# Genuine Paracomplete Logics

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**Abstract**

In 2016 Béziau, introduces a restricted notion of paraconsistency, the so-called *genuine paraconsistency*. A logic is *genuine paraconsistent* if it rejects the laws $\varphi, \neg\varphi \vdash \psi$ and $\vdash \neg(\varphi \wedge \neg\varphi)$. In that paper the author analyzes, among the three-valued logics, which of them satisfy this property. If we consider multiple-conclusion consequence relations, the dual properties of those above mentioned are $\vdash \varphi, \neg\varphi$ and $\neg(\psi \vee \neg\psi) \vdash$. We call *genuine paracomplete logics* those rejecting the mentioned properties. We present here an analysis of the three-valued genuine paracomplete logics. A very natural twist structures semantics for these logics is also found in a systematic way. This semantics produces automatically a simple and elegant Hilbert-style characterization for all these logics. Finally, we introduce the logic **LGP** which is genuine paracomplete, is not genuine paraconsistent, not even paraconsistent, and cannot be characterized by a single finite logical matrix.

*Keywords:* Three-valued logics; Paraconsistent logics; Paracomplete logics; Dual logic; Twist structures semantics.

## 1 Introduction: from genuine paraconsistency to genuine paracompleteness

Classically, a negation $\neg$ for a given logic **L** is semantically characterized by two properties: (1) for no sentence $\varphi$ it is the case that $\varphi$ and $\neg\varphi$ are simultaneously true; and (2) for no sentence $\varphi$ it is the case that $\varphi$ and $\neg\varphi$ are simultaneously false. Principle (1) is known as the *law of non-contradiction* (**NC**), while (2) is usually called the *law of excluded middle* (**EM**). In terms of multiple-conclusion consequence relations,[1] both laws can be represented as follows:

$$(\mathbf{NC})\quad \varphi, \neg\varphi \vdash \qquad \text{and} \qquad (\mathbf{EM})\quad \vdash \varphi, \neg\varphi.$$

This is why both laws are usually considered as being *dual* one from the other. If **L** has a conjunction $\wedge$ (which corresponds to commas on the left-hand side of $\vdash$) and a disjunction $\vee$ (which corresponds to commas on the right-hand side of $\vdash$), then both laws can be written as[2]

$$(\mathbf{NC})\quad \varphi \wedge \neg\varphi \vdash \qquad \text{and} \qquad (\mathbf{EM})\quad \vdash \varphi \vee \neg\varphi.$$

Let **L** be a logic with a negation $\neg$. If it satisfies (**NC**), then the negation $\neg$ is said to be *explosive*, and **L** is *explosive* (w.r.t. $\neg$). On the other hand, **L** is said to be *paraconsistent* (w.r.t. $\neg$) if (**NC**) does not hold in general, that is: $\varphi, \neg\varphi \nvdash$ in general. This means that there are formulas $\varphi$ and $\psi$ such that $\varphi, \neg\varphi \nvdash \psi$ (or $\varphi \wedge \neg\varphi \nvdash \psi$, if **L** has a conjunction). Dually, a logic **L** is *paracomplete* (w.r.t. $\neg$) if (**EM**) does not hold in general, that is: $\nvdash \varphi, \neg\varphi$ in general. That is, there are formulas $\varphi$ and $\psi$ such that $\psi \nvdash \varphi, \neg\varphi$ (or $\psi \nvdash \varphi \vee \neg\varphi$, if **L** has a disjunction).

As observed in [2], (**NC**) is sometimes expressed as follows:

$$(\mathbf{NC}')\quad \vdash \neg(\varphi \wedge \neg\varphi).$$

However, as the authors have shown in [2], both principles are independent. Moreover, they show that many paraconsistent logics (for instance, several three-valued paraconsistent logics such as Priest's logic **LP**) validate (**NC**$'$), which is arguably counterintuitive or undesirable. This motivates the definition of a *strong paraconsistent logic* as being a logic in which both principles, (**NC**) and (**NC**$'$), are not valid in general. In subsequent papers (see, for instance, [1]) strong paraconsistent logics were rebaptized as *genuine paraconsistent logics*. Thus, a logic **L** with negation and conjunction is genuine paraconsistent if, for some formulas $\varphi$ and $\psi$,

$$(\mathbf{GP}1)\quad \varphi \wedge \neg\varphi \nvdash \qquad \text{and} \qquad (\mathbf{GP}2)\quad \nvdash \neg(\psi \wedge \neg\psi).$$

