The conbinatorics of Stoic Conjunction: Hipparchus refuted, Chrysippus vindicated

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THE COMBINATORICS OF STOIC CONJUNCTION: HIPPARCHUS REFUTED, CHRYSIPPUS VINDICATED

SUSANNE BOBZIEN

Dieser Aufsatz ist dem Andenken von Michael Frede gewidmet — einem wahren Philosophen und wahren Freund. Ohne sein grundlegendes Werk über die stoische Logik und ihre Entwicklung in der Antike wäre dieser Aufsatz nicht möglich.

1. Introduction

[Chrysippus] says that the number of conjunctions [constructible] from ten assertibles exceeds one million . . . Yet many mathematicians have refuted Chrysippus; among them is Hipparchus, who has demonstrated that his error in the calculation is huge, since the affirmative produces 103,049 conjoined assertibles, and the negative 310,952. (Plut. *Stoic. repugn.* 1047 C–E)

Chrysippus says that the number of conjunctions [constructible] from only ten assertibles exceeds one million. However, Hipparchus refuted this, demonstrating that the affirmative encompasses 103,049 conjoined assertibles and the negative 310,952. (Plut. *Quaest. conv.* 732 F–733 A)

Chrysippus, third head of the Stoa and one of the two greatest logicians in antiquity, made his statement about the number of conjunctions in the third century BCE. Hipparchus, famous astronomer and writer of a work in which he discussed combinatorics, flourished in the second half of the second century BCE. The claims of

This paper has profited greatly from detailed written comments by Fabio Acerbi and István Bodnár. An early version was presented at the Seminar 'Hellenistic Science and Scholarship' at the Radcliffe Institute, Harvard, in May 2006, and I am grateful to the participants for their helpful remarks, in particular to Alexander Jones and the late Ian Mueller. Thanks go also to my brother Matthias Bobzien for first bringing the tree-structure method to my attention. Finally, I wish to acknowledge support from the Institute for Advanced Studies, Princeton, the National Endowment for the Humanities, and the Mellon Foundation.

these two thinkers are incompatible. Who is right, and why does it matter?

Recently, some mathematicians and historians of mathematics, most notably Fabio Acerbi, have provided a reconstruction of Hipparchus' calculations that allows us to explain how Hipparchus came up with almost exactly[1] the numbers Plutarch gives us.[2] Their result is important for the history of mathematics, since it shows how far combinatorics had been developed in the second century BCE. Acerbi's reconstruction is admirable and my purpose in this paper is not to find fault with it. Hipparchus, it seems, got his mathematics right.[3] What I suggest in this paper is that he got his Stoic logic wrong; moreover, that Chrysippus not only got his Stoic logic right (which would not be that surprising), but also got his mathematics right; in other words, that, within the context of Stoic logic, 'the number of conjunctions [constructible] from ten assertibles exceeds one million'.

The motivation to vindicate Chrysippus is not pure righteousness. Rather, by showing how—from the perspective of Stoic logic—Chrysippus was right and Hipparchus was wrong, new light can be shed on the Stoic notions of conjunction and assertible (proposition, *axiōma*); perhaps also on the state of the art of combinatorics in the third century BCE. Furthermore, by utilizing evidence regarding the development of logic from the third to the second centuries BCE, in particular on the amalgamation of Peripatetic and Stoic theories, it can be explained what *logical* notions Hipparchus must have used to obtain his two numbers, and why he would have used those notions. Thus we close an explanatory gap in the accounts of Stanley, Habsieger *et al*., and

[1] 'Almost exactly', since, as Acerbi and Habsieger *et al*. have convincingly argued, instead of '2' as the last numeral in the case of negations, there should be a '4'.

[2] The significance of Hipparchus' first number was first recognized by D. Hough and published by R. P. Stanley, 'Hipparchus, Plutarch, Schröder, and Hough', *American Mathematical Monthly*, 104 (1997), 344–50; the significance of Hipparchus' second number was explained by L. Habsieger, M. Kazarian, and S. Lando, 'On the Second Number of Plutarch' ['Plutarch'], *American Mathematical Monthly*, 105 (1998), 446. These discoveries triggered a brilliant and exemplary study on ancient combinatorics by Fabio Acerbi: 'On the Shoulders of Hipparchus: A Reappraisal of Ancient Greek Combinatorics' ['Hipparchus'], *Archive of the History of the Exact Sciences*, 57 (2003), 465–502. I urge everyone with an interest in the present paper to read Acerbi's paper, to which mine is indebted on many points.

[3] The occurrence of the numeral '2' instead of '4' is more likely to be a transmission error in our sources than a calculation error on the part of Hipparchus; see e.g. Acerbi, 'Hipparchus', 475.

Acerbi. The paper has three main parts: a first, reconstructing the logical notions of conjunction and negation that Hipparchus used (Section 2); a second, providing a reconstruction of Chrysippus' calculations based on the relevant surviving evidence on Stoic logic (Sections 3–5); a third, explaining the developments in logic which may have made Hipparchus think that the logical notions he used in his calculations were Stoic (Section 6).

2. The logical basis of Hipparchus' calculations

It will be helpful to provide some idea about what Hipparchus did when trying to prove Chrysippus wrong, and on what logical assumptions he based his proof. Stanley, Habsieger *et al.*, and in particular Acerbi have explained by what calculations Hipparchus may have obtained the numbers which Plutarch relates. Acerbi also makes some ingenious suggestions as to how Hipparchus could have arrived at his numbers by invoking certain elements of Stoic logic.[4] I accept—by and large—Acerbi's proposal regarding Hipparchus' method of obtaining his numbers, but differ in my assessment of how and how far Stoic logic played a role in it.

From the two Plutarch passages quoted above we can gather that Hipparchus divides the relevant conjunctions$_H$[5] into two groups: those which *the affirmative* produces, and those which *the negative* produces. He comes up with a number for ten atomic assertibles for either group. Presumably we are to add those two numbers and realize that the sum falls short of one million. I focus on each of the two groups separately. My assumption is that in either case Hipparchus has a general method which works for conjunctions$_H$ with fewer or more than ten assertibles, and that this method involves what I call a basic pattern and variations of this pattern obtained by different ways of grouping together the elements in this pattern by means of some—actual or envisaged—bracketing device. I use p, q, r,… for atomic assertibles,[6] '$\wedge$' as conjunctor, and square

[4] Acerbi, 'Hipparchus', 469–75.

[5] I use the subscript 'H' when talking about conjunctions so classified by Hipparchus, since—as will become clear—his notion of conjunction squares neither fully with that of the Stoics nor with that of contemporary logic.

[6] In this paper I use the terms 'atomic' and 'molecular' in the sense in which modern propositional logic does. With these I contrast 'simple' and 'non-simple', the Stoic terms for assertibles that are/are not constructed from a plurality of assertibles

brackets instead of the respective Stoic notations (for which see below).

For the conjunctions$_{H}$ *from the affirmative*, I assume that Hipparchus took the basic pattern for some n and calculated the ways in which one can partition the pattern by bracketing, including no bracketing.[7] The resulting numbers correspond to the (little) Schröder numbers (or super-Catalan numbers) for $n \geq 2$.[8] Thus the basic pattern for $n=2$ is $p \wedge q$, and we obtain the conjunction

$p \wedge q$.

For $n=3$, the basic pattern is $p \wedge q \wedge r$, and we obtain

$p \wedge q \wedge r$
$[p \wedge q] \wedge r$
and $p \wedge [q \wedge r]$.

For the basic pattern with $n=4$ we obtain eleven possibilities, and with $n=10$ the desired 103,049.

For the conjunctions$_{H}$ *from the negative* there is no equally obvious method. If we want to get the (corrected) number from Plutarch, we have to make the following assumptions. For any n Hipparchus included only molecular expressions with the basic pattern of a single negation sign followed by the basic pattern for the affirmative with n atomic sentences. He allows for the presence or absence of an opening bracket between the negation sign and the rest of the expression. In the case of the absence of such a bracket, the scope of the negation sign is the first atomic assertible. (All this was realized by Acerbi.[9])

(or the same assertible used more than once). For this latter pair of terms see below, sect. 2.

[7] Here and throughout when I talk about Hipparchus' bracketing, I do not assume that Hipparchus had actual syntactic conventions for bracketing. Unlike Chrysippus, he may have just used abstract mathematical considerations without any exact syntactic rules being spelt out.

[8] These numbers are given by the recurrence relation $s(n)=3(2n-3)s(n-1)-(n-3)s(n-1)/n$, with s for 'Schröder number'. They can be interpreted as the total number of bracketings of a string of n letters (excluding brackets around single letters and around the entire expression). Of course, Hipparchus need not have used precisely the formula given. There are alternative ways of obtaining the same numbers, and Acerbi notes rightly that it is possible to use fairly basic mathematical methods to get the same results, as long as one assumes Hipparchus had an adequate grasp of the recursive element in the calculations (Acerbi, 'Hipparchus', 497–8).

