VAGUENESS
SUSANNE BOBZIEN AND ROSANNA KEEFE

I—SUSANNE BOBZIEN

COLUMNAR HIGHER-ORDER VAGUENESS, OR VAGUENESS IS HIGHER-ORDER VAGUENESS

Most descriptions of higher-order vagueness in terms of traditional modal logic generate so-called higher-order vagueness paradoxes. The one that doesn't (Williamson's) is problematic otherwise. Consequently, the present trend is toward more complex, non-standard theories. However, there is no need for this.

In this paper I introduce a theory of higher-order vagueness that is paradox-free and can be expressed in the first-order extension of a normal modal system that is complete with respect to single-domain Kripke-frame semantics. This is the system $\mathrm{QS4M}_{+\mathrm{BF}+\mathrm{FIN}}$. It corresponds to the class of transitive, reflexive and final frames. With borderlineness (unclarity, indeterminacy) defined logically as usual, it then follows that something is borderline precisely when it is higher-order borderline, and that a predicate is vague precisely when it is higher-order vague.

Like Williamson's, the theory proposed here has no clear borderline cases in Sorites sequences. I argue that objections that there must be clear borderline cases ensue from the confusion of two notions of borderlineness—one associated with genuine higher-order vagueness, the other employed to sort objects into categories—and that the higher-order vagueness paradoxes result from superimposing the second notion onto the first. Lastly, I address some further potential objections.

This paper proposes that vagueness is higher-order vagueness. At first blush this may seem a very peculiar suggestion. It is my hope that the following pages will dispel the peculiarity and that the advantages of the proposed theory will speak for themselves. The vagueness in question is that of Sorites susceptibility. A linguistic expression is Sorites-susceptible, and thus vague, if it is in principle possible to construct a Sorites paradox with it. As is commonly done, vagueness is set out in terms of borderlineness. First and foremost, this paper is concerned with the *structural* properties of vagueness and borderlineness: the logical skeleton on which a full theory of vagueness still needs to be fleshed out. Accordingly, other philosophical elements are introduced on a need-to-know basis only. So, for example, border-

Proceedings of the Aristotelian Society Supplementary Volume LXXXIX
doi: 10.1111/j.1467-8349.2015.00244.x

lineness is assumed to be context-sensitive, but this context-sensitivity is presupposed and not discussed.

The theory of higher-order vagueness offered is called columnar higher-order vagueness. In its most basic case, it consists of a first-order extension of a normal modal system that defines the logical structure of borderlineness, with its modal operators given a factive-cognitive interpretation, and supplemented by two plain assumptions that provide the link between the logic and Sorites sequences. Its defining characteristic is that if something is borderline, it is borderline so.

The introduction of columnar higher-order vagueness is complemented by the uncovering of the distinct logical structures of two notions of borderlineness: one is associated with genuine higher-order vagueness and serves as the basis for the technical notion of columnar higher-order vagueness; the other is employed to categorize objects as being borderline. This second notion is compatible with the existence of clear borderline cases, and iterations here lead to the nesting of ever finer-grained categories of borderline cases, or borderline nestings. The unearthing of the structural difference of these two common notions of borderlineness (one leading to higher-order vagueness, one to borderline nestings) is key in removing the air of oddity that accompanies the identification of vagueness with higher-order vagueness. Besides, it reveals that the so-called paradoxes of higher-order vagueness are simply the result of superimposing the notion of distributing objects into categories onto that of higher-order vagueness.

The identification of vagueness with columnar higher-order vagueness has some major advantages. It yields a demarcation of vagueness from other phenomena that have prompted philosophers to relinquish classical logic or bivalent semantics, such as partially defined predicates, future contingents, and statements like Goldbach's conjecture. It is *versatile* in that it is compatible with, but does not require, bivalence and classical logic. It provides the basis for a straightforward solution to the Sorites paradox (which is the topic of Bobzien 2016). It does justice to our intuition that natural language expressions are ineliminably vague and that there appears to be a *seamless transition* in Sorites paradoxes. It obviates all higher-order vagueness paradoxes. Finally, it has an attractive simplicity.

The paper is structured as follows. §1 juxtaposes hierarchical and

Proceedings of the Aristotelian Society Supplementary Volume LXXXIX
doi: 10.1111/j.1467-8349.2015.00244.x

columnar higher-order vagueness and gives an informal explanation of the latter. §II presents the logic of columnar higher-order vagueness at the propositional level and establishes its coherence and the completeness of its basic case, *normal* columnar higher-order vagueness. §III introduces the first-order extension of normal columnar higher-order vagueness. §IV shows how, in logical terms, columnar higher-order vagueness is related to Sorites sequences. §V defines the vagueness of sentences and predicates in terms of higher-order vagueness and explicates in what ways vagueness is higher-order vagueness. §VI explains why vagueness, thus defined, avoids all—known—so-called higher-order vagueness paradoxes, and how these have resulted from superimposing the notion of borderline nestings onto that of higher-order vagueness. §VII offers replies to several common objections to the proposal that vagueness is higher-order vagueness.

I

Columnar Higher-Order Vagueness and Hierarchical Higher-Order Vagueness. *Hierarchical higher-order vagueness* is characterized by a hierarchy of consecutively higher orders of borderline cases of a vague predicate (i) that include clear (definite, determinate) borderline cases, and (ii) whose extensions do not overlap: there are borderline cases between the clear cases, borderline borderline cases between the clear and the clear borderline cases, etc. Hierarchical higher-order vagueness is generally taken to lead to incoherence. (For more details see Sainsbury 1991, pp. 168–9 and §VI below.) *Columnar higher-order vagueness* differs from hierarchical higher-order vagueness in that, extensionally, it contains just one kind of borderline cases, and that each borderline case is radically higher-order, or radically borderline, i.e. borderline borderline …, ad infinitum. Columnar higher-order vagueness also maintains that if something is a clear case, it is radically clear, i.e. clearly clearly … clearly clear,[1] and that if there is something that is borderline, it is borderline that this is so. As a result, there are no clear borderline cases and no borderline clear cases, and it is not clear whether there are any borderline cases (but see §VI). Columnar higher-order

[1] I use ellipsis to indicate the indefinite number of repetitions of the relevant expression.

Proceedings of the Aristotelian Society Supplementary Volume LXXXIX
doi: 10.1111/j.1467-8349.2015.00244.x

vagueness is called *columnar* because the depiction of higher orders with regard to a vague predicate (e.g. 'tall') and a dimension (e.g. height) in a Sorites sequence results in a columnar shape for the borderline cases, and contrasts with the pyramidal shape that a corresponding depiction of hierarchical higher-order vagueness would exhibit (Bobzien 2013, pp. 1–3, 13–16).

Columnar higher-order vagueness compares favourably to other theories that examine higher-order vagueness. Its chief advantage over hierarchical higher-order vagueness is its immunity to higher-order vagueness paradoxes (see §VI). Its main advantages over theories that deny the existence of higher-order vagueness are that it can explain the boundarylessness intuition (see §IV) and that it does justice to the everyday assumption that there are borderline borderline cases. Its advantages over Williamson's theory of higher-order vagueness (Williamson 1994, appendix; 1999) are that it does not require a minimum of two borderline cases in a Sorites sequence; that it faces no conjunction-agglomeration problem for non-vague sentences; that non-borderlineness (preciseness) is closed under uniform substitution; and that polar cases are not borderline cases at any order. In addition to sporting these advantages, columnar higher-order vagueness is significant, because it may be the only plausible and coherent theory of radical higher-order vagueness that includes modal axiom 4. It puts the lie to the long and ongoing tradition that discredits the possibility that axiom 4 is part of a viable theory of higher-order vagueness.[2]

The logic of columnar higher-order vagueness can in principle be interpreted epistemically, semantically or ontically. This paper puts forward a factive-cognitive (i.e. epistemic in the wider sense) interpretation.[3] This is because every solution of the Sorites has to explain people's move from competent judgement to hedging behaviour and back when walked through a Forced March Sorites, and thus needs a cognitive or epistemic element. It is not thereby precluded that this interpretation is ultimately grounded in some other interpretation. On the factive-cognitive interpretation, borderline-

[2] For details see also Bobzien (2012). The tradition starts with Dummett (1975, p. 311). It is continued in Wright (1987, 1992) and Williamson (1994, pp. 159, 271–2), and is still thriving. Keefe (2000) retains the general idea of axiom 4 by introducing an indefinite hierarchy of metalanguages.

[3] I borrow the term 'factive-cognitive' from linguistics. Note that 'epistemic' is not the same as 'epistemicist'.

doi: 10.1111/j.1467-8349.2015.00244.x

ness of some *a* with regard to some predicate *F* is cashed out as a type of cognitive inaccessibility, expressed as 'one cannot tell whether'. Some *a* is borderline *F* if relevantly qualified individuals cannot tell whether *Fa*. Relevantly qualified individuals are those humans who are in no way handicapped with regard to assessing whether *Fa*. So, *when it is borderline whether Fa, the reason does not lie in any shortcomings of the individuals, but in Fa*. (This is all I say in this paper about the factive-cognitive interpretation of borderlineness. The paper is about the structural properties of borderlineness and vagueness. For these, no further interpretational details are required. The phrase 'one cannot tell whether' is used as a natural language stand-in for borderlineness as defined modally in this paper. I have no interest in providing a semantics for the natural language expression 'can tell'. Furthermore, I say nothing about the relation between ability to know and ability to tell, or tellability, beyond mentioning here that both are factive and that neither entails the other.)

