Die Grundlagen der Arithmetik §§82–83

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Reductions of arithmetic, whether to set theory or to a theory formulated in a higher-order logic, must prove the infinity of the sequence of natural numbers. In his *Was sind und was sollen die Zahlen?*, Dedekind attempted, in the notrious proof of Theorem 66 of that work, to demonstrate the existence of infinite systems by examining the contents of his own mind. The axioms of General Set Theory, a simple set theory to which arithmetic can be reduced, are those of Extensionality, Separation ("Aussonderung"), and Adjunction:

$$\forall w \forall z \exists y \forall x [x \in y \equiv x \in z \vee x = w)$$

It is Adjunction that guarantees that there are at least two, and indeed infinitely many, natural numbers. The authors of *Principia Mathematica*, after defining zero, the successor function, and the natural numbers in a way that made it easy to show that the successor of any natural number exists and is unique, were obliged to assume an axiom of infinity on those occasions on which they needed the proposition that different natural numbers have different successors.

In §§70–83 of *Die Grundlagen der Arithmetik*, Frege outlines the derivations of some familiar laws of the arithmetic of the natural numbers from principles he takes to be "primitive" truths of a general logical nature. In §§70–81, he explains how to define zero, the natural numbers, and the successor *relation*; in §78 he states that it is to be proved that this relation is one-one and adds that it does not follow that every natural number *has* a successor; thus, by the end of §78, the existence, but not the uniqueness, of the successor remains to be shown. Frege sketches, or attempts to sketch, such an existence proof in §§82–83, which would complete his proof that there are infinitely many natural numbers.

§§82–83 offer severe interpretive difficulties. Reluctantly and hesitantly, we have come to the conclusion that Frege was at least somewhat confused in these two sections and that he cannot be said to have outlined, or even to have intended, any correct proof there. We will discuss two (correct) proofs of the statement that every natural number has a successor which might be extracted from §§82–83. The first is quite similar to a proof of this proposition that Frege provides in

Grundgesetze der Arithmetik, differing from it only in notation and other relatively minor respects. We will argue that fidelity to what Frege wrote in *Die Grundlagen* and in *Groundgesetze* requires us to reject the charitable suggestion that it was this (beautiful) proof that he had in mind when he wrote *Die Grundlagen*. The second proof we discuss conforms to the outline Frege gives in §§82–83 more closely than does the first. But if it had been the one he had in mind, the proof-sketch in these two sections would have contained a remarkably large gap that was never filled by any argument found in *Grundgesetze*. In any case, it is certain that Frege did not know of this proof.

We begin by discussing §§70–81.

In §70, Frege begins the definition of equinumerosity by explaining the notion of a relation, arguing that like (simple) concepts, relational concepts belong to the province of pure logic. In §71, he defines "the objects falling under F and G are correlated with each other by the relation ϕ". Using modern notation, but strictly following Frege's wording, we would write:

$$\forall a \neg (Fa \wedge \neg \exists b(a\phi b \wedge Gb)) \wedge \forall a \neg (Ga \wedge \neg \exists b(Fb \wedge b\phi a)).$$

To put the definition slightly more transparently, the objects falling under F and G are correlated by ϕ iff

$$\forall x(Fx \rightarrow \exists y(Gy \wedge x\phi y)) \wedge \forall y(Gy \rightarrow \exists x(Fx \wedge x\phi y)).$$

In §72, Frege defines what it is for the relation ϕ to be one-one ("beiderseits eindeutig", "single-valued in both directions"): It is for it, as we should say, to be a function, i.e., $\forall d \forall a \forall e(d\phi a \wedge d\phi e \rightarrow a = e)$, that is one-one, i.e., $\forall d \forall a \forall e(d\phi a \wedge b\phi a \rightarrow d = b)$. Frege then defines "equinumerous" ("gleichzahlig"): F is equinumerous with G iff there is a relation that correlates the objects falling under F one-one with those falling under G:

$$\exists \phi [\forall x(Fx \rightarrow \exists y(Gy \wedge x\phi y)) \wedge \forall y(Gy \rightarrow \exists x(Fx \wedge x\phi y)) \wedge \\ \forall d \forall a \forall e(d\phi a \wedge d\phi e \rightarrow a = e) \wedge \forall d \forall a \forall e(d\phi a \wedge b\phi a \rightarrow d = b)].$$

We abbreviate this formula: $F \approx G$.

At the end of §72, Frege defines the number that belongs to F as the extension of the concept "equinumerous with the concept F". He also defines "n is a (cardinal) number": there is a concept F such that n is the number that belongs to F. His next task, attempted in §73, is to prove a principle that Crispin Wright (1983) once called $N^{=}$ (for numerical equality), Michael Dummett (1991) calls "the original equivalence", and we call "HP": the number belonging to F is identical with that belonging to G iff F is equinumerous with G.

The trouble with the definition of number given in §72 and the proof of HP given in §73 is that they implicitly appeal[1] to an inconsistent theory of extensions of second-level concepts. Russell of course demonstrated the inconsistency of Frege's theory, presented in *Grundgesetze der Arithmetik*, of extensions of first-level concepts; a routine jacking-up of Russell's argument shows that of the theory Frege tacitly appeals to in *Die Grundlagen*.[2] It is by now well-known, however, that Frege Arithmetic, i.e., the result of adjoining a suitable formalization of HP to axiomatic second-order logic, is consistent if second-order arithmetic is, and is strong enough to imply second-order arithmetic (as of course Frege can be seen as attempting to prove in *Die Grundlagen*). Indeed, Frege Arithmetic and second-order arithmetic are equi-interpretable; in Appendix 2, we should how to interpret Frege Arithmetic in second-order arithmetic.

Writing: $\#F$ to mean: the number belonging to the concept F, we may symbolize HP: $\#F = \#G \equiv F \approx G$.

The development of arithmetic sketched in §§74–81 makes use only of Frege Arithmetic and can thus be reconstructed in a consistent theory (or one we believe to be so!). Nothing will be lost and much gained if we henceforth suppose that Frege's background theory is Frege Arithmetic.

In §74, Frege defines 0 as the number belonging to the concept "not identical with itself": $0 = \#[x : x \neq x]$. ($[x : \ldots x \ldots]$is the concept *being an object x such that* $\ldots x \ldots$.) Frege notes that it can be shown on logical grounds that nothing falls under $[x : x \neq x]$. In §75, he states that $\forall x(\neg Fx) \to [\forall x(Gx) \equiv F \approx G]$ has to be proved, from which $\forall x(\neg Fx) \equiv 0 = \#[x : Fx]$ follows. These have easy proofs. Frege outlines that of the former in detail.

§76 contains the definition of "n follows immediately after m in the 'natürliche Zalenreihe'":

$$\exists F \exists x (Fx \wedge \#F = n \wedge \#[y : Fy \wedge y \neq x] = m)$$

It is advisable, we think, to regard the relation so defined in this section as going

[1] The appeal is made when Frege writes "In other words:" at the end of the second paragraph of §73.

