Necessity modals, disjunctions, and collectivity[1]

Richard Jefferson BOOTH — *Columbia University*

Abstract. Upward monotonic semantics for necessity modals give rise to Ross's Puzzle: they predict that $\Box\phi$ entails $\Box(\phi \vee \psi)$, but common intuitions about arguments of this form suggest they are invalid. It is widely assumed that the intuitive judgments involved in Ross's Puzzle can be explained in terms of the licensing of 'Diversity' inferences: from $\Box(\phi \vee \psi)$, interpreters infer that the truth of each disjunct (ϕ, ψ) is compatible with the relevant set of worlds. I introduce two pieces of data that this analysis fails to explain. Analyzing this data, I argue, suggests that necessity modals with embedded disjunctions license 'Independence' inferences: from the truth of $\Box(\phi \vee \psi)$, interpreters infer that ϕ-without-ψ and ψ-without-ϕ are each compatible with the relevant set of worlds. I outline a bilateral inquisitive semantics for necessity modals that predicts the validity of the Independence inferences. I then argue that the resulting theory should be understood as one on which disjunctions denote *pluralities* of propositions, and necessity modals behave like *collective predicates* applied to these pluralities.

Keywords: necessity, modal, disjunction, bilateral, inquisitive semantics, plural, collectivity

1. Ross's Puzzle

On the dominant theory of natural language necessity modals like English 'must', 'ought', 'have to', etc. (I use '$\Box$' as an arbitrary necessity modal), due to Kratzer (1991, 2012a, b), they express the subset relation between a set of relevant worlds $R(f,g,w)$ (determined by the interaction of two contextually-determined parameters: a *modal base* f, and an *ordering source* g) and the set of worlds that make the complement clause true. Abstracting from some complications, a modal base at a world $f(w)$ is a set of propositions (sets of worlds), and the semantics is partly a function of the *conjunction* of these propositions ($\bigcap f(w)$).[2] An ordering source at a world ($g(w)$) is a set of propositions that determines a partial ordering of worlds:

$$v \geqslant_{g(w)} u := \forall P \in g(w): \text{ if } u \in P \text{ then } v \in P$$

This partial order in turn determines, relative to a set of worlds P, a set of *maximal* or *undominated* P-worlds:

$$\mathsf{max}(g(w),P) := \{v \in P \mid \text{ for all } u \in P: \text{ if } u \geqslant_{g(w)} v \text{ then } u = v\}$$

That is, $\mathsf{max}(g(w),P)$ is the set of P-worlds that are undominated by other P-worlds, relative to the ordering source $g(w)$.

The relevant set of worlds a modal quantifies over ($R(f,g,w)$), given a modal base f, ordering source g, and world of evaluation w, is defined as follows:

$$R(f,g,w) := \mathsf{max}(g(w), \bigcap f(w))$$

[1] I am very grateful to the organizers of *Sinn und Bedeutung 26*, to the anonymous reviewers of this paper, and the audience for my presentation at *Sinn und Bedeutung 26*, who provided very helpful feedback. I would also like to thank Jessica Collins, Melissa Fusco, Ben Holguín, Ezra Keshet, Karen Lewis, Janum Sethi, Achille Varzi, and the participants of the 2021 Dianoia Institute of Philosophy Language Workshop.

[2] One issue I ignore (harmlessly, I believe), is the limit assumption for ordering sources. See Kratzer's papers cited above for discussion.

Putting these elements together, we can state the Kratzerian semantics as follows:

Definition 1 (Kratzerian Necessity Semantics)**.**

$$[\![\Box\phi]\!]^{f,g,w} = 1 \text{ iff } R(f,g,w) \subseteq [\![\phi]\!]^{f,g}$$

Where $[\![\Box\phi]\!]^{f,g,w}$ is the truth value of ϕ at a world w, modal base f, and ordering source g; and $[\![\phi]\!]^{f,g}$ is the set of worlds that make ϕ true given f and g.

Now suppose $[\![\phi]\!]^{f,g} \subseteq [\![\psi]\!]^{f,g}$. Then, since the subset relation is transitive, we have that:

$$\text{if } R(f,g,w) \subseteq [\![\phi]\!]^{f,g}, \text{ then } R(f,g,w) \subseteq [\![\psi]\!]^{f,g}$$

Substituting equivalents gives us:

$$\text{if } [\![\Box\phi]\!]^{f,g,w} = 1, \text{ then } [\![\Box\psi]\!]^{f,g,w} = 1$$

Fact 1 (Kratzerian Semantics is Truth-Preservationally Upward Monotonic)**.** *Say that* $\phi \models^+ \psi$ *iff for any* f,g, $[\![\phi]\!]^{f,g} \subseteq [\![\psi]\!]^{f,g}$. *Then:*

$$\textit{If } \phi \models^+ \psi, \textit{ then } \Box\phi \models^+ \Box\psi$$

This is mostly a welcome result. First, consider lexical generalizations:

(1) a. Bolt was sprinting.
b. So, Bolt was running.

(2) a. Bolt must have been sprinting.
b. So, Bolt must have been running.

Also, consider *conjunction elimination* (since standardly, $\phi \wedge \psi \models^+ \phi$):[3]

(3) a. Bolt was sprinting and laughing.
b. So, Bolt was sprinting.

(4) a. Bolt must have been sprinting and laughing.
b. So, Bolt must have been sprinting.

These arguments are intuitively valid, and predicted to be so by any semantics that makes 'must' upward monotonic.

There are, however, some well-known and puzzling counterexamples to the upward monotonicity prediction of standard semantics for natural language necessity modals. This paper is centrally concerned with one type of putative counterexample, which trades on the fact that on standard semantics for disjunction, $\phi \models^+ \phi \vee \psi$:

[3] See Jackson and Pargetter (1986), Cariani (2013), and Blumberg and Hawthorne (2021) for discussion of some potential counterexamples.

(5) a. Bolt sprints.
b. So, Bolt either sprints or laughs.

(6) a. Bolt ought to sprint.
b. # So, Bolt ought to either sprint or laugh.

If (5) is valid and 'must' is upward monotonic, (6) is also valid. But readers usually balk at the inference in (6); it appears to be invalid. Call any argument of the form $\Box\phi \therefore \Box(\phi \vee \psi)$, where $\Box$ is a necessity modal, a *Ross argument*. *Ross's Puzzle* is the challenge of reconciling the apparent invalidity of Ross arguments with the prediction of standard semantics for necessity modals and disjunction that they are valid.

1.1. The diversity analysis

Semantic solutions to Ross's puzzle rewrite the semantics of necessity modals and/or disjunction so that Ross arguments are not valid.[4] *Pragmatic* solutions to Ross's puzzle typically accept that Ross arguments are valid, but appeal to pragmatic mechanisms in order to explain the appearance of invalidity.[5]

A widespread assumption among both types of solution in the literature is that Ross arguments appear (or are) invalid primarily because interpreters draw what I will call *Diversity inferences* from the conclusion that are not supported by the premise.

Definition 2 (Diversity inferences: Meta-language)**.** Where $[\![\phi]\!]^{f,g}$ is the truth set of ϕ, and '$\rightsquigarrow$' is ambiguous between truth-conditional entailment, implicature, or some other robust form of licensing:

$$\begin{aligned}\Box(\phi \vee \psi) &\rightsquigarrow R(f,g,w) \cap [\![\phi]\!]^{f,g} \neq \varnothing \\ &\rightsquigarrow R(f,g,w) \cap [\![\psi]\!]^{f,g} \neq \varnothing\end{aligned}$$

The Diversity inferences have an object-language correlate, given the following Kratzerian semantics for possibility modals (e.g. English 'may', 'might', 'could', etc.):

Definition 3 (Kratzerian Possibility Semantics)**.**

$$[\![\Diamond\phi]\!]^{f,g,w} = 1 \text{ iff } R(f,g,w) \cap [\![\phi]\!]^{f,g} \neq \varnothing$$

Under certain conditions, the meta-linguistic Diversity inferences are equivalent to the following object-language inferences:

Definition 4 (Diversity inferences: Object-Language)**.**

$$\begin{aligned}\Box(\phi \vee \psi) &\rightsquigarrow \Diamond\phi \\ &\rightsquigarrow \Diamond\psi\end{aligned}$$

[4] See, for example, Simons (2005a).
[5] See, for example, von Fintel (2012) and Wedgwood (2006).