[9] See Acerbi, 'Hipparchus', 474.

For n=2, the basic pattern is ¬p∧q and we obtain the two cases

(i) ¬p∧q
(ii) ¬[p∧q].

For n=3, the basic pattern is ¬p∧q∧r and we obtain seven cases:

(i) ¬p∧q∧r
(ii) ¬[p∧q∧r]
(iii) ¬[p∧q]∧r
(iv) ¬p∧[q∧r]
(v) ¬[[p∧q]∧r]
(vi) ¬[p∧[q∧r]]
(vii) [¬p∧q]∧r.

For the basic pattern with n=4 we obtain 28 possibilities, and with n=10 the desired 310,954.[10] A general formula which produces these numbers is $(s(n)+s(n+1))/2$, with s for 'Schröder number'. Of course this does not entail that Hipparchus himself obtained his number(s) by using precisely this formula.[11] On the assumption that the reconstruction of Hipparchus' numbers is in all important respects accurate, we can infer several things about Hipparchus' notions of negation and conjunction.[12]

[10] For convenience, here is the list of numbers we obtain for Hipparchus' conjunctions$_H$ *from the affirmatives* and *from the negatives* for 2 to 10:

	from the affirmative	from the negative
n=2	1	2
n=3	3	7
n=4	11	28
n=5	45	121
n=6	197	550
n=7	903	2591
n=8	4279	12536
n=9	20793	61921
n=10	103049	310954

The values for n=1 are left out, since there are no one-atomic-assertible 'conjunctions', and it is thus likely that Hipparchus did not assign any values for the cases with n=1.

[11] Expanding on Habsieger *et al*., 'Plutarch', Acerbi, 'Hipparchus', 474, suggests that and explains how Hipparchus may have come to use the formula given in the text. But there are mathematically more basic (and more cumbersome) ways of calculating Hipparchus' negative numbers, and we do not know what method Hipparchus actually used.

[12] In sect. 3 I provide further reasons for this assumption.

Focusing on what he says about affirmatives, we can see the following: Hipparchus is aware of the possibility of alternative bracketing. He counts the same sequence of conjuncts but with different bracketing as different conjunctions$_H$. For example, using modern notation, he counts $[p \wedge q] \wedge r$ and $p \wedge [q \wedge r]$ as different assertibles. Unlike modern propositional logic, Hipparchus assumes that a conjunction can consist of two *or more* conjuncts. For example, $p \wedge q \wedge r$ would be a well-formed expression denoting a conjunction with three conjuncts.

Now we can make sense of Plutarch's report that according to Hipparchus 'the affirmative produces 103,049 conjoined assertibles'. The affirmative with ten simple assertibles is of the kind $p \wedge q \wedge r \wedge s \wedge t \wedge u \wedge v \wedge w \wedge x \wedge y$. By collecting all the 'partitions' of such an affirmative which we obtain by grouping together by bracketing in any way that results in a well-formed expression, we get 103,049 different conjunctions. We can infer a further point: in order to get to this number, Hipparchus had to take the *order* of the ten atomic assertibles as fixed.[13]

Zooming in on what Hipparchus says about conjunctions$_H$ *from negatives*, we see that his notion of a conjunction appears somewhat peculiar and his choice of the basic pattern somewhat arbitrary. According to Plutarch, Hipparchus stated that 'the negative produces 310,95⟨4⟩'.[14] The context makes it clear that these are meant to be 310,954 conjunctions. Hipparchus' criterion for what is to count as 'a negative that produces a conjunction' seems to have been this: he lines up the respective number of atomic assertibles with a conjunctive particle between any two of them and prefixes a negation particle to the entire expression. Or, put differently, each time he takes the corresponding unbracketed affirmative and prefixes a negator to it thus:

(n=2) $\neg p \wedge q$
(n=3) $\neg p \wedge q \wedge r$
...
(n=10) $\neg p \wedge q \wedge r \wedge s \wedge t \wedge u \wedge v \wedge w \wedge x \wedge y$.

In this way, Hipparchus obtains (basic) negatives: one kind for n=2, one for n=3, etc. A (basic) negative is here characterized by having a negator in front of the respective unbracketed affirma-

[13] Thus for purposes of calculation it would not have been required to use different letters for different assertibles.

[14] Cf. nn. 1 and 3 above.

tive. Then Hipparchus calculates all the 'partitions' of the (basic) negative which one obtains by adding brackets in whichever way that results in a well-formed expression.[15] The presence of opening brackets between the negator and the first assertible is allowed but not required for well-formedness.[16] Again, we need to assume that Hipparchus regards the order of the atomic assertibles as fixed. Thus for $n=10$ Hipparchus gets 310,954 well-formed expressions of which all but one differ from the unbracketed $\neg p \wedge q \wedge r \wedge s \wedge t \wedge u \wedge v \wedge w \wedge x \wedge y$ in that they have additional bracketing.

According to our evidence in Plutarch, Hipparchus took all these expressions to be conjunctions. Take $n=2$ as an example: $\neg p \wedge q$ and $\neg[p \wedge q]$ would each count as conjunctions. This is somewhat peculiar, since in contemporary (and in Stoic) logic the second would be a negation, not a conjunction. Note that Hipparchus' view does not imply that any of these are negations. They are conjunctions produced from a negative. Leaving for the moment all historical considerations out of account, a reasonable interpretation of Hipparchus' procedure would be the following. His goal is to list all conjunctions produced with ten assertibles. His expression 'from a negative' refers to a certain syntactic structure, but is not intended to carve out a certain kind of assertible. Rather, this structure *together with* that 'from the affirmative' is meant to give the totality of all conjunctions$_H$. The syntactic structure 'from a negative' *de facto* produces two kinds of conjunctions$_H$: ordinary affirmative conjunctions$_H$, which differ from those from the affirmative in the fact that one of their conjuncts is negative; and negative conjunctions$_H$, which differ from affirmative conjunctions$_H$ in that their main connector is of the form $\neg[\ldots \wedge \ldots \wedge \ldots$ etc.], where either atomic affirmative assertibles or affirmative conjunctions can take the place of '…'. Then, for instance for $n=3$, cases (i), (iii), (iv), and (vi) from above would be affirmative conjunctions$_H$, whereas (ii), (v), and (vi) would be negative conjunctions$_H$. From the perspective of contemporary logic, calling these negative conjunctions$_H$ conjunctions would be a cavalier way of speaking. More import-

[15] For details see Acerbi, 'Hipparchus', 473–5.

[16] Hipparchus' treatment of conjunctions$_H$ from the negative shows that if he had syntactic conventions for two-or-more-place connectives, they would have required conjunctors placed only between the conjuncts of a conjunction, and not both between and in front of the conjuncts of a conjunction, as the Stoics required.

antly, in Stoic logic such expressions would not be conjunctions, and Hipparchus' argument thus appears to fail.

A second peculiarity of Hipparchus' conjunctions$_H$ *from a negative* is that they allow for only one (significant) negation sign in the entire expression, and this has to be at the beginning. This seems the only way to get the Plutarch's second number (in its corrected version). Why did Hipparchus not consider basic patterns such as $\neg p \wedge \neg q \wedge \neg r$ or $p \wedge q \wedge \neg r$ and similar? If he had considered the possibility of the presence or absence of a negation sign for any constituent atomic assertible, his total sum would arguably have exceeded one million. From the point of view of Stoic logic, there seems to be no reason to exclude cases such as those just mentioned. So, again, Hipparchus' argument seems to fail, this time because it appears to beg the question by artificially cutting down on the negation signs allowed in the basic pattern. I return to these two points of criticism of Hipparchus' calculation after consideration of Chrysippus' own claim.

3. What went into Chrysippus' calculations? A textual point

In order to exonerate Chrysippus, one could submit that he just used the number one million as another way of saying 'a very large number', and that Hipparchus' accusations that he was wrong are therefore unjustified. This kind of suggestion is, however, implausible, for at least three reasons. First, in ancient Greek there is no one word for 'million', which was expressed as 'one hundred myriads'. This makes it less likely that the expression is used to mean 'large number'. And even if it very occasionally is used in this way,[17] when used as part of phrases which express a lower limit, such as 'exceeds *x*' or 'more than *x*', as in our passages, the implicature of such a precise term is generally that the number given is indeed the lower limit.[18] Second, Hipparchus and those 'many mathematicians' hinted at by Plutarch—if they existed—said that Chrysippus made a mistake in his calculations. This implies that there were at the very

[17] It does occur a few times in Greek literature used to express an unspecified large number, including twice in Plutarch (*Pomp.* 67 and *Reg. et imp.* 180 C). I owe both passages to Alexander Jones, who also first brought to my attention the possibility of Chrysippus intending an unspecified large number.