[2] Let (V) be $\forall \mathscr{F} \forall \mathscr{G} (\hat{\mathscr{F}} = \hat{\mathscr{G}} \equiv \forall X(\mathscr{F}X \equiv \mathscr{G}X))$. Then (V) is inconsistent (in third-order logic). For let $\mathscr{F}$ be $[X : \forall \mathscr{H} (\forall x(Xx \equiv x = \hat{\mathscr{H}}) \to \neg \mathscr{H}X)]$ and let X be $[x : x = \hat{\mathscr{F}}]$. Suppose $\mathscr{F}X$. Then $\forall \mathscr{H} (\forall x(Xx \equiv x = \hat{\mathscr{H}}) \to \neg \mathscr{H}X)$. So $\forall x(Xx \equiv x = \hat{\mathscr{F}}) \to \neg \mathscr{F}X$, whence $\neg \mathscr{F}X$ by the definition of X. Thus $\neg \mathscr{F}X$. So for some $\mathscr{H}$, $\forall x(Xx \equiv x = \hat{\mathscr{H}})$ and $\mathscr{H}X$, and then $X(\hat{\mathscr{F}}) \equiv \hat{\mathscr{F}} = \hat{\mathscr{H}}$. By the definition of X again, $\hat{\mathscr{F}} = \hat{\mathscr{F}} \equiv \hat{\mathscr{F}} = \hat{\mathscr{H}}$, $\hat{\mathscr{F}} = \hat{\mathscr{H}}$, and by (V), $\forall X(\mathscr{F}X \equiv \mathscr{H}X)$, contra $\neg \mathscr{F}X$ and $\mathscr{H}X$. (We use '^' to mean "the extension of" and "[: …]" to denote concepts (of whatever level).)

from m to n, despite the order of 'n' and 'm' in both the definiens and the definiendum of "n immediately follows m in the natural seris of numbers". We shall thus symbolize this relation: mPn ('P' for "(immediately) precedes").

Call a concept *Dedekind infinite* if it is equinumerous with a proper subconcept of itself; equivalently, if it has a subconcept equinumerous with the concept *being a natural number*. With the aid of the equivalence of these definitions of Dedekind infinity, it is not difficult to see that nPn if and only if n is the number belonging to a Dedekind infinite concept. Thus the number of finite numbers, which Frege designates roughly: ∞,[3] but which we shall as usual denote: $\aleph_0$, follows itself in the "natürliche Zahlenreihe", in symbols: $\aleph_0 P \aleph_0$. Since $\aleph_0$ is not a finite, i.e., natural, number, we shall translate "in der natürliche Zahlenreihe" as "in the natural sequence of numbers".[4]

§77 contains the definition of 1, as $\#[x : x = 0]$, and a proof that $0P1$. In §78, Frege lists a number of propositions to be proved:

1. $0Pa \rightarrow a = 1$;
2. $1 = \#f \rightarrow \exists x(Fx)$;
3. $1 = \#F \rightarrow (Fx \wedge Fy \rightarrow x = y)$;
4. $\exists x(Fx) \wedge \forall x \forall y(Fx \wedge Fy \rightarrow x = y) \rightarrow 1 = \#F$;
5. P is one-one ("beiderseits eindeutig"), i.e., $mPn \wedge m'Pn' \rightarrow (m = m' \equiv n = n')$.[5]

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[3] This is not quite the symbol Frege uses, which looks like a very open omega with a frown across the top.

[4] Timothy Smiley (1988) observed that "in the nautral series of numbers" is to be preferred as a translation of "in der natürliche Zahlenreihe" to Austin's "in the series of natural numbers". We have substitutes "sequence" for "series" throughout.

[5] Frege does not indicate what proof of 78.5 he might have intended. Here is an obvious one that he might have had in mind.

Suppose mPn and $m'Pn'$. Then for some F, F', x, x', Fx, $F'x'$, $\#F = n$, $\#F' = n'$, $\#[y : Fy \wedge y \neq x] = m$, and $\#[y' : F'y' \wedge y' \neq x'] = m'$.

Assume $m = m'$. Then by HP, there is a one-one correspondence ϕ between the objects y such that Fy and $y \neq x$ and the objects y' such that $F'y'$ and $y' \neq x'$. We may assume that if $y\phi y'$, then Fy, $y \neq x$, $F'y'$, and $y' \neq x'$. Let $y\psi y'$ iff $(y\phi y' \vee [y = x \wedge y' = x'])$. Then ψ is a one-one correspondence between the objects falling under F and those falling under F', and so by HP, $n = n'$.

Assume $n = n'$. By HP, let ψ be a one-one correspondence between the obejcts falling under F and those falling under F'. We may assume that if $y\phi y'$, then Fy, and $F'y'$. Let $y\phi y'$ iff $(Fy \wedge y \neq x \wedge F'y' \wedge y' \neq x' \wedge [y\psi y' \vee (y\psi x' \wedge x\psi y')])$. Then ψ is a one-one correspondence between the objects y such that Fy and $y \neq x$ and the objects y' such that $F'y'$ and $y' \neq x'$, and so by HP, $m = m'$.

Frege observes that it has not yet been stated that every number immediately follows or is followed by another. He then states:

6. Every number except 0 immediately follows a number in the natural sequence of numbers.

It is clear from §44 of *Grundgesetze*[6] that Frege did not take (6) to imply that 0 does not immediately follow a number, that $\neg xP0$. This proposition is proved separately in *Grundgesetze*, as Theorem 108, and will be used later on here, at a key point in the argument.

§79 contains the definition of the strong ancestral of ϕ, "x precedes y in the ϕ-sequence" or y follows x in the ϕ-sequence":

$$\forall F\,(\forall a\,(x\phi a \to Fa) \to \forall d\forall a\,(Fd \to d\phi a \to Fa) \to Fy)$$

which was Definition (76) of the *Begriffsschrift*. Frege will use this definition in §81 to define "member of the natural sequence of numbers ending with n". We shall use the standard abbreviation: $x\phi^*y$ for the strong ancestral. To prove that if $x\phi^*y$, then $\ldots y\ldots$, it suffices, by the comprehension scheme $\exists F\forall a(Fa \equiv \ldots a\ldots)$ of second-order logic, to show that $\forall a(x\phi a \to \ldots a\ldots)$ and $\forall d\forall a(\ldots d\cdots \to d\phi a \to \ldots a\ldots)$. We call this method of proof *Induction 1*. (Induction 2 and Induction 3 will be defined below.)

Here and below, we associate iterated conditionals to the right. Thus, e.g., "$A \to B \to C$" abbreviates "$(A \to (B \to C))$". This convention provides an easy way to reproduce in a linear symbolism one major notational device of both *Begriffsschrift* and *Grundgesetze*.

Frege mentions in §80 that it can be deduced from the definition of "follows" that if b follows a in the ϕ-sequence and c follows b, then c follows a; the transitivity of the strong ancestral is Proposition (98) of the *Begriffsschrift*. The proof Frege gives there can be formalized in second-order logic only wiht the aid of the comprehension schema (or something to the same effect); however, there is an easier proof that makes uses only of the ordinary quantifier rules, applied to the universal quantifier in the definition of ϕ^* (Boolos, 1998, pp. 158–9). For the proof in §§82–83, Frege will also need Propsition (95) of *Begriffsschrift*: if $x\phi y$, then $x\phi^*y$, which easily follows from the definition of ϕ^*.