Given the duality between (**NC**) and (**EM**), it makes sense to consider (in a logic with disjunction) the dual property of (**NC**$'$), namely

$$(\mathbf{EM}')\quad \neg(\varphi \vee \neg\varphi) \vdash .$$

This motivates the following definition, which is the subject of the present paper:

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[1] We can consider a multiple-conclusion consequence relation $\vdash$ as a binary relation between sets of formulas $\Gamma$ and $\Delta$, such that $\Gamma \vdash \Delta$ means that any model of every $\gamma \in \Gamma$ is also a model for some $\delta \in \Delta$ [21].

[2] It should be observed that in [2] the authors use **NC** for representing that, for any set $\Gamma$ of formulas, it is the case that $\Gamma \vdash \neg(\varphi \wedge \neg\varphi)$.

**Definition 1.1** *A logic* **L** *with negation and disjunction is said to be a* genuine paracomplete logic *(or a* strong paracomplete logic*) if neither (***EM***) nor (***EM′***) are valid, that is: for some formulas* $\varphi$ *and* $\psi$*,*

$$(\mathbf{GP}1_D) \quad \nvdash \varphi \vee \neg\varphi \qquad \textit{and} \qquad (\mathbf{GP}2_D) \quad \neg(\psi \vee \neg\psi) \nvdash .$$

Observe that, in terms of a Tarskian (single-conclusion) consequence relation (see Definition 2.1), $(\mathbf{GP}2_D)$ is equivalent to the following:

$$(\mathbf{GP}2^*_D) \quad \neg(\psi \vee \neg\psi) \nvdash \varphi \quad \text{for some formulas } \varphi, \psi.$$

In semantical terms, if $(\mathbf{GP}2_D)$ holds then $\neg(\psi \vee \neg\psi)$ is satisfiable, that is: it has some model.

The aim of this paper, which is a continuation of our previous work [14], is the study of some basic systems of genuine paracomplete logics. An interesting start point is considering all the three-valued systems that can be defined, taking into account some reasonable restrictions, this will be the subject of Section 3. Another genuine paracomplete system called **LGP**, which cannot be characterized by a single finite-valued logic matrix, will be presented in the remaining sections.

The notions of genuine paraconsistent and genuine paracomplete logics can be analyzed in view of the logic principles above mentioned, as well as its connection with the basic properties of connectives.

Let **L** be a logic with negation $\neg$ and conjunction $\wedge$. In the case $\neg$ satisfies the right-introduction rule:

$$\Gamma, \varphi \vdash \Delta \quad \text{implies that} \quad \Gamma \vdash \neg\varphi, \Delta$$

it is easy to see that (**EM**) is valid, hence **L** is not paracomplete. Indeed, from $\varphi \vdash \varphi$ it follows that $\vdash \varphi, \neg\varphi$, by the right-introduction rule for $\neg$. Moreover, if **L** additionally satisfies (**NC**) then clearly it will satisfy (**NC′**): from $\varphi, \neg\varphi \vdash$ it follows that $\varphi \wedge \neg\varphi \vdash$ and then $\vdash \neg(\varphi \wedge \neg\varphi)$.

If **L** satisfies the right-introduction rule and satisfies (**GP**2) for some formula, then it is genuine paraconsistent. Indeed, if (**GP**2) holds for some formula $\varphi$ then $\nvdash \neg(\varphi \wedge \neg\varphi)$ and so, by contraposition of the right-introduction rule for $\neg$, we infer that $\varphi \wedge \neg\varphi \nvdash$. That is, (**GP**1) also holds for $\varphi$.