II

Columnar Higher-Order Vagueness in Propositional Logic. Readers who wish to get to the philosophical gist of the paper before ingesting the dry exposition of axiomatic modal logic can skip all but the last paragraph of §II and of §III, and return to the whole of these sections later.

2.1. *The Logical Core of Columnar Higher-Order Vagueness*. The logical core of columnar higher-order vagueness is a propositional modal logic with a first-order extension. It is set out here as an axiomatic modal system. I use 'axiom' and 'theorem' as short for 'axiom schema' and 'theorem schema'.[4] This section explicates the propositional portion of the logic of columnar higher-order vagueness.

The syntax is as follows. $p, p_1, \ldots, p_s$ are used for atomic sentences; the connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$ and $\leftrightarrow$ are those from classical logic and square brackets ([,]) serve for bracketing in the usual manner. The modal operator C ('it is clear that') is modelled on the necessity operator $\Box$. The syntax of C is that of normal modal systems with $\Box$.

[4] Except for the use of axiom and theorem schemata instead of axioms and theorems, '$\Box$' for 'L', 'A_1', 'A_2' for 'α', 'β', I follow the conventions and terminology of Hughes and Cresswell (1996).

Proceedings of the Aristotelian Society Supplementary Volume LXXXIX
doi: 10.1111/j.1467-8349.2015.00244.x

A second operator B ('it is borderline whether') is *nominally* defined in terms of clarity so that, for any arbitrary formula A,

$$\text{(2.1)}\quad \mathrm{B}A \leftrightarrow \neg\mathrm{C}A \wedge \neg\mathrm{C}\neg A \qquad \text{(df B)}$$

Thus the borderline operator B stands to the C-operator as the contingency operator ▽ stands to □. *Substantively*, B is defined by the syntax, rules and axioms of the modal system. The operators C and B are not metalinguistic. To give an illustration, it would be 'It is borderline whether Tallulah is tall', not 'It is borderline whether "Tallulah is tall" is true', 'It is borderline true whether "Tallulah is tall"', or the like. I assume borderlineness to be the central notion that underlies (both hierarchical and) columnar higher-order vagueness. Accordingly, I take the meaning of 'it is clear that' to be specified in terms of borderlineness rather than the other way about: it is clear that A precisely if both A and it is non-borderline that A. (I am not interested in the semantics of the natural language expression 'it is clear'.) In line with the tellability interpretation of borderlineness from §1, interpreted, the B-operator reads 'one cannot tell whether', and the C-operator reads 'one can tell that'.

All systems of propositional columnar higher-order vagueness are then characterized by the combination of the following rules and theorems:

(2.2) If A_1 and A_2 are theorems, then $A_1 \wedge A_2$ is a theorem. ($\wedge$-introduction)

(2.3) If $A_1 \rightarrow A_2$ and A_1 are theorems, A_2 is a theorem. (MP)

(2.4) $\mathrm{C}A \rightarrow A$ (T)

(2.5) $[\mathrm{C}A_1 \wedge \mathrm{C}A_2] \rightarrow \mathrm{C}[A_1 \wedge A_2]$ (K2)

(2.6) $\mathrm{C}A \rightarrow \mathrm{C}^2A$ (Axiom 4)

(2.7) $[\neg\mathrm{C}A \wedge \neg\mathrm{C}\neg A] \rightarrow [\neg\mathrm{C}[\neg\mathrm{C}A \wedge \neg\mathrm{C}\neg A] \wedge \neg\mathrm{C}\neg[\neg\mathrm{C}A \wedge \neg\mathrm{C}\neg A]]$ (V)

(2.8) The axioms and rules of the system ensure that $\neg\mathrm{C}A \rightarrow \mathrm{C}\neg A$ is not a theorem.

The meta-rule (2.8) guarantees that the existence of borderlineness is not logically precluded. (2.7) is the distinctive axiom of columnar higher-order vagueness (with V for 'vagueness'). It expresses that if something is borderline, it is borderline borderline. This is more obvious in terms of the B-operator:

doi: 10.1111/j.1467-8349.2015.00244.x

$$(2.9)\quad \mathrm{B}A \rightarrow \mathrm{B}^2 A$$

A basic theorem of borderlineness that is likewise more easily expressed in terms of B is the mirror axiom, that it is borderline that A precisely if it is borderline that $\neg A$, which is already captured in the formulation 'borderline *whether*':

$$(2.10)\quad \mathrm{B}A \leftrightarrow \mathrm{B}\neg A$$

The resulting modal fragment, that is, the fragment consisting of all and only the modalized formulae, is coherent (see below). We call any modal system that contains this fragment and has non-collapsing modalities a *logic of columnar higher-order vagueness*. Logics of columnar higher-order vagueness are characterized by (2.2)–(2.8), with (2.1) being optional.[5] They can be normal or non-normal, classical or non-classical, and their semantics can be bivalent, trivalent or multivalent, as long as any semantic status beyond truth and falsehood has its origin in non-modalized formulae A of which BA is true. Depending on what additional axioms or rules such a logic encompasses, it can be used as the logical backbone for a variety of familiar conceptions of vagueness.

2.2. *Normal Columnar Higher-Order Vagueness*. Next I set out the simplest case: columnar higher-order vagueness for normal modal systems, or *normal columnar higher-order vagueness*. It contains classical logic and is bivalent.

A normal modal system of modal propositional logic can be defined as 'a class S of wff of modal propositional logic which contains all PC-valid wffs and K, and has the property that if α and β are in S then so is anything obtainable from them by the use of MP and N' (Hughes and Cresswell 1996, p. 111), where the wff are of a language $\mathcal{L}$ of modal PC. Accordingly, I add to the requirements for columnar higher-order vagueness axiom **K**, the rule of necessitation (N), and the rule that all tautologies of propositional calculus are axioms (PC). **K** makes **K2** obsolete. The result is the modal system S4 along with axiom **V**. System S4 ensures that $\neg\mathrm{C}A \rightarrow \mathrm{C}\neg A$ is not a theorem (2.8). It also ensures substitutivity of logical equivalents in the scope of the C-operator (*Subst. Equiv.*).

The completeness of normal columnar higher-order vagueness

[5] Alternative axiomatization with B instead of C is possible.

doi: 10.1111/j.1467-8349.2015.00244.x

with respect to the class of transitive, reflexive and final Kripke frames can be demonstrated as follows.[6] Axiom **V** is logically equivalent in system S4 to the McKinsey axiom **M**:

$$(2.11)\quad \mathrm{C}\neg\mathrm{C}\neg A \rightarrow \neg\mathrm{C}\neg\mathrm{C}A \qquad (\mathbf{M})$$

The proof is given in appendix 1. It follows that *normal* columnar higher-order vagueness corresponds to the normal modal system S4M (or $\mathrm{KT4G_C}$) and is thus complete. Accordingly, the modal fragment that defines the core of columnar higher-order vagueness is coherent. For a completeness proof of S4M I refer the reader to Hughes and Cresswell (1996, pp. 131–3). Regarding consistency, normal columnar higher-order vagueness is evidently consistent; that is, not all its wff are theorems. In particular the converse of **M** (G1, i.e. $\neg\mathrm{C}\neg\mathrm{C}A \rightarrow \mathrm{C}\neg\mathrm{C}\neg A$) and the Brouwerian axiom (**B**, i.e. $A \rightarrow \mathrm{C}\neg\mathrm{C}\neg A$) are not.

The philosophical significance of columnar higher-order vagueness, normal or other, at the propositional level is captured best by the following pair of key principles that can be derived in its logic:

$$(2.12)\quad \mathrm{C}A \leftrightarrow \mathrm{C}^n A \quad \text{for any } n$$

$$(2.13)\quad \mathrm{B}A \leftrightarrow \mathrm{B}^n A \quad \text{for any } n$$

These principles say, respectively, that clarity and radical higher-order clarity are co-extensive and that borderlineness and radical higher-order borderlineness are co-extensive. For reasons of simplicity, in the following I limit myself to the case of normal columnar higher-order vagueness.

III

Columnar Higher-Order Vagueness in First-Order Logic. A logic of higher-order vagueness as such does not require an extension to first-order logic or the lower predicate calculus (LPC). However, (i) to express that something is a borderline case of something, (ii) to define the vagueness of predicates, and (iii) to show that columnar higher-order vagueness avoids the so-called higher-order vagueness paradoxes that are formulated in modal LPC, such an extension is needed.

[6] A final frame is one in which every world can access a world that can access only itself.

Proceedings of the Aristotelian Society Supplementary Volume LXXXIX
doi: 10.1111/j.1467-8349.2015.00244.x

It also makes it more straightforward (iv) to express relations between objects of a Sorites sequence, and (v) to formulate a solution to the Sorites paradox. This section provides the simplest case, which is the first-order extension of normal columnar higher-order vagueness.