Necessity modals, like other quantifiers, are usually assumed to presuppose that their domains are non-empty. This means that when $\Box(\phi \vee \psi)$ is true on the Kratzerian Semantics, at least one of the two Diversity conclusions will always be true. But the other might easily fail. For example, if $R(f,g,w) \neq \varnothing$ and $R(f,g,w) \subseteq [\![\phi]\!]^{f,g} \backslash [\![\psi]\!]^{f,g}$, then $\Box(\phi \vee \psi)$ and $\Diamond\phi$ are true but $\Diamond\psi$ is false.

The Diversity inferences are supposed to explain the (apparent) invalidity of Ross arguments as follows. Interpreters begin by assuming the truth of the premise, $\Box\phi$. They then process the conclusion $\Box(\phi \vee \psi)$ and, realizing that the truth of the premise does not ensure that the licensed ψ-directed Diversity inference holds, judge that $\Box(\phi \vee \psi)$ itself does not follow from the premise. Thus, they judge the argument *invalid*. Call this the *Diversity Analysis* of Ross's Puzzle. Now, on a semantic account of the Diversity inferences, on which they are genuine entailments, and entailment is transitive, this explanation clearly suffices to explain why Ross arguments would be judged invalid: since the conclusion entails the Diversity inferences, but the premise does not, the premise cannot entail the conclusion. I want to note, however, that whether a pragmatic account is also sufficient is not as straightforward. Pragmatic inferences are usually thought to be licensed only when plausible in a context. A pragmatic account of Ross's Puzzle that takes the form just outlined suggests that when processing the conclusion, an interpreter of a Ross argument draws the ψ-directed Diversity inference, senses it is unsupported, and rather than simply withdrawing it, mistakenly deems the argument invalid. Pragmatic accounts of Ross's Puzzle thus have the burden of explaining why interpreters would stick to the pragmatically derived Diversity inferences even when they are felt to be implausible. Semantic accounts of the Diversity inferences do not share this explanatory burden, since they hold that interpreters must be committed to the Diversity inferences just by virtue of supposing the truth of $\Box(\phi \vee \psi)$.

2. Independence

The Diversity analysis makes a prediction that is not borne out. If the reason that the Ross argument $\Box\phi \therefore \Box(\phi \vee \psi)$ is judged to be invalid is that the conclusion licenses $\Diamond\psi$, which is not supported by the premise, then adding $\Diamond\psi$ as a premise should suffice to make interpreters judge the resulting argument ($\Box\phi, \Diamond\psi \therefore \Box(\phi \vee \psi)$) valid. But consider the following examples:

(7) a. Bolt has to sprint.
    b. Bolt may/is allowed to laugh.
    c. So, Bolt has to either sprint or laugh.

(8) a. Bolt must have been sprinting.
    b. Bolt might have been happy.
    c. So, Bolt must have either been sprinting or been happy.

Let us call any argument of the form $\Box\phi, \Diamond\psi, \therefore \Box(\phi \vee \psi)$ an *Extended Ross Argument*. Extended Ross arguments are valid on the Kratzerian semantics for necessity modals, and furthermore, the premises ensure that the Diversity inferences are supported. Yet, I submit, they appear to be just as invalid as the original Ross inference. If that is correct, then the Diversity analysis seems to fall short: it cannot explain the data we see in Extended Ross arguments.[6]

What is more, if Extended Ross arguments are invalid like Ross arguments, then the Diversity analysis seems to be off the mark: the reason the original Ross argument is judged invalid is not that the Diversity inferences are unsupported. Therefore, the Diversity inferences must not be the full explanation of intuitions in the original Ross arguments. There must be something else going on.

A second piece of data the Diversity analysis does not explain will help us identify what is missing. Necessity modals with disjunctive complement clauses license what I will call *Independence Conditionals*:

Definition 5 (Independence Conditionals)**.**

$$\begin{aligned}\Box(\phi \vee \psi) &\rightsquigarrow \neg\phi \to \Box\psi \\ &\rightsquigarrow \neg\psi \to \Box\phi\end{aligned}$$

Consider the following examples. First, ‘must’ on an epistemic reading:

(9) a. Pim must have either been trudging or talking.
b. So, If Pim was not trudging, he must have been talking.
c. So, If Pim was not talking, he must have been trudging.

Second, ‘ought’ on a deontic reading:

(10) a. Pim ought to either give up or accept his fate. $\Box(\textsf{give-up} \vee \textsf{accept})$
b. So, If Pim does not give up, he ought to accept his fate. $\neg\textsf{give-up} \to \Box\textsf{accept}$
c. So, If Pim does not accept his fate, he ought to give up. $\neg\textsf{accept} \to \Box\textsf{give-up}$

In my judgment, these arguments appear valid. The Kratzerian semantics, even supplemented with the licensed Diversity inferences, does not predict this.[7] Let me explain why.

The truth conditions of (10a) say that all worlds in $R(f,g,w)$ are worlds in which either Pim gives up or accepts his fate. The Diversity inferences require that there is a $v \in R(f,g,w)$ such that in v Pim gives up, and there is a $u \in R(f,g,w)$ u, Pim accepts his fate. Importantly, the Diversity analysis does not require that in v, Pim does not accept his fate; nor that in u Pim does not give up. So it leaves open the possibility that, for example, Pim can only accept his fate by giving up — that all worlds in the modal base where Pim accepts his fate are also worlds where he gives up.

On the Kratzerian ‘restrictor’ analysis of conditionals, the antecedent clause serves to restrict the modal base that a modal scoping over the consequent clause quantifies over to worlds in which the antecedent is true. Thus, the truth conditions of (10b) are: there are worlds in $R(f \cup \{[\![\neg\textsf{give-up}]\!]^{f,g}\}, g, w)$, and in all of these, Pim accepts his fate. Suppose that all

[6]See Sayre-McCord (1986), Fusco (2015), and Booth (2022) for further discussion of this data.
[7]See Fusco (2015) for this data in the case of ‘ought’ and Booth (2022) for a more general discussion of this data.

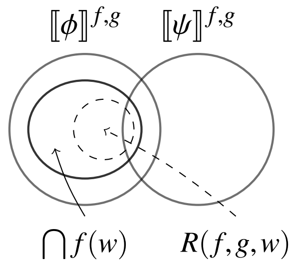

Figure 1: Diversity without Independence

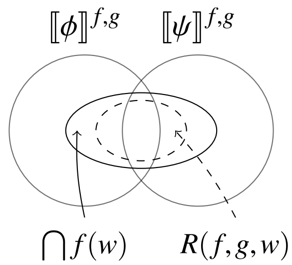

Figure 2: Independence

worlds in $\bigcap f(w)$ where Pim accepts his fate are worlds where he gives up. Then even if $R(f,g,w)$ contains worlds where Pim accepts his fate (given the Diversity inferences of (10)), $\bigcap(f \cup \{[\![\neg\textsf{give-up}]\!]^{f,g}\})$ does not. In that case, $R(f \cup \{[\![\neg\textsf{give-up}]\!]^{f,g}\}, g, w)$ contains no worlds where Pim accepts his fate. Thus, the conditional (10c) would be false. Similar reasoning shows that the other inferences in (9-10) may fail.

