possible to tell that it's possible to tell" or "it's unknowable that it's unknowable" or "it's definite that it's definite", etc. leads automatically to a hierarchy of borderline cases of the kind for which the nestings are paradigmatic.

[77] If the modal operators are given a non-epistemic interpretation, there is still no requirement for a hierarchy as set out in Section 1. A non-epistemic interpretation is to provide an *explanation* of what happens in the borderline zone,

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and if the phenomenon or *explanandum* is not hierarchical, there is no reason for the *explanans* to be. Likewise, if the purpose of the investigation of 'definitely' or 'clearly' is to cover ordinary language uses of these expressions, there is no reason to expect a hierarchical pattern, see e.g. Barker 2002, Raffman 2010.

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