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Maxwell, Helmholtz, and the Unreasonable Effectiveness of the Method of Physical Analogy

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Abstract:
The fact that the same equations or mathematical models reappear in the descriptions of what are otherwise disparate physical systems can be seen as yet another manifestation of Wigner's "unreasonable effectiveness of mathematics." James Clerk Maxwell famously exploited such formal similarities in what he called the "method of physical analogy." Both Maxwell and Hermann von Helmholtz appealed to the physical analogies between electromagnetism and hydrodynamics in their development of these theories. I argue that a closer historical examination of the different ways in which Maxwell and Helmholtz each deployed this analogy gives further insight into debates about the representational and explanatory power of mathematical models.

I. Introduction: Wigner's Puzzles

Eugene Wigner, in his classic paper "The Unreasonable Effectiveness of Mathematics in the Natural Sciences," poses two challenges: The first concerns the subject of his title most directly, namely the challenge of understanding how "mathematical concepts turn up in entirely unexpected connections" (Wigner 1960, p. 2). The second challenge is how we can "know whether a theory formulated in terms of mathematical concepts is uniquely appropriate" (p. 2), or what he later describes as the remarkable accuracy and (prima facie) explanatory power of false theories. There is, however, a third puzzle that lies at the intersection of Wigner's two challenges, and that is understanding how the same equations or mathematical models can sometimes reappear in the descriptions of what are otherwise very different sorts of physical systems. This puzzle not only raises questions about the unreasonable effectiveness of mathematics but also Wigner's worries about non-uniqueness and the prima facie explanatory power of false models. Moreover, this formal similarity between two distinct domains of science, can give rise to a methodology whereby the results obtained in elaborating the models in one domain can then be imported into the other domain to also solve problems there.

This third puzzle, involving the same mathematical equations reappearing in the descriptions of what are otherwise very different physical systems, is most strikingly illustrated in the works of James Clerk Maxwell and Hermann von Helmholtz. Maxwell famously exploited these formal similarities between two distinct domains of science in what he called the method of physical analogy. An early articulation of this methodology occurs in his 1855 article “On Faraday’s Lines of Force”, where he writes,

> By a physical analogy I mean that partial similarity between the laws of one science and those of another which makes each of them illustrate the other. . . . [W]e find the same resemblance in mathematical form between two different phenomena. (Maxwell [1855/56] 1890, p. 156)

Maxwell used this methodology repeatedly in his development of the theory of electromagnetism, such as by drawing physical analogies between fluid dynamics (hydrodynamics) and electromagnetic phenomena. He, for example, conceives of Faraday's lines of force as thin tubes carrying an imaginary incompressible fluid, though explicitly notes that this fluid should not be thought of as a physical hypothesis, but rather simply as a useful fiction.

Interestingly Helmholtz independently exploited this very same physical analogy; but rather than using hydrodynamics for the further development of electromagnetism, Helmholtz used electromagnetism for the further development of hydrodynamics.[1] In his seminal 1858 paper on vortex motions, he writes that there is a

> remarkable analogy between the vortex-motion of fluids and the electro-magnetic action of electric currents. . . . I shall therefore frequently avail myself of the analogy of the presence of magnetic masses or of electric currents, simply to give a briefer and more vivid representation. (Helmholtz [1858], p.43; 1867, p. 486-487)

By using this physical analogy and fictitious representation, Helmholtz was able to derive three fundamental theorems of fluid dynamics that are still accepted today.[2] While the fertility of this method of physical analogy is, I believe, indisputable, its philosophical grounding and implications are still not fully understood. To many, Maxwell's and Helmholtz's remarkable successes using this method are just another indication of Wigner's "unreasonable effectiveness of mathematics."

In what follows I shall take a closer look at the different ways in which Maxwell and Helmholtz each deployed this physical analogy between hydrodynamics and electromagnetism, and offer a more nuanced historical understanding of how this methodology works. My aim in this paper is not to reconstruct Maxwell's logic of scientific discovery (for this see, for example, Buchwald's (1985) rigorous and detailed book); rather, my aim is to use Maxwell's own reflections on the method of physical

[1] There are some interesting parallels between Maxwell's and Helmholtz's methodology here and what I have called the "reciprocal correspondence principle methodology" in the context of classical and quantum mechanics. Paul Dirac, for example, frequently used this latter method to solve problems in quantum theory by first translating them into the classical context, solving them there, and then using relations such as the correspondence principle, to import the solution back into quantum theory (see Bokulich 2008, Section 3.2). More generally, both methods illustrate an important, but often overlooked, "horizontal" dimension to model building (see, for example, Bokulich 2003).

[2] Lydia Patton (2009) has an excellent discussion of Helmholtz's work on fluid dynamics as presaging the *Bild* (picture) theory, and its subsequent influence on Hertz and Wittgenstein.

analogy as a framework for thinking about Wigner's puzzles and the representational power of mathematics. Maxwell does not simply employ these physical analogies and fictional posits with a naive opportunism, but rather engages in a philosophical reflection on both the legitimacy of such a methodology and its broader metaphysical implications. There are three points in Maxwell's reflections on this methodology that I wish to call attention to: The first concerns Maxwell's views on how mathematical models represent reality; the second, his views on the explanatory power of mathematical models; and the third, the version of scientific structuralism that Maxwell believes underlies this method of physical analogy. I shall conclude by showing the relevance of Maxwell's reflections for Wigner's puzzles and what I call the unreasonable effectiveness of the method of physical analogy.