Diagrammatically, the Independence conditional inferences may fail in cases like the one illustrated in Figure (1), where $\Box(\phi \vee \psi)$ is true on the Kratzerian semantics, the Diversity inferences are supported, but the Independence conditional $\neg\phi \rightarrow \Box\psi$ fails, since there are no $\neg\phi$-worlds that are also ψ-worlds in the modal base. The Kratzerian semantics, even supplemented so that it licenses the Diversity inferences, thus fails to predict the validity of (9-10) because while $\Box(\phi \vee \psi)$ conveys that each disjunct denotes a relevant alternative, it leaves open that one disjunct's truth in the modal base is *dependent* on the truth of the other in the modal base. In order to validate the Independence conditional inferences, we need to give a theory that predicts that $\Box(\phi \vee \psi)$ conveys that ϕ-without-ψ and ψ-without-ϕ are each relevant alternatives. In other words, we need to ensure that when $\Box(\phi \vee \psi)$ is true, it licenses the inference that the relevant set of worlds overlaps with the relative complements of the disjuncts: $[\![\phi]\!]^{f,g} \backslash [\![\psi]\!]^{f,g}$ and $[\![\psi]\!]^{f,g} \backslash [\![\phi]\!]^{f,g}$.

If this latter thought is correct, then it is also sufficient to explain what is wrong with the (Extended) Ross arguments above (Figure 1 is a model of the premises of such arguments). Neither the premise of the Ross argument $\Box\phi$ nor the premises of the Extended Ross argument $\Box\phi, \Diamond\psi$ ensure that ϕ and ψ are *independently* relevant alternatives, i.e. that ϕ-without-ψ and ψ-without-ϕ are both compatible with the relevant set of worlds. Diagrammatically, we need the relationship between the disjuncts and the relevant set of worlds to look more like Figure 2 when $\Box(\phi \vee \psi)$ is true.

Thus, I argue that necessity modals with disjunctive complements license *Independence inferences*:

Definition 6 (Independence Inferences: Meta-Language)**.** Where R is the relevant set of worlds $\Box$ quantifies over, $[\![\phi]\!]$ is the truth set of ϕ, and '$\rightsquigarrow$' is ambiguous between truth-conditional entailment, implicature, or some other robust form of licensing:

$$\begin{aligned}\Box(\phi \vee \psi) &\rightsquigarrow R \cap ([\![\phi]\!] \backslash [\![\psi]\!]) \neq \varnothing \\ &\rightsquigarrow R \cap ([\![\psi]\!] \backslash [\![\phi]\!]) \neq \varnothing\end{aligned}$$

Under certain conditions, we can recast the Independence inferences in the object-language as follows:

Definition 7 (Independence inferences: Object-Language)**.**

$$\begin{aligned}\Box(\phi \vee \psi) &\rightsquigarrow \Diamond(\phi \wedge \neg\psi) \\ &\rightsquigarrow \Diamond(\psi \wedge \neg\phi)\end{aligned}$$

Of course, the Independence inferences only make sense when the complement $\phi \vee \psi$ is *non-Hurford*: in other words, when neither ϕ entails ψ nor ψ entails ϕ. Following recent work on the subject, I assume that when for example, ϕ entails ψ and $\Box(\phi \vee \psi)$ is interpretable, it contains an exhaustification operator at the level of logical form so that it amounts to something like $\phi \vee (\psi \wedge \neg\phi)$, and the relevant Independence inferences licensed by $\Box(\phi \vee \psi)$ would thus be:[8]

$$R \cap ([\![\phi]\!] \backslash [\![\psi \wedge \neg\phi]\!]) \neq \varnothing$$
$$R \cap ([\![\psi \wedge \neg\phi]\!] \backslash [\![\phi]\!]) \neq \varnothing$$

Thus, to reduce complexity, I will harmlessly, I believe, focus on non-Hurford disjunctions.

Before moving on, I want to note some precedents to the Independence inferences I have defended here. Menéndez Benito (2005, 2010) argues that similar inferences are licensed by free choice 'any' and Spanish 'cualquiera'. Aloni and Ciardelli (2013) draws on Menéndez Benito's work to license similar inferences for imperatives. In Booth (2022), I discuss similar inferences for attitude verbs and disjunctions under possibility modals.

2.1. Minimal coverings

The Independence inferences are tied to a certain type of *covering* relation between the truth sets of the disjuncts (call these the *alternatives* of the disjunction) and the relevant set of worlds.

$$C \text{ is a cover of } S \text{ iff } S \subseteq \bigcup C$$

[8]See Hurford (1974), Gazdar (1979), Simons (2001), Katzir and Singh (2013), Meyer (2013, 2014), and Ciardelli and Roelofsen (2017) for recent discussion.

Take, for example, the disjunction of two simple sentences '$p \vee q$'. The Kratzer semantics says that $\Box(p \vee q)$ is true iff the set of the truth sets of the disjuncts $\{[\![p]\!], [\![q]\!]\}$ is a cover of the relevant set of worlds:

$$\Box(p \vee q) \text{ is true iff } R \subseteq \bigcup\{[\![p]\!], [\![q]\!]\}$$

The Diversity inferences require that $\{[\![p]\!], [\![q]\!]\}$ is a *super cover* of R (Simons, 2005a):

$$C \text{ is a super cover of } S \text{ iff } C \text{ is a cover of } S \text{ and } \forall c \in C : \ c \cap S \neq \varnothing$$

The Independence inferences, by contrast, require that $\{[\![p]\!], [\![q]\!]\}$ is a *minimal cover* (abbreviated: m-cover) of R (Booth, 2022):

$$C \text{ is a minimal cover of } S \text{ iff } C \text{ is a cover of } S \text{ and } \neg\exists C' \subset C : \ C \text{ is a cover of } S$$

When $\{[\![p]\!], [\![q]\!]\}$ m-covers R, then since it is a cover, every world is one where the disjunction $p \vee q$ is true. Since it is *minimal*, each element plays a necessary role in making $\{[\![p]\!], [\![q]\!]\}$ a cover of R. Thus, for each element of $\{[\![p]\!], [\![q]\!]\}$, there is an element $w \in R$ that is not contained in any other element of $\{[\![p]\!], [\![q]\!]\}$. In other words, there is a world where p is true but q is not, and a world where q is true but p is not.

The notion of a minimal covering is thus the set-theoretic correlative of the Independence inferences; it will play a key role in the semantics I give in the next section, as well as the interpretation of the semantics I outline in the final section.

3. Bilateral minimal covering semantics

The theory I outline in the present section is, with some minor differences, the same as that given in Booth (2022). In that paper, I offer more background on the treatment of possibility modals, and defend the *bilateral* nature of the theory — the fact that the falsity conditions of ϕ are not a function of the truth conditions of ϕ.[9] In this paper, I generalize some of the results stated in that paper, and offer more detailed proofs of them.

Definition 8 (Language)**.** From a countable set of atomic sentences At, our language $\mathcal{L}$ is built from the following grammar (where $p \in \mathsf{At}$ and $\phi, \psi \in \mathcal{L}$):

$$p \mid \neg\phi \mid \phi \wedge \psi \mid \phi \vee \psi \mid \phi \rightarrow \psi \mid \Box\phi \mid \Diamond\phi$$

[9] In particular, the bilateral version of the theory allows us to retain duality between necessity and possibility modals; this also has the consequence of validating what I call below 'unnecessity' and impossibility distribution over disjunctions. See the 'radical inquisitive semantics' of Groenendijk and Roelofsen (2010) and Aher (2012), the dual update semantics in Willer (2018), the bilateral 'state-based' semantics of Aloni (2018), and the bilateral truthmaker semantics of Yablo (2014) and Fine (2017a, b) for other bilateral semantics.