At the very end of §80 Frege states that only by means of the definition of following in a sequence is it possible to reduce the method of inference ("Schlussweise",

[6] All reference to sections of *Grundgesetze* are to Volume I.

which Austin mistranslates as "argument") from n to $n+1$ to the general laws of logic. Of course, the method of inference from n to $n+1$ is what we call mathematical induction; Frege's remarks may be taken to be a claim that mathematical induction can be proved with the aid of the definition of the ancestral of P.

In §81, Frege defines the weak ancestral: "y is a member of the ϕ-sequence beginning with x" and "x is a member of the ϕ-sequence ending with y" are to mean: $x\phi y \vee y = x$. We shall use the abbreviation: $x\phi^{*=}y$. He states at the beginning of the section that if ϕ is P, then he will use the term "natural sequence of numbers" instead of "P-sequence". We thus have five terms: "y follows x in the natural sequence of numbers", "x precedes $y\ldots$", "y immediately follows $x\ldots$", "x is a member of the natural sequence of numbers ending with y", and "y is a member... beginning with x". We shall abbreviate these as: $xP^{*}y$, $xP^{*}y$, xPy, $xP^{*=}y$, and $xP^{*=}y$, respectively.

Induction 2 is the following method of proof, in which weak ancestrals occur as hypotheses: To prove that if $x\phi^{*=}y$, then $\ldots y\ldots$, it suffices to prove:

(i) $\ldots x \ldots$

(ii) $\forall d \forall a(\ldots d \cdots \to d\phi a \to \ldots a \ldots)$.

Induction 2 follows quickly from Induction 1: If (i) and (ii) hold, then so does $\forall a(x\phi a \to \ldots a \ldots)$; thus, if $x\phi^{*}y$, then $\ldots y \ldots$, by Induction 1. But if $x = y$, then by (i), $\ldots y \ldots$ again. Frege proves Induction 2 as Theorem 144 of *Grundgesetze*.

A basic fact about the weak ancestral, to which we shall repeatedly appeal, is that $x\phi^{*}a$ and thus $x\phi^{*=}a$, provided that $x\phi^{*=}d$ and $d\phi a$, fo rthem either $x\phi^{*}d\phi a$, $x\phi^{*}d\phi^{*}a$, and $x\phi^{*}a$, or $x = d\phi a$, $x\phi a$, and $x\phi^{*}a$, by (95) and (98) of *Begriffsschrift*. That $x\phi^{*}a$ if $x\phi^{*=}d$ and $d\phi a$ is Theorem 134 of *Grundgesetze der Arithmetik*; that $x\phi^{*=}a$ if $x\phi^{*}a$ is Theorem 136.

Frege has not yet defined finite, or natural, number. He will do so only at the end of §83, where "n is a finite number" is defined as "n is a member of the natural sequence of numbers beginning with 0", i.e., as: $0P^{*=}n$. By Induction 2, to prove that $\ldots n \ldots$ if n is finite, it suffices to prove $\ldots 0 \ldots$ and $\forall d \forall a(\ldots d \cdots \to dPa \to \ldots a \ldots)$.

In the formalism in which we are supposing Frege to be working the existence and uniqueness of 0, defined in §74 as $\#[x : x \neq x]$, are given by the comprehension scheme for second-order logic and the standard convention of logic that function signs denote *total* functions. Thus # denotes a total function from second-order to first-order entities and the existence of $\#[x : x = x]$, that of $\#[x : x \neq x]$, and that of $\#[x : x = \#[x : x \neq x]]$ will count as truths of logic. The propositions that 0 is a natural number and that any successor of a natural number is a natural number follow immediately from the definition of "natural number"; 78.5 says that P is functional and one-one. So apart from the easily demonstrated statement that nothing precedes zero, by the end of §81 Frege can be taken to have established the Dedekind-Peano axioms for the natural numbers, except for the statement that every natural number *has* a successor.

Using the notation we have introduced, we may condense §§82–83 as follows:§82. It is now to be shown that—subject to a condition still to be specified—

$$(0)\ P(n, Nx : P^{*=}xn)$$

And in thus proving that there exists a Number that follows in the series of natural numbers directly after n, we shall have proved at the same time that there is no last member of this series. Obviously, this proposition cannot be established on empirical lines or by [enumerative] induction.

> §82. It is now to be shown that—subject to a condition still to be specified—(0) $nP\#[x : xP^{*=}n]$. And in thus proving that there exists a Number k such that nPk, we shall have proved at the same time that there is no last member of the natural sequence. . . .
>
> . . . It is to be proved that (1) $dPa \wedge dP\#[x : xP^{*=}d] \rightarrow aP\#[x : xP^{*=}a])$.
>
> It is then to be proved, secondly, that (2) $0P\#x[xP^{*=}0]$. And finally, it is to be deduced that $(0')$ $0P^{*=}n \rightarrow nP\#x : xP^{*=}n$. The method of inference ("Schlussweise") here is an application of the definition of the expression "y follows x in the natural sequence of numbers", taking [the strong ancestral], taking $\#[y : yP\#[x : xP^{*=}y]]$ for our concept F.[7]
>
> §83. In order to prove (1), we must show that (3) $a = \#[x : xP^{*=}a]$. And for this again it is necessary to prove that (4*) $[x : xP^{*=}a \wedge x \neq a]$ has the same extension as $[x : xP^{*=}d]$. For this we need the proposition $(5')$ $\forall a(0P^{*=}a \rightarrow \neg aPa)$. And this must once again be proved by mean of our definition of following in a sequence, along the lines indicated above.

[7] This sentence seems to throw Austin. But we take its last half to mean: when one takes for the concept F what is common to the statements about d and about a, about 0 and about n, and thus that the concept in question is $\#[y : yP\#[x : xP^{*=}y]]$. Austin's translation makes it sound as if some binary relation holding between d and a and also between 0 and n were meant. However good his German and English may have been, Austin was no logician. It is time for a reliable English translation of *Die Grundlagen*.

> We are obliged hereby to attach a condition to the proposition that $nP\#[x : xP^{*=}n]$, the condition that $0P^{*=}n$. For this there is a convenient abbreviation. . . : n is a finite number. We can thus formulate $(5')$ as follows: no finite number follows itself in the natural sequence of numbers.

(We have added some reference numbers; (1) is Frege's own. Primes indicate the presence of a finiteness condition in the antecedent; the asterisk in (4*) indicates (what at least appears to be) a reference to extensions.)

It might appear that Frege proposes in these two sections to prove, not (0), but $(0')$, as follows: First, prove $(5')$ by an appeal to the definition of P^*. Then derive (4*) from $(5')$ and (3) from (4*). From (3) derive (1). Prove (2). Then, finally, infer $(0')$ from (2) and (1), by a similar appeal to the definition of P^*.

However, it will turn out that this precise strategy cannot succeed. It cannot be (4*) and (3) that Frege wishes to derive—(3), e.g., is false if $a = \aleph_0$, as we shall see—but certain conditionals $(4')$ and $(3')$, whose consequents are (4*) (or rather an equivalent of it) and (3).