II. Maxwell's Method of Physical Analogy

As Maxwell himself describes it, the most immediate source of inspiration for his method of physical analogy, was William Thomson's (Lord Kelvin) use of the analogy between heat and electrostatics, as presented in an 1842 paper.[3] In a letter to Thomson in the spring of 1855 Maxwell acknowledges this influence:

> I am trying to construct two theories, mathematically identical, in one of which the elementary conceptions shall be about fluid particles attracting at a distance while in the other nothing (mathematical) is considered but various states of polarization tension &c existing at various parts of space. The result will resemble your analogy of the steady motion of heat. Have you patented that notion with all its applications? for I intend to borrow it for a season without mentioning anything about heat...but applying it in a somewhat different way to a

[3] Thomson, W. (1842) "On the Uniform Motion of Heat in Homogeneous Solid Bodies, and its Connection with the Mathematical Theory of Electricity", *Cambridge Mathematical Journal* 3: 71-84.

> more general case. (Maxwell letter to Thomson, 15 May, 1855; in Harman 1990, pp. 306-307).[4]

As we will see below, Maxwell greatly expanded Thomson's method of physical analogy, using it in a more general way. Nancy Nersessian explains the difference between Thomson's and Maxwell's use of the method of analogy as follows:

> Thomson's method was to take an existing mathematical representation of a known physical system . . . as an analogical source. . . .That is, Thomson proceeded directly to the mathematical structures using a formal analogy between the two real-world domains. . . . What makes Maxwell's [approach]. . . different is that the analogical sources to be mapped to the domain of electromagnetism were not ready to hand, but had to be constructed. (Nersessian 2008, p. 51)

This more creative use of the method of analogy has been remarked on by many, such as Giora Hon and Bernard Goldstein who argue that it anticipated a very modern approach to modeling:

> Unlike Thomson, Maxwell described an artifact--an imaginary scheme--which he set into an analogical relation with the newly discovered electromagnetic phenomena. . . . this shift constitutes a new methodology: the application of contrived analogy, which may be considered the harbinger of the modern methodology of modeling. (Hon and Goldstein 2012, p. 246)

In order to better understand these innovations, let us turn to a closer examination of Maxwell's method.

Maxwell introduces his physical analogy as a middle path between what he calls a "purely mathematical formula" on the one hand and a "physical hypothesis" on the other.[5] He notes that if one adopts a purely mathematical approach, conceiving of these

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[4] In this same letter, Maxwell also mentions to Thomson that he has been investigating hydrodynamics and gives the example of vortex motion: "I have been investigating fluid motion with reference to stability and I have got results when the motion is confined to the plane of xy. I do not know if the method is new. It only applies to an incompressible fluid moving in a plane" (Maxwell letter to Thomson, 15 May, 1855; Larmor 1937, p.12). Helmholtz's treatise on vortex motion would appear just two years later.

[5] Turner (1955), for example, has talked extensively about this feature of Maxwell's methodology.

equations as nothing more than a string of mathematical symbols, then "we entirely lose sight of the phenomena to be explained; and though we may trace out the consequences of given laws, we can never obtain more extended views of the connexions of the subject" (Maxwell [1855/56] 1890, p. 155). According to Maxwell, an approach that conceives of these equations simply as a piece of mathematics is impoverished, and unable to generate scientific explanations. On the other hand, he also warns against the dangers of trying to investigate and explain phenomena through what he describes as the distorting medium of a physical hypothesis, which can lead to a sort of blindness and rashness of conclusions (Maxwell [1855/56] 1890, p. 156). The proper methodology, according to Maxwell, is that of physical analogy, which he describes as a way of getting physical ideas without actually adopting a full physical hypothesis.

A physical analogy is a resemblance in the form of the equations between what are otherwise different sorts of phenomena. In his 1870 address to the Mathematical and Physical Sections of the British Association, Maxwell describes the foundation of his method of physical analogy as follows:

> [T]he mathematical processes and trains of reasoning in one science resemble those in another so much that his knowledge of the one science may be made a most useful help in the study of the other. When he examines into the reason of this, he finds that . . . the mathematical forms of the relations of the quantities are the same in both systems, though the physical nature of the quantities may be utterly different. (Maxwell [1870] 1890, p. 218)

He goes on to note that these formal similarities provide the basis for a powerful methodology, which he describes as,

> a method to enable the mind to grasp some conception or law in one branch of science, by placing before it a conception or law in a different branch, and directing the mind to lay hold of that mathematical form which is common to the corresponding ideas in the two sciences, leaving out of the account for the present

> the difference between the physical nature of the real phenomena. (Maxwell [1870] 1890, p. 219)

Note in these quotations that Maxwell distinguishes rather sharply between what he calls the *form of the relations* between quantities and the *physical nature of the quantities themselves*. His method of physical analogy—despite the adjective *physical*—is not between the physical nature of quantities in the two sciences, which as he emphasizes can be quite different in the two systems being compared. It is rather an analogy between the relational or structural features of the two domains of phenomena.

Maxwell defends this methodology not simply as a lazy scientist's shortcut, but rather as a method that can lead to novel insights into a system that would be missed by simply studying that system alone:

> [T]he recognition of the formal analogy between the two systems of ideas leads to a knowledge of both, more profound than could be obtained by studying each system separately. (Maxwell 1870, p.219; Garber et al. 1986, p. 94)

Maxwell's own seminal discoveries using such a methodology furnishes some striking evidence in favor of this claim.

One might think that Maxwell's emphasis on formal correspondences, that ignore the physical differences between the systems being compared would lead him to defend a very abstract, mathematical characterization of scientific theories and models. As we will see, however, he does not advocate investigating these analogies between the formal relations in an abstract, purely mathematical way. Rather, it is the method of *physical* analogy because these formal relations are to be investigated in what Maxwell calls their "embodied form."