Definition 9 (Models)**.** A model $\mathcal{M}$ is a triple $\langle W, R, V\rangle$ such that:

- W is a set of worlds.
- R is a function from worlds to sets of relevant worlds $R : W \mapsto \wp(W)$.
- V is a valuation function from atomic sentences to sets of worlds $V : \mathsf{At} \mapsto \wp(W)$

Definition 10 (Bilateral Inquisitive Propositions)**.** Given a model $\mathcal{M} = \langle W_{\mathcal{M}}, R_{\mathcal{M}}, V_{\mathcal{M}}\rangle$. Let $\mathcal{P}$ be the set of bilateral inquisitive propositions over $W_{\mathcal{M}}$ defined as follows. $P = \langle P^+, P^-\rangle \in \mathcal{P}_{\mathcal{M}}$ iff (where $P^\circ \in \{P^+, P^-\}$):

- $P^\circ \subseteq \wp(W)$ (set of sets of worlds)
- For any $s, t \subseteq W$, if $t \in P^\circ$ and $s \subseteq t$, then $s \in P^\circ$ (subset closed)
- $P^+ \cap P^- = \{\varnothing\}$ (no substantive overlap)

Definition 11 (Downward Closure)**.** Given a set of sets of worlds $S \subseteq \wp(W)$:

$$\downarrow S := \{t \in \wp(W) \mid \exists s \in S : t \subseteq s\}$$

Definition 12 (Informative Content)**.** Given a set of sets of worlds $S \subseteq \wp(W)$:

$$\mathsf{info}(S) := \bigcup S$$

Definition 13 (Alternatives)**.** Given a bilateral inquisitive proposition $P = \langle P^+, P^-\rangle$:

$$\mathsf{alt}^+(P) = \{s \in P^+ \mid \forall t \in P^+, \text{ if } s \subseteq t, \text{ then } s = t\}$$
$$\mathsf{alt}^-(P) = \{s \in P^- \mid \forall t \in P^-, \text{ if } s \subseteq t, \text{ then } s = t\}$$

Definition 14 (Semantics)**.** Given a model $\mathcal{M}$, the assignment of bilateral inquisitive propositions $[\![\phi]\!]_{\mathcal{M}} = \langle [\![\phi]\!]^+_{\mathcal{M}}, [\![\phi]\!]^-_{\mathcal{M}}\rangle$ to $\phi \in \mathcal{L}$ goes as follows (with reference to the model suppressed when uninteresting):

$$[\![p]\!]^+ = \downarrow \{V(p)\} \qquad [\![p]\!]^- = \downarrow \{W \backslash V(p)\}$$
$$[\![\neg\phi]\!]^+ = [\![\neg\phi]\!]^- \qquad [\![\neg\phi]\!]^- = [\![\neg\phi]\!]^+$$
$$[\![\phi \vee \psi]\!]^+ = [\![\phi]\!]^+ \cup [\![\psi]\!]^+ \qquad [\![\phi \vee \psi]\!]^- = [\![\phi]\!]^- \cap [\![\psi]\!]^-$$
$$[\![\Box\phi]\!]^+ = \downarrow \{\{w \in W \mid R(w) \neq \varnothing \text{ and } \mathsf{alt}^+([\![\phi]\!]) \text{ m-covers } R(w)\}\}$$
$$[\![\Box\phi]\!]^- = \downarrow \{\{w \in W \mid \exists R' \subseteq R(w) : R' \neq \varnothing \text{ and } \mathsf{alt}^-([\![\phi]\!]) \text{ m-covers } R'\}\}$$

Let:

$$[\![\phi \wedge \psi]\!] = [\![\neg(\neg\phi \vee \neg\psi)]\!]$$
$$[\![\Diamond\phi]\!] = [\![\neg\Box\neg\phi]\!]$$

The semantics of conditionals will depend on the notion of a model update:[10]

[10]See van Ditmarsch et al. (2008) and other research in dynamic epistemic logic for similar operators.

Definition 15 (Accessibility Update)**.** Given a model $\mathcal{M} = \langle W_{\mathcal{M}}, R_{\mathcal{M}}, V_{\mathcal{M}} \rangle$, and a sentence ϕ, we define the update of $R_{\mathcal{M}}$ with ϕ ($R^{\phi}_{\mathcal{M}}$) as follows:

$$R^{\phi}_{\mathcal{M}} := \{\langle w, S\rangle \mid w \in W_{\mathcal{M}} \text{ and } S = R_{\mathcal{M}}(w) \cap \mathsf{info}([\![\phi]\!]^{+}_{\mathcal{M}})\}$$

So $R^{\phi}_{\mathcal{M}}$ applied to w delivers $R(w) \cap \mathsf{info}([\![\phi]\!]^{+}_{\mathcal{M}})$. With this notion in hand, we define model update as follows:

Definition 16 (Model Update)**.** Given a model $\mathcal{M} = \langle W_{\mathcal{M}}, R_{\mathcal{M}}, V_{\mathcal{M}} \rangle$, and a sentence ϕ, we define the update of $\mathcal{M}$ with ϕ, ($\mathcal{M}^{\phi}$) as follows:

$$\mathcal{M}^{\phi} := \langle W_{\mathcal{M}}, R^{\phi}_{\mathcal{M}}, V_{\mathcal{M}} \rangle$$

Fact 2 (Adequacy of Model Update)**.** *Let $\mathcal{M}$ be a model, and $[\![\phi]\!]_{\mathcal{M}}$ a bilateral inquisitive proposition. Then, $\mathcal{M}^{\phi}$ is a model.*

Proof. Since $[\![\phi]\!]_{\mathcal{M}}$ a bilateral inquisitive proposition, $\mathsf{info}([\![\phi]\!]^{+}_{\mathcal{M}}) \subseteq W_{\mathcal{M}}$. Thus, for each $w \in W_{\mathcal{M}}$, $R_{\mathcal{M}}(w) \cap \mathsf{info}([\![\phi]\!]^{+}_{\mathcal{M}}) \in \wp(W_{\mathcal{M}})$. So $R^{\phi}_{\mathcal{M}}$ is an accessibility relation over $W_{\mathcal{M}}$, so $\mathcal{M}^{\phi} = \langle W_{\mathcal{M}}, R^{\phi}_{\mathcal{M}}, V_{\mathcal{M}} \rangle$ is a model. □

Finally, we can give a semantics for our restrictor conditional as follows:

$$[\![\phi \to \psi]\!]_{\mathcal{M}} = [\![\psi]\!]_{\mathcal{M}^{\phi}}$$

3.1. Important characteristics

Note that the present semantics retains duality for necessity and possibility modals (by definition of $\Diamond$), and in contrast to standard inquisitive semantics (see, e.g., Ciardelli et al. (2018)) it also generates the equivalence of $[\![\phi]\!]$ and $[\![\neg\neg\phi]\!]$:

Fact 3 (Double Negation)**.** *For any $\phi \in \mathcal{L}$ and model $\mathcal{M}$:*

$$[\![\phi]\!]^{+} = [\![\neg\phi]\!]^{-} = [\![\neg\neg\phi]\!]^{+}$$
$$[\![\phi]\!]^{-} = [\![\neg\phi]\!]^{+} = [\![\neg\neg\phi]\!]^{-}$$

It is simple to prove that the semantic clauses given above do assign bilateral inquisitive propositions for every sentence ϕ of our language:

Fact 4 (Adequacy)**.** *For any sentence $\phi \in \mathcal{L}$, if $\mathcal{M}$ is a model, $[\![\phi]\!]_{\mathcal{M}}$ is a bilateral inquisitive proposition.*

Fact 5 (Compactness of Alternatives)**.** *For every sentence ϕ of $\mathcal{L}$, if $\mathcal{M}$ is a model:*

$$\mathsf{alt}^{+}([\![\phi]\!]) \text{ is finite, and } [\![\phi]\!]^{+} = \downarrow \mathsf{alt}^{+}([\![\phi]\!])$$
$$\mathsf{alt}^{-}([\![\phi]\!]) \text{ is finite, and } [\![\phi]\!]^{-} = \downarrow \mathsf{alt}^{-}([\![\phi]\!])$$

3.2. Predictions

Definition 17 (Truth and Falsity)**.** Given a model $\mathcal{M}$ and world $w \in W_{\mathcal{M}}$, we define $\mathcal{M}, w \vDash \phi$ (ϕ is true) and $\mathcal{M}, w \dashv \phi$ (ϕ is false) as follows:

$$\mathcal{M}, w \vDash \phi := w \in [\![\phi]\!]^+_{\mathcal{M}}$$
$$\mathcal{M}, w \dashv \phi := w \in [\![\phi]\!]^-_{\mathcal{M}}$$