We do not, of course, know how Frege might have tried to fill in the details of this proof-sketch at the time of composition of *Die Grundlagen*. In particular, we do not know exactly how he would have proved $(5')$. (We can be reasonably certain that his proof of (2), however, would have been at least roughly like the proof we shall give below.) But, since he later proved a version of the following lemma as Theorem 141 of *Grundgesetze*, it seems plausible to us to speculate that he might have intended to appeal to something rather like it in his proof of $(5')$. The lemma is a logicized version of the arithmetical truth: if $i < k$, then for some j, $j+1 = k$ and $i \leq j$.

Lemma. $xP^{*=}y \rightarrow \exists z(zPy \wedge xP^*z)$

Proof. Let $Fa \equiv \exists z(zPa \wedge xP^{*=}z)$. Then $xPa \rightarrow Fa$, for if xPa, then certainly Fa: take $z = x$. And $Fd \wedge dPa \rightarrow Fa$: Suppose Fd and dPa. Then for some z, zPd and $xP^{*=}z$. By the basic fact about the weak ancestral, $xP^{*=}d$. But since dPa, Fa. The lemma follows by Induction 1. □

With the aid of the lemma, we can now use Induction 2 to prove $(5')$:

Proof. $0 = \#[x : x \neq x]$. By HP and the definition of P, $\forall z(\neg zP0)$, and therefore, by the lemma, $\neg 0P^*0$.

Now suppose dPa and aP^*a. Then by the lemma, for some z, zPa and $aP^{*=}z$, i.e., either aP^*z or $a = z$, and therefore either $zPaP^*z$ or $zPa = z$. In either case, zP^*z. Since dPa and zPa, $z = d$ by 78.5, and so dP^*d. Thus, $\neg dP^*d \wedge dPa \rightarrow \neg aP^*a$.

(5′) now follows by Induction 2. □

(5′) merits a digression. The part of *Die Grundlagen der Arithmetik* entitles "Our definition completed and its worth proved" begins with §70 and ends with §83; the concluding sentence of §83 reads: "We can thus formulate the last proposition above as follows: No finite number follows itself in the natural sequence of numbers." Apart from its position in the book and the fact that Frege mentions it in both the table of contents and the recapitulation of the book's argument at the end of *Die Grundlagen*, there are a number of reasons for thinking that Frege regarded this proposition as especially significant..

First, there is, according to Frege, and interesting connection with *counting*. When we count, he points out in §108 of *Grundgesetze*, we correlate the objects falling under a concept $\Phi(\xi)$ with the number words in their normal order from "one" up to a certain one, "*N*"; *N* is then the number of objects falling under $\Phi(\xi)$. Since correlating relations between concepts are not in general unique,

> the question arises whether one might arrive at a different number word '*M*' with a different choice of this relation. By our stipluations, *M* would then be the same number as *N*, but at the same time one of the two number words would follow after the other, e.g., '*N*' after '*M*'. Then *N* would follow in the sequence of numbers after *M*, i.e., after itself. Our Proposition [(5′)] excludes this for finite numbers.

We find this argument of considerable interest, but will not enter into a discussion of its correctness here.

Second, one of Frege's major philosophical aims, as is well known, was to show that reasons, under the aspect of logic, could yield conclusions for which many philosophers of his day might have supposed some sort of Kantian intuition to be necessary. The proof of (5′) is a paradigm illustration of how the role of intuition in delivering knowledge can be played by logic instead.

One might think that the truth of (5′) could be seen by the following sort of mixture of reason and intuition: (5′) says that there is no (non-null) loop of *P*-steps leading from *a* back to *a* whenver *a* is a finite number. So if *a* is finite but not zero and there is a loop from *a* to *a*, then within the loop, there is some number *x* that (immediately) precedes *a*, and therefore there is a loop from *x* (through *a*, back) to x. But since a is finite, there is a finite sequence of P-steps from zero to some number d preceding a; since *precedes* is one-one, $d = x$, and therefore there is a loop from d to d. Thus a loop "rolls back" from a to d, and then all the way back to zero. But there is no loop from zero to zero; otherwise, some number would precede zero, and that is impossible.

Of course, Frege's proof of Theorem 145 avoids any appeals to intuition like those found in the foregoing argument.

Finally, in the proof of Theorem 263 of *Grundgesetze*, Frege shows that any structure satisfying a certain set of four conditions is isomorphic to that of the natural numbers. We find it quite plausible to think that Frege realized that the statement that the natural numbers satisfy these conditions constitutes an axiomatization of them and regarded them as *the* basic laws of arithmetic.[8] Since one of these conditions is the one $(5')$ shows to be satisfied, there is considerable reason to think that Frege regarded $(5')$ as one of the basic laws of arithmetic.

End of digression.

(4*) at least appears to mention extensions of (first-level) concepts and may well do so. But (4*) is unlike the definition of cardinal number and proof of HP in that any mention of extensions it contains is readily eliminable without loss: Frege could have written to exactly the same point, "a member of the natural sequence of numbers ending with a, but not identical with a, is a member of the natural sequence of numbers ending with d, and vice versa".

It is evident that Frege cannot be proposing to derive (4*) or the equivalent

(4) $$\forall x\,([xP^{*=}a \wedge x \neq a] \equiv xP^{*=}d)$$

from $(5')$ since both (4*) and (4) contain free occurrences of 'd'. Since the supposition of §82 that dPa is still clearly in force, it might be thought that Frege wishes to derive

(4†) $$dPa \rightarrow \forall x\,([xP^{*=}a \wedge x \neq a] \equiv xP^{*=}d)$$

from $(5')$.

However, if $d = a = \aleph_0$, then, as we have observed, dPa; and then, since $aP^{*=}a$, (4†) has a true antecedent and fals consequent. Thus it cannot be (4†) that Frege is proposing to derive from $(5')$.

We may note, though, that $\forall x\,([xP^{*=}a \wedge x \neq a] \equiv xP^{*=}d)$ can be derived from dPa and $\neg aP^{*}a$. So we may take it that Frege is proposing to derive

[8] For elaboration of this suggestion, see Heck (1995).

$$(4') \qquad 0P^{*=}a \rightarrow dPa \rightarrow \forall x\,([xP^{*=}a \wedge x \neq a] \equiv xP^{*=}d)$$

from $(5')$.

Proof. Suppose $0P^*a$ and dPa. Assume $xP^{*=}a \wedge x \neq a$. Then xP^*a. By the lemma, for some c, cPa and $xP^{*=}c$. By 78.5, $c = d$. Thus, $xP^{*=}d$. Conversely, assume $xP^{*=}d$. Since dPa, xP^*a, by the basic fact about the weak ancestral, and so $xP^{*=}a$. If $\neg aP^*a$, then also $x \neq a$. But since $0P^{*=}a$, it follows from $(5')$ that indeed $\neg aP^*a$. Thus $(4')$ is proved. □

Nor could Frege be proposing to derive (3) $a = \#[x : xP^{*=}a \wedge x \neq a]$ from any proposition he takes himself to have demonstrated. For (3) is false if 'a' has $\aleph_0$ as value. In fact, $\#[x : xP^{*=}\aleph_0 \wedge x \neq \aleph_0] = 0$. For if $xP\aleph_0$, then since $\aleph_0 P \aleph_0$, $x = \aleph_0$, by 78.5. Let S be the converse of P. Then if $\aleph_0 Sx$, $x = \aleph_0$. Thus if $\aleph_0 S^* x$, $x = \aleph_0$. (Let $Fa \equiv a = \aleph_0$ in the definition of S^*.) But the ancestral is the converse of the ancestral of the converse. So if $xP^*\aleph_0$,then $x = \aleph_0$. Thus $xP^{*=}\aleph_0$ iff $x = \aleph_0$, and therefore for *no* x, $xP^{*=}\aleph_0 \wedge x \neq \aleph_0$. By a proposition given in §75, $\#[x : xP^{*=}\aleph_0 \wedge x \neq \aleph_0] = 0$.