III. "Embodied" Mathematical Models

Despite Maxwell's emphasis on formal relations, he argues that we should remain grounded as far as possible in a physical interpretation of these mathematical quantities and relations. He writes, for example, in "On Faraday's Line of Force",

> My aim has been to present the mathematical ideas to the mind in an embodied form . . . not as mere symbols, which convey neither the same ideas, nor readily adapt themselves to the phenomena to be explained. (Maxwell [1855/56] 1890, p. 187)

He emphasizes this notion of embodied mathematics again in his discussion of Lagrangian mechanics:

> The aim of Lagrange was, as he tells us himself, to bring dynamics under the power of the calculus, and therefore he had to express dynamical relations in terms of the corresponding relations of numerical quantities. . . . We must therefore avail ourselves of the labours of the mathematician, and selecting from his symbols those which correspond to conceivable physical quantities, we must retranslate them into the language of dynamics. In this way our words will call up the mental images, not of certain operations of the calculus, but of certain characteristics of the motion of bodies. (Maxwell [1876] 1890, p. 308)

Maxwell is quite dismissive of formalist approaches that conceive of mathematical models in physics as simply a piece of pure mathematics. He instead calls attention to the complicated back and forth process by which physical considerations and mathematical formalisms develop in tandem, each one continually being re-tailored to the other.

In these passages, Maxwell is also calling attention to an important distinction that we often loose sight of when thinking about the representational power of mathematics. When we write down an equation or mathematical model there are really three different models we can intend by this expression: First, it might be conceived of as just a piece of pure mathematics--a string of symbols. Second, there is what Maxwell calls the mathematics in its embodied form. Here the quantities and relations are given a dynamical or physical interpretation, but it need not be in terms of the physical system one is investigating; the emphasis is rather on the general dynamical relations and properties, which can be instantiated by a number of different physical systems. Third, there is the equation as an expression of a particular physical hypothesis, where the physical interpretation is specifically in terms of the concrete system one is describing. With his notion of an embodied mathematical model, Maxwell is trying to carve out a middle ground between a mathematical model interpreted simply as a piece of abstract mathematics and a mathematical model interpreted as the expression of some particular mechanical or physical hypothesis.

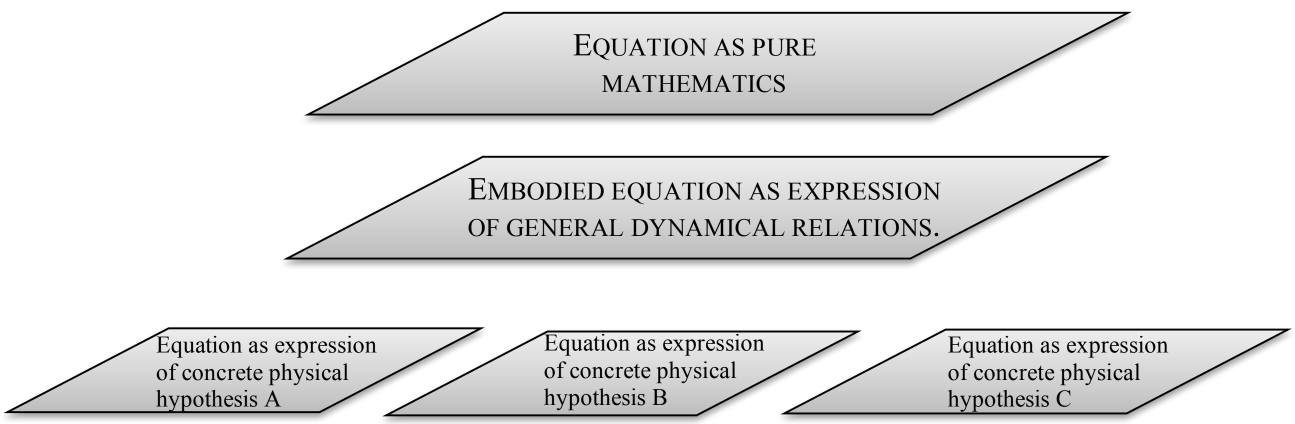

Figure 1: "Embodied" mathematical models as an intermediate level between the equation interpreted as pure mathematics (at the highest level of abstraction) and the equation interpreted as the expression of a particular physical hypothesis (at the most concrete level). Note that there can be multiple different kinds of concrete systems (A, B, C) which are instantiations of the general dynamical relations and properties expressed at the embodied level. Thus, for example, "A" can be used as an analogous or fictional representation for the investigation of physical system "B".

In her discussion of Maxwell's methodology, Nancy Nersessian similarly calls attention to these various levels of abstraction in terms of what she describes as the process of generic abstraction in model-based reasoning. She writes,

> Maxwell was able to formulate the laws of the electromagnetic field by abstracting from specific mechanical models the dynamical properties and relations. . . . [T]hese common dynamical properties and relations were separated from the specific instantiations provided in the models through which they had been rendered concrete. The generic mechanical relationships represented by the imaginary systems of the models served as the basis from which he abstracted a mathematical structure of sufficient generality. (Nersessian 2002, p.157)

What she describes as the model understood as a representation of "generic mechanical relationships" (apart from various possible concrete realizations) corresponds to this second "embodied" level, which Maxwell is emphasizing as so important. It can be distinguished both from the model as an abstract mathematical structure at one end, and from those mechanical relations as realized in a particular concrete system, at the other end. Recognizing the model as an expression of generic mechanical or dynamical relations allows one to use a particular concrete instantiation of those relations—even if a fictional or imaginary one—to stand in as a proxy for the particular physical system one is investigating.

IV. A Role for Fictions

Both Maxwell's and Helmholtz's use of the method of physical analogy lead them to embrace the legitimacy of fictional modeling. Helmholtz, for example, justifies his use of fictional magnetic masses and fictional electrical currents in his studies of hydrodynamics on the grounds that,

> [b]y means of these theorems a series of forms of motion, concealed in the class of the unexamined integrals of the hydrodynamic equations, at least becomes accessible to the imagination even if the complete integration is possible only in a few of the simplest cases. (Helmholtz 1858, p.43)

By thinking through these mathematical equations in a fictional embodied form, Helmholtz is able to arrive at results that would be difficult to extract through purely formal methods.