The syntax is expanded as follows. *F*, *G* are used for vague predicates of a natural language, a, a_1, ..., a_s for designators, x, y, x_1, x_2, ..., x_s for variables, and $\exists$ and $\forall$ for quantifiers in the usual manner, with brackets as above. B*Fa* is to be read as '*a* is borderline *F*' or '*a* is a borderline case of being *F*'. Higher orders of borderlineness are expressed thus: *a* is a *first-order* borderline case of *F*, written B^1Fa, iff B*Fa*. And *a* is an $(n+1)$*th-order* borderline case (for $n \geqslant 1$), written $\mathrm{B}^{n+1}Fa$, iff BB^nFa. For example, using (2.1) twice, we obtain for an arbitrary predicate ϕ,

$$(3.1)\ \mathrm{B}^2\phi a \leftrightarrow \neg\mathrm{C}[\neg\mathrm{C}\phi a \wedge \neg\mathrm{C}\neg\phi a] \wedge \neg\mathrm{C}\neg[\neg\mathrm{C}\phi a \wedge \neg\mathrm{C}\neg\phi a]$$

Quantified columnar higher-order vagueness has the following three distinctive valid principles. The first two carry over from propositional columnar higher-order vagueness and are no surprise:

$$(3.2)\quad \forall x[\mathrm{B}\phi x \rightarrow \mathrm{B}^2\phi x] \qquad (\mathrm{V_Q})$$

which says that if something is a first-order borderline case of ϕ, it is a second-order borderline case of ϕ, and

$$(3.3)\quad \forall x[\mathrm{C}\phi x \rightarrow \mathrm{C}^2\phi x] \qquad (4_\mathrm{Q})$$

which says that if something is a first-order non-borderline case of ϕ, it is a second-order non-borderline case of ϕ. The third principle spells out the implications of columnar higher-order vagueness for the existence of borderline cases:

$$(3.4)\quad \exists x\mathrm{B}\phi x \rightarrow \mathrm{B}\exists x\mathrm{B}\phi x \qquad (\mathrm{V}_\exists)$$

Roughly, $\mathrm{V}_\exists$ says that if there is something that is borderline ϕ, then it is borderline that there is something that is borderline ϕ.

One can show that both $\mathrm{V_Q}$ and $\mathrm{V}_\exists$ are part of a first-order extension of S4M, abbreviated $\mathrm{QS4M}_{+\mathrm{BF}+\mathrm{FIN}}$. Complementing the syntax from above, here are first the rules and axioms for the system. The wff are now wff of a language $\mathcal{L}$ of modal LPC.

S4M′ If *A* is an LPC substitution instance of a theorem of S4M, then *A* is an axiom of $\mathrm{QS4M}_{+\mathrm{BF}+\mathrm{FIN}}$.

doi: 10.1111/j.1467-8349.2015.00244.x

$\forall 1$ If A is any wff and x and y are variables and $A[x/y]$ is A with free y replacing every free x, then $\forall x A \rightarrow A[x/y]$ is an axiom of $\mathrm{QS4M}_{+\mathrm{BF}+\mathrm{FIN}}$ (not employed in this paper).

N If A is a theorem of $\mathrm{QS4M}_{+\mathrm{BF}+\mathrm{FIN}}$, then so is $\mathrm{C}A$.

MP If A_1 and $A_1 \rightarrow A_2$ are theorems of $\mathrm{QS4M}_{+\mathrm{BF}+\mathrm{FIN}}$, then so is A_2.

$\forall 2$ If $A_1 \rightarrow A_2$ is a theorem of $\mathrm{QS4M}_{+\mathrm{BF}+\mathrm{FIN}}$ and x is not free in A_1, then $A_1 \rightarrow \forall x A_2$ is a theorem of $\mathrm{QS4M}_{+\mathrm{BF}+\mathrm{FIN}}$.

BF $\forall x \mathrm{C}A \rightarrow \mathrm{C}\forall x A$

FINAX $\neg\mathrm{C}\neg\forall x_1, \ldots, \forall x_n[A \rightarrow \mathrm{C}A]$

(This follows Hughes and Cresswell 1996, p. 244, except for the use of the clarity operator in lieu of the necessity operator, and Cresswell 2001, p. 160 for axiom **FINAX**.) $\mathrm{QS4M}_{+\mathrm{BF}+\mathrm{FIN}}$ is complete with respect to the quantificational single domain Kripke frame semantics with the class of transitive, reflexive and final frames (Cresswell 2001, pp. 159–64), that is, with *the same class of frames* with respect to which S4M is complete.

V_Q and 4_Q are simply the quantified versions of axioms V and 4 for one-place predicates. $\mathrm{V}_\exists$ can also be demonstrated in $\mathrm{QS4M}_{+\mathrm{BF}+\mathrm{FIN}}$. A sketch of the proof can be found in appendix II (i). All three are thus theorems of $\mathrm{QS4M}_{+\mathrm{BF}+\mathrm{FIN}}$. Finally, the converse of $\mathrm{V}_\exists$,

$$(3.5)\quad \mathrm{B}\exists x \mathrm{B}\phi x \rightarrow \exists x \mathrm{B}\phi x \qquad (\mathrm{V}_{\exists\mathrm{c}})$$

can also be demonstrated in $\mathrm{QS4M}_{+\mathrm{BF}+\mathrm{FIN}}$ (see appendix II (ii)).

The philosophical significance of normal columnar higher-order vagueness at the predicate level is captured best by the following three key principles.

$$(3.6)\quad \forall x[\mathrm{B}\phi x \leftrightarrow \mathrm{B}^n\phi x] \text{ for any } n$$

$$(3.7)\quad \forall x[\mathrm{C}\phi x \leftrightarrow \mathrm{C}^n\phi x] \text{ for any } n$$

These principles say that every borderline case is radically borderline (and vice versa) and that every non-borderline case is radically non-borderline (and vice versa). They can be validated in $\mathrm{QS4M}_{+\mathrm{BF}+\mathrm{FIN}}$. With the tellability interpretation, it is a consequence of (3.6) that we can tell of no borderline case whether it is borderline or non-border-

doi: 10.1111/j.1467-8349.2015.00244.x

line. So of no borderline case can we rule out that it is not a borderline case.

Third, from (3.4) and (3.5) plus (2.13) we obtain the biconditional

$$(3.8)\quad \exists x \mathrm{B}\phi x \leftrightarrow \mathrm{B}^n \exists x \mathrm{B}\phi x \quad \text{for any } n \qquad (\mathrm{V}_{\exists\leftrightarrow})$$

as a valid principle: there is a borderline case of ϕ precisely if it is radically borderline whether there is a borderline case of ϕ. The philosophical significance becomes clearer in a hybrid formulation: if there is a borderline case of ϕ, then one can't tell whether ... one can tell whether there is a borderline case of ϕ (and vice versa). For this paper, the purpose of $\mathrm{V}_{\exists\leftrightarrow}$ lies in (ii) and (iii) from above.

IV

Sorites Sequences and Two Basic Assumptions of Columnar Higher-Order Vagueness. As developed so far, columnar higher-order vagueness has been described as the logic $\mathrm{QS4M}_{+\mathrm{BF}+\mathrm{FIN}}$ with the operators C and B for $\Box$ and $\triangledown$, with a tellability interpretation of those operators, but without any reference to Sorites sequences. This section draws the connection to vagueness qua Sorites susceptibility and thus to the Sorites paradox. It uses the following definition of a Sorites sequence. A *Sorites sequence* is a finite sequence of objects a_1 to a_n (i) that is ordered with respect to some dimension (e.g. height) and some predicate (e.g. 'short'), (ii) with the ordering being total and strict, (iii) that displays tolerance, i.e. it *appears* that we cannot have a_i but not a_{i+1} satisfy the predicate, because they seem indistinguishable with respect to the predicate, and (iv) for which the following two principles hold. First, a principle that concerns the polar (that is, first and last) cases of Sorites sequences a_1, ..., a_n with regard to some predicate *F*. It states that the polar cases are clear cases of *F* and $\neg F$ respectively, formally for arbitrary predicates ϕ:

$$(4.1)\quad \mathrm{C}\phi a_1 \wedge \mathrm{C}\phi\neg a_n$$

The second principle expresses a continuity relation for non-borderline cases: any a_i with a lower index than an *a* that is *F* is itself *F* and any a_i with a higher index than an *a* that is not-*F* is itself not-*F*, formally for arbitrary predicates ϕ:

Proceedings of the Aristotelian Society Supplementary Volume LXXXIX
doi: 10.1111/j.1467-8349.2015.00244.x

(4.2) $[\mathrm{C}\phi a_i \rightarrow \mathrm{C}\phi a_{i-1}] \wedge [\mathrm{C}\phi\neg a_i \rightarrow \mathrm{C}\neg\phi a_{i+1}]$[7]

The elements of this account of Sorites sequences should find approval from most, and minor discrepancies and notational variants such as the use of the successor function ‘′’ instead of a_n, a_{n+1}, and alternative formulations of (4.1) and (4.2), should not matter in what follows.

With Sorites sequences thus defined, columnar higher-order vagueness can be linked to the Sorites paradox and to the higher-order vagueness paradoxes by the addition of two generally accepted assumptions, (4.3) and (4.5). The first postulates that:

> (4.3) For any Sorites sequence a_1 to a_n of a predicate ϕ, one cannot rule out that it contains a borderline case of ϕ.