However, it is important to observe that at this point it is not only the conjunct dPa of the antecedent of (1) that is assumed to be in force; the other conjunct $dP\#[x : xP^{*=}d]$ is also assumed to hold. (It is easy to oblivious to this further assumption since (3) does not mention d. But it is supposed at this point that a is such that dPa, and it is likely also supposed that d is such that $dP\#[x : xP^{*=}d]$.) Since (3) follows from these two conjuncts and the consequent of $(4')$, we may take it that Frege wishes to prove:

$$(3') \qquad 0P^{*=}a \rightarrow dPa \rightarrow dP\#[x : xP^{*=}d] \rightarrow a = \#[x : xP^{*=}a \wedge x \neq a]$$

Proof. Suppose dPa and $dP\#[x : xP^{*=}d]$. Then by 78.5 (the other way), $a = \#[x : xP^{*=}d]$. Suppose further that $0P^{*=}a$. Then by $(4')$, $\forall x\,([xP^{*=}a \wedge x \neq a] \equiv xP^{*=}d)$. By HP, $\#[x : xP^{*=}a \wedge x \neq a] = \#[x : xP^{*=}d]$. Thus $a = \#[x : xP^{*=}a \wedge x \neq a]$. □

We come now to the difficult question how Frege proposes to derive (1) from $(3')$. Frege tells us that to prove (1), we must show (3). But (3) is not unconditionally true. However, $(3')$, whose consequent is (3) and whose antecedent contains a conjunct stating that the value of 'a' satisfies the condition of finiteness, can be proved. Thus it might seem reasonable to think that Frege may be proposing, as in the case of (4) and (3), not to derive (1) from (3), but some conditional whose antecedent expresses a finiteness condition and whose consequent is (1). Moreover, since dPa is one of the clauses of the antecedent, it we take $0P^{*=}d$ as another conjunct of the antecedent, we need not also take $0P^{*=}a$. So we have

(1′) $$0P^{*=}d \rightarrow dPa \rightarrow dP\#[x : xP^{*=}d] \rightarrow aP\#[x : xP^{*=}a]$$

(1′) readily follows from (3′).

Proof. Suppose that $0P^{*=}d$, dPa, and $dP\#[x : xP^{*=}d]$. By the basic fact about the weak ancestral, $0P^{*=}a$. By (3′), $a = \#[x : xP^{*=}a \wedge x \neq a]$. Since $a = a$, $aP^{*=}a$. By the definition of P, $\#[x : xP^{*=}a \wedge x \neq a]\, P\, \#[x : xP^{*=}a]$. Thus $aP\#[x : xP^{*=}a]$. □

It may be useful to recapitulate here our (somewhat intricate) derivation of (1′) from (5′) and the other propositions to which Frege appeals.

Proof. Suppose $0P^{*=}d$, dPa, and $dP\#[x : xP^{*=}d]$. By the basic fact about the weak ancestral, $0P^{*=}a$, and thus by (5′), $\neg aP^{*}a$. If $xP^{*=}a \wedge x \neq a$, then $sP^{*}a$, and so by the lemma, for some z, $xP^{*=}z$ and zPa. By one half of 78.5, $z = d$, and so $xP^{*=}d$; conversely, if $xP^{*=}d$, then by the basic fact, $xP^{*}a$, whence $x \neq a$ (since $\neg aP^{*}a$) and $xP^{*=}a$. Thus $\forall x\,([xP^{*=}a \wedge x \neq a] \equiv xP^{*=}d)$, which is (4), and so by HP $\#[x : xP^{*=}a \wedge x \neq a] = \#[x : xP^{*=}d]$, and therefore $dP\#[x : xP^{*=}a \wedge x \neq a]$. By the other half of 78.5, $a = \#[x : xP^{*=}a \wedge x \neq a]$, which is (3). Since $aP^{*=}a$ (trivially), by the definition of P, $\#[x : xP^{*=}a \wedge x \neq a]\, P\, \#[x : xP^{*=}a]$, and therefore $aP\#[x : xP^{*=}a]$. □

(2) is proved much more easily.

(2) $$0P\#[x : xP^{*=}0]$$

Proof. $0 = \#[x : x \neq x]$. By HP and the definition of P, $\forall z(\neg zP0)$. By the lemma, $\forall x(\neg xP^{*}0)$, and so $\forall x\neg(xP^{*=}0 \wedge x \neq 0)$. By a result of §75 mentioned above, $\#[x : xP^{*=}0 \wedge x \neq 0] = 0$. But $0P^{*=}0$, whence $\#[x : xP^{*=}0 \wedge x \neq 0]\, P\, \#[x : xP^{*=}0]$, and therefore $0P\#[x : xP^{*=}0]$. □

(0′) must now be derived from (1′) and (2). It is not possible to appeal to Induction 2 because of the presence of '$0P^{*=}d$' in the antecedent of (1′). But, it might be supposed, Frege can appeal here to *Induction 3*, which he explicitly demonstrated in *Grundgesetze* as Theorem 152: To prove that if $x\phi^{*=}y$, then $\ldots y \ldots$, it suffices to prove:

(i) $\ldots x \ldots$

(ii) $\forall d \forall a (x\phi^{*=}d \to \ldots d \cdots \to d\phi a \to \ldots a \ldots)$.

Note the formula $x\phi^{*=}d$, whose presence weakens (ii) and thereby strengthens the method. The derivation of Induction 3 from Induction 2 is significantly more interesting than that of Induction 2 from Induction 1. It appeals to the comprehension scheme of second-order logic and uses a technique sometimes called "loading the inductive hypothesis". (At the beginning of §116 of *Grundgesetze*, Frege writes, "To prove proposition (γ) of §114, we replace the function mark '$F(\xi)$' with '$\neg(aP^{*=}x \to \neg F(\xi))$'.")