Maxwell similarly embraces the legitimacy of such fictional modeling, as long as one does not lose sight of the fact that it is only a fiction. When, for example, he conceives of Faraday's lines of force as fine tubes carrying an imaginary incompressible fluid, he warns,

> The substance here treated of must not be assumed to possess any of the properties of ordinary fluids except those of freedom of motion and resistance to compression. It is not even a hypothetical fluid. . . . It is merely a collection of imaginary properties. . . . The use of the word "Fluid" will not lead us into error, if we remember that it denotes a purely imaginary substance (Maxwell [1855/56] 1890, p. 160).

He explicitly distinguishes here between a fictional posit and a physical hypothesis. He argues that as long as we don't lose sight of the fictional nature of the posit, then we won't be mislead in reasoning with such fictional models.

Maxwell dismisses the idea that the same results could be achieved by simply restricting oneself to the formal equations alone. In his discussion of the analogy between light and vibrations of an elastic medium. He writes,

> The other analogy, between light and vibrations of an elastic medium extends farther, but, though its importance and fruitfulness cannot be over-estimated, we must recollect that it is founded only on a resemblance *in form* between the laws of light and those of vibrations. By stripping it of its physical dress and reducing it to a theory of 'transverse alternations,' we might obtain a system of truth strictly founded on observation, but probably deficient both in the vividness of its conceptions and the fertility of its method. (Maxwell [1855/56] 1890, p. 156)

In this passage we again see that Maxwell's willingness to make use of fictional models is tied to his belief in the importance of working with mathematical models in an *embodied* form.

Maxwell's use of fictions to provide a physical embodiment for his mathematical modeling culminates in his famous vortex-idle wheel model of the electromagnetic medium.

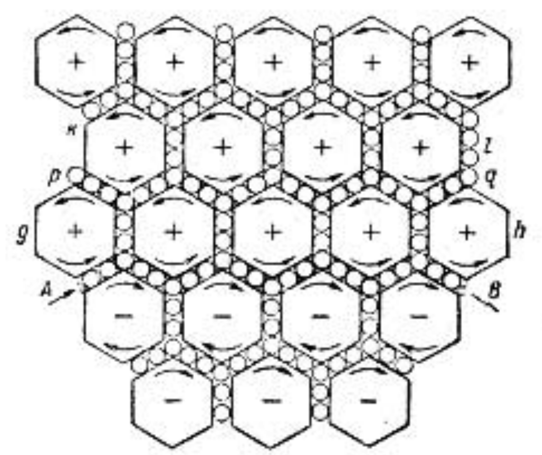

Figure 2: Maxwell's (1861/62) vortex-idle wheel model of the electromagnetic medium (from plate VIII, figure 2, facing p. 488).

As we see in the figure here, the model consists of rows of vortices, that is, cells of rotating fluid, that are stacked on top of each other. In order to keep the vortices rotating in the right directions, Maxwell introduces a layer of round "idle wheel" particles in between adjacent cells to allow the vortices to rotate in same direction. The rotating vortices are taken to represent the magnetic field, while the translation of the idle wheel particles represents the electric current. After working through all the details of this vortex model, Maxwell reminds his reader that it should not be interpreted literally:

> The conception of a particle having its motion connected with that of vortex by perfect rolling contact may appear somewhat awkward. I do not bring it forward as a mode of connexion existing in nature, or even as that which I would willingly

> assent to as an electrical hypothesis. It is, however, a mode of connexion which is mechanically conceivable, . . . and it serves to bring out the actual mechanical connexions between the known electro-magnetic phenomena; so that I venture to say that any one who understands the provisional and temporary character of this hypothesis, will find himself rather helped than hindered by it in his search after the true interpretation of the phenomena. (Maxwell [1861/62] 1890, p. 486)

Maxwell in this passage describes three possible ways of interpreting his model: first, it might be thought of as a literal description of what is actually going on in electromagnetic phenomena—an option he rejects outright; second, it could be interpreted as an hypothesis—a possible candidate for what is going on in an electromagnetic medium—an interpretation which he rejects as well. The third interpretation, which Maxwell endorses, is that it is a model which is consistent with certain mechanical/dynamical laws and, moreover, is able to capture in its fictional representation the correct mechanical/dynamical relations of electromagnetic phenomena. After introducing his fictional vortex model, Maxwell concludes, "we have now shewn in what way electro-magnetic phenomena may be imitated by an *imaginary* system of molecular vortices" (Maxwell [1861/62] 1890, p. 488; emphasis added). Once again, he notes that as long as we keep the fictional status of this model in mind, we will not be misled by it.

Maxwell's use of fictional models such as this highlights the point, often forgotten in contemporary philosophy of science, that fictional models can embody true physical information. This is a point I have defended at greater length in the context of fictional models of quantum systems (Bokulich 2008, 2009) and that William Wimsatt (1987), for example, has defended in the context of evolutionary biology. The striking successes of Maxwell's use of imaginary mechanical models in his development of electromagnetism offers a nice illustration of what Wimsatt calls the use of "false models as a means to truer theories" (Wimsatt 1987, 23). Hence, as Jordi Cat has argued, "the [traditional] dichotomy between realism and instrumentalism proves inadequate for making sense of Maxwell's position" (Cat 2001, p. 398).[6]

V. Maxwell, Helmholtz, and the Hydrodynamic-Electromagnetic Analogy

As we have seen, there is a striking similarity between Maxwell's and Helmholtz's uses of the method of physical analogy. Not only do they take a similar view of the importance of embodied fictional models, but they also both apply this methodology to the exact same pair of theories: electromagnetism and hydrodynamics. Nonetheless, a closer look at their writings reveals that they did not just differ on which way they ran the analogy. Interestingly, they also differed on how they thought the particular physical quantities hooked up with the mathematical formalism.