(Here and below, the phrase ‘one cannot rule out’ is short for ‘one cannot tell that it is not the case’.) This assumption has its justification in the fact that, if one could rule out the existence of borderline cases in a Sorites sequence, no Sorites paradox would arise in the first place, since one could tell that in the sequence a clear case of ϕ bordered a clear case of $\neg\phi$. In order for columnar higher-order vagueness to be compatible with the existence of Sorites paradoxes, nothing stronger is required.

Given that the vagueness under discussion is that of Sorites susceptibility, and Sorites susceptibility of ϕ entails that it is in principle possible to construct a Sorites paradox with ϕ, the following more general assumption derives from (4.3):

> (4.4) If a predicate is vague, one cannot rule out that it has borderline cases.

The second basic assumption postulates that:

> (4.5) For any two adjacent objects a_i, a_{i+1} in a Sorites sequence of ϕ it holds that if a_i is non-borderline ϕ, then one can't rule out that a_{i+1} is ϕ.

The job of (4.5) is to supply a satisfactory logical underpinning for the persuasiveness of the conditional Sorites premiss that for any a_i, a_{i+1} in a Sorites sequence of some predicate F, it holds that $Fa_i \rightarrow Fa_{i+1}$. It

[7] From (2.1) and (4.2), it follows that in a Sorites sequence there also holds a continuity relation for borderline cases: if $\mathrm{B}\phi a_n$ and $\mathrm{B}\phi a_m$, then any a_i, with $n \leqslant i \leqslant m$, is also $\mathrm{B}\phi$.

Proceedings of the Aristotelian Society Supplementary Volume LXXXIX
doi: 10.1111/j.1467-8349.2015.00244.x

takes the place that logical principles like Williamson's $\mathrm{K}Fa_n \rightarrow Fa_{n+1}$ and versions of $\mathrm{D}^k[\mathrm{D}^m Fa_i \rightarrow \neg\mathrm{D}\neg\mathrm{D}^{m-1}Fa_{i+1}]$ with $m \geqslant 1$ and $k \geqslant 0$ (e.g. Wright 1992) have in other modal theories of higher-order vagueness. This second assumption, in C-terms $\mathrm{C}\phi a_i \rightarrow \neg\mathrm{C}\neg\phi a_{i+1}$, is relevant to the Sorites solution that columnar higher-order vagueness provides. (This is discussed in detail in Bobzien 2016.)

It is important to keep in mind that the two basic assumptions of columnar higher-order vagueness, (4.3) and (4.5), are not logical theorems. In particular, although the theory puts them forward as (assumed to be) true, in the logic of columnar higher-order vagueness they can neither be derived nor refuted, nor can they be empirically proved or disproved. (Williamson's claim that there is a sharp border between the true and the false sentences in a Sorites sequence appears to be an assumption of his theory in this sense.) The logic of columnar higher-order vagueness defines borderlineness. In the basic case of *normal* columnar higher-order vagueness, it is $\mathrm{QS4M}_{+\mathrm{BF}+\mathrm{FIN}}$ with the interpreted C-operator that defines borderlineness. In contrast, (4.3) and (4.5) are not part of what defines borderlineness. They are the component of the theory that relates borderlineness to Sorites sequences.

It is now possible to express how columnar higher-order vagueness satisfies two philosophical desiderata mentioned above. First, it must help explain the intuition that there appears to be a *seamless transition* in a Sorites sequence from the cases where the vague predicate applies to those where it doesn't (cf. Fara 2003, p. 197). This desideratum is met by the fact that with respect to non-borderlineness, every borderline case is indistinguishable from adjacent non-borderline cases. (See also Bobzien 2010, esp. pp. 19–20; 2013, n. 37.) Second, a theory of higher-order vagueness must tally with the *boundarylessness intuition* that there is no determinable boundary that marks the non-borderline cases from the borderline cases (cf. Sainsbury 1990). This desideratum is met by the fact that it is impossible to ascertain of any borderline case of any order that it is borderline rather than non-borderline. This is not some independent stipulation. Rather, it is *part of the logical structure* of columnar higher-order vagueness that we have no access to a boundary that marks the non-borderline from the borderline cases. (As in Williamson's theory the *boundarylessness intuition* is satisfied by the fact that the non-borderline cases that border the borderline cases are not clearly non-borderline, so in my theory it is satisfied by

Proceedings of the Aristotelian Society Supplementary Volume LXXXIX
doi: 10.1111/j.1467-8349.2015.00244.x

the fact that the borderline cases that border the non-borderline cases are not clearly borderline.)

At this point we have everything needed for defining vagueness and for showing exactly how columnar higher-order vagueness avoids the so-called higher-order vagueness paradoxes.

V

Defining Vagueness.

5.1. *Borderlineness*. First, we can now say more clearly what borderlineness is. As mentioned earlier, borderlineness can be relative to contexts (such as comparison classes), so that what is borderline in one context may not be borderline in another. That granted, in the case of normal columnar higher-order vagueness the logical structure of borderlineness is defined by $\mathrm{QS4M}_{+\mathrm{BF}+\mathrm{FIN}}$. Borderlineness itself is defined by $\mathrm{QS4M}_{+\mathrm{BF}+\mathrm{FIN}}$, with the contingency operator ∇ replaced by the borderlineness operator B, and B interpreted as 'one cannot tell whether' in the way explained above. If we simplify this by singling out the principal characteristics of B — (2.10) and (2.13) from above—we can say:

(5.1) It is borderline whether A precisely if we have $\mathrm{B}^n A \wedge \mathrm{B}^n \neg A$ for any $n \geqslant 1$.

We can simplify further by omitting the second conjunct, since it seems generally agreed that it is borderline that A if and only if it is borderline that $\neg A$.

(5.2) It is borderline whether A precisely if we have $\mathrm{B}^n A$ for any $n \geqslant 1$.

Given the above assumption that if it is borderline whether Tallulah is tall, then Tallulah is borderline tall and is a borderline case of being tall, etc., the syntax of $\mathrm{QS4M}_{+\mathrm{BF}+\mathrm{FIN}}$ makes it possible to move from 'it is borderline whether A' to a definition of 'a is borderline ϕ' or 'a is a borderline case of being ϕ' as follows:

(5.3) a is borderline ϕ precisely if we have $\mathrm{B}^n \phi a$ for any $n \geqslant 1$.

Since in $\mathrm{QS4M}_{+\mathrm{BF}+\mathrm{FIN}}$ it holds that $\mathrm{B}^n A \leftrightarrow \mathrm{B}A$ for any n, it also comes out as true that it is borderline whether A precisely if we have

Proceedings of the Aristotelian Society Supplementary Volume LXXXIX
doi: 10.1111/j.1467-8349.2015.00244.x

BA, and that a is borderline ϕ precisely if we have $B\phi a$. Thus, in some sense, borderlineness *is* radical higher-order borderlineness and, where 'vagueness' is used to denote borderlineness (as is standardly the case in discussions of hierarchical higher-order vagueness), that vagueness *is* higher-order vagueness.

5.2. *Vagueness*. The borderlineness operator B is not metalinguistic. It tells us nothing about sentences. To say of a sentence that it is borderline is without sense in the definition given.[8] The operator B can, however, be employed to define the vagueness and preciseness of sentences. For example, if one wishes to remove relativization to any specific context in the vagueness of sentences, the following is an option:

(5.4) A sentence A is vague if and only if there is a context in which $B^n A$ for any $n \geqslant 1$.

(5.5) A sentence A is precise if and only if in every context $C^n A \vee C^n \neg A$ for any $n \geqslant 1$.

Since (2.12) and (2.13) are valid principles of $\mathrm{QS4M}_{+\mathrm{BF}+\mathrm{FIN}}$, it results that a sentence A is vague if and only if it is not precise.

Of greater significance than the vagueness of sentences is that of predicates. A common view is that a predicate is vague if it has borderline cases and is otherwise precise. The vagueness of predicates is thus definable without reference to context. After the relevant adjustments of the above simplifications for predicates, $\mathrm{QS4M}_{+\mathrm{BF}+\mathrm{FIN}}$, with the contingency operator ∇ replaced by the interpreted borderlineness operator B, provides:

(5.6) A first-order predicate ϕ is vague precisely if $B^n \exists x B\phi x$ for any $n \geqslant 1$.

(5.7) A first-order predicate ϕ is precise precisely if $\neg B^n \exists x B\phi x$ for any $n \geqslant 1$.

$B^n \exists x B\phi x$ entails $\exists x B\phi x$ (3.8) and $\neg B^n \exists x B\phi x$ entails $\neg \exists x B\phi x$.[9] Thus (5.6) and (5.7) square with the common understanding of

[8] Except that it could refer to the fact that one can't tell whether it is a sentence in some way that allows the construction of a Sorites paradox of the predicate 'is a sentence', of course.

[9] By contraposition from: for any n, $\exists x B\phi x \rightarrow B^n \exists x B\phi x$, via (3.5) and (3.6).

Proceedings of the Aristotelian Society Supplementary Volume LXXXIX
doi: 10.1111/j.1467-8349.2015.00244.x

what counts as the vagueness of first-order predicates. In keeping with the general character of columnar higher-order vagueness, one can tell of precise predicates that they are precise, but of a vague predicate one cannot tell whether one can tell … that it is vague. (Does (5.6) prevent us from identifying vague predicates in any way that matters? Compare the target group of Sorites-susceptible natural language predicates with the natural language predicates that remain when you remove all those of which you know that you can tell that they are precise and judge for yourself.)