[18] Compare in English 'there were over ten thousand people there' with 'there were thousands of people there' and 'there were over a million people there'.

least assumed *to be* some such calculations, including a resulting number. Third, there is sufficient evidence that basic combinatorial calculations were used by several philosophers in the hundred years or so before Chrysippus wrote his logical works.[19]

So what were Chrysippus' calculations? It is likely that Hipparchus had at least a tad more information about Chrysippus' mathematical endeavour than the half sentence Plutarch provides us with. But he would not have had Chrysippus' calculations themselves. For if he did, they would have prevented him from making his own un-Stoic choice of the cases that go into the calculation (see also Section 3). One possibility is that Chrysippus did perform some basic combinatoric calculations, but that in his writings he recorded solely the results, or just 'over one hundred myriads' as a lower bound established by his results.

Can we infer anything about his calculations from what Plutarch says about Hipparchus? It has been suggested that Hipparchus added the number of conjunctions obtained from 'the negative' to show, just in case, that *if* Chrysippus had had both affirmatives and negatives in mind, he still would not reach a million.[20] This is an honourable attempt at making sense of the text. However, an explanation that integrates Hipparchus' calculations of 'the negative' more naturally is preferable. In this light, the best explanation seems to be that in the information Hipparchus had about Chrysippus' combinatorial efforts, both affirmative and negative assertibles were either mentioned or implied by Chrysippus. They could have been *mentioned* like this:

> The number of conjunctions [constructible] from ten assertibles *and their negations* exceeds one million.

This makes Hipparchus' response, as reported by Plutarch, immediately plausible:

> However, Hipparchus refuted this, demonstrating that the affirmative makes 103,049 conjoined assertibles and the negative 310,952.

[19] See below, sect. 5, and for a detailed account Acerbi, 'Hipparchus', esp. 477, 482–3.

[20] This idea is entertained as a possibility by István Bodnár in 'Notice 1: Chrysippus and Hipparchus on the Number of Conjunctive Propositions' ['Notice'], in I. Bodnár and R. Netz, 'Hipparchus' Numbers in Two Polemical Contexts: Two Notices' (unpublished manuscript).

All we have to assume is that Hipparchus misunderstood Chrysippus' claim in the following way. Chrysippus intended that the constituents would be ten affirmative assertibles plus the ten negations of these ten assertibles (details below). By contrast, Hipparchus understood that what was to be calculated were first the bracketing 'partitions' of the expression obtained by stringing together ten affirmative assertibles with conjunctive connectors, and second the bracketing 'partitions' of the negation of that very expression, i.e. the bracketings of that expression with a negation particle prefixed.

The English I have provided contains an ambiguity which makes such a misunderstanding plausible: 'their' can refer either to the ten individual assertibles (which would have been Chrysippus' intention) or to all the relevant ordered sets of ten individual affirmative assertibles, i.e. all those which embody the basic pattern of the conjunctions from the affirmative (which would have been Hipparchus' reading). I believe this ambiguity could be reproduced in ancient Greek.[21] Thus, one possibility is that Chrysippus said something like:

> The number of conjunctions [constructible] from ten assertibles, *and their negations*, exceeds one million.

But even if what Chrysippus said was simply

> The number of conjunctions [constructible] from ten assertibles exceeds one million,

it is possible that he assumed that negations formed by prefixing a negator to a simple affirmation were included, especially since for the Stoics a simple assertible with a negation prefixed is still a simple assertible (see below).[22]

[21] In this case, the original Greek may have been *τὰς διὰ δέκα ἀξιωμάτων ⟨καὶ ἀποφάσεων αὐτῶν⟩ συμπλοκὰς ἑκατὸν μυριάδας πλήθει ὑπερβάλλουσιν*, with *sauter du même au même* from the first *των* to the second in the phrase. Note that in the Greek 'their' cannot refer to the conjunctions and I therefore do not consider this option.

[22] Either way (mentioned or intended), if the Boethius passages mentioned in sect. 5 below go back to a pre-Chrysippean Peripatetic, then there was a precedent for calculating the number of complex propositions from both affirmative and negative simple propositions.

4. The relevant elements from Chrysippus' logic

In order to be able to reconstruct Chrysippus' calculations, we first need to look at the relevant basic elements from Stoic logic: simple and non-simple assertibles, negations, conjunctions, scope and bracketing. The most fundamental distinction among assertibles is that between *simple and non-simple assertibles* (cf. S.E. *M*. 8. 93). Here are the definitions:

Non-simple [assertibles] are those that are, as it were, double [διπλᾶ], that is, which are put together by means of a connecting particle or connecting particles from an assertible that is taken twice [δίς], or from different assertibles, such as . . . 'both it is day and it is light'. (S.E. *M*. 8. 95)[23]

Simple [assertibles] are those that are neither put together from one assertible taken twice [δίς], nor from different assertibles by means of a connecting particle or connecting particles. (S.E. *M*. 8. 93; cf. D.L. 7. 68)

Chrysippus' claim in Plutarch is usually taken to concern the conjunctions that can be built from ten *simple* assertibles.[24] This may be correct, though it is in fact immaterial for the calculations.

In Stoic logic, *negations* can be either simple or non-simple assertibles. A negation (*ἀποφατικόν*) is formed by prefixing to an assertible the negation particle 'not:' (*οὐχί*):

The Stoics call only those [assertibles] negation which have a negation particle put at the beginning. (Apul. *Int*. 191. 6–11 Moreschini)

An example of a negative [assertible] is 'not: it is day'. (D.L. 7. 69)

And an example for a non-simple negation can be found in the description of the Stoic third indemonstrables, e.g. in S.E. *M*. 8. 226:

. . . *ἐξ ἀποφατικοῦ συμπλοκῆς . . . οὐχί καὶ ἡμέρα ἔστι καὶ νύξ ἔστι.*

. . . from the negation of a conjunction . . . not: both it is day and it is night.

The Stoic account of negation is such that one obtains a negation

[23] Similarly at S.E. *M*. 8. 108: 'Non-simple are the assertibles . . . which are put together by means of a doubly taken [*διφορουμένου*] assertible or of different assertibles and which are governed by a connective particle or connective particles' (cf. also D.L. 7. 68–9). For the expression *διφορούμενον* see below, n. 43.

[24] *ἐκ δέκα μόνων ἀξιωμάτων* (Plut. *Quaest. conv*. 732 F) is sometimes translated as 'from ten simple assertibles' rather than 'from only ten assertibles'. This may fit the context, but is unlikely to be the correct translation, since the Stoic technical term for simple assertibles is *ἁπλᾶ ἀξιώματα*, and there is no other known occurrence of *μόνος* with this meaning.

whenever one prefixes the negation particle to a well-formed expression, i.e. an assertible. Stoic logic thus allows in principle for negations of any complexity. The Stoics insisted on prefixing the negation particle to the entire assertible, since in that way it is an indicator of the scope of the negation particle. They express this by saying that it *governs* (*κυριεύει*, *ἐπικρατεῖ*) the whole assertible:

They say, [the assertibles that form a pair of contradictories][25] are opposed on this condition that the negative [particle] is prefixed to one of the two [assertibles]; for then it *governs* the whole assertible; whereas in the case of 'it is day and not: it is light' it does not *govern* the whole [assertible] so as to make it a negation. (S.E. *M*. 8. 90)[26]

When the negation particle is prefixed to a simple assertible, the resulting negation is simple.[27] When it is prefixed to a non-simple assertible, the resulting negation presumably counted as non-simple.

That the Stoics were aware of the importance of *scope* in propositional logic generally is clear from the fact that not just the negation, but also all non-simple assertibles (conjunction, disjunction, conditional) and various other simple assertibles (cf. D.L. 7. 69–70), were defined in such a way that their defining logical particle is the first word of the assertible. In the case of non-simple assertibles, the scope is indicated by the governance of the connecting particle or particles. If there is only one logical particle, as in the conditional, this is prefixed to the entire assertible and said to govern it. If there are two they together govern the assertible, indicating the scope of each component assertible (cf. S.E. *M*. 8. 95, 108).[28]

In our Plutarch passages we find the two Greek expressions for *conjunctions* which the Stoics used. *Sumplokē* is the Stoic technical

[25] 'Contradictories are those [assertibles] of which the one exceeds the other by a negation particle, such as "It is day"—"Not: it is day"' (S.E. *M*. 8. 89; cf. D.L. 7. 3).

[26] Similarly, by the prefixing requirement, the Stoics avoid ambiguity regarding existential import in ordinary-language formulations of simple assertibles such as 'Diotima doesn't walk'; the Stoics count 'Diotima doesn't walk' as an affirmation, since in their view—unlike 'Not: Diotima walks'—for its truth it presupposes Diotima's existence (cf. Apul. *Int*. 191. 6–11 Moreschini; Alex. Aphr. *In An. Pr*. 402. 8–12 Wallies).