In his 1858 paper on the hydrodynamic equations, Helmholtz justifies drawing an analogy between hydrodynamics and electromagnetism on the grounds that both phenomena satisfy equations with the same formal structure. He explains,

> The mathematical similarity of these two classes of natural phenomena rests upon this, that in the case of water vortices, for those parts of the water mass that have no rotation, a velocity potential exists that satisfies the equation:
>
> $$\frac{d^2\varphi}{dx^2} + \frac{d^2\varphi}{dy^2} + \frac{d^2\varphi}{dz^2} = 0$$
>
> which equation holds everywhere except within the vortex filaments. If, however, we consider the vortex filaments as always closed . . . then the space for which the differential equation for φ is valid is multiply connected. . . . Such also is the case with the electromagnetic effects of a closed electric current. . . . the force it exerts on a magnetic particle can be considered as the differential quotients of a potential function V that satisfies the equation:
>
> $$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} = 0$$

[6] While I agree with Cat's conclusion here, our motivations are somewhat different.

> . . . Thus, in the case of vortex motions of water as in the case of electromagnetic effects, the velocities or forces outside the space traversed by vortex filaments or electric currents depend upon multivalued potential functions. (Helmholtz [1858], p. 57)

Although both sets of phenomena satisfy equations with the same formal structure, there is still the question of the proper physical interpretation of the terms, and more specifically, which particular physical quantities are going to be identified in the analogy. Helmholtz identifies the specific physical analogs as follows:

> Each rotating water particle *a* thus determines in every other particle *b* . . . a velocity whose direction is perpendicular to the plane through the axis of rotation. . . .The magnitude of this velocity is directly proportional to the volume of *a*, its velocity of rotation . . . and inversely proportional to the square of the distance between both particles. Exactly the same law holds for the force that would be exerted by an electric current at *a*, parallel to the axis of rotation, on a magnetic particle at *b*. (Helmholtz [1858], p. 56)

In other words, Helmholtz takes the rotating water particles to be analogous to the electric current, and those water particles whose motion is perpendicular to the plane of rotation are taken to be analogous to the motion of a magnetic particle.

Hydrodynamics	**Electromagnetism**
Rotation in Fluid	Electric Currents
Lines of Fluid Motion	Lines of Magnetic Force

Figure 3: Helmholtz on the analogs in the hydrodynamic-electromagnetic analogy

Maxwell seems to have first become aware of Helmholtz's paper and use of this analogy while writing Part II of "On Physical Lines of Force." He argues that although they are essentially using the same physical analogy, he believes that Helmholtz has incorrectly identified the corresponding analogs. Maxwell writes,

> Professor Helmholtz has investigated the motion of an incompressible fluid, and has conceived lines drawn so as to correspond at every point with the instantaneous axis of rotation of the fluid there. He has pointed out that the lines of fluid motion are arranged according to the same laws with respect to the lines of rotation, as those by which the lines of magnetic force are arranged with respect to electric currents. On the other hand, in this paper I have regarded magnetism as a phenomenon of rotation, and electric currents as consisting of the actual translation of particles, thus assuming the inverse of the relation between the two sets of phenomena. (Maxwell [1861/62] 1890, p. 503)

Maxwell appeals to several recent experiments, such as those by Ampère and by Thomson, in defense of his view that it is magnetism that should be identified as the phenomenon depending on rotation, while electric currents are to be regarded as a "species" of translation.

Hydrodynamics	**Electromagnetism**
Rotation in Fluid	Electric Currents
Lines of Fluid Motion	Lines of Magnetic Force

Figure 4: Maxwell's alternative view of the proper analogs in the hydrodynamic-electromagnetic analogy.

I want to argue that this difference between Helmholtz and Maxwell on how to identify the analogs in the physical analogy raises a number of interesting philosophical issues. First, even when one has specified the type of physical system that is correctly described by a system of equations, the meanings of the terms in the equation cannot be read off the mathematical equation alone—how those mathematical terms should be "embodied" (to use Maxwell's phrase) is still a nontrivial task. Mathematical equations do not wear their interpretations on their sleeves. This distinction is often forgotten when looking at a familiar equation in science—the physical embodiment of the equation has come to appear indistinguishable to us from its formal expression. Keeping this distinction in mind can help make sense of what are to some "puzzling" episodes in the history of science, when a scientist has successfully written down an equation expressing a new law, but fails to recognize its proper physical interpretation or physical implications.

Second, I want to argue that Maxwell's experience, seeing Helmholtz deploy the same physical analogy, but with a different physical interpretation of the mathematical terms led Maxwell to start thinking about what he called the "mathematical classification of physical quantities." Maxwell begins to articulate these ideas in Part IV of "On Physical Lines of Force", where he describes several different kinds of physical phenomena that all satisfy the same structural or formal relations. He begins by describing the electromagnetic case:

> The connexion between the distribution of lines of magnetic force and that of electric currents may be completely expressed by saying that the work done on a unit of imaginary magnetic matter, when carried around any closed curve, is proportional to the quantity of electricity which passes through the closed curve. The mathematical form of this law may be expressed as in equations [below], which I here repeat, where α, β, γ are the rectangular components of magnetic intensity, and p, q, r are the rectangular components of steady electric currents,

$$p = \frac{1}{4\pi}\left(\frac{d\gamma}{dy} - \frac{d\beta}{dz}\right)$$
$$q = \frac{1}{4\pi}\left(\frac{d\alpha}{dz} - \frac{d\gamma}{dx}\right)$$
$$r = \frac{1}{4\pi}\left(\frac{d\beta}{dx} - \frac{d\alpha}{dy}\right).$$

Maxwell then goes on to note that this same system of equations can also be used to describe the relation between other sets of phenomena in physics:

> (1) If α, β, γ represent displacements, velocities, or forces, then p, q, r will be rotatory displacements, velocities of rotation, or moments of couples producing rotation, in the elementary portions of the mass.
> (2) If α, β, γ represent rotatory displacements in a uniform and continuous substance, then p, q, r represent the *relative* linear displacement of a particle with respect to those in its immediate neighbourhood.[7]
> (3) If α, β, γ represent rotatory velocities of vortices whose centres are fixed, then p, q, r represent the velocities with which loose particles placed between them would be carried along.[8]

Maxwell then concludes from these different examples,

> It appears from all these instances that the connexion between magnetism and electricity has the same mathematical form as that between certain pairs of phenomena, of which one has a linear and the other a rotatory character. (Maxwell [1861/62] 1890, pp. 502-503)

The fact that the same mathematical relation can be found in the description of otherwise different physical phenomena led him to reflect on the utility of a mathematical classification of quantities.