Definition 18 (Truth-Preservational Entailment)**.** Given a model $\mathcal{M}$, we say that $\phi_0, ..., \phi_n \models^+_{\mathcal{M}} \psi$ iff for every $w \in W_{\mathcal{M}}$:

$$\text{if } \mathcal{M}, w \vDash \phi_0 \text{ and ..., and } \mathcal{M}, w \vDash \phi_n, \text{ then } \mathcal{M}, w \vDash \psi$$

Definition 19 (Validity)**.** Given a class of models $\mathfrak{M}$, we say that $\phi_0, ..., \phi_n \models^+_{\mathfrak{M}} \psi$ (the argument $\phi_0, ..., \phi_n \therefore \psi$ is valid over $\mathfrak{M}$) iff for every $\mathcal{M} \in \mathfrak{M}$:

$$\phi_0, ..., \phi_n \models^+_{\mathcal{M}} \psi$$

Definition 20 (Invalidity)**.** Given a class of models $\mathfrak{M}$, we say that $\phi\_0, ..., \phi\_n \not\models \_\mathfrak{M}^+ \psi$ (the argument $\phi\_0, ..., \phi\_n \therefore \psi$ is invalid over $\mathfrak{M}$) iff for some $\mathcal{M} \in \mathfrak{M}$, there is a $w \in W\_\mathcal{M}$ such that:

$$\mathcal{M}, w \vDash \phi\_0 \text{ and ..., and } \mathcal{M}, w \vDash \phi\_n, \text{ and } \mathcal{M}, w \nvDash \psi$$

Definition 21 (Strong Invalidity)**.** Given a class of models $\mathfrak{M}$, we say that the argument $\phi\_0, ..., \phi\_n \therefore \psi$ is *strongly invalid* over $\mathfrak{M}$ iff there is *no* $\mathcal{M} \in \mathfrak{M}$, such that for some world $w \in W\_\mathcal{M}$:

$$\mathcal{M}, w \vDash \phi\_0 \text{ and ..., and } \mathcal{M}, w \vDash \phi\_n, \text{ and } \mathcal{M}, w \vDash \psi$$

Definition 22 (Non-Hurford Disjunctions)**.** Given a model $\mathcal{M}$ we say that a disjunction $\phi \vee \psi$ is non-Hurford iff:

$$[\![\phi]\!]^+_{\mathcal{M}} \not\subseteq [\![\psi]\!]^+_{\mathcal{M}}$$
$$[\![\psi]\!]^+_{\mathcal{M}} \not\subseteq [\![\phi]\!]^+_{\mathcal{M}}$$

Fact 6 (Independence Inferences: Meta-Language)**.** *Let $\mathfrak{M}$ be the set of models such that $\phi \vee \psi$ is non-Hurford. Then for each $\mathcal{M} \in \mathfrak{M}$:*

$$\textit{if } \mathcal{M}, w \vDash \Box(\phi \vee \psi) \textit{ then } R(w) \cap (\textsf{info}([\![\phi]\!]^+_{\mathcal{M}}) \backslash \textsf{info}([\![\psi]\!]^+_{\mathcal{M}})) \neq \emptyset \textit{ and}$$
$$R(w) \cap (\textsf{info}([\![\psi]\!]^+_{\mathcal{M}}) \backslash \textsf{info}([\![\phi]\!]^+_{\mathcal{M}})) \neq \emptyset$$

Proof. Suppose that in $\mathcal{M}$, $\phi \vee \psi$ is non-Hurford, and that for some $w \in W_{\mathcal{M}}$, $\mathcal{M}, w \vDash \Box(\phi \vee \psi)$.

Suppose for every $a \in \textsf{alt}^+([\![\phi]\!]^+)$, there is a $b \in \textsf{alt}^+([\![\psi]\!]^+)$ such that $a \subseteq b$. Then, by downward closure, $[\![\phi]\!]^+ \subseteq [\![\psi]\!]^+$, contradicting the fact that $\phi \vee \psi$ is non-Hurford. So there is an

$a \in \mathsf{alt}^+(\llbracket\phi\rrbracket^+)$ such that for no $a' \in \mathsf{alt}^+(\llbracket\psi\rrbracket^+)$, $a \subseteq a'$. By compactness, $a \notin \llbracket\psi\rrbracket^+$. By the semantics of disjunction, $a \in \llbracket\phi \vee \psi\rrbracket^+$, and by compactness, $a \in \mathsf{alt}^+(\llbracket\phi \vee \psi\rrbracket)$.

Since $\mathcal{M}, w \vDash \Box(\phi \vee \psi)$, $\mathsf{alt}^+(\llbracket\phi \vee \psi\rrbracket^+)$ m-covers $R(w)$. Thus there is a world $v \in R(w)$ such that $v \in a$ and for all $b \in \mathsf{alt}^+(\llbracket\phi \vee \psi\rrbracket^+) - a$ $v \notin b$. Thus, $v \notin \bigcup(\mathsf{alt}^+(\llbracket\phi \vee \psi\rrbracket) - a)$. Since $v \in a$, $v \in \mathsf{info}(\llbracket\phi\rrbracket^+)$. Since $(\mathsf{alt}^+(\llbracket\phi \vee \psi\rrbracket) - a) \supseteq \mathsf{alt}^+(\llbracket\psi\rrbracket)$, by compactness, $\bigcup(\mathsf{alt}^+(\llbracket\phi \vee \psi\rrbracket) - a) \supseteq \mathsf{info}(\llbracket\psi\rrbracket^+)$. Thus, it follows that $v \notin \mathsf{info}(\llbracket\psi\rrbracket^+)$. Thus:

$$R(w) \cap (\mathsf{info}(\llbracket\phi\rrbracket^+_{\mathcal{M}}) \backslash \mathsf{info}(\llbracket\psi\rrbracket^+_{\mathcal{M}})) \neq \varnothing$$

Parallel reasoning with ψ traded for ϕ shows that:

$$R(w) \cap (\mathsf{info}(\llbracket\psi\rrbracket^+_{\mathcal{M}}) \backslash \mathsf{info}(\llbracket\phi\rrbracket^+_{\mathcal{M}})) \neq \varnothing$$

□

Fact 7 (Ross Inference Strongly Invalid)**.** *Let $\mathfrak{M}$ be the set of models such that $\phi \vee \psi$ is non-Hurford. Then there is no $\mathcal{M} \in \mathfrak{M}$ such that for some $w \in W_{\mathcal{M}}$,*

$$\mathcal{M}, w \vDash \Box\phi \textit{ and } \mathcal{M}, w \vDash \Box(\phi \vee \psi)$$

Proof. Take a model $\mathcal{M} \in \mathfrak{M}$ and world w such that $\mathcal{M}, w \vDash \Box\phi$. So $\mathsf{alt}^+(\llbracket\phi\rrbracket)$ m-covers $R(w)$. Since $\phi \vee \psi$ is non-Hurford, by compactness there is an alternative, let it be $a \in \mathsf{alt}^+(\llbracket\psi\rrbracket)$, such that for no alternative $b \in \mathsf{alt}^+(\llbracket\phi\rrbracket)$, $b \subseteq a$. Clearly, $a \in \mathsf{alt}^+(\llbracket\phi \vee \psi\rrbracket)$.

By the semantics of disjunction, $\mathsf{alt}^+(\llbracket\phi\rrbracket) \subseteq \llbracket\phi \vee \psi\rrbracket^+$ and by compactness, for every $c \in \mathsf{alt}^+(\llbracket\phi\rrbracket)$, there is a $d \in \mathsf{alt}^+(\llbracket\phi \vee \psi\rrbracket)$ such that $c \subseteq d$. Clearly, for each such c, there is more specifically a $d \in (\mathsf{alt}^+(\llbracket\phi \vee \psi\rrbracket) - a)$ such that $c \subseteq d$. Thus, $\mathsf{alt}^+(\llbracket\phi \vee \psi\rrbracket) - a$ is a cover of $R(w)$, and since $\mathsf{alt}^+(\llbracket\phi \vee \psi\rrbracket) - a \subset \mathsf{alt}^+(\llbracket\phi \vee \psi\rrbracket)$, $\mathsf{alt}^+(\llbracket\phi \vee \psi\rrbracket)$ is not an m-cover of $R(w)$. Thus, $\mathcal{M}, w \nvDash \Box(\phi \vee \psi)$.