[27] Thus the addition of the negative particle does not make a simple assertible non-simple. The negative particle 'not:' is not a Stoic connective (*συνδεσμός*). Stoic connectives bind together parts of speech. The negation particle does not do that.

[28] For the question of scope with simple and non-simple assertibles see also M. Frede, *Die stoische Logik* [*Logik*] (Göttingen 1974), 51–105, and S. Bobzien, 'Stoic Logic' ['Logic'], in K. Algra, J. Barnes, J. Mansfeld, and M. Schofield (eds.), *The Cambridge History of Hellenistic Philosophy* (Cambridge, 1999), 92–157 at 96–111.

expression for conjunction (e.g. D.L. 7. 77), and the Greek *sumpeplegmenon axiōma* is the Stoic technical expression for conjunctive assertible (e.g. D.L. 7. 72).[29] Chrysippus' statement in Plutarch thus undoubtedly concerns conjunctions. Moreover, it is concerned with conjunctions generally, not merely with true ones. What we need to explore is what a well-formed expression of the kind Chrysippus called a conjunction is. Syntax is at issue, not semantics.

The Stoics classify conjunctions as non-simple assertibles (e.g. D.L. 7. 68–9, quoted above). A definition has survived:

A conjunctive [assertible] is an assertible that is conjoined by conjunctive connectors, as for example 'both it is day and it is light'. (D.L. 7. 72)

The conjunctive connector is 'both . . . and ---' (*καὶ . . . καὶ* ---).[30] Any free-standing well-formed assertible that has 'both' as its first word is a conjunction. The 'both' indicates the scope of the conjuncts, or governs the conjunction, as the Stoics would say:

In [non-simple assertibles] the connective particle or particles govern [*ἐπικρατοῦσιν*] [sc. the assertible]. (S.E. *M*. 8. 108)

The definition of the conjunction has to be read in tandem with the definition of non-simple assertibles (see above). Thus it becomes clear that one obtains a conjunction whenever one constructs an expression of the form *καὶ*-assertible-*καὶ*-assertible.[31] Thus it is possible to construct conjunctions of any length, since any two assertibles, simple or non-simple, make up a conjunction if they are conjoined by the conjunctive connectors. This latter fact is made explicit in another passage:

The conjunction must either be put together from simple assertibles, or from non-simple assertibles, or from mixed ones.[32] (S.E. *M*. 8. 124)

[29] H. Cherniss, in his Loeb edition of *De Stoicorum repugnantiis* (Plutarch, *Moralia*, xiii/2 (London and Cambridge, Mass., 1976), 527–8), suggested that Chrysippus may have used the term 'conjunction' (*συμπλοκή*) in its broad meaning, covering connectives generally. However, this seems unlikely. First, such a use of *συμπλοκή* is not recorded elsewhere in Stoic logic. Second, *συμπλοκή* is used by the Stoics regularly as an alternative for *συμπεπλεγμένον ἀξίωμα* as a technical term for conjunctions in the narrow, logical, sense (see e.g. D.L. 7. 77, 80; S.E. *PH* 2. 137).

[30] Literally 'and . . . and ---'. As in English, in ancient Greek it is also possible to form a conjunction of sentences by just having a conjunctive particle between the conjuncts (. . . *καὶ* ---), and in ordinary language this is more frequent than the alternative with an additional *καί* at the beginning.

[31] I consider the question whether Chrysippus had multi-conjunct conjunctions below.

[32] That is, from a mixture of simple and non-simple assertibles.

Thus 'both p and q', 'both p and both p and q' and 'both both both p and q and r and s' are all conjunctions.

Here we can see that the Stoic language regimentations (i) of indicating the type of assertible to which an individual assertible belongs by its first word, and (ii) of prefixing all conjuncts with an identifying particle, provide them with a natural bracketing method that is similar to that of Polish notation: the 'both' (i.e. the first *καί*) indicates the scope of the conjunction, the 'and' (i.e. the second *καί*) indicates the beginning of the first conjunct.[33]

Now we have all the information required for considering Stoic *combinations of negations and conjunctions*. From the scope and bracketing conventions, it follows that in Stoic logic one and the same assertible cannot be both a negation and a conjunction. Which one it is is determined by whether the negation particle 'not' or the conjunctive particles 'both . . . and ---' have the largest scope. Thus 'both not: p and q' is a conjunction, but 'not: both p and q' is a negation. This fact is reflected in our sources, most clearly in Stoic terminology. For example, in the account of the Stoic third indemonstrables an expression corresponding to

τρίτος δὲ ἐστι λόγος ἀναπόδεικτος ὁ ἐξ ἀποφατικοῦ συμπλοκῆς . . . (S.E. *M*. 8. 226)

A third indemonstrable argument is one which . . . from a negation of a conjunction . . .[34]

[33] Thus the above examples could be represented as

καὶ p *καὶ* q	=	Kpq	=	p∧q
καὶ p *καὶ καὶ* p *καὶ* q	=	KpKpq	=	p∧[p∧q]
καὶ καὶ καὶ p *καὶ* q *καὶ* r *καὶ* s	=	KKKpqrs	=	[[p∧q]∧r]∧s

[34] Cf. 'from a negation of a conjunction' (*ἐξ ἀποφατικοῦ συμπλοκῆς* . . .) in S.E. *PH* 2. 158 and [Galen], *Hist. phil.* 15, 607 Diels. D.L. 7. 80 has *ἀποφατικοῦ* BPF, *ἀποφατικῆς vulgo*. In the transmission of Stoic logic we repeatedly find both readings recorded for the same text. This confirms what we know from many other instances, namely that later thinkers often no longer understood how Stoic logic worked. As the majority of texts presenting the accounts of the Stoic indemonstrables in their original form have the genitive neuter *ἀποφατικοῦ*, and only this reading tallies with the definitions of Stoic logic, it must be the correct reading in D.L. 7. 80. Similarly, modern translators often do not preserve the formal rigour of the Stoic formulation (e.g. 'negative conjunction', J. Annas and J. Barnes (trans.), *Sextus Empiricus:* Outlines of Scepticism (Cambridge, 1994) 109, and R. Bett (trans.), *Sextus Empiricus:* Against the Logicians (Cambridge, 2005), 133; 'negative Konjunktion', K. Hülser (ed.), *Die Fragmente zur Dialektik der Stoiker*, 4 vols. [*FDS*] (Stuttgart and Bad Cannstatt, 1987–8), 1529, 1531, 1533; or my past self (oh shame!) 'negated conjunction', Bobzien, 'Logic', 128).

Similarly, where Cicero reports that Chrysippus distinguished between his conditional and the negation of a conjunction, the assertibles are clearly labelled as negations:

. . . negationes infinitarum coniunctionum. (Cic. *Fat*. 15; cf. *Fat*. 16)

. . . negations of indefinite conjunctions.

Some of the examples and the presentation of the mode or schema of the third indemonstrable support the same point:

Not: both it is day and it is night. (S.E. *M*. 8. 226)[35]

Not: both the first and the second. (S.E. *M*. 8. 227)

Both display the required form for negations of conjunctions, i.e. *ouchi: kai . . . kai* ---.

Before we move on to the reconstruction of Chrysippus' calculation, we need to settle three issues that are relevant to the algorithm to be used in the reconstruction.

1. Could any of the ten affirmative assertibles occur more than once in one of the ten-assertible conjunctions Chrysippus counted? For example, could we have (i) p, p, q, q, r, s, t, u, v, and w instead of (ii) p, q, r, s, t, u, v, w, x, and y? Textual evidence suggests the answer is 'no'. Look again at the definition of non-simple assertibles and at the Plutarch passage:

Non-simple [assertibles] are those that are, as it were, double, that is, which are put together by means of a connecting particle or connecting particles from an assertible that is taken twice, or from different assertibles. (S.E. *M*. 8. 95)

Chrysippus says that *the number of conjunctions* [*constructible*] *from ten assertibles* exceeds one million. (Plut. *Stoic. repugn*. 1047 C)

You can see that the Stoics would have described (i) as 'combined from eight different assertibles, of which two are taken twice', not as 'from ten assertibles'. We can also rule out that Chrysippus counted expressions with ten different assertibles of which one or more are used more than once, as in (iii) p, p, q, q, r, s, t, u, v, w, x, and y. For in this case it would have been trivial that the number of conjunctions exceeds a million, since Chrysippus could simply

[35] Cf. [Galen], *Hist. phil*. 15, 607 Diels. Sometimes the first καί is missing, or replaced by ἅμα. This may be the result of lax formulations or scribal omission.

have added the equivalent to $\wedge$p between one and one million times to cases of (ii).

2. Did Chrysippus count only two-place conjunctions or also multi-conjunct conjunctions with conjuncts that are logically on a par? The question is controversial and details are discussed in a separate paper. Here I argue only (i) that all evidence for Stoic logic is compatible with the assumption that Chrysippus' contained only two-place conjunctions and (ii) that there are certain factors that make this interpretation preferable.