[7] He cites Thomson's (1847) paper "On a Mechanical Representation of Electric, magnetic, and Galvanic Forces".

[8] He cites Part II of his "On Physical Lines of Force" where he introduces vortex-idle wheel model.

In an 1871 article Maxwell tackles this issue head on, and begins by explaining the different ways in which one might try to classify physical quantities. He writes, "One very obvious classification of quantities is founded on that of the sciences in which they occur. . . . But the classification which I now refer to is founded on the mathematical or formal analogy of the different quantities, and not on the matter to which they belong" (Maxwell [1871] 1890, p. 257). He argues that a mathematical classification of quantities is important in allowing us to make more efficient use of physical analogies in science.

Ten years later Maxwell again brings up what we might refer to as "Helmholtz's mistake," and explicitly connects it to this issue of the mathematical classification of quantities. He suggests that one useful way of classifying physical vectors is in terms of those which are defined with respect to a translation versus those defined with reference to rotation. He notes once again that Helmholtz misconstrues this physical analogy by representing the magnetic force by the velocity of the fluid (a species of translation) and representing the electric current by the rotation of the elements of the fluid (a species of rotation). For Maxwell not just any fictional embodiment of the mathematics will do—the physical quantities identified in the physical analogy must be of the same mathematical type. Simply having the same mathematical equations describing the two sets of phenomena is not enough. Knowing, for example, whether the rotational component should be identified with electricity or magnetism cannot be read off the equations themselves, and instead requires bringing a whole body of experimental evidence to bear. The fact that Maxwell brings up Helmholtz's mistake in several different articles, spanning more than ten years, I argue, is not in any way to undermine Helmholtz's genius (for which Maxwell had great respect); rather, for Maxwell, this episode served to bring out a subtle set of philosophical issues that he was struggling to articulate.

VI. Maxwell's Scientific Structuralism

The striking successes of his method of physical analogy led Maxwell at several points in his career to reflect on what it was about the world—or our knowledge of the world—that undergirded the fertility of this method. Early on in his career, Maxwell devotes an entire paper to precisely this question, which he titles, "Are There Real Analogies in Nature?" Maxwell begins the essay by dismissing the obvious objection that "no question exists as to the possibility of an analogy without a mind to recognise it—that is rank nonsense" (Maxwell [1856] 1882, p. 236). But he continues,

> Now, if in examining the admitted truths in science and philosophy, we find certain general principles appearing throughout a vast range of subjects, and sometimes re-appearing in some quite distinct part of human knowledge . . . are we to conclude that these various departments of nature in which analogous laws exist, have a real inter-dependence; or that their relation is only apparent and owing to the necessary conditions of human thought? (Maxwell [1856] 1882, p. 236)

Maxwell offers two possible answers here: one is a broadly Kantian answer, that we find the same principles throughout nature because they are conditions of human thought; that is, these analogies are, in a sense, an artifact of the nature of our own minds, and not in the world. Maxwell rejects this Kantian answer in favor of the view that there is a real interdependence in these various branches of science, which exists independently of our minds.

To understand Maxwell's view that there are real analogies in nature, it is necessary to take a closer look at what I will refer to as Maxwell's version of scientific structuralism. In this same article on analogies in nature, he writes,

> [A]lthough pairs of things may differ widely from each other, the *relation* in the one pair may be the same as that in the other. Now, as in a scientific point of view the *relation* is the most important thing to know, a knowledge of the one thing leads us a long way towards a knowledge of the other. (Maxwell [1856] 1882, p. 243)

There are two points worth highlighting in this passage. First, he argues that although the nature of the physical quantities—the relata in the two analogs—can be quite different, the *form of the relations* between those quantities in the two cases can be the same. Maxwell seems to be emphasizing here the structural—rather than physical—nature of the phenomena being compared. Second, Maxwell further emphasizes that it is these structural features of the phenomena that are the most important for science to know.

As Richard Olson (1975) has recounted in great detail, Maxwell's views here have their roots in Scottish Common Sense philosophy, which emphasized the importance of analogical thinking more generally, and in the philosophy of Sir William Hamilton, who was Maxwell's metaphysics and logic teacher at the University of Edinburgh. Of particular interest to our discussion here, one can also see elements of Maxwell's structuralism in the views of some of the prominent Scottish mathematicians who preceded Maxwell at Edinburgh. For example, Colin MacLaurin (1698-1746), in his *Treatise on Fluxions* writes, "[O]ur ideas of relations are often clearer and more distinct than of the things to which they belong; and to this we may ascribe, in some measure, the peculiar evidence of the mathematics" (MacLaurin [1742] 1801, p. 52).

Similarly, the Edinburgh mathematician and philosopher Dugald Stewart (1753-1828) writes, "the knowledge of the philosopher is more extensive than that of other men in consequence of the attention which he gives, not merely to objects and events, but to the relations which different objects and different events bear to each other" (Stewart 1792, p. 433). Stewart then connects this emphasis on a knowledge of the relations to the unreasonable effectiveness of mathematics when he writes, "there are various relations existing among physical events and various connexions existing among these relations. It is owing to this circumstance that mathematics is so useful an instrument in the hands of the physical inquirer" (Stewart 1792, p. 437). Like Maxwell, Stewart connects this emphasis on relations to the fertility of analogical thinking: "[T]hings which have no resemblance to each other may nevertheless be analogous; analogy consisting in a resemblance or correspondence of relations" (Stewart 1829, p. 275).[9] It is quite plausible, as Olson and others have suggested, that Maxwell's views were shaped by these philosophically inclined mathematicians as well.[10]

In his *Treatise on Electricity* Maxwell distinguishes sharply between the relations between the phenomena and the phenomena themselves. In presenting his method of physical analogy he writes,

> In many cases the relations of the phenomena in two different physical questions have a certain similarity . . . The similarity which constitutes the analogy is not between the phenomena themselves, but between the *relations* of these phenomena. (Maxwell 1888, p. 51; emphasis added)

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[9] Stewart here cites the work of his teacher at Edinburgh, the philosopher Adam Ferguson.