□

Fact 8 (Extended Ross Inference Strongly Invalid)**.** *Let $\mathfrak{M}$ be the set of models such that $\phi \vee \psi$ is non-Hurford. Then there is no $\mathcal{M} \in \mathfrak{M}$ such that for some $w \in W_{\mathcal{M}}$,*

$$\mathcal{M}, w \vDash \Box\phi, \mathcal{M}, w \vDash \Diamond\psi, \textit{ and } \mathcal{M}, w \vDash \Box(\phi \vee \psi)$$

Proof. By the previous fact, if $\mathcal{M}, w \vDash \Box\phi$, $\mathcal{M}, w \nvDash \Box(\phi \vee \psi)$. □

The following properties concern the set of models $\mathfrak{N}$ such that for $p, q \in \mathsf{At}$, '$p \vee q$' is non-Hurford. They can be generalized in interesting ways, but I unfortunately do not have the space to explore these generalizations. Relative to a model $\mathcal{N} \in \mathfrak{N}$, since $p \vee q$ is non-Hurford:

$$\mathsf{alt}^+(\llbracket p \vee q\rrbracket) = \{V(p), V(q)\} \qquad \mathsf{alt}^-(\llbracket p \vee q\rrbracket) = \{W \backslash (V(p) \cup V(q))\}$$
$$\mathsf{alt}^+(\llbracket p\rrbracket) = \{V(p)\} \qquad \mathsf{alt}^-(\llbracket p\rrbracket) = \{W \backslash V(p)\}$$
$$\mathsf{alt}^+(\llbracket q\rrbracket) = \{V(q)\} \qquad \mathsf{alt}^-(\llbracket q\rrbracket) = \{W \backslash V(q)\}$$

Fact 9 (Independence Inferences: Object-Language)**.** *For each $\mathcal{N} \in \mathfrak{N}$:*

$$\Box(p \vee q) \models^{+}_{\mathfrak{N}} \Diamond(p \wedge \neg q)$$
$$\Box(p \vee q) \models^{+}_{\mathfrak{N}} \Diamond(q \wedge \neg p)$$

Proof. Assume $\mathcal{N} \in \mathfrak{N}$, and let $w \in W_{\mathcal{N}}$ be a world such that $\mathcal{N}, w \vDash \Box(p \vee q)$. Since $p \vee q$ is non-Hurford, $\mathsf{alt}^{+}(p \vee q) = \{V(p), V(q)\}$ and $V(p) \neq V(q)$. Since $\mathcal{N}, w \vDash \Box(p \vee q)$, $\{V(p), V(q)\}$ m-covers $R(w)$. Thus, there is a $v \in R(w)$ such that $v \in V(p) \cap (W \backslash V(q))$. Now, $\mathsf{alt}^{+}(p \wedge \neg q) = \{V(p) \cap (W \backslash V(q))\}$, and clearly $\mathsf{alt}^{+}(p \wedge \neg q)$ m-covers $\{v\}$. Since $\mathsf{alt}^{+}(p \wedge \neg q) = \mathsf{alt}^{-}(\neg(p \wedge \neg q))$, $\mathsf{alt}^{-}(\neg(p \wedge \neg q))$ m-covers $\{v\}$. Thus $\Box\neg(p \wedge \neg q)$ is false at w, so $\neg\Box\neg(p \wedge \neg q)$ is true at w, i.e. $\Diamond(p \wedge \neg q)$ is true. Parallel reasoning shows the same for $\Diamond(q \wedge \neg p)$. □

Fact 10 (Free Choice)**.**

$$\Diamond(p \vee q) \models^{+}_{\mathfrak{N}} \Diamond p$$
$$\Diamond(p \vee q) \models^{+}_{\mathfrak{N}} \Diamond q$$

Proof. Assume $\mathcal{N} \in \mathfrak{N}$, and let $w \in W_{\mathcal{N}}$ be a world such that $\mathcal{N}, w \vDash \Diamond(p \vee q)$. Thus, there is a non-empty $R' \subseteq R(w)$ such that $\mathsf{alt}^{-}([\![\neg(p \vee q)]\!]) = \mathsf{alt}^{+}([\![p \vee q]\!]) = \{V(p), V(q)\}$ m-covers R'. Thus there is a world $v \in R' \cap (V(p) \backslash V(q))$. Clearly, $\{v\} \subseteq V(p)$, so there is a non-empty subset of $R(w)$, namely $\{v\}$, that is minimally covered by $\{V(p)\}$. Since $\mathsf{alt}^{-}([\![\neg p]\!]) = \{V(p)\}$, so $\mathcal{N}, w \vDash \neg\Box\neg p$. In other words, $\mathcal{N}, w \vDash \Diamond p$. Parallel reasoning shows the same for $\Diamond q$. □

Fact 11 (Independence Conditionals)**.** *Let $\mathfrak{N}$ be the set of models such that for $p, q \in \mathsf{At}$, '$p \vee q$' is non-Hurford. Then:*

$$\Box(p \vee q) \models^{+}_{\mathfrak{N}} \neg p \rightarrow \Box q$$
$$\Box(p \vee q) \models^{+}_{\mathfrak{N}} \neg q \rightarrow \Box p$$

Proof. Let $\mathcal{N} \in \mathfrak{N}$. Suppose $\mathcal{N}, w \vDash \Box(p \vee q)$. Then $\{V(p), V(q)\}$ m-covers $R(w)$, so $R(w) \subseteq (V(p) \cup V(q))$ and there is a $w_q \in R(w) \backslash V(p)$, and $R(w) \backslash V(p) \subseteq V(q)$.

Clearly, $\mathsf{info}([\![\neg p]\!]^{+}) = \mathsf{info}([\![p]\!]^{-}) = W \backslash V(p)$. Thus, $R^{\neg p}(w) = R(w) \cap W \backslash V(p) = R(w) \backslash V(p)$. Thus, $R^{\neg p}(w)$ is non-empty, and $\mathsf{alt}^{+}([\![q]\!])$ m-covers $R^{\neg p}(w)$. Thus, $\langle W_{\mathcal{N}}, R^{\neg p}_{\mathcal{N}}, V_{\mathcal{N}} \rangle, w \vDash \Box q$, so $\mathcal{N}^{\neg p}, w \vDash \Box q$. Thus, $\mathcal{N}, w \vDash \neg p \rightarrow \Box q$. Parallel reasoning shows the same for $\neg q \rightarrow \Box p$. □

Although I did not discuss this data in the main section, the present *bilateral* version of our minimal covering semantics retains some desirable results of the Kratzerian semantics for modals. First, negated necessity modals seem to distribute over disjunctions:

Fact 12 (Unnecessity Distribution)**.**

$$\neg\Box(p \vee q) \models^{+}_{\mathfrak{N}} \neg\Box p$$
$$\neg\Box(p \vee q) \models^{+}_{\mathfrak{N}} \neg\Box q$$

Proof. Let $\mathcal{N} \in \mathfrak{N}$ and suppose $\mathcal{N}, w \vDash \neg\Box(p \vee q)$. Then there is a non-empty subset $R' \subseteq R(w)$ such that $\mathsf{alt}^-(\llbracket p \vee q \rrbracket) = \{W \backslash (V(p) \cup V(q))\}$ m-covers R'. Since $W \backslash (V(p) \cup V(q)) \subseteq W \backslash V(p)$, $\{W \backslash V(p)\}$ also m-covers R'. Thus, there is a non-empty subset of $R(w)$, namely, R', such that $\mathsf{alt}^-(\llbracket p \rrbracket) = \{W \backslash V(p)\}$ m-covers it. So $\mathcal{N}, w \vDash \neg\Box p$. Parallel reasoning shows the same for $\Box q$. □

Similarly, negated possibility modals seem to distribute over disjunctions:

Fact 13 (Impossibility Distribution)**.**

$$\neg\Diamond(p \vee q) \models^+_{\mathfrak{N}} \neg\Diamond p$$
$$\neg\Diamond(p \vee q) \models^+_{\mathfrak{N}} \neg\Diamond q$$

Proof. Let $\mathcal{N} \in \mathfrak{N}$ and suppose $\mathcal{N}, w \vDash \neg\Diamond(p \vee q)$. Then $\mathcal{N}, w \vDash \Box\neg(p \vee q)$, so $R(w)$ is non-empty and $R(w)$ is m-covered by $\{W \backslash (V(p) \cup V(q))\}$. Since $W \backslash (V(p) \cup V(q)) \subseteq W \backslash V(p)$, $R(w)$ is m-covered by $\{W \backslash V(p)\} = \mathsf{alt}^+(\llbracket \neg p \rrbracket)$. Thus, $\mathcal{N}, w \vDash \Box\neg p$. Since $\neg\Diamond p$ abbreviates $\neg\neg\Box\neg p$, which is clearly equivalent to $\Box\neg p$, $\mathcal{N}, w \vDash \neg\Diamond p$. Parallel reasoning shows the same for $\neg\Diamond q$. □

4. Collectivity

The semantics of the last section, on which the Independence inferences are licensed, was motivated by the data concerning Extended Ross Arguments and Independence Conditional Arguments I adduced above. Over and above this data, however, it would be desirable to have some explanation for *why* the Independence inferences are licensed. In this section, I want to propose what I think is a plausible interpretation of the semantic interaction between necessity modals and disjunctions that might provide such an explanation. In particular, I want to suggest that we might think of disjunctions as denoting something like *pluralities* of propositions, and that we might think of necessity modals as behaving like *collective predicates* of these pluralities.

Connections between puzzles surrounding the disjunction-modal interaction and the plural term-predicate interaction have been suggested before.[11] In part, it is unsurprising, since most semantic solutions to the puzzles surrounding the disjunction-modal interaction (free choice, Ross's Puzzle, Simplification of Disjunctive Antecedents) adopt semantic frameworks in which disjunctions denote something like a set of multiple alternatives, each corresponding to a proposition denoted by a disjunct. Most treatments of plural noun/determiner phrases adopt a framework in which such terms denote sets (or sums) of multiple individuals.[12] The two frameworks thus make comparison tempting. Those who have gone in for the comparison, however, have usually focused on comparing the logic of possibility modals and disjunctions, on the one hand, with *distributive* predicates and plural terms, on the other (see the works cited in fn. 11). But since necessity modals do not distribute over disjunctions, the focus on distributivity ensures

[11] See, for example, Simons (2005b) which draws the connection to homogeneity in the plural domain, Goldstein (2019) which works out this connection systematically, and Santorio (2018) for connections between disjunctions, modal accounts of conditionals, and the logic of plurals.

[12] See for example, Link (1983) and a vast amount research following.

that the comparison is narrow. I wish to expand the comparison by considering non-distributive, or *collective*, predicates and necessity modals.[13]

To begin, I want to note that there is a connection between the notion of a minimal cover and the notion of a state that obtains only at the *collective* or *group* level. Suppose C is a set of several elements, and that C is a minimal cover of S. Then there is a sense in which C's covering of S obtains essentially collectively: it is accomplished by the elements of C *only* when they are taken *together*. Another way to put this is that each member of C plays a necessary role in making it so that C is a cover of S. This idea — that each member of the group must play a necessary role for the performance of an action or the obtaining of a state to be *collective* — has been a recurring way of cashing out the notion of collectivity in distinct literatures: for example, on the logic of agency (see the notion of 'strictly stit' in (Belnap and Perloff, 1993: 41), explored further by Sergot (2021)) and others; and on the semantics of 'together' (see, e.g., Lasersohn (1990) and Schwarzschild (1993), and Moltmann (2004) for criticism and an alternative).

Additionally, notice that while semantic solutions to Ross's Puzzle that adopt the Diversity Analysis must make reference to the multiple *alternatives* denoted by a non-Hurford disjunction $p \vee q$, they still allow that the set of relevant worlds contains a *single* world witnessing these multiple alternatives. If $R = \{w\}$ and p and q are both true at w, then $\Box(p \vee q)$ is true and the Diversity inferences are supported. A minimal covering semantics like the one I outlined in the last section, by contrast, requires that when $\Box(p \vee q)$ is true, there are at least *two* worlds in R: one which makes p but not q true, and one which makes q but not p true. In this sense, then, a minimal covering semantics requires that the set of relevant worlds with respect to which $\Box(p \vee q)$ is evaluated contains as many worlds as there are alternatives offered by the complement clause. This means that only licensing the Independence inferences ensures that the 'plurality' of the embedded disjunction is mirrored by a 'plurality' of relevant worlds.

Besides these conceptual connections between a minimal covering semantics, collectivity, and plurality, I want to note that the logic of sentences like $\Box(p \vee q)$ shares some characteristic marks of collective predicates applied to plural terms.

First, collective predicates applied to plural terms are non-distributive:

(11) a. Alicia and Bulmaro performed *Happy Days*. $P(a+b)$
b. $\not\Rightarrow$ Alicia performed *Happy Days*. $P(a)$
c. $\not\Rightarrow$ Bulmaro performed *Happy Days*. $P(b)$

Similarly, when sentences of the form $\Box(p \vee q)$ are true, the necessity modal does not distribute over its embedded disjuncts:

(12) a. Pim must trudge or talk. $\Box(p \vee q)$
b. $\not\Rightarrow$ Pim must trudge. $\Box(p)$
c. $\not\Rightarrow$ Pim must talk. $\Box(q)$

[13] 'Collective' is used in many different ways in the literature on plurals. Here I adopt one of its weakest uses: to pick out predicates that are not as a rule distributive when applied to plural-denoting terms.

Second, even though collective predicates do not usually distribute over their plural terms, there are usually some 'involvement' or 'participation' inferences one can derive that *do* distribute (Dowty (1987); Link (1983)):

(13) a. Alicia and Bulmaro performed *Happy Days*. $P(a+b)$
b. $\Rightarrow$ Alicia played a role in a performance of *Happy Days*. $P^*(a)$
c. $\Rightarrow$ Bulmaro played a role in a performance of *Happy Days*. $P^*(a)$

Similarly, the Independence inferences $\Box(p \vee q)$ licenses can be thought of as the inference that p and q each are involved, or participate in, covering the relevant set of worlds:

(14) a. Pim must trudge or talk. $\Box(p \vee q)$
b. $\Rightarrow$ Pim may trudge (without talking). $\Diamond(p \wedge \neg q)$
c. $\Rightarrow$ Pim may talk (without trudging). $\Diamond(q \wedge \neg p)$

Third, because of such involvement inferences, predicates that can apply collectively tend to exhibit what I will call *upward failure*:

(15) a. Alicia lifted the piano. $Q(a)$
b. $\not\Rightarrow$ Alicia and Bulmaro lifted the piano. $Q(a+b)$

The analogous inference for disjunctions and necessity modals is just a Ross inference, which also fails:

(16) a. Alicia must lift the piano. $\Box p$
b. $\not\Rightarrow$ Alicia or Bulmaro must lift the piano. $\Box(p \vee q)$

Fourth and finally, both collective predicates applied to plural terms and necessity modals applied to disjunctions exhibit strong negations (the negation of ϕ entails but is not equivalent to the failure of the truth of ϕ), a phenomenon usually called 'homogeneity' (see Schwarzschild (1993); Simons (2005b); Alonso-Ovalle (2006); Križ (2015); Goldstein (2019)). Consider the case of a collective predicate applied to a plural term:

(17) a. Alicia and Bulmaro didn't perform *Happy Days*. $\neg P(a+b)$
b. $\Rightarrow$ Alicia didn't perform *Happy Days*. $\neg P(a)$
c. $\Rightarrow$ Bulmaro didn't perform *Happy Days*. $\neg P(b)$

(17a), the negation of (13a), does not seem to allow that Alicia but not Bulmaro performed the play, which would be expected if the negation were simply the denial of the truth. Rather, it seems to require that neither Alicia nor Bulmaro performed the play.