Proof of Induction 3. Suppose $x\phi^{*=}y$ and, moreover, (i) and (ii). Let $Ga \equiv \ldots a \cdots \wedge x\phi^{*=}a$ (second-order comprehension). Now, $x\phi^{*=}x$ trivially; thus by (i), Gx. We now show $\forall d \forall a (Gd \to d\phi a \to Ga)$: Suppose $d\phi a$ and Gd, i.e., $\ldots d \ldots$ and $x\phi^{*=}d$. By (ii), $\ldots a \ldots$. By the basic fact about the weak ancestral, $x\phi^{*=}a$. Thus, $\forall d \forall a (Gd \to d\phi a \to Ga)$. By Induction 2, Gy, whence $\ldots y \ldots$. □

We believe that no one will seriously dispute that this proof of $(0')$, which features a derivation of $(1')$ from $(5')$ and an appeal to Induction 3, is Fregean in spirit, ingenious, and of a structure that fits the proof-sketch found in §§82–83 rather well. But there are a number of strong reasons for doubting that Frege had *this* proof in mind while writing these two sections. Accordingly, we shall refer to it as the *conjectural* proof.[9]

First of all, Frege twice *says* that (1) is to be proved, once in §82 and again in §83. He says, moreover, "The method of inference here is an application of the definition of the expression 'y follows x in the natural sequence of numbers', taking $[y : yP\#[x : xP^{*=}y]]$ for our concept F". It would thus seem natural to take Frege as arguing by appeal to Induction 1 or Induction 2 (with P as ϕ). Frege mentions the condition that n be finite, but does not also mention, as he might easily have done, the need to assume that d (or a) is finite as well. Thus it would seem overly charitable to assume that the argument he really intended proceeds via Induction 3.

Second, notice that Frege says in §83 that $(5')$, which he proves in *Grundgesetze* by appeal to Induction 2, "must likewise ('ebenfalls') be proved by means of our definition of following in a series, as indicated above". It seems plain that Frege does not intend to use Induction 3 to prove $(5')$; "ebenfalls" suggests that the induction used to prove $(0')$ would be like the one used for $(5')$.

The most telling objections to the suggestion that Frege was intending to sketch the conjectural proof in *Die Grundlagen*, however, arise from a close reading of

[9] Of course, what is conjectural is whether the proof is Frege's not whether it is a (correct) proof.

Section H (Eta) of Part II of *Grundgesetze*. WE quote and comment upon part of Section H.[10]

H. Proof of the Proposition

$$0P^{*=}b \rightarrow bP\#[x : xP^{*=}b]$$

> **§114. Analysis**
>
> We wish to prove the proposition that the Number that belongs to the concept
>
> *member of the number-series ending with b*
>
> follows after b in the number-series if b is a finite number. Herewith, the conclusion that the number-series is infinite follows at once; i.e., it follows at once that there is, for each finite number, one immediately following after it.
>
> We first attempt to carry out the proof with the aid of Theorem (144) ⟦viz., $aq^{*=}b \rightarrow \forall d(Fd \rightarrow \forall a(dqa \rightarrow Fa) \rightarrow (Fa \rightarrow Fb)$⟧, replacing the function-mark '$F\xi$' with '$\xi P\#[x : xP^{*=}\xi]$'. For this we need the proposition '$dP\#[x : xP^{*=}d] \rightarrow dPa \rightarrow aP\#[x : xP^{*=}a]$'.

That is to say, one's "first" idea might be to prove $(0')$ by applying Induction 2 to the concept $[y : yP\#[x : xP^{*=}y]]$, which would, among other things, require a proof of (1). (A footnote, to which we shall return, is attached to this last sentence.)

> Substituting... in (102) ⟦viz., $\#[x : Fx \wedge x \neq b] = c \rightarrow Fb \rightarrow cP\#F]$⟧, ... we thus obtain
>
> $$\#[x : P^{*=}xa \wedge x \neq a] = a \wedge aP^{*=}a \rightarrow a = \#[x : xP^{*=}a],$$
>
> from which we can remove the subcomponent '$aP^{*=}a$' by means of (140) ⟦viz., $aP^{*=}a$⟧. The question arises whether the subcomponent '$\#[x : xP^{*=}a \wedge x \neq a] = a$' can be established as a consequence of 'dPa' and '$dP\#[x : P^{*=}xd]$'.

Put differently, the problem reduces to that of proving

[10] The present translation is based upon one due to Richard Heck and Jason Stanley. We have changed Frege's notation to ours and added some material in brackets.

$$(3\dagger) \qquad dPa \to dP\#[x : xP^{*=}d] \to a = \#[x : xP^{*=}a \wedge x \neq a]$$

which is $(3')$ without the finiteness condition $0P^{*=}a$, and which, together with the relevant instance of Frege's Theorem 102, implies (1).

> By the functionality of progression in the number-series..., we have
>
> $$dP\#[x : P^{*=}xd] \wedge dPa \to a = \#[x : P^{*=}xd]$$
>
> ...We thus attempt to determine whether
>
> $$\#[x : xP^{*=}a \wedge x \neq a] = \#[x : xP^{*=}d]$$
>
> can be shown to be a consequence of '*dPa*'. ...For this it is necessary to establish
>
> $$[bP^{*=}a \wedge b \neq a] \equiv bP^{*=}d$$
>
> as a conseqence of '*dPa*'....

That is, $(3\dagger)$ will follow from

$$(4\dagger) \qquad dPa \to \forall x\,([xP^{*=}a \wedge x \neq a] \equiv xP^{*=}d)\,,$$

an easy consequence of HP, and the one-one-ness of *P*.

> For this it is necessary to establish
>
> $$bP^{*=}a \wedge b \neq a \to bP^{*=}d$$
>
> and
>
> $$bP^{*=}d \to bP^{*=}a \wedge b \neq a$$
>
> as consequences of '*dPa*'. But it turns out that another condition must be added if '$b \neq a$' is to be shown to be a consequence of '*bPd*' and '*dPa*'. By (134) we have
>
> $$bP^{*=}d \wedge dPa \to bP^{*}a$$
>
> If *b* coincided with *a*, then the main component ⟦consequent⟧ would transform into '$aP^{*}a$'. By (145) ⟦our $(5')$⟧, this is impossible if *a* is a finite number. Thus the subcompent '$0P^{*=}a$' is also added.
>
> Admittedly, the desired application of (144) thereby becomes impossible; but, with (137) ⟦viz., $aq^{*=}e \to eqm \to aq^{*=}m$⟧, we can replace

this subcomponent with '$0P^{*=}d$' and derive from (144) the Proposition ⟦(152)⟧

$$(aq^{*=}b \rightarrow \forall d\,(Fd \rightarrow aq^{*=}d \rightarrow \forall a\,(dqa \rightarrow Fa))) \rightarrow (Fa \rightarrow Fd)$$

which takes us to our goal.

That is, to establish the first half of $(4')$, we need to know that $\neg aP^{*}a$; this will follow from $(5')$ and the additional assumption that a is finite. However, this new assumption must then be carried along throughout the proof, transforming $(4\dagger)$ into $(4')$, $(3\dagger)$ into $(3')$, and (1) into '$0P^{*=}a \rightarrow dP\#[x : xP^{*=}d] \rightarrow dPa \rightarrow aP\#[x : xP^{*=}a]$, from which $(1')$ easily follows. The attempt to prove $(0')$ via Induction (2) then fails, since we simply have not proved (1), though we can still complete the proof by making use of Induction 3 instead.