[10] See also Suárez's (forthcoming) Chapter 2 for a discussion of these influences on Maxwell.

He warns that just because two sets of phenomena share the same formal relations or are described by the same system of equations does not mean that they share the same physical causes. He continues,

> We must not conclude from the partial similarity of some of the relations . . . that there is any real physical similarity between the causes of these phenomena. The similarity is a similarity between relations, not a similarity between the things related. (Maxwell 1888, p. 52)

As we will see next, the fact that two sets of phenomena may have very different physical causes operating, despite being described by the same mathematical model has important implications for Maxwell's views on scientific explanation.

VII. Physical Analogies and Scientific Explanation

The question remains whether Maxwell believed that the discovery of a physical analogy between a new domain of phenomenon and one already mathematically described carried any explanatory force. If by explanation we mean something like a causal explanation, then Maxwell's answer is clearly no. In his discussion of the physical analogy between electromagnetism and hydrodynamics, he denies that this analogy should be taken as a physical or causal explanation. He writes approvingly of Helmholtz's discussion of this same analogy, "He does not propose this as an explanation of electro-magnetism; for though the analogy is perfect in form, the dynamics of the two systems are different" (Maxwell [1871] 1890, p.263). Although the mathematical models or equations describing these two set of phenomena are the same, the "physics" of the two systems is quite different, hence it should not be construed as anything like a causal explanation.

Interestingly, Maxwell makes precisely the same point with regard to his vortex model. Referring back to his work in "On Physical Lines of Force", he writes in 1865[11]

> I have on a former occasion attempted to describe a particular kind of motion and a particular kind of strain, so arranged as to account for the phenomena. In the present paper I avoid any hypothesis of this kind; and in using such words as electric momentum and electric elasticity in reference to the known phenomena . . . I wish merely to direct the mind of the reader to mechanical phenomena which will assist him in understanding the electrical ones. All such phrases in the present paper are to be considered illustrative, not as explanatory. (Maxwell [1865] 1890, pp. 563-564).

In a post card written to his close colleague and friend Peter Guthrie Tait in 1867, Maxwell tries to further explain this difference between his approach in the paper just quoted from and his approach in "On Physical Lines of Force". Maxwell writes,

> There is a difference between a vortex theory ascribed to Maxwell . . . and a dynamical theory of Electromagnetics by the same author in Phil Tran 1865. The former is built up to show that the phenomena are such as can be explained by a mechanism. The nature of this mechanism is to the true mechanism what an orrery is to the solar system. The latter is built on Lagranges Dynamical Equation and is not wise about vortices. (Maxwell to Tait, December, 1867; quoted in Harman 1998, p. 118)

An orrery does not purport to be an explanation of the causes of the motion of the solar system, only a representation of the dynamical relations between the elements in the solar system. Finally he makes this point yet again in the *Treatise*, where he explicitly describes it as a *model* that is not to be interpreted literally:

> The attempt which I then made to imagine a working model of this mechanism must be taken for no more than it really is, a demonstration that mechanism may be imagined capable of producing a connexion mechanically equivalent to the actual connexion of the parts of the electromagnetic field. (Maxwell 1873, pp. 416-417)[12]

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[11] "A Dynamical Theory of the Electromagnetic Field"

[12] *Treatise* Volume 2. He goes on to say "The problem of determining the mechanism required to establish a given species of connexion between the motions of the parts of a system always admits of an infinite number of solutions" (Maxwell 1873, p. 417).

Thus, it is clear that Maxwell does not take his physical analogies and fictional models to be explanatory in the straightforward sense of providing a literal mechanistic or causal explanation.

Moreover, both Maxwell and Helmholtz also recognize that these physical analogies are typically only partial, and argue that it is just as important to study where these analogies break down, as where they hold. In "On Physical Lines of Force", for example, Maxwell argues that these "coincidences in the mathematical expressions" of two sets of phenomena are in fact only partial coincidences, and "that they are only partial is proved by the divergence of the laws of the two sets of phenomena in other respects" (Maxwell [1861/62] 1890, p. 488). [13] Thus it is not just in their physical nature that these analogs differ, they will also typically differ with respect to other formal relations. Nonetheless Maxwell holds out the possibility that "[w]e may chance to find, in the higher parts of physics, instances of more complete coincidence, which may require much investigation to detect their ultimate divergence (Maxwell [1861/62] 1890, p. 488). Maxwell seems to envision different levels of abstraction at which various formal correspondences might be found. By going to a higher level of abstraction, physicists and mathematicians can continue to rework and extend these analogies, making them look more complete than they did at a lower level of abstraction.

While physical analogies are not meant to provide a causal-mechanical explanation of the phenomena to which they are applied, there is another sense in which Maxwell seems to suggest that such physical analogies do carry some explanatory force.

[13] In the *Treatise* he gives the example that "it is only a particular class of cases of the conduction of heat that have analogous cases in electrostatics" (Maxwell 1888, p. 52).

Maxwell accounts for the success of the method of physical analogy by noting that "all analogies of this kind depend on principles of a more fundamental nature" (Maxwell [1871] 1890, p. 258). He elaborates on the relevance of these principles for his views on scientific explanation in a later work where he writes,

> When any phenomenon can be described as an example of some general principle which is applicable to other phenomena, that phenomena is said to be explained. Explanations, however, are of very various orders, according to the degree of generality of the principle which is made use of. (Maxwell [1875] 1986, p. 217)

Maxwell is not only highlighting the importance of abstract principles in scientific explanation, but also articulating a notion of explanatory depth, according to which some scientific explanations may be counted as "deeper" than others.