The above-quoted definition of non-simple assertibles (S.E. *M*. 8. 95) implies that every non-simple assertible consists of precisely two main component assertibles, which, in turn, can be simple or non-simple.[36] All other surviving definitions are compatible with this claim. By contrast, some texts have been taken to suggest that some Stoics allowed for multi-conjunct conjunctions. Here are the passages typically adduced in support of this point:

. . . when [the logicians] say that a conjunctive [assertible] is sound when all those [parts] it has in it are true . . . (S.E. *M*. 8. 125)

But they say, just as . . . so also with the conjunctive [assertible]: even if it contains only one false [assertible] and several true ones, the whole will be named ['false'] after the one false one. (S.E. *M*. 8. 128)

These passages seem to talk about multi-conjunct conjunctions. However, they do not prove that the Stoics allowed for conjunctions with more than two conjuncts in the largest scope. Both passages make perfect sense if what is at issue are two-conjunct conjunctions of which one or more conjuncts are again conjunctions.[37] At first sight, an example in Gellius may seem stronger evidence.

And what they [i.e. the Stoics] call a *sumpeplegmenon*, we call a conjunctive or connective, which is of this kind: 'Publius Scipio, son of Paulus, both was consul twice and was triumphant, and had the censorship, and was in

[36] The scope of 'double, as it were' is both those in which the same assertible is taken twice and those composed from different assertibles.

[37] In fact, the first passage is followed by an example of a two-conjunct conjunction and a Sextan argument which assumes that the conjunctions at issue are two-conjunct. Moreover, it is not without parallel in ancient logic that a proposition that is connected from a simple and non-simple proposition (containing two simple ones) is considered as a proposition in which three simple (or categorical) propositions are connected. See Boeth. *Hyp. syll.* 1. 8. 3: 'nam quae ex categorica et conditionali constant, vel e diverso, haec tribus cagetoricis iunctae sunt.' For context see below, sect. 5, and Acerbi, 'Hipparchus', 483.

his censorship colleague of Lucius Mummius.' But in this whole conjunction, if one [part] is false, even if all the others are true, the whole is said to be false. (Gell. 16. 8. 10–11)

However, several points need to be taken into account here. First, the example is not early Stoic. Second, it does not have the required syntactic form. Stoic conjunctive particles conjoin whole assertibles, not predicates, even if their subject is identical.[38] Third, any Greek text that may have contained a complex example of the kind '*καὶ* p *καὶ καὶ* q *καὶ* r' or '*καὶ καὶ* p *καὶ* q *καὶ* r', would almost inevitably have been shortened at some point in the transmission process to '*καὶ* p *καὶ* q *καὶ* r' on the scribe's assumption that the text contained dittography. This would probably have happened to the Greek counterpart of Gellius' example—if there was one. Thus any grouping by way of Stoic 'bracketing' would have been lost.[39] Hence the discussed texts leave it an open question whether Chrysippus' logic included multi-conjunct conjunctions.[40]

The fact that Chrysippus admitted multi-disjunct disjunctions for the case of a special indemonstrable argument with such a disjunction as first premiss (e.g. S.E. *PH* 1. 69) is insufficient reason for assuming that he admitted multi-conjunct conjunctions as well. They are never mentioned, there are no Stoic examples, there is no special indemonstrable, and they do not occur anywhere in Chrysippus' syllogistic.[41] Moreover, whereas multi-disjunct disjunctions increase the expressive power of Stoic logic,[42] multi-conjunct conjunctions would not.

A positive argument for the restriction to two-component assertibles in standard Stoic logic (with components of any complexity)

[38] Cf. e.g. Gell. 16. 8. 9; S.E. *M*. 8. 113–14; Cic. *Fat*. 12; Galen, *Inst. log*. 4. 1.

[39] The reconstruction of Hipparchus' two numbers is only successful on the assumption that Hipparchus did not include the initial *καί* of Stoic conjunctions. Thus at his time the syntactic bracketing conventions may already have been lost or loosened.

[40] Fabio Acerbi has adduced as evidence for Chrysippean more-than-two-conjunct conjunctions the passage Plut. *Stoic. repugn.* 1084 C–D (Acerbi, 'Hipparchus' 470). However, it can be shown that in this passage, rather than multi-conjunct conjunctions, we have two abbreviated Sorites-type arguments. In any event, the passage does not contain any fully formulated multi-conjunct conjunctions, and points two and three stated with regard to the Gellius passage would hold for it, too.

[41] Michael Frede showed that the introduction of the fifth multi-disjunct indemonstrable does not entail multi-conjunct conjunctions. See Frede, *Logik*, 156–7.

[42] This is so because the truth-conditions for the disjunction—which is exclusive—include a modal element, but those for the conjunction do not.

is the fact that the first parts of the definitions of non-simple assertibles (those with the same assertible taken twice) allow for only two-component assertibles.[43] It would be decidely odd if the second parts would allow for multi-component assertibles. Another positive reason is the advantage of having a unified interpretation of all Stoic accounts of (or involving) non-simple assertibles and the way the ancient sources understood them, leaving only the multi-disjunct disjunctions as an 'out-of-the-ordinary' case with its own special deduction rule.[44] In the same vein, if Stoic standard logic had allowed for multi-component non-simple assertibles, their syntactic bracketing system would have broken down,[45] which is improbable, given the care the Stoics put into avoiding lexical ambiguities and the advanced status of Chrysippus' logic.

It is thus more likely than not that Chrysippus' logical system did not contain multi-conjunct conjunctions, and I shall reconstruct Chrysippus' calculations accordingly. It is evident that by adding the possibility of multi-conjunct conjunctions, the resultant number of conjunctions increases. So if we get over a million without them, we also get over a million with them.

3. Did Chrysippus take the order of the simple assertibles in the conjunction as fixed or as variable for the purpose of his calculations? We saw above that for Hipparchus the order was fixed. If we can show that for the Stoics 'both p and q' was the same conjunction as 'both q and p', we can assume with some certainty that in his

[43] Michael Frede argued convincingly that the term used to describe these assertibles, which is transmitted in two versions (*διφορούμενον* and *διαφορούμενον*), must be read as *διφορούμενον*, meaning 'taken twice' (Frede, *Logik* 50; cf. Apul. *Int*. 201. 6–7 Moreschini 'geminantes', and Alex. Aph. *In Top*. 10. 7–10 Wallies; *In An. Pr*. 18. 15–19, 20. 10–12 Wallies; Ammon. *In. An. Pr*. 27. 35–28. 5, 32. 13 Wallies).

[44] The use of *διφορούμενον* (see previous note), the definition at S.E. *M*. 8. 95, the occurrence of *ἕτερον* in the definition of the disjunction at D.L. 7. 72, and the understanding of the definition by several sources as including only two-component assertibles would otherwise force us to accept a (non-evidenced) variety of definitions or else multiple misrepresentations of the definitions in the sources. (Passages such as Philop. *In An. Pr*. 244. 1–246. 14 Wallies, Anon. *Logica et quadrivium*, 38 (30. 16–32. 7 Heiberg), Galen, *Inst. log*. 5. 4, 6. 7, 15. 1–11, and scholia to Ammon. *In An. Pr*., xi. 30 ff. Wallies, do not represent Stoic but Peripatetic or Platonist syllogisms: see S. Bobzien, 'Hypothetical Syllogistic in Galen: Propositional Logic off the Rails?' ['Hypothetical Syllogistic'], *Rhizai*, 2 (2004), 57–102, and 'Propositional Logic in Ammonius', in H. Linneweber-Lammerskitten and G. Mohr (eds.), *Interpretation und Argument* (Würzburg, 2002), 103–19 . Nor does Cic. *Top*. 53–7, for which see below, sect. 6.)

[45] Acerbi shows this compellingly in 'Hipparchus', 472.

calculations Chrysippus too took the order of the simple assertibles to be fixed. For in that case many of the permutations one gets by varying the position of the simple assertibles would be just another way of stating one of the conjunctions already counted. Chrysippus would then have to subtract these cases, and as a result the calculation would be rather cumbersome and anything but straightforward.

We are thus led to ask about the Stoic identity criteria for conjunctions. The Stoic definitions of non-simple assertibles (S.E. *M*. 8. 95) and of conjunctions (D.L. 7. 72), and the Stoic account of the truth-conditions for conjunctions (S.E. *M*. 8. 125, 128; Gell. 16. 8. 11) are all compatible with the assumption that for the Stoics 'both p and q' and 'both q and p' were the same assertible.[46] This is an interesting fact. But we want more than compatibility.