It is, we think, difficult to read these passages without supposing that they reveal Frege's *second thoughts* about his idea in *Die Grundlagen* of applying Induction 2 to prove $0P^{*=}n \rightarrow nP\#[x : xP^{*=}n]$ by substituting $[y : yP\#[x : xP^{*=}y]]$ for F. The attempt won't work, he says, because we need the hypothesis that a is finite in order to derive $\neg aP^{*}a$, which is needed for $bP^{*=}d \rightarrow dPa \rightarrow b \neq a$, which is in turn necessary for the rest of the proof. Read side by side with §§82–83 of *Die Grundlagen*, Frege's discussion in these paragraphs strikes us as penetrating and direct criticism of his earlier work. Moreover, the criticism suggests a way in which the conjectural proof can be regarded as Frege's after all: it is the proof obtained on amending the proof-sketch of §§82–83 in the way suggested in this section of *Grundgesetze*.

It is striking that the formal proof Frege actually gives in *Grundgesetze*, though closely related to the conjectural proof, is not quite the same proof. The formal proof,[11] given in §§115, 117, and 119, does proceed by deriving $(0')$ by means of Induction 3 (Frege's Proposition 152), from $(1')$, which is (150ε),[12] and (2), which is (154). And the proof of $(1')$ does begin with a derivation of $(4')$, which is 149ε, from $(5')$, which is (145). But $(1')$ is not derived from $(4')$ via $(3')$; the argument is slightly different.

This part of the *Grundgesetze* proof, translated into English plus our notation, runs as follows. By the basic fact about the weak ancestral it suffices to show that if $0P^{*=}a$, dPa, and $dP\#[x : xP^{*=}d]$, then $aP\#[x : xP^{*=}a]$. By $(4')$ and (an easy

[11] For a fuller account, see Heck (1993).
[12] By proposition nx we mean the proposition labeled with Greek letter x that occurs *during*, as opposed to *after*, the proof of proposition number n.

consequence of) HP, we have that $\#[x : xP^{*=}a \wedge x \neq a] = \#[x : xP^{*=}d]$ (cf. 149). But substituting into Proposition (102) quoted above, we have

$$\#[x : xP^{*=}a \wedge x \neq a] = \#[x : xP^{*=}d] \rightarrow aP^{*=}a \rightarrow \#[x : xP^{*=}d]\, P\, \#[x : xP^{*=}a].$$

Hence, by (140), $\#[x : xP^{*=}d]\, P\, \#[x : xP^{*=}a]$ (cf. 150β). Since dPa and $dP\#[x : xP^{*=}d]$, $a = \#[x : xP^{*=}d]$ (cf. 150γ), whence $aP\#[x : xP^{*=}a]$ (cf. 150δ) and we are done.

Comparing this argument with the relevant portion of the conjectural proof, one sees immediately how little they differ from each other; one might therefore overlook (or ignore) the fact that (3′) does not actually appear in the proof given in *Grundgesetze*. But the omission of (3′) is significant, since the "proof" discussed in §114 explicitly highlight (3†) as what must be proved if (1) is to be derived from (4†). The typical point of a section of *Grundgesetze* headed "Analysis" is to describe a formal proof found in "Construction" sections that follow it. Thus on reading §114, one would naturally expect the following proof to include, not just proofs of the results of adding a finiteness condition to (4†) and to (1), but also, as part of the derivation of the latter from the former, a proof of a proposition similarly related to (3†). As we said, however, the derivation of (1′) from (4′) in §115 does not go via (3′). That (3†) is so much as mentioned in §114 is therefore bound to seem mysterious unless one reads it as we have suggested: as criticism of Frege's *own* "first attempt" to prove (0′) in §§82–83 of *Die Grundlagen*, for (3†) or (3′) is indeed an intermediate step in *that* proof.

This observation concerning how the *Grundgesetze* proof differs from the conjectural proof also suggests a plausible explanation of the original of the mistake of which we have accused Frege. Consider the two lists of propositions in Table 1 on page 18. As we have seen, (4′) follows from (5′), (3′) from (4′), and (1′) from (3′). But notice also that (4†) follows from (5†), (3†) from (4†), and (1) from (3†), as obvious modifications of our proofs show. Frege, able to prove (5′) and desirous of proving (1), may well have lost sight of the need for a finiteness condition somewhere in the middle of his argument—perhaps he had not yet fully written out the argument in his conceptual notation—and mistakenly concluded that he could deduce (1) from (5′). If forced to guess, we would suppose that it was between (4′) and (1), i.e., at (3†) or (3′), that the finiteness condition vanished, for it is there that the *Grundgesetze* proof differs from the conjectural proof.

The first sentence of the second paragraph of §83 calls for some discussion. Frege writes there that we are obliged "hereby" ("hierdurch") to attach to the proposition that $nP\#[x : xP^{*=}n]$ the condition that $0P^{*=}n$. One might be forgiven for thinking that, in so stating, Frege is indicating that this condition is required by the presence

(1′)	$0P^{*=}d \to dPa \to dP\#[x : xP^{*=}d] \to aP\#[x : xP^{*=}a]$
(3′)	$0P^{*=}a \to dPa \to dP\#[x : xP^{*=}d] \to a = \#[x : xP^{*=}a \land x \neq a]$
(4′)	$0P^{*=}a \to dPa \to \forall x\,([xP^{*=}a \land x \neq a] \equiv xP^{*=}d)$
(5′)	$0P^{*=}a \to \neg aP^{*}a$
(1)	$dPa \to dP\#[x : xP^{*=}d] \to aP\#[x : xP^{*=}a]$
(3†)	$dPa \to dP\#[x : xP^{*=}d] \to a = \#[x : xP^{*=}a \land x \neq a]$
(4†)	$dPa \to \forall x\,([xP^{*=}a \land x \neq a] \equiv xP^{*=}d)$
(5†)	$\neg aP^{*}a$

Tab. 1: The Origin of Frege's Mistake

of the finiteness condition in (5′), since it is with an indication of how (5′) is to be proven that the previous paragraph ends. But this thought cannot be right. Frege says in §82 that, once (1) and (2) are proved, "it is to be deduced that $0P^{*=}n \to nP\#[x : xP^{*=}n]$" by means of Induction 2. Thus what subjects n in (0) to a finiteness condition is not the presence of such a condition in (5′), but the kind of proof (0′) being given in the first place. "Hierdurch" refers to the use in the proof of (0′) of the "definition of following in a series, on the lines indicated above", that is, as was discussed in §82.

There is one final piece of textual evidence to which we should like to draw attention. As we said earlier, a footnote is attached to (1) when it is first mentioned in §114: "This proposition is, as it seems, unprovable, but it is not here being asserted as true, since it stands in quotation marks". The natural explanation for this remark of Frege's is that he once *did* believe (1) to be provable, namely when he wrote *Die Grundlagen*, and any defender of the view that Frege was outlining the conjectural proof in §§82–83 will have the occurrence of this remark to explain away.

Apart from the light it may throw on the question whether Frege made a reparable error, the footnote is astonishing. Note that Frege says, not that (1) seems to be false, but that it "seems to be *unprovable*" [emphasis ours]. There is, moreover, reason to suppose Frege believed (1) to be not false, but *true*. For one thing, had Frege believed it to be false, he presumably would have said so. Furthermore, Frege's difficulty was probably not that he did not know how to prove (1), but rather that he did not know how to prove it *in his formal system*. There is a very simple proof of (1) that depends only upon $(1')$, Dedekind's claim that every infinite number is (the number of a concept that is) Dedekind infinite, and the observation, made earlier, that 'd is Dedekind infinite' is equivalent to 'dPd'. We may take $(1')$ to be one half of a dilemma, the other half of which is:

$$\neg 0P^{*=}d \rightarrow dPa \rightarrow dP\#[x : xP^{*=}d] \rightarrow aP\#[x : xP^{*=}a].$$

This proposition may be proved as follows. Suppose the antecedent. Since d is not finite, it is Dedkind infinite. So dPd, and since dPa, $d = a$, and now the consequent is immediate.