5.3. *The Vagueness of 'Vague'*. The account of the vagueness of first-order predicates can be naturally extended to second-order predicates.

(5.8) A second-order predicate Φ is vague precisely if we have $B^n \exists \phi B \Phi \phi$ for any $n \geqslant 1$.

(5.9) A second-order predicate Φ is precise precisely if we have $\neg B^n \exists \phi B \Phi \phi$ for any $n \geqslant 1$.

The question of how to define the vagueness of second-order predicates must not be confused with the quite different question of whether there are borderline Sorites sequences, an issue not considered in this paper.

As 'vague' is itself a second-order predicate, we can also define the vagueness of 'vague'.

(5.10) The predicate 'vague' is vague precisely if we have $B^n \exists \phi B \text{vague} \phi$ for any $n \geqslant 1$.

It appears that with columnar higher-order vagueness one cannot tell whether one can tell … whether 'vague' is vague. There is thus another sense in which vagueness turns out to be higher-order vagueness.

VI

Columnar Higher-Order Vagueness and the Higher-Order Vagueness Paradoxes. The so-called higher-order vagueness paradoxes are all meant to show that introducing higher-order vagueness to solve the Sorites by avoiding determinable sharp boundaries leads to inco-

doi: 10.1111/j.1467-8349.2015.00244.x

herence.[10] The arguments all make some questionable assumption about the position to be refuted. The assumption may be (i) that there are clear (definite, determinate) borderline cases ($\exists x \mathrm{CB}Fx$), or (ii) that it is clear (definite, determinate) that there are borderline cases ($\mathrm{C}\exists x \mathrm{B}Fx$), or, in the most sophisticated version, (iii) that it is clear that there are borderline clear cases ($\mathrm{C}\exists x \mathrm{BC}Fx$).[11] (Here 'B' is used generically as an operator for whatever logic of borderlineness is assumed or intended by the authors.)

In $\mathrm{QS4M}_{+\mathrm{BF}+\mathrm{FIN}}$ the negations of each of these three assumptions are valid. This shows that columnar higher-order vagueness is immune to those so-called paradoxes.

(i) $\exists x \mathrm{CB}Fx$ is equivalent to $\exists x[\mathrm{CB}Fx \wedge \mathrm{B}Fx]$ and thus incompatible with the $\mathrm{QS4M}_{+\mathrm{BF}+\mathrm{FIN}}$ theorem $\forall x[\mathrm{B}Fx \rightarrow \neg\mathrm{CB}Fx]$ (which we get via $\mathrm{B}A \rightarrow \neg\mathrm{CB}A$ from V, df B, PC).

(ii) $\mathrm{C}\exists x \mathrm{B}Fx$ is incompatible with the $\mathrm{QS4M}_{+\mathrm{BF}+\mathrm{FIN}}$ theorem $\neg\mathrm{C}\exists x \mathrm{B}Fx$ (for which see line 23 of appendix II(i)).

(iii) $\mathrm{C}\exists x \mathrm{BC}Fx$ is incompatible with the $\mathrm{QS4M}_{+\mathrm{BF}+\mathrm{FIN}}$ theorem $\neg\mathrm{C}\exists x \mathrm{BC}Fx$, which can be derived from $\neg\mathrm{C}\exists x \mathrm{B}Fx$ in (ii) by substitution of $\mathrm{C}F$ for F.

Thus the arguments and proofs which purport to demonstrate the paradoxicality of higher-order vagueness tell us nothing about the coherence of *columnar* higher-order vagueness. Higher-order vagueness per se is neither paradoxical nor incoherent; only hierarchical higher-order vagueness is.

For those who may find this result puzzling, I here add and explicate my view that at bottom all higher-order vagueness paradoxes rest on a confusion between higher-order borderlineness and what I

[10] For reasons of space I cannot set out the various versions of the paradox here, and refer the reader to their representations in Fara (2003, pp. 196–200), Sainsbury (1991, pp. 167–70), Shapiro (2005, pp. 147–51), Wright (1992, pp. 129–33, 137), and Greenough (2005, pp. 182–3).

[11] For assumption (i) see, for example, Wright (2010, p. 529) in conjunction with Bobzien (2013, p. 40), Greenough (2005, pp. 183–4) with Bobzien (2013, pp. 38–9), and Shapiro (2005, pp. 147–9). For assumption (ii) see, for example, Shapiro (2005, pp. 147–9), Sainsbury (1991, p. 170), and Raffman (2010, p. 510). Generally (ii) is entailed by any theory that assumes the weakened Sorites premiss (WSP) $\mathrm{C}\neg\exists x[\mathrm{C}Fx \wedge \mathrm{C}\neg Fx']$. Given the definition of 'Sorites sequence', including (4.1) and (4.2), (WSP) entails that for any Sorites sequence, $\mathrm{C}\exists x(\neg\mathrm{C}Fx \wedge \neg\mathrm{C}\neg Fx]$ or, what is the same, $\mathrm{C}\exists x \mathrm{B}Fx$. By the same token, (iii) is entailed by any higher-order vagueness theory that assumes for Sorites sequences that $\mathrm{C}\neg\exists x[\mathrm{CC}Fx \wedge \mathrm{C}\neg\mathrm{C}Fx']$, as, for example, Wright (1992, pp. 131–2) does. This can be seen if one substitutes $\mathrm{C}F$ for F in (WSP).

doi: 10.1111/j.1467-8349.2015.00244.x

call borderline nestings, that is, the distribution of the objects of a Sorites sequence into categories that correspond to extensionally non-overlapping classes. I have argued for this position in detail in Bobzien (2013), of which the following paragraphs are just the briefest summary.

People use the expression 'borderline' (and thus 'borderline borderline') to refer to two quite different things. On the one hand, a case is called borderline if it is undecidable whether an object *a* is *F* or *G*. ('There's no way to decide whether this patch is blue or green.' 'I can't tell whether or not she is tall; it's borderline.') Such undecidability could be a matter of cognitive inaccessibility or of insufficient evidence, or have other grounds. Whatever the ultimate reason, *a* is borderline *F*/*G* in this sense if it can be determined neither that *a* is *F* nor that *a* is *G*; and in the special case in which instead of *G* we have $\neg F$, if it can be determined neither that *Fa* nor that $\neg Fa$. Call this kind of being borderline *borderline by undecidability*. By contrast, an object may be called borderline as a way of categorizing it as belonging in neither of two categories C_F, C_G of the same kind (e.g. neighbouring colour categories), but somewhere in between. ('This patch is borderline blue. It is neither blue nor green, but of a colour somewhere in between. Let's say it's blue/green borderline.') Here, an object *a* is borderline *F*/*G* if it can be determined that it is neither *F* nor *G*, and it is taken to fall into a third category, say $C_{F/G}$, introduced in order to accommodate it. Call this kind of being borderline *borderline as category* or *in-between borderlineness*.

The logical structures of these two kinds of borderlineness are quite different. In modal-speak, with $\Diamond$ for 'it is possible':

> Undecidability borderlineness of *a* regarding *F* requires that [$\neg\Diamond$ to determine that *Fa*] $\wedge$ [$\neg\Diamond$ to determine that *Ga*].
>
> In-between borderlineness of *a* regarding categories C_F, C_G requires that $\Diamond$ to determine that [$\neg Fa \wedge \neg Ga$].

From these formulations one sees the following substantial dissimilarites between the two ways of being borderline. Undecidability borderlineness (i) allows something to be both *F* and borderline *F*/*G*, and (ii) allows for the things that are *F* to border those that are *G*. In-between borderlineness (i) does not allow something to be both *F* and borderline *F*/*G*, and (ii) does not allow the things

doi: 10.1111/j.1467-8349.2015.00244.x

that are F to border those that are G. These differences remain if we consider the special case in which instead of G we have $\neg F$:

> Undecidability borderlineness of a regarding F would require that [$\neg\Diamond$ to determine that Fa] $\wedge$ [$\neg\Diamond$ to determine that $\neg Fa$].
>
> In-between borderlineness of a regarding categories C_F, $C_{\neg F}$ would require that $\Diamond$ to determine that [$\neg Fa \wedge \neg\neg Fa$].

With in-between borderlineness, we don't see this special case in ordinary discourse, though it does occur in discussions of vagueness (Bobzien 2013, pp. 30–1).

Each kind of borderlineness has associated with it a kind of borderline-borderlineness. In the first case, it is undecidable whether something is undecidable. Its paradigm case is of $F/\neg F$ borderlineness. In the second case, something is borderline-borderline if it is characterized as belonging in neither of two categories C_F, $C_{F/G}$ (or $C_{F/G}$, C_G) of the same general kind, but somehow in between. Its paradigm is of F/G borderlineness. Again, in modal speak:

> Undecidability borderline-borderlineness of Fa requires that [$\neg\Diamond$ to determine that it is undecidable-borderline whether Fa] $\wedge$ [$\neg\Diamond$ to determine that $\neg$[it is undecidable-borderline whether Fa]].
>
> In-between borderline-borderlineness of a regarding C_F, $C_{F/G}$ requires that $\Diamond$ to determine that [$\neg Fa \wedge \neg Ga \wedge \neg$[$a$ is in-between-borderline F/G]].