It is also not sufficient for us to show that Chrysippus thought 'both p and q', 'both q and p', etc. to be logically equivalent: that is, that they had the same truth-value in all possible situations. We can safely assume that Chrysippus thought they were. But we can also show that this is insufficient for the identity of the assertibles. For example, for the Stoics the truth-values of p and ¬¬p are the same (D.L. 7. 69), but one is an affirmation, the other a negation, so they cannot be the same assertible.

The identity criteria must be stronger than logical equivalence. The most promising passage is perhaps the following one about the position of the antecedent and consequent in a conditional:

> Of the assertibles in the conditional the one placed after the connective particle 'if' or 'if indeed' is called the antecedent and the first, and the other is called the consequent or the second, *even if the whole conditional is uttered in reverse* [*καὶ ἐὰν ἀναστρόφως ἐκφέρηται τὸ ὅλον συνημμένον*], e.g. in this way: 'it is light if indeed it is day'. For in this, too, the 'it is light' is called consequent, even though it is uttered first, and the 'it is day' is called antecedent, even though it is said second, because it is placed after the connective particle 'if indeed'. (S.E. *M*. 8. 110)

This passage implies that 'if it is day, it is light' and 'it is light, if it is day' are the same assertible, just expressed in different ways.

I now produce analogous deliberations for Stoic conjunctions. For this, it is useful first to remember that in Greek, rather than having two different connecting particles like 'both' and 'and', we

[46] All five passages have been quoted above.

have twice the same particle: 'and' (*καί*) and 'and' (*καί*). Thus, if we take the Stoic example for a conjunctive assertible (i) 'and it is day and it is light', and utter it in reverse (*ἀναστρόφως ἐκφέρω*), we get (ii) 'and it is light and it is day'. (ii) should be the same assertible as (i), just uttered in reverse. If we generalize from here, the Stoics would have understood '$p \wedge q$' as being the same conjunction as '$q \wedge p$', just uttered in reverse, and so for all cases. Thus there is no need to count both '$p \wedge q$' and '$q \wedge p$', because this would be, in fact, counting the same assertible twice. This seems good evidence for proceeding in the reconstruction with the assumption that in his calculation Chrysippus took the order of the atomic assertibles to be fixed.

In addition, as a logician, Chrysippus is more likely to have been interested in *types* of conjunction, rather than in the actual conjunctions that one gets by combining ten simple assertibles. That is, we would expect him, for the purpose of his calculations, to think of the assertibles as something like schematic letters. Then, whether one writes 'p' for 'q', or 'r' for 's', etc., would be irrelevant.

Thus, all things considered, it is most likely that Chrysippus took the order of the atomic assertibles to be fixed. In any case, it holds that if we get over a million conjunctions in this way, we will also get over a million conjunctions if the order is variable.

Putting together the elements of Stoic logic, we get the following results relevant to the reconstruction of Chrysippus' calculations:

- Most probably, Chrysippus allowed affirmative and negative simple assertibles in his calculation. (≠Hipparchus)
- Chrysippus had well-worked-out conventions about scope and bracketing that make it possible to determine unequivocally of every well-formed expression whether it is a conjunction or a negation. Some of the expressions Hipparchus counted are not conjunctions in Stoic logic. (≠Hipparchus)
- The ten atomic assertibles that go in the calculation have to be ten different assertibles. (=Hipparchus)
- Most probably, the conjunctions and component conjunctions in the expressions Chrysippus counted all had precisely two conjuncts. (≠Hipparchus)
- Most probably, the order of the atomic assertibles was fixed. (=Hipparchus)

5. A reconstruction of Chrysippus' calculations

With the results of the previous section, we can now proceed to a reconstruction of Chrysippus' estimation.

Chrysippus' claim. The number of different conjunctions one obtains with ten atomic assertibles exceeds one million.

Rules for the formation of conjunctions

(i) The order of the ten atomic assertibles is fixed.
(ii) The ten atomic assertibles are ten different assertibles.
(iii) A conjunction has precisely two conjuncts.
(iv) A conjunction can have as conjuncts (*a*) simple assertibles, affirmative or negative; (*b*) conjunctions of these.

Method of proof. First we calculate the number of conjunctions without any negation particle anywhere (call these 'positive conjunctions'). Then we calculate the number of possibilities with negations and integrate them.

1. *Positive conjunctions*. In a sequence of atomic assertibles we have combinations either of two atomic assertibles or of one atomic assertible and one conjunction or of two conjunctions. We can represent this by using binary trees,[47] with dark circles (or leaves) for atomic assertibles and light circles (nodes) for conjunctions, as in Figure 1.

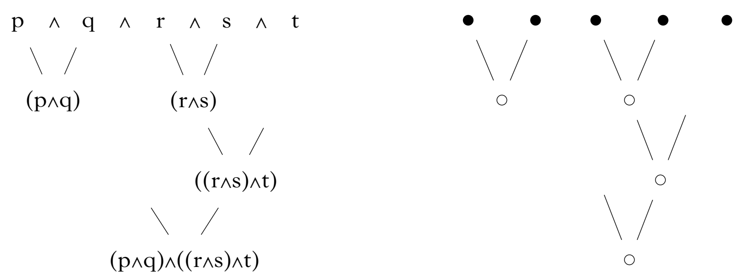

Figure 1

[47] See e.g. D. E. Knuth, *The Art of Computer Programming*, i. *Fundamental Algorithms* [*Algorithms*], 3rd edn. (Reading, Mass., 1997), sect. 2.3.4.5

The problem can now be reduced to the question how many trees there are with n leaves. For example, for $n=3$ there are two trees (Figure 2).

FIGURE 2

A binary tree with n leaves has always $n-1$ nodes (or conjunctions, in our case). This is so because you pass from n leaves to a single node, and every node reduces two items to one. Hence we can further reduce our problem to the question of how many binary trees with $n-1$ nodes there are. The answer to this question is known.[48] The number of binary trees with n nodes is the so-called Catalan number C_n:

$$C_n \equiv \frac{(2n)!}{(n+1)!n!}$$

Since n atomic assertibles correspond to n leaves and hence to $n-1$ nodes, one obtains

$$b_n = C_{n-1} = \frac{2(n-1)!}{n!(n-1)!}$$

different conjunctions. For $n=10$ we have b_{10}, which is 4,862.

2. *Negative simple assertibles integrated*. Every atomic assertible can be negated. Hence there are n expressions that can be negated. Thus we have n places at which a negation particle can be added. At each of these places we may or may not have a negator. Thus we get 2^n possibilities.

Now all that is left to do is to multiply this number of all possible combinations of placings of the negation particles with the number of possible positive conjunctions from our first step. Thus we get

$$b_n \times 2^n$$

[48] Cf. e.g. Knuth, *Algorithms*, sect. 2.3.4.4.

possible different conjunctions for n atomic assertibles.[49] For n= 10 we get $b_{10} \times 2^{10}$, which is 4,862×1,024, which is 4,978,688. This is more than a million. Chrysippus' claim is thus correct. Even better, this is less than ten million. Chrysippus' claim is thus not only correct, but our resultant number also squares with the implicature of Chrysippus' statement that the number is somewhere above one million but below ten million. The implicature would be even stronger in ancient Greek. Ten million would most probably have been expressed as a thousand myriads. Chrysippus expressed a million as one hundred myriads. Thus we can assume that, if his calculations had taken him above ten million, he would have stated the Greek equivalent of 'the number of conjunctions [constructible] from only ten assertibles exceeds ten million'. This speaks in favour of my construction, compared with all those which result in a number above one million but also above ten million.

Could Chrysippus himself have calculated this number? Catalan numbers and Schröder numbers are related. Both can be generated with tree structures. If Hipparchus was able to calculate the 10th (or 11th) Schröder number, for which Acerbi, 'Hipparchus', makes a good case, then it seems at least not impossible that Chrysippus made it to the 10th Catalan number in one way or another. This claim can be supported in two ways: first, by showing that some combinatorial calculations were done by philosophers and logicians before Chrysippus, and second, by showing how Chrysippus could have performed the calculations using very elementary mathematical procedures.