Insofar as physical analogies depend on abstract principles, and these abstract principles carry a certain degree of explanatory force, then the physical analogies can be thought of as providing a kind of explanation in this limited sense. On this view, the physical analogy can be thought of as a representation—an embodied representation—of this abstract principle, which is itself explanatory.

VIII. The Unreasonable Effectiveness of the Method of Physical Analogy

Maxwell ends Part II of "On Physical Lines of Force" by reflecting on the philosophical significance of the non-uniqueness of such fictional modeling, and more generally the striking successes of his method of physical analogy. He writes,

> The facts of electro-magnetism are so complicated and various, that the explanation of any number of them by several different hypotheses must be interesting, not only to physicists, but to all who desire to understand how much evidence the explanation of phenomena lends to the credibility of a theory, or how far we ought to regard a coincidence in the mathematical expression of two sets of phenomena as an indication that these phenomena are of the same kind. (Maxwell [1861/62] 1890, p. 488)

Maxwell is raising two interesting philosophical issues here. The first is essentially Wigner's second puzzle about the uniqueness of theories in physics and the apparent success of false theories. Given that we can construct multiple—mutually inconsistent—models of the phenomena, each of which can successfully account for the observable phenomena, then how do we know which hypothesis or model is correct? The apparent "explanation" of a set of phenomena by a particular physical or mechanical model can no longer naively be taken as a direct confirmation of that model.

The second philosophical issue that Maxwell is raising, concerns the method of physical analogy more specifically: namely, can we infer from the fact that two sets of phenomena obey the same mathematical equations that they are the same *kind* of physical phenomena? For example, can we conclude from the fact that electro-magnetic phenomena and hydrodynamic phenomena can be accurately described by means of the same mathematical models, that electricity is essentially a fluid? The answer, as Maxwell makes clear, is no—the similarity in question is between the form of the relations, not the physical nature of the relata. As we have seen, however, there is for Maxwell a second, subtler sense in which two phenomena can be "of the same kind": they can be two different physical tokens representable by the same type of mathematical quantity.[14] This is the beginning of Maxwell's answer to the unreasonable effectiveness of the method of physical analogy.

Maxwell's more complete answer to Wigner's puzzles—regarding both the applicability of mathematics in general and the effectiveness of his methodology more

[14] For example, both the magnetic field and vorticity are quantities that transform as pseudovectors, gaining a sign change under transformations such as reflections.

specifically—seems to rely on the familiar "Book of Nature" metaphor. In his article on "Real Analogies in Nature," Maxwell considers two metaphysical possibilities:

> Perhaps the 'book,' as it has been called, of nature is regularly paged; if so, no doubt the introductory parts will explain those that follow, and the methods taught in the first chapters will be taken for granted and used as illustrations in the more advanced parts of the course; but if it is not a 'book' at all, but a magazine, nothing is more foolish to suppose that one part can throw light on another. (Maxwell [1856] 1882, p. 243)

In order for his method of physical analogy to work, Maxwell assumes that the same structures must reappear throughout nature as variations on a theme. If there is not in fact a structural continuity across different areas of science, then as Maxwell says, it would be foolish to expect the method of physical analogy to work. What was posed as an open question in this early article, becomes a confident assertion 17 years later: "The Book of Nature, in fact, contains elementary chapters, and, to those who know where to look for them, the mastery of one chapter is a preparation for the study of the next" (Maxwell [1873] 1986, p. 126). Did Maxwell think that this Book of Nature was written in the language of mathematics? From his emphasis on the importance of having an embodied mathematics, I suspect that while he believed that the syntax of nature was mathematical, he would say that its semantics was not (cf. McMullin 1985).

There is, however, another answer to Wigner's puzzles, that begins to emerge from a close examination of this historical episode involving Maxwell and Helmholtz. First, the way in which mathematical models hook up with physical quantities is not something that is given, but rather is forged with difficulty. As "Helmholtz's mistake" makes clear, even once one has established that a physical system can be adequately described by some equation, there is still the nontrivial question of how the particular mathematical quantities are to be hooked up with the physical ones. It might be clear from the mathematical equation that it describes a pair of phenomena, one of which has a rotatory and the other a linear character, as Maxwell puts it; but the equation itself does not tell us which one represents electricity and which one represents magnetism, for example. That can only be determined after many experiments.

Second, this suggests not only that we need to be careful in trying to read the physics off of the mathematics, but that there is also a more complicated process by which mathematics comes to represent the world. Another way to put this might be to say that mathematics doesn't represent the world, it represents our physical models of the world, and it is those physical models that in turn represent the world.[15] Conceptually, one can think of there being multiple layers (as in Fig. 1), and Maxwell, with his emphasis on embodied mathematics is calling our attention to the importance of this middle layer.

Third, the analogies between different sets of phenomena, which are described by the same mathematical models, are only partial. It is only by reworking the analogies—both from the side of mathematics and from the side of physics, through ever higher levels of abstraction that they eventually come to look the same. As we saw in this historical episode, Helmholtz built the theory of hydrodynamics on the backbone of electromagnetism and Maxwell developed electrodynamics while drawing analogies to hydrodynamics. The fact that these two different domains of physics can be described by

[15] I owe this way of putting the point to John Stachel (personal communication). I have not offered a theory of representation here, though I suspect something like Mauricio Suárez's (2004) inferential conception is largely right.

some of the same mathematical models should not, perhaps, strike us as quite so surprising after all.[16]

[16] Ivor Grattan-Guinness (2008) also cites the fact that theories are often built in analogy with other theories in the explanation of what he calls "the reasonable (though perhaps limited) effectiveness of mathematics in the natural sciences" (p.7)

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