In terms of the B-operator, we can say that undecidability borderline-borderlineness is the result of substituting BF for F in the account of BFa. A formal representation in terms of modal logic is natural, and the question whether the semantics of sentences with vague predicates should be two or more valued is at least prima facie open. In-between borderline-borderlineness is the result of introducing a new category, say $C_{F/(F/G)}$, which does not overlap with any of the previous categories and is situated somehow in between categories C_F and $C_{F/G}$. The semantics here is that of first-order logic, since we have nothing but the distribution of objects into categories describing non-empty extensionally non-overlapping classes. For purposes of distinction, we say that the former iteration of 'bor-

doi: 10.1111/j.1467-8349.2015.00244.x

derline' expresses *higher-order borderlineness* and the latter *borderline nestings*.

It is easy to hop from one kind of borderlineness to the other, for instance by introducing a new category $C_{(F/G)}$ for those cases that are F/G undecidable, or a new category $C_{(F/\neg F)}$ for those cases that are $F/\neg F$ undecidable.[12] This illustrates how an unintentional shift from one kind of borderlineness to the other may readily occur. In the second case, with such a shift *the categories* C_F *and* $C_{\neg F}$ *would become mere contraries, despite the fact that their names still indicate contradictoriness*. (Either way such introduction of a new category $C_{(F/G)}$ or $C_{(F/\neg F)}$ requires that one assume one *can decide* that the cases to go into the category are undecidable.) Those who describe hierarchical higher-order vagueness commonly vacillate between the two kinds of borderlineness without acknowledging this (cf. Bobzien 2013, pp. 33–41).

VII

Replies to Some Common Objections to Columnar Higher-Order Vagueness.

Objection 1: The vagueness axiom **V** is incompatible with the evidence. There *are* clear borderline cases. *Objection* 1a: There are clear borderline cases. Here, this *a* is a clear borderline case of *F*. So axiom **V** is incompatible with the evidence. *Reply*: For you to be able to present me with an *a* that is a clear (and hence non-borderline) borderline case of *F*, you must be able to distinguish *a* from the non-borderline *F* cases and the non-borderline $\neg F$ cases. But this means, I maintain, that you have, perhaps inadvertently, and at least temporarily, shifted to the above-described in-between borderlineness, or still another kind of borderlineness, and that you equivocate on 'borderline'.

An alternative version of objection 1 focuses on the interpretation of the C-operator. *Objection* 1b: I can tell that I can't tell whether this object is *F*. So this object is clearly borderline *F* and axiom **V** is

12 Bobzien (2013, §5) lists five different possible ways of merging the two notions of borderline. The first is the one with the special case with $\neg F$ instead of *G*. This is common with undecidability borderlineness, but not with in-between borderlineness, where it introduces a Third Kind, in Crispin Wright's terms (Wright 2003).

Proceedings of the Aristotelian Society Supplementary Volume lxxxix
doi: 10.1111/j.1467-8349.2015.00244.x

incompatible with the evidence. *Reply*: Given the interpretation of operators C and B (§§I and II above), if you *actually* can tell that you can't tell (as opposed to mistakenly believing you can), there are two possibilities: (i) you use a notion of tellability that includes lack of qualification of individuals and thus is not the one suggested. ('I can tell that *I* can't tell whether *Fa*, but don't rule out that someone better qualified might be able to tell that *Fa*'); (ii) you use the notion of tellability introduced above. That notion abstracts from all individual human handicaps. As a result, if you can tell that you can't tell, there must be something *in a with regard to the predicate F* that you pick up on and that allows you to distinguish *a* from those cases that are non-borderline *F* and non-borderline not-*F*. In that case, again, I maintain that you have, perhaps inadvertently and at least temporarily, shifted to using 'borderline' for in-between borderlineness, and not as it is used in this paper, that is, for undecidability borderlineness, and that you may be equivocating on 'borderline'.

Objection 2: Axiom 4 is incompatible with the evidence. There *are* unclear clear cases. More precisely, there are cases *a* of vague predicates *F* where it is clear that *Fa* but it is not clear that this is so. *Reply*: Unlike Williamson's notion of knowability, the notion of clarity or tellability that is part of columnar higher-order vagueness is luminous. That is, it is used in the sense in which 'it is clear that this is blue, but it is not clear that this is so' seems absurd (Bobzien 2012, pp. 194–6). In terms of tellability: since the relevant subjects who are the criterion for tellability of C*Fa* do not suffer from any handicaps regarding C*Fa*, and this includes that they fully understand the interpreted C-operator as set out in this paper, there is nothing that prevents their being able to tell that they can tell that *Fa*. (I have argued this point slightly differently in Bobzien 2010, pp. 6–10; 2012, pp. 204–10.)

Objection 3: Columnar higher-order vagueness introduces sharp boundaries for vague predicates, *in the sense that there is a last clear case*, and thus is no improvement over other theories that do so. *Reply*: In the case of *natural language* vague expressions *F*, 'clearly *F*', 'definitely *F*', etc., sharp boundaries are counterintuitive even if they are for some reason indeterminable. This is not so for *technical* terms introduced to represent the structural elements of vague expressions that give rise to the Sorites. Here intuitions and empirical evidence play no direct role. The notion of borderlineness defined in terms of $\mathrm{QS4M}_{+\mathrm{BF}+\mathrm{FIN}}$ and tellability is a technical term of this

Proceedings of the Aristotelian Society Supplementary Volume LXXXIX
doi: 10.1111/j.1467-8349.2015.00244.x

kind, and crucially, it is a component of that notion that any borderline/non-borderline boundary is indeterminable. There is nothing untoward about boundaries between objects that do and objects that don't satisfy a technical term where that term itself makes those boundaries indeterminable, and has been selected precisely for that reason. (A more detailed reply can be found in Bobzien 2016.)

Objection 4: Normal columnar higher-order vagueness is no advance over epistemicism, since it introduces a sharp boundary between the true Fa_1, ..., Fa_i and the false Fa_{i+1}, ..., Fa_n. *Reply*: No, it doesn't. $\mathrm{QS4M}_{+\mathrm{BF}+\mathrm{FIN}}$ columnar higher-order vagueness is compatible with a sharp border between the true cases of *F* and the false cases of *F*, as we find it in, for example, Williamson's epistemicism, but it does not entail such a sharp border.

Objection 5: $\mathrm{QS4M}_{+\mathrm{BF}+\mathrm{FIN}}$ with the interpreted C-operator is incompatible with the possible existence of borderline cases in Sorites-susceptible predicates. The argument goes like this. It is a theorem of $\mathrm{QS4M}_{+\mathrm{BF}+\mathrm{FIN}}$ that (1) if there is a borderline case of some predicate, it is not clear that there is a borderline case of that predicate. It is another theorem of $\mathrm{QS4M}_{+\mathrm{BF}+\mathrm{FIN}}$ that (2) if one cannot rule out that there is a borderline case, there is a borderline case. It is an assumption of the theory of columnar higher-order vagueness that (3) for any Sorites-susceptible predicate, one cannot rule out that there are borderline cases. Hence, by *modus ponens*, (4) there are borderline cases for Sorites-susceptible predicates. Hence, (5) we can tell that (and that is, it is clear that) there are such borderline cases: we have just shown this. But by (1), also (6) one cannot tell (and that is, it is not clear) that there is such a borderline case. Given (5) and (6), it follows that (7) the theory of columnar higher-order vagueness is incoherent.

(1)	$\exists x \mathrm{B}Fx \rightarrow \neg \mathrm{C} \exists x \mathrm{B}Fx$	theorem of $\mathrm{QS4M}_{+\mathrm{BF}+\mathrm{FIN}}$
(2)	$\neg \mathrm{C} \neg \exists x \mathrm{B}Fx \rightarrow \exists x \mathrm{B}Fx$	theorem of $\mathrm{QS4M}_{+\mathrm{BF}+\mathrm{FIN}}$
(3)	$\neg \mathrm{C} \neg \exists x \mathrm{B}Fx$	assumption of CHOV
(4)	$\exists x \mathrm{B}Fx$	(2), (3) MP
(5)	$\mathrm{C} \exists x \mathrm{B}Fx$	since (4) just shown
(6)	$\neg \mathrm{C} \exists x \mathrm{B}Fx$	(1), (4) MP
(7)	$\perp$ (contradiction)	(5), (6)

Reply: To start, two remarks. (3) is an assumption, not a theorem; and (5) is not derived by Rule N, but from the tacit additional, and

doi: 10.1111/j.1467-8349.2015.00244.x

perfectly acceptable, assumption that if one has just shown that A, one can tell that A. Now, in order to infer from A that one can tell that A *because one has just shown that* A (i.e. derived A from some premisses $P_1, \ldots, P_n$), one needs to be able to tell that $P_1, \ldots$, and P_n. If P but one can't tell that P, probabilistic logic aside, one can't infer anything from P. So there is a second tacit assumption, (8), *that one can tell* that one can't rule out that there are borderline cases:

(8) $\mathrm{C}\neg\mathrm{C}\neg\exists x \mathrm{B}Fx$

Without (8), (5) cannot be derived. (5) requires that one has just shown (4) by deriving it from (2) and (3), which in turn requires that one can tell that (3).