There is good evidence that philosophers before Chrysippus were acquainted with and used basic ideas of combinatorics. Plato in the *Laws* (737 E–738 A), and Aristotle in his *Politics* (1290^b25–39, 1300^a31–1301^a15), use combinatorial ideas.[50] Plutarch tells us,

[49] For n from 2 to 10 we have then:

n=2	p, q	4
n=3	p, q, r	16
n=4	p, q, r, s	80
n=5	p, q, r, s, t	448
n=6	p, q, r, s, t, u	2688
n=7	p, q, r, s, t, u, v	16896
n=8	p, q, r, s, t, u, v, w	109824
n=9	p, q, r, s, t, u, v, w, x	732160
n=10	p, q, r, s, t, u, v, w, x, y	4978688

[50] For details see Acerbi, 'Hipparchus', 477 and 482.

immediately after his remarks on Hipparchus, that Xenocrates (head of the Academy from 339 to 314 BCE) concerned himself with the question of what the number of syllables was which the letters produce when combined with each other.[51] Thus it would not be surprising if Chrysippus was acquainted with basic questions and methods of a combinatorial nature. Most interestingly, there is a passage in Boethius' *De hypotheticis syllogismis* (1. 8. 1–7= 244. 1–248. 55 Obertello) which reports calculations about how many conditional (or hypothetical) sentences one can construct from all the different types of affirmative and negative simple (or categorical) ones.[52] This is a question very similar to the one reported by Plutarch, i.e. how many molecular sentences of a certain kind (conditionals in the Boethius passage) one can obtain by combining simple sentences, including affirmative and negative ones. Although Boethius wrote in the sixth century CE, the basis of *De hypotheticis syllogismis* seems to go back to a pre-Chrysippean Peripatetic source, possibly Theophrastus, and possibly via the early Neoplatonists, perhaps Porphyry.[53] If *Hyp. syll.* 1. 8. 1–7 is based on such a pre-Chrysippean Peripatetic text, it suggests that Chrysippus' calculations were part of an existing practice of using basic combinatorial ideas for calculating the number of certain

[51] 'Xenocrates claimed that the number of syllables which the letters make when combined with each other is 1,002,000,000,000 (a myriad-and-twenty times a myriad-myriad)' (Plut. *Quaest. conv.* 733 A).

[52] Acerbi, 'Hipparchus', 482–3, quotes part of the text and paraphrases the rest thus: '"If someone is inquiring the number of all conditional propositions, he can find it from categorical [propositions]; and first one must inquire the [conditionals] made up of two simple [. . . .]". The answer runs thus: there are five affirmative categorical propositions and five correlated negative propositions: ten in all. An hypothetical proposition is made of two categorical propositions: one hundred combinations result. Considering also the propositions composed of one categorical and one hypothetical, or of two hypothetical, one obtains one thousand and ten thousand respectively.' For further details see Acerbi, ibid.

[53] Boethius wrote commentaries on Porphyry's *Isagoge* and was influenced by Porphyry in his logical writings; according to Boethius, *In De int.* 1, 16^a1, 2nd edn., 1. 1, Porphyry wrote commentaries on Theophrastus' *On Affirmation and Denial*; Porphyry wrote on hypothetical syllogisms; Boethius refers repeatedly to Theophrastus in *De hypotheticis syllogismis*. (See e.g. J. Magee, 'On the Composition and Sources of Boethius' Second *Peri hermeneias* Commentary', *Vivarium*, 48 (2010), 7–54 at 46; H. Chadwick, 'Boethius: Logic', in E. Craig (ed.), *The Routledge Encyclopedia of Philosophy* ⟨http://www.rep.routledge.com/article/B016SECT3⟩ [accessed 19 Oct. 2010]; S. Bobzien 'Pre-Stoic Hypothetical Syllogistic in Galen', in V. Nutton (ed.), *The Unknown Galen* (*Bulletin of the Institute of Classical Studies*, suppl. 77; London, 2002), 57–72 at 70–1.)

kinds of non-simple assertibles or sentences that one obtains from simple affirmative and negative ones.

If need be, Chrysippus could have constructed and counted the 4,862 positive cases one by one; alternatively, he could have used some semi-formal method. Here is one which I used myself, trying not to rely on any mathematical means beyond addition and multiplication. I wrote out the cases for n=1 to n=5 on a piece of paper. Then, for each n, I collected the different types in which bracketing was possible, looked how often they occur, and how they increase with increasing n, and added them up for each n. The following pattern occurred:[54]

n=3	2	=	[1×1]+1×1
n=4	5	=	1×1 +2×2
n=5	14	=	2×2+2×5
n=6	42	=	2×2+2×5+ 2×14
n=7	132	=	4×5+ 2×14 + 2×42
n=8	429	=	5×5+ 4×14 + 2×42 +2×132
n=9	1430	=	10×14+ 4×42 +2×132+2×429
n=10	4862	=	14×14+10×42+4×132+2×429+2×1430

This took me a few hours—like Chrysippus, I am not a mathematician but a logician. After calculating the Catalan numbers up to n=10, the second step of integrating the negative cases (see above) should, in any case, have been easy for Chrysippus.[55]

Before returning to Hipparchus, I should stress the following point. The goal of the present section is not to prove beyond doubt

[54] In the first line, the [1×1] would of course be 0×0 in the case of conjunctions, so this home-made procedure only really works from n=4 onwards.

[55] As Fabio Acerbi notes (in correspondence), the basic conception of the calculations I ascribe to Chrysippus are very similar to those Acerbi worked with when reconstructing possible methods of calculation of Hipparchus. Does this render it implausible that Hipparchus performed his calculations without having those by Chrysippus at his disposal? (If so, this would jeopardize either my or Acerbi's reconstruction, since if Hipparchus *did* have Chrysippus' calculations, it becomes very hard to explain why his are at variance with Chrysippus' in the way they are.) But I do not think that it is implausible at all. To support my point, I note that I did my 'Chrysippean' calculations without using any knowledge of combinatorics, and without knowledge of Acerbi's reconstruction of his 'Hipparchan' calculations (he does not spell them out in his article), so we can assume that Hipparchus could have done his calculations without knowing Chrysippus'. Trying to answer a question like 'How many *x* do you get from n *y*?', where the *x*s are obtained by combining the *y*s in some way or other, invites calculations that fall broadly into the area of combinatorics, even for someone who has little or no idea what combinatorics are. Moreover, there is a difference between simply performing such calculations (as Chrysippus may have done) and gaining awareness of some of the combinatoric principles underlying such calculations (as Hipparchus may have).

that Chrysippus produced the numbers and calculations presented here. Rather, I have shown that there is a plausible reconstruction of what Chrysippus might have calculated that produces the desired result (a number above one million but below ten million) and squares with all the surviving evidence, including Stoic logic. There are quite a few alternative reconstructions that tally with Stoic logic which produce a number above one million.[56] These include the options of allowing for permutations,[57] or for more multiple-conjunct conjunctions, or for non-simple negative conjuncts, or for negations of conjunctions also to be counted. It is less easy to produce alternative reconstructions that also both lead to a number below ten million and accord with what Hipparchus took into account in his calculations according to Plutarch, and have the simplicity one expects to come with a claim such as that recorded for Chrysippus in Plutarch. Still, it cannot be ruled out that even such alternative reconstructions can be found—so much the better.

6. Hipparchus revisited

We saw above (i) that Hipparchus' conjunctions$_H$ *from an affirmative* are all conjunctions, whereas his conjunctions$_H$ *from a negative* include both conjunctions and negations of conjunctions; and (ii) that Hipparchus' conjunctions$_H$ *from a negative* allow for only one (significant) negation sign in the entire expression, and that this has to be at the beginning. I remarked that both (i) and (ii) are odd and at odds with Stoic logic, but left it at that. From the perspective of combinatorics, one might see some attraction in Hipparchus' choice of (i) and (ii). Yet Hipparchus' intent was not idle calculation but proving Chrysippus wrong. So we have to assume that he somehow thought he had added up *what Chrysippus claimed would exceed a million when added up*. Thus we still need explanations.

We can make full sense of (the reconstruction of) Hipparchus' calculations if we make one simple assumption: rather than having a full understanding of Stoic logic as it was developed by Chrysippus in his works, Hipparchus gained his acquaintance with Stoic logic from texts or teachers that presented it from a Peripatetic or

[56] István Bodnár considers some in Bodnár, 'Notice'.

[57] If Chrysippus took the order of the ten assertibles to be variable rather than fixed, the resultant number would far exceed one million even if no negation signs were allowed anywhere in the conjunctions formed from ten assertibles.

Peripatetic-influenced perspective. The purpose of this last section is to explicate and support this point.

To start with, it is worth noting that we have no evidence regarding how Hipparchus got knowledge of Chrysippus' over-one-million claim or how he got acquainted with Stoic logic; in particular we have no evidence that he studied Chrysippus' writings directly. In fact, we know nothing about any philosophical or logical education of Hipparchus. On the other hand, we do know that at least since the first century BCE, but most probably earlier, the Stoics had taken notice of Peripatetic logic, and the Peripatetics of Stoic logic. Moreover, at least from the first century BCE, but most probably earlier, early Peripatetic hypothetical syllogistic and Stoic syllogistic were taken to express (more or less) the same underlying theory, a theory concerned with—among other things—simple and non-simple propositions.[58] As a result, eclectic theories feeding on both Stoic and Peripatetic ideas came into existence, and generally a conflation of both terminology and theories can be observed.