Likewise, $\neg\Box(p \vee q)$ does not allow that p is necessary but not q as would be expected if it simply required that $\Box(p \vee q)$ was not true. Rather, it seems to make the stronger claim that neither p nor q is necessary:

(18) a. Pim does not have to either trudge or talk. $\neg\Box(p \vee q)$
b. $\Rightarrow$ Pim does not have to trudge. $\neg\Box p$
c. $\Rightarrow$ Pim does not have to talk. $\neg\Box q$

This 'unnecessity' distribution of necessity modals over disjunctions is predicted by the bilateral minimal covering semantics of the last section.

To summarize, I think a number of features common to the necessity modal-disjunction interaction, on the one hand, and the collective predicate-plural term interaction, on the other, recommend that we interpret the bilateral minimal covering semantics as a framework in which disjunctions denote *pluralities* of propositions, and necessity modals with disjunctive complements behave like *collective predicates* applied to those pluralities. Of course, there are important differences between the two domains, and these differences could ultimately strain the analogy. Still, the several respects in which they are similar means, I believe, that it may be fruitful to explore the comparison in more detail in future research.

References

Aher, M. (2012). Free choice in deontic inquisitive semantics (dis). In M. Aloni, V. Kimmelman, F. Roelofsen, G. W. Sassoon, K. Schulz, and M. Westera (Eds.), *Logic, Language and Meaning*, Berlin, Heidelberg, pp. 22–31. Springer Berlin Heidelberg.

Aloni, M. (2018). FC disjunction in state-based semantics. Unpublished Manuscript.

Aloni, M. and I. Ciardelli (2013). A logical account of free choice imperatives. In M. F. M. Aloni and F. Roelofsen (Eds.), *The Dynamic, Inquisitive, and Visionary Life of ϕ, ?ϕ , and $\Diamond\phi$: a Festschrift for Jeroen Groenendijk, Martin Stokhof, and Frank Veltman*, pp. 1–17. Amsterdam: ILLC.

Alonso-Ovalle, L. (2006). *Disjunction in Alternative Semantics*. Amherst: PhD Dissertation, University of Massachusetts Amherst.

Belnap, N. and M. Perloff (1993). In the realm of agents. *Annals of Mathematics and Artificial Intelligence 9*, 25–48.

Blumberg, K. and J. Hawthorne (2021). Inheritance: Professor procrastinate and the logic of obligation. *Philosophy and Phenomenological Research*.

Booth, R. J. (2022). Independent alternatives: Ross's puzzle and free choice. *Philosophical Studies 179*, 1241–1273.

Cariani, F. (2013). *Ought* and resolution semantics. *Noûs 47*(3), 534–558.

Ciardelli, I., J. Groenendijk, and F. Roelofsen (2018). *Inquisitive Semantics*. Oxford: Oxford University Press.

Ciardelli, I. and F. Roelofsen (2017). Hurford's constraint, the semantics of disjunction, and the nature of alternatives. *Natural Language Semantics 25*(3), 199–222.

Dowty, D. (1987). Collective predicates, distributive predicates, and *All*. In F. Marshall (Ed.), *Proceedings of the 3rd ESCOL, Ohio State University, Ohio*, pp. 97–115. The University of Pittsburgh.

Fine, K. (2017a). A theory of truthmaker content i: Conjunction, disjunction and negation. *Journal of Philosophical Logic 46*(6), 625–674.

Fine, K. (2017b). A theory of truthmaker content ii: Subject-matter, common content, remainder and ground. *Journal of Philosophical Logic 46*(6), 675–702.

Fusco, M. (2015). Deontic modality and the semantics of choice. *Philosophers' Imprint 15*.

Gazdar, G. (1979). *Pragmatics: Implicature, presupposition, and logical form*. New York: Academic Press.

Goldstein, S. (2019). Free choice and homogeneity. *Semantics and Pragmatics 12*, 1–48.

Groenendijk, J. and F. Roelofsen (2010). Radical inquisitive semantics. Unpublished Manuscript.

Hurford, J. R. (1974). Exclusive or inclusive disjunction. *Foundations of Language 11*(3), 409–411.

Jackson, F. and R. Pargetter (1986). Oughts, options, and actualism. *Philosophical Review 95*(2), 233–255.

Katzir, R. and R. Singh (2013). Hurford disjunctions: embedded exhaustification and structural economy. In U. Etxeberria, A. I. Anamaria Fălăuş, and B. Leferman (Eds.), *Sinn und Bedeutung 18*, pp. 201–216.

Kratzer, A. (1991). Modality. In A. von Stechow & Dieter Wunderlich (Ed.), *Semantics: An International Handbook of Contemporary Research*, pp. 639–650. Berlin, New York: Berlin: de Gruyter.

Kratzer, A. (2012a). The notional category of modality. In *Modals and Conditionals*, pp. 27–69. Oxford: Oxford University Press.

Kratzer, A. (2012b). What 'must' and 'can' must and can mean. In *Modals and Conditionals*, pp. 4–20. Oxford: Oxford University Press.

Križ, M. (2015). *Aspects of Homogeneity in the Semantics of Natural Language*. Vienna: PhD Thesis, University of Vienna.

Lasersohn, P. (1990). Group action and spatio-temporal proximity. *Linguistics and Philosophy 13*(2), 179–206.

Link, G. (1983). The logical analysis of plurals and mass terms: A lattice-theoretic approach. In P. Portner and B. H. Partee (Eds.), *Formal Semantics - the Essential Readings*, pp. 127–147. Oxford, Malden: Blackwell.

Menéndez Benito, P. (2005). *The Grammar of Choice*. Amherst: Ph.D. thesis, University of Massachusetts, Amherst.

Menéndez Benito, P. (2010). On universal free choice items. *Natural Language Semantics 18*(1), 33–64.

Meyer, M.-C. (2013). *Ignorance and Grammar*. Cambridge, MA: PhD Thesis, MIT.

Meyer, M.-C. (2014). Deriving Hurford's Constraint. In M. W. Todd Snider, Sarah D'Antonio (Ed.), *Proceedings of SALT 24*, pp. 577–596. LSA and CLC Publications.

Moltmann, F. (2004). The semantics of together. *Natural Language Semantics 12*(4), 289–318.

Santorio, P. (2018). Alternatives and truthmakers in conditional semantics. *Journal of Philosophy 115*(10), 513–549.

Sayre-McCord, G. (1986). Deontic logic and the priority of moral theory. *Noûs 20*(2), 179–197.

Schwarzschild, R. (1993). Plurals, presuppositions and the sources of distributivity. *Natural Language Semantics 2*(3), 201–248.

Sergot, M. (2021). Some forms of collectively bringing about or 'seeing to it that'. *Journal of Philosophical Logic 50*(2), 249–283.

Simons, M. (2001). Disjunction and alternativeness. *Linguistics and Philosophy 24*(5), 597–

619.
Simons, M. (2005a). Dividing things up: The semantics of or and the modal/or interaction. *Natural Language Semantics 13*(3), 271–316.
Simons, M. (2005b). Semantics and pragmatics in the interpretation of or. In E. Georgala and J. Howell (Eds.), *Proceedings of SALT 15*, pp. 205–222.
van Ditmarsch, H., W. van Der Hoek, and B. Kooi (2008). *Dynamic Epistemic Logic*. Dordrecht: Springer.
von Fintel, K. (2012). The best we can (expect to) get? challenges to the classic semantics for deontic modals. *Paper presented in a session on Deontic Modals at the Central APA, February 17, 2012*.
Wedgwood, R. (2006). The meaning of 'ought'. In R. Shafer-Landau (Ed.), *Oxford Studies in Metaethics: Volume 1*, pp. 127–160. Oxford: Clarendon Press.
Willer, M. (2018). Simplifying with free choice. *Topoi 37*(3), 379–392.
Yablo, S. (2014). *Aboutness*. Princeton: Princeton University Press.