This proof is one Frege might well have known. It is not at all difficult and once $(1')$ has been proved, a proof of (1) by dilemma suggests itself. Moreover, Frege was familiar with Dedekind's claim and, at least while he was working on Part II of *Grundgesetze*, believed it to be true (Frege, 1984, op. 271).[13] As for the observation, not only is it easily proved, it is natural, in Frege's system, just to use 'dPd' as a *definition* of 'd is Dedekind infinite' (cf. *Grundgesetze*, Proposition 426). We conclude that Frege believed (1) to be a true but unprovable formula of Frege Arithmetic.

Frege's belief that (1) is unprovable in Frege Arithmetic is mistaken, however. A proof of (1) can be given that makes use of techniques that are different from any found in §§82–83 of *Die Grundlagen* or in relevant sections of *Grundgesetze*, but with which Frege was familiar. What we shall prove is that the hypothesis $0P^{*=}d$ of $(1')$, that d is finite, is dispensable. More precisely, we shall prove that if $dP\#[x : xP^{*=}d]$, then $\#[x : xP^{*=}d]$ is finite, from which it follows that d is finite, since, by Proposition (143) of *Grundgesetze* (viz, $dPb \rightarrow aP^{*}b \rightarrow aP^{*=}d$), any predecessor of a finite number is finite.

Theorem (FA)**.** *Suppose* $dP\#[x : xP^{*=}d]$*. Then* $\#[x : xP^{*=}d]$ *is finite.*[14]

Proof. In FA, define $h : [x : 0P^{*=}x] \rightarrow [x : xP^{*=}d]$ by:

$$h(0) = d; \quad h(n+1) = \begin{cases} y & \text{if } yPh(n) \\ h(n) & \text{if } \neg\exists y(yPh(n)) \end{cases}$$

The definition is OK since P is one-one.

[13] Of course, if Frege did know of our proof and believed (1) to be unprovable, then he must have believed Dedekind's result too to be unprovable, which he (rightly) did. For further discussion, see Heck (1995).

[14] This result is due to Heck; the present proof to Boolos.

Since in general $yR^*z \equiv z(R^\cup)^*y$,[15] $\forall x(xP^{*=}d \equiv d(P^\cup)^{*=}x)$, and so h is onto. Therefore $[x : xP^{*=}d]$ is countable, i.e., either finite or countably infinite. If the latter, then $\#[x : xP^{*=}d] = \aleph_0$, and by the supposition of the theory, $dP\aleph_0$. But as we saw just after the proof of $(4')$, $xP^{*=}\aleph_0 \rightarrow x = \aleph_0$. Since $dP\aleph_0$, $dP^{*=}\aleph_0$, $d = \aleph_0$, and $\#[x : xP^{*=}d] = 1$, contra $\#[x : xP^{*=}d] = d$. Therefore $\#[x : xP^{*=}d]$ is finite. □

Thus Frege could have proved (1) after all and thus appealed to Induction 2 to prove $(0')$. Of course the technology borrowed from second-order arithmetic used in the proof just given, particularly the inductive definition of h, is considerably more elaborate than that needed to derive Induction 3 from Induction 2. The conjectural proof is unquestionably to be preferred to this new one on almost any conceivable grounds.

So, Frege erred in §§82–83 of *Die Grundlagen*, where an oversight marred the proof he outlined of the existence of the successor. Mistakes of that sort are hardly unusual, though, there are four or five ways the proof can be patched up, and Frege's way of repairing it cannot be improved on. But even if one ought not to make too much of Frege's mistake, there is lots to be made of his belief that (1) was true but unprovable in his system. One question that must have struck Frege is: If there are truths about numbers unprovable in the system, what becomes of the claim that the truths of arithmetic rest solely upon definitions and general logical laws? Another that may have occurred to him is: Can the notion of a truth of logic be explained otherwise than via the notion of provability?

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[15] $R^\cup$ is the converse of R.

Appendix 1: Counterparts in *Grundgesetze* of some propositions of *Die Grundlagen*

Proposition of this chapter	Proposition of *Grundgesetze*
HP	32, 49
$\forall x(\neg Fx) \rightarrow 0 = \#[x : Fx]$	94, 97
781.	114
78.2	113
78.3	117
78.4	122
78.5	71, 89, 90
78.6	107
$\neg zP0$	108
Induction 1	123
The basic fact about the weak ancestral	134, 136
The Lemma	141
Induction 2	144
$(5')$	145
$(4')$	149α
$(1')$	150
Induction 3	152
(2)	154
$(0')$	155

References

Boolos, G. (1998). ‘Reading the *Begriffsschrift*’, in *Logic, Logic, and Logic*. Cambridge MA, Harvard University Press. 155–170.

Dedekind, R. (1963). ‘The nature and meaning of numbers’, in *Essays on the theory of numbers*. New York, Dover. Tr by W.W. Beman.

Dummett, M. (1991). *Frege: Philosophy of Mathematics*. Cambridge MA, Harvard University Press.

Frege, G. (1962). *Grundgesetze der Arithmetik*. Hildesheim, Georg Olms Verlagsbuchhandlung. Translations are based upon those in Frege, 1964, for Part I, and in Frege, 1970, for Part III. Extracts are in Frege, 1997, pp. 194–223 and 258–89.

—— (1964). *The Basic Laws of Arithmetic: Exposition of the System*, ed. and tr. by M. Furth. Berkeley CA, University of California Press.

—— (1970). *Translations from the Philosophical Writings of Gottlob Frege*, Geach, P. and Black, M., eds. Oxford, Blackwell.

—— (1980). *The Foundations of Arithmetic*, 2d revised edition, tr. by J. L. Austin. Evanston IL, Northwestern University Press.

—— (1984). 'Review of Georg Cantor, *Zum Lehre vom Transfiniten*', tr. by H. Kaal, in B. McGuiness (ed.), *Collected Papers on Mathematics, Logic, and Philosophy*. Oxford, Basil Blackwell. 178–81.

—— (1997). *The Frege Reader*, Beaney, M., ed. Oxford, Blackwell.

Heck, R. G. (1993). 'The development of arithmetic in Frege's *Grundgesetze der Arithmetik*', *Journal of Symbolic Logic* 58: 579–601.

—— (1995). 'Definition by induction in Frege's *Grundgesetze der Arithmetik*', in W. Demopoulos (ed.), *Frege's Philosophy of Mathematics*. Cambridge MA, Harvard University Press. 295–333.

Smiley, T. J. (1988). 'Frege's "series of natural numbers"', *Mind* 97: 388–389.

Wright, C. (1983). *Frege's Conception of Numbers as Objects*. Aberdeen, Aberdeen University Press.