With this background provided, back to Hipparchus. Here is a list of the aspects of logic that got into his calculations which seem to differ from Chrysippus' logic:

(1) The connecting particle 'both' (i.e. the first *καί*) of the regimented form of Stoic conjunction is not an essential part of Hipparchus' notion of conjunction. If it was, he would not have counted e.g. $[\neg p \wedge q] \wedge \ldots$, since the required formulation $[\neg \wedge p \wedge q] \wedge \ldots$ would not have been well formed. But he did count such cases.

(2) The pair of terms 'affirmative' (*καταφατικόν*) and 'negative' (*ἀποφατικόν*) in the sentence in which Plutarch presents Hipparchus' response to Chrysippus is not Stoic.

(3) Hipparchus counted expressions of the form $\neg[p \wedge q]$ (and $\neg[p \wedge q \wedge r]$, and $\neg[[p \wedge q] \wedge r]$, etc.) as conjunctions; more precisely, as negative conjunctions.

(4) Hipparchus took multi-conjunct conjunctions to be well

[58] e.g. Cic. *Top.* 53–7; Galen, *Inst. log.* 5; 6; 15. 1–11; Alex. Aphr. *In An. Pr.* 262–5 Wallies; *In Top.* 165–6, 174–5 Wallies. However, the hypothetical syllogistic was much more elementary than Chrysippean syllogistic. Throughout antiquity, it never reached the status of a propositional logic. There is no evidence of truth-functionality of negation and conjunction, of worked-out rules for bracketing, or of rules for reducing all complex syllogisms to axiomatic syllogisms, three features that characterize Stoic logic. For details cf. e.g. Bobzien, 'Hypothetical Syllogistic'; 'Propositional Logic in Ammonius'.

formed. Otherwise, he would not have counted e.g. cases starting [p∧q∧r]∧[…, but he did.

(5) Hipparchus appears to have assumed that to the general affirmative pattern of propositions p∧q∧r∧… a general negative pattern of propositions ¬p∧q∧r∧… corresponded, rather than allowing for negation signs in front of any of the ten simple affirmative assertibles.

For each of these five aspects in which Hipparchus' understanding of logic seems to differ from Chrysippus' own there are several parallels in Peripatetic and Peripatetic-influenced sources.

(Ad 1) Other than the early Stoics, no ancient logician is known to have insisted on, or consistently used, a 'both . . . and ---' (*καὶ . . . καὶ* ---) connective for Łukasiewicz-style bracketing purposes. In fact, Galen and later Peripatetics such as Alexander generally derided the Stoic language regimentations as unhealthy formalism. In addition, as mentioned above, even in Stoic texts, any two adjacent *καί*s required by the bracketing conventions were almost certainly lost when the texts were transcribed, which would have contributed to the demise of this feature in the transmission of Stoic logic.

(Ad 2) The expressions 'affirmative' (*καταφατικόν*) and 'negative' (*ἀποφατικόν*) in the sentence presenting Hipparchus' view in Plutarch reflect standard Peripatetic terminology, going back to Aristotle.[59]

(Ad 3) In the Peripatetic tradition, we regularly encounter the notion of a negative conjunction (*τὸ ἀποφατικὸν συμπεπλεγμένον, ἡ ἀποφατικὴ συμπλοκή*)[60] as opposed to the Stoic 'negation of a conjunction' (*τὸ ἀποφατικὸν συμπλοκῆς*). It appears that the Peripatetics applied Aristotle's basic distinction between affirmation (*κατάφασις*) and negation (*ἀπόφασις*) to conjunctions—something unthinkable in the context of Stoic logic. This is in line with the general Peripatetic assumption that for any type of affirmative

[59] Cf. e.g. Arist. *Cat.* 12^b6–10; *Int.* 5 and 6; *Pr. An.* 25^a1–5; Galen, *Inst. log.* 6. 5–9; Alex. Aphr. *In An. Pr.* 10. 18, 13. 24–5, 19. 6–8 Wallies; Ammon. *In Cat.* 16. 19–21 Busse. Theophrastus wrote a work entitled *On Affirmation and Negation* (*Περὶ καταφάσεως καὶ ἀποφάσεως*).

[60] Cf. e.g. [Ammon.] *In An. Pr.* 68. 28–9 Wallies; scholia on Ammon. *In An. Pr.*, xi. 30 Wallies; Alex. Aph. *In An. Pr.* 390 Wallies; Philop. *In An. Pr.* 245 Wallies; Anon. *Logica et quadrivium* 38, p. 31 Heiberg=*FDS* 1134. 30.

proposition there is a corresponding type of negative proposition.[61] Thus, from a Peripatetic perspective, all the expressions Hipparchus counted in his calculation would have been conjunctions, which would give Hipparchus' method a uniformity and consistency it does not have when looked at from the point of view of Stoic logic.

(Ad 4) In Peripatetic and Peripatetic-influenced texts we find, on a regular basis, the notion of negated conjunctions with more than two conjuncts.[62] (This is never remarked upon and may have been simply considered a given, since in ordinary—ancient Greek—language more than two sentences or predicates can be, and were, connected with repeated *καί*s.)

(Ad 5) In their hypothetical syllogistic (which in antiquity was understood as the Peripatetic counterpart to Stoic syllogistic) the Peripatetics used certain forms of complex propositions as fixed structures that would feature in the major premisses of the small number of syllogistic figures that they admitted. In addition to four figures very roughly corresponding to the *modi ponens* and *tollens*, the Peripatetics had a syllogistic figure with a negative conjunction as major premiss, which is almost always defined in such a way that it allows for multiple conjuncts.[63] Those major premisses thus had the structures: $\neg(p \wedge q)$, $\neg(p \wedge q \wedge r)$, $\neg(p \wedge q \wedge r \wedge s)$, etc. So there would have been an almost exact model for the basic patterns of Hipparchus' negatives, in particular since (unlike in Stoic logic) there would not have been any syntactic conventions for bracketing.

Most of the sources made use of in response to points (1)–(5) are later than Hipparchus. However, there is good evidence that Peripatetics of the second century BCE discussed Stoic philosophy in some detail, and that they focused among other things on rhetoric,[64]

[61] e.g. Arist. *Pr. An.* 25^{a}1–5. Following a Peripatetic source (possibly Theophrastus), Boethius also distinguishes affirmative and negative conditionals (*Hyp. syll.* 1. 9. 7).

[62] Implied by the accounts of syllogisms with negated conjunction in Cic. *Top.* 54; Galen, *Inst. log.* 5. 4; Philop. *In An. Pr.* 245 Wallies; scholia on Ammon. *In An. Pr.*, xi. 30 ff. Wallies; Anon. *Logica et quadrivium* 38, p. 31 Heiberg = *FDS* 1134; examples in Philop. *In An. Pr.* 245 Wallies; Galen, *Inst. log.* 5. 4.

[63] For passages see previous note.

[64] Most famously Critolaus: see D. Hahm, 'Critolaus and Late Hellenistic Peripa-

which is also the context in which, in the first century BCE, Cicero's evidence of a Peripatetic-influenced notion of a negative multi-conjunct conjunction as part of a syllogistic form is to be found. It has been convincingly shown that Cicero draws on a rhetorical tradition that is influenced by Aristotle (*Rhetoric* and *Topics*) and early Peripatetics, in particular Theophrastus.[65] Moreover, Galen, Alexander, and Boethius frequently draw at least indirectly on Theophrastus' logical works. And for the first century BCE there is evidence that some Peripatetics had absorbed parts of Stoic propositional logic,[66] and there is no reason to think that Peripatetics had not been acquainted with and accommodated elements of Stoic logic already in the previous century.

There is thus nothing impossible about the assumption that Hipparchus gained his outlook on logic, including Stoic logic, via works or teachers of Peripatetic heritage or influence. If this is so, we have a comprehensive explanation of the non-Stoic ways in which Hipparchus devised his calculations of the number of 'Stoic' conjunctions one obtains from ten assertibles.

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[65] W. W. Fortenbaugh, 'Cicero's Knowledge of the Rhetorical Treatises of Aristotle and Theophrastus', in W. W. Fortenbaugh and P. Steinmetz (eds.), *Cicero's Knowledge of the Peripatos* (New Brunswick, 1989), 39–60; id., 'Cicero, *On Invention* 1. 51–77: Hypothetical Syllogistic and Early Peripatos', *Rhetorica*, 16 (1998), 32–5; id., 'Cicero as a Reporter of Aristotelian and Theophrastean Rhetorical Doctrine', *Rhetorica*, 23.1 (2005), 37–64; P. Huby, 'Cicero's *Topics* and its Peripatetic Sources', in Fortenbaugh and Steinmetz (eds.), *Cicero's Knowledge of the Peripatos*, 61–76.

[66] Cf. Galen, *Inst. log.* 7. 2, on Boethus (i.e. Boethus of Sidon). See also S. Bobzien, 'The Development of *modus ponens* in Antiquity', *Phronesis*, 47 (2002), 359–94, and 'Pre-Stoic Hypothetical Syllogistic in Galen'.

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