Columnar higher-order vagueness assumes (3) but *not* (8).[13] Nor does it follow from postulating (3) that (8). (If I assume that one can't rule out that A, I don't thereby assume that one can tell that one can't rule out that A.) In fact, it is *part of the theory of columnar higher-order vagueness* that, if $\exists x \mathrm{B}Fx$, then it is borderline whether (8), that is,

(9) $\exists x \mathrm{B}Fx \rightarrow \mathrm{BC}\neg\mathrm{C}\neg\exists x \mathrm{B}Fx$

Hence by the definition of B together with *modus ponens* and (4),

(10) $\neg\mathrm{CC}\neg\mathrm{C}\neg\exists x \mathrm{B}Fx \wedge \neg\mathrm{C}\neg\mathrm{C}\neg\mathrm{C}\neg\exists x \mathrm{B}Fx$

and hence by axiom 4 (via $\neg\mathrm{CC}A \leftrightarrow \neg\mathrm{C}A$) together with $\wedge$-elimination,

(11) $\neg\mathrm{C}\neg\mathrm{C}\neg\exists x \mathrm{B}Fx$

which is incompatible with the implicit assumption (8) of the objection.

Related arguments that aim to prove that with the theory of columnar higher-order vagueness one can show of any borderline case in a Sorites sequence that it is a borderline case fail along the same lines. Using X as meta-metavariable, one can say that they all at some point infer $\mathrm{B}X$ from $\neg\mathrm{C}\neg\mathrm{B}X \rightarrow \mathrm{B}X$ and $\neg\mathrm{C}\neg\mathrm{B}X$, and then infer $\mathrm{CB}X$ from the fact that they just inferred $\mathrm{B}X$. Each time, this second inference requires the tacit assumption that one can tell that $\neg\mathrm{C}\neg\mathrm{B}X$ (i.e. that $\mathrm{C}\neg\mathrm{C}\neg\mathrm{B}X$). But $\mathrm{C}\neg\mathrm{C}\neg\mathrm{B}X$ is not available to the

[13] If one were to assume (8) in any modal system with **K** and **T** (of which S4M is one) this would be like adding axiom **E** [$\Diamond A \rightarrow \Box\Diamond A$], in other words, a patently unwise move.

doi: 10.1111/j.1467-8349.2015.00244.x

objector. In fact, since $BX \rightarrow \neg C\neg C\neg BX$ is part of $\mathrm{QS4M}_{+\mathrm{BF}+\mathrm{FIN}}$, from B*X* the negation of the tacit assumption can be inferred.[14]

Appendix I: Sketch of a proof that the McKinsey axiom **M** is logically equivalent to the vagueness axiom **V** in system T, and hence in S4.

V entails **M**

(1) $[\neg CA \wedge \neg C\neg A] \rightarrow [\neg C[\neg CA \wedge \neg C\neg A] \wedge \neg C\neg[\neg CA \wedge \neg C\neg A]]$ **V**
(2) $[\neg CA \wedge \neg C\neg A] \rightarrow \neg C[\neg CA \wedge \neg C\neg A]$ (1) PC
(3) $C[\neg CA \wedge \neg C\neg A] \rightarrow \neg[\neg CA \wedge \neg C\neg A]$ (2) ctrp
(4) $C[\neg CA \wedge \neg C\neg A] \rightarrow [CA \vee C\neg A]$ (3) DeMorgan
(5) $[C\neg CA \wedge C\neg C\neg A] \rightarrow [CA \vee C\neg A]$ (4) K2
(6) $[C\neg CA \wedge C\neg C\neg A] \rightarrow [\neg C\neg CA \vee C\neg A]$ (5) **T**
(7) $[C\neg C\neg A \wedge C\neg C\neg\neg A] \rightarrow [\neg C\neg CA \vee C\neg A]$ (6) *Subst. Equiv.*
(8) $[C\neg C\neg A \rightarrow \neg C\neg CA] \vee [C\neg C\neg\neg A \rightarrow C\neg A]$ (7) PC
(9) $[C\neg C\neg A \rightarrow \neg C\neg CA] \vee [C\neg C\neg\neg A \rightarrow \neg C\neg C\neg A]$ (8) **T**
(10) $[C\neg C\neg A \rightarrow \neg C\neg CA] \vee [C\neg CA \rightarrow \neg C\neg C\neg A]$ (9) *Subst. Equiv.*
(11) $[C\neg C\neg A \rightarrow \neg C\neg CA] \vee [C\neg C\neg A \rightarrow \neg C\neg CA]$ (10) PC
(12) $C\neg C\neg A \rightarrow \neg C\neg CA$ (11) PC

M entails **V**

(i) Sketch of proof that **V** is logically equivalent to **V′**, that is, $[\neg C\neg A \wedge \neg CA] \rightarrow \neg C[\neg C\neg A \wedge \neg CA]$, in system T. Lines (1)–(3) show that **V** entails **V′**, lines (2)–(4) that **V′** entails **V**.

(1) $[\neg C\neg A \wedge \neg CA] \rightarrow [\neg C[\neg C\neg A \wedge \neg CA] \wedge \neg C\neg[\neg C\neg A \wedge \neg CA]]$ **V**
(2) $[\neg C\neg A \wedge \neg CA] \rightarrow \neg C\neg[\neg C\neg A \wedge \neg CA]]$ **T** ctrp
(3) $[\neg C\neg A \wedge \neg CA] \rightarrow \neg C[\neg C\neg A \wedge \neg CA]$ (1) PC
(4) $[\neg C\neg A \wedge \neg CA] \rightarrow [\neg C[\neg C\neg A \wedge \neg CA] \wedge \neg C\neg[\neg C\neg A \wedge \neg CA]]$ (2), (3) PC

(ii) Proof sketch from **M** to **V′**

(1) $C\neg C\neg A \rightarrow \neg C\neg CA$ **M**
(2) $\neg C\neg[\neg C\neg A \rightarrow CA]$ (1) K7
(3) $\neg C\neg[[\neg C\neg A \wedge \neg CA] \rightarrow CA]$ (2) PC, **T**
(4) $C[\neg C\neg A \wedge \neg CA] \rightarrow \neg C\neg CA$ (3) K7

[14] Thanks go to Andrew Bacon, Jennifer Hornsby, Marko Malink, Daniel Rothschild, Ian Rumfitt and Bruno Whittle for valuable comments on individual sections of this paper, to Geoffrey Moseley for Teutonism removal, to Crispin Wright for helpful discussion of some central ideas, and to Rosanna Keefe for her patience.

Proceedings of the Aristotelian Society Supplementary Volume LXXXIX
doi: 10.1111/j.1467-8349.2015.00244.x

(5)	$\neg\neg C\neg CA \rightarrow \neg C[\neg C\neg A \wedge \neg CA]$	(4) ctrp
(6)	$C\neg CA \rightarrow \neg C[\neg C\neg A \wedge \neg CA]$	(5) PC
(7)	$[C\neg C\neg A \wedge C\neg CA] \rightarrow \neg C[\neg C\neg A \wedge \neg CA]$	(6) PC
(8)	$C[\neg C\neg A \wedge \neg CA] \rightarrow \neg C[\neg C\neg A \wedge \neg CA]$	(7) K1
(9)	$[\neg C\neg A \wedge \neg CA] \rightarrow \neg C[\neg C\neg A \wedge \neg CA]$	(8) PC—see subproof

(i)	$C[\neg C\neg A \wedge \neg CA] \rightarrow \neg C[\neg C\neg A \wedge \neg CA]$	(8)
(ii)	$[C[\neg C\neg A \wedge \neg CA] \rightarrow \neg C[\neg C\neg A \wedge \neg CA]] \rightarrow \neg C[\neg C\neg A \wedge \neg CA]$	(i) PC
(iii)	$[C[\neg C\neg A \wedge \neg CA] \rightarrow \neg C[\neg C\neg A \wedge \neg CA] \wedge [\neg C\neg A \wedge \neg CA]] \rightarrow \neg C[\neg C\neg A \wedge \neg CA]$	(ii) PC
(iv)	$[C[\neg C\neg A \wedge \neg CA] \rightarrow \neg C[\neg C\neg A \wedge \neg CA]] \rightarrow [[\neg C\neg A \wedge \neg CA] \rightarrow \neg C[\neg C\neg A \wedge \neg CA]]$	(iii) PC
(v)	$[\neg C\neg A \wedge \neg CA] \rightarrow \neg C[\neg C\neg A \wedge \neg CA]$	(i), (iv) MP

Appendix II: Sketch of proof that Principle $V_\exists$ and its converse are valid in $QS4M_{+BF+FIN}$.

(i) Sketch of how $V_\exists$, that is, $\exists x BFx \rightarrow B\exists x BFx$, can be obtained in $QS4M_{+BF+FIN}$.

(1)	$\forall x[BFx \rightarrow B^2Fx]$	V, ∀
(2)	$\neg\exists x[BFx \wedge \neg B^2Fx]$	(1) PC, df ∃
(3)	$\forall x[CFx \rightarrow C^2Fx]$	axiom 4, ∀
(4)	$\neg\exists x[CFx \wedge \neg C^2Fx]$	(3) PC, df ∃
(5)	$\forall x[C\neg Fx \rightarrow C^2\neg Fx]$	axiom 4, ∀
(6)	$\neg\exists x[C\neg Fx \wedge \neg C^2\neg Fx]$	(5) PC, df ∃
(7)	$\forall x[CFx \vee C\neg Fx \vee BFx]$	df B, T, PC, ∀
(8)	$B^2A \leftrightarrow [\neg CBA \wedge \neg C\neg BA]$	df B
(9)	$B^2A \rightarrow \neg CBA$	(8) PC
(10)	$\neg\exists x[BFx \wedge CBFx]$	(1), (9) LPC
(11)	$CBA \rightarrow \neg C^2A$	T, df B, PC, N, K
(12)	$CBA \rightarrow \neg C^2\neg A$	T, df B, PC, N, K
(13)	$\neg\exists x[CFx \wedge CBFx]$	(5), (11)
(14)	$\neg\exists x[C\neg Fx \wedge CBFx]$	(6), (12)
(15)	$\neg\exists x CBFx$	(7), (10), (13), (14)
(16)	$C\neg\exists x CBFx$	(15) rule N
(17)	$\neg C\neg\forall x[Fx \rightarrow CFx]$	**FINAX** for one-place predicates with C-operator
(18)	$\forall x[Fx \rightarrow CFx] \rightarrow [\exists x Fx \rightarrow \exists x CFx]$	QPC
(19)	$\neg C\neg[\exists x Fx \rightarrow \exists x CFx]$	(17), (18) $A \rightarrow B \vdash \Diamond A \rightarrow \Diamond B$, MP

doi: 10.1111/j.1467-8349.2015.00244.x

(20) $\mathrm{C}\exists x Fx \rightarrow \neg\mathrm{C}\neg\exists x \mathrm{C}Fx$ (19) K7
(21) $\mathrm{C}\exists x \mathrm{B}Fx \rightarrow \neg\mathrm{C}\neg\exists x \mathrm{CB}Fx$ (20) BF/F
(22) $\mathrm{C}\neg\exists x \mathrm{CB}Fx \rightarrow \neg\mathrm{C}\exists x \mathrm{B}Fx$ (21) ctrp
(23) $\neg\mathrm{C}\exists x \mathrm{B}Fx$ (16), (22) MP
(24) $\exists x \mathrm{B}Fx \rightarrow \neg\mathrm{C}\neg\exists x \mathrm{B}Fx$ T ctrp
(25) $\exists x \mathrm{B}Fx \rightarrow \neg\mathrm{C}\exists x \mathrm{B}Fx$ (23) PC
(26) $\exists x \mathrm{B}Fx \rightarrow \mathrm{B}\exists x \mathrm{B}Fx$ (24), (25) PC, df B

(ii) Sketch of how $\mathrm{V}_{\exists c}$, the converse of $\mathrm{V}_{\exists}$, that is, $\mathrm{B}\exists x\mathrm{B}Fx \rightarrow \exists x\mathrm{B}Fx$, can be obtained in $\mathrm{QS4M}_{+\mathrm{BF}+\mathrm{FIN}}$.

(1) $\forall x[\mathrm{C}Fx \vee \mathrm{C}\neg Fx] \rightarrow \forall x[\mathrm{C}Fx \vee \mathrm{C}\neg Fx]$ PC
(2) $\forall x[\mathrm{C}Fx \vee \mathrm{C}\neg Fx] \rightarrow \forall x[\mathrm{CC}Fx \vee \mathrm{CC}\neg Fx]$ (1) axiom 4
(3) $\forall x[\mathrm{C}Fx \vee \mathrm{C}\neg Fx] \rightarrow \forall x\mathrm{C}[\mathrm{C}Fx \vee \mathrm{C}\neg Fx]$ (2) K4
(4) $\forall x[\mathrm{C}Fx \vee \mathrm{C}\neg Fx] \rightarrow \mathrm{C}\forall x[\mathrm{C}Fx \vee \mathrm{C}\neg Fx]$ (3) **BF** trans
(5) $\forall x[\mathrm{C}Fx \vee \mathrm{C}\neg Fx] \rightarrow [\mathrm{C}\forall x[\mathrm{C}Fx \vee \mathrm{C}\neg Fx] \vee \mathrm{C}\exists x\mathrm{B}Fx]$ (4) PC
(6) $\forall x[\mathrm{C}Fx \vee \mathrm{C}\neg Fx] \rightarrow [\mathrm{C}\neg\exists x\mathrm{B}Fx \vee \mathrm{C}\exists x\mathrm{B}Fx]$ (5) df B, df $\exists$
(7) $\neg[\mathrm{C}\neg\exists x\mathrm{B}Fx \vee \mathrm{C}\exists x\mathrm{B}Fx] \rightarrow \neg\forall x[\mathrm{C}Fx \vee \mathrm{C}\neg Fx]$ (6) ctrp
(8) $[\neg\mathrm{C}\neg\exists x\mathrm{B}Fx \wedge \neg\mathrm{C}\exists x\mathrm{B}Fx] \rightarrow \neg\forall x[\mathrm{C}Fx \vee \mathrm{C}\neg Fx]$ (7) DeMorgan
(9) $\mathrm{B}\exists x\mathrm{B}Fx \rightarrow \exists x\mathrm{B}Fx$ (8) df $\exists$, df B $\times 2$

All Souls College
University of Oxford
Oxford OX1 4AL
UK
susanne.bobzien@philosophy.ox.ac.uk

References

Beall, JC (ed.) 2003: *Liars and Heaps: New Essays on Paradox*. Oxford: Oxford University Press.

Bobzien, Susanne 2010: 'Higher-Order Vagueness, Radical Unclarity and Absolute Agnosticism'. *Philosophers' Imprint*, 10(10), pp. 1–30.

——2012: 'If It's Clear, then It's Clear that It's Clear, or Is It?—Higher-Order Vagueness and the S4 Axiom'. In Benjamin Morison and Katerina Ierodiakonou (eds.), *Episteme, etc.: Essays in Honour of Jonathan Barnes*, pp. 189–212. Oxford: Oxford University Press.

——2013: 'Higher-Order Vagueness and Borderline Nestings: A Persistent Confusion'. *Analytic Philosophy*, 54(1), pp. 1–43.

——2016: 'How to Solve the Sorites Paradox with Normal Modal Logic'.

doi: 10.1111/j.1467-8349.2015.00244.x

In Otávio Bueno (ed.), *The Sorites Paradox*. Dordrecht: Springer.

Cresswell, Max J. 2001: 'How to Complete Some Modal Predicate Logics'. In Michael Zakharyaschev, Krister Segerberg, Maarten de Rijke and Heinrich Wansing (eds.), *Advances in Modal Logic*, vol. 2, pp. 155–78. Stanford, CA: CSLI Publications.

Dietz, Richard, and Sebastiano Moruzzi (eds.) 2010: *Cuts and Clouds: Vagueness, Its Nature and Its Logic*. Oxford: Oxford University Press.

Dummett, Michael 1975: 'Wang's Paradox'. *Synthèse*, 30. pp. 301–24. Reprinted in his *Truth and Other Enigmas*, pp. 248–68. London: Duckworth, 1978. Also reprinted in Keefe and Smith 1997, pp. 99–118.

Fara, Delia Graff 2003: 'Gap Principles, Penumbral Consequence, and Infinitely Higher-Order Vagueness'. In Beall 2003, pp. 195–221. Originally published under the name Delia Graff.

Greenough, Patrick 2005: 'Higher-Order Vagueness'. *Proceedings of the Aristotelian Society Supplementary Volume* 79, pp. 167–90.

Hughes, G. E., and M. J. Cresswell 1996: *A New Introduction to Modal Logic*. London: Routledge.

Keefe, Rosanna 2000: *Theories of Vagueness*. Cambridge: Cambridge University Press.

——and Peter Smith (eds.) 1997: *Vagueness: A Reader*. Cambridge, MA: MIT Press.

Raffman, Diana 2010: 'Demoting Higher-Order Vagueness'. In Dietz and Moruzzi 2010, pp. 509–22.

Sainsbury, R. M. 1990: 'Concepts Without Boundaries'. Inaugural lecture, King's College London. Reprinted in Keefe and Smith 1997, pp. 251–64.

——1991: 'Is There Higher-Order Vagueness?'. *Philosophical Quarterly*, 41, pp. 167–82.

Shapiro, Stewart 2005: 'Context, Conversation, and So-called "Higher-Order Vagueness"'. *Proceedings of the Aristotelian Society Supplementary Volume* 79, pp. 147–65.

Williamson, Timothy 1994: *Vagueness*. London: Routledge.

——1999: 'On the Structure of Higher-Order Vagueness'. *Mind*, 108, pp. 127–42.

Wright, Crispin 1987: 'Further Reflections on the Sorites Paradox'. *Philosophical Topics*, 15, pp. 227–90. Reprinted in Keefe and Smith 1997, pp. 204–50.

——1992: 'Is Higher-Order Vagueness Coherent?'. *Analysis*, 52, pp. 129–39.

——2003: 'Vagueness: A Fifth Column Approach'. In Beall 2003, pp. 84–105.

——2010: 'The Illusion of Higher-Order Vagueness'. In Dietz and Moruzzi 2010, pp. 523–49.

Proceedings of the Aristotelian Society Supplementary Volume LXXXIX
doi: 10.1111/j.1467-8349.2015.00244.x