Dually, if **L** is a logic with negation $\neg$ and disjunction $\vee$ such that $\neg$ satisfies the left-introduction rule:

$$\Gamma \vdash \varphi, \Delta \quad \text{implies that} \quad \Gamma, \neg\varphi \vdash \Delta$$

then (**NC**) is valid in **L**, that is, **L** is not paraconsistent. Indeed, note that from $\varphi \vdash \varphi$ it follows that $\varphi, \neg\varphi \vdash$ for every $\varphi$. Moreover, if **L** additionally satisfies (**EM**) then it satisfies (**EM′**).

Now, if **L** satisfies the left-introduction rule and ($\mathbf{GP}2_D$) for some formula, then it is genuine paracomplete. Indeed, if ($\mathbf{GP}2_D$) holds for some $\varphi$, then ($\mathbf{GP}1_D$) also holds for $\varphi$, by taking contraposition of the left-introduction rule of $\neg$.

**Example 1.2**

1. *Propositional intuitionistic logic* **IPL** *is paracomplete, but it is not genuine paracomplete: the formula* $\neg(\varphi \vee \neg\varphi)$ *is unsatisfiable.*

2. *The Belnap-Dunn logic* $FOUR$ *(with the standard truth ordering) is both genuine paraconsistent and genuine paracomplete.*

3. *Nelson logic* **N4** *(see [19]) is both genuine paraconsistent and genuine paracomplete.*

4. *The three-valued logic* **MH***, introduced in [3], is genuine paracomplete and explosive. As we shall see, it is one of the possible three-valued genuine paracomplete logics which extend the 2-valued truth functions of classical propositional logic* **CPL***, to be studied in Section 3.*

The organization of the paper is as follows: after introducing in Section 2 some basic concepts and notation to be used along the paper, in Section 3 all the possible three-valued genuine paracomplete logics (satisfying certain basic requirements) will be characterized. In Section 4 we show in detail how to characterize one of the three-valued logics described in the previous section by means of a useful algebraic semantics known as twist structures semantics. This semantics can be systematically extended to the other logics, as we shall see. Moreover, it is possible to find from such semantics a simple Hilbert-style axiomatization for all these logics, as we show in Section 6. In Section 7 an example of a genuine paracomplete (not paraconsistent) logic called **LGP** is given, showing in Section 8 that this logic cannot be characterized by a single finite logical matrix. Finally, in Section 9 we present some concluding remarks.

# 2 Basic concepts

We consider propositional signatures $\Theta$, which are families of connectives together with their arities. By simplicity, we will only consider signatures with a finite set of connectives. Consider, from now on, an infinite denumerable set $\mathcal{V} = \{p_n \ : \ n \in \mathbb{N}\}$ of propositional variables. The (absolutely free) algebra of formulas over $\Theta$ generated by $\mathcal{V}$ will be denoted by $For(\Theta)$, and its elements will be called *formulas (over* $\Theta$*)*. The propositional variables are also called *atoms* (or *atomic formulas*). Formulas will be denoted by lowercase Greek letters, while *theories*, which are sets of formulas, will be denoted by uppercase Greek letters.

**Definition 2.1** *A* (Tarskian) consequence relation $\vdash$ *between theories and formulas is a relation satisfying the following properties, for every theory* $\Gamma \cup \Delta \cup \{\varphi\}$*:*

> ***(Reflexivity)*** *if* $\varphi \in \Gamma$*, then* $\Gamma \vdash \varphi$*;*
>
> ***(Monotonicity)*** *if* $\Gamma \vdash \varphi$ *and* $\Gamma \subseteq \Delta$*, then* $\Delta \vdash \varphi$*;*
>
> ***(Transitivity)*** *if* $\Delta \vdash \varphi$ *and* $\Gamma \vdash \psi$ *for every* $\psi \in \Delta$*, then* $\Gamma \vdash \varphi$*.*

$\vdash$ *is called* structural *if, in addition, it holds:* $\Gamma \vdash \varphi$ *implies that* $\rho[\Gamma] \vdash \rho(\varphi)$*, for every substitution* $\rho$ *over* $\Theta$*.*[3] *If there exists some non-empty theory* $\Gamma$ *and some* $\varphi$ *such that* $\Gamma \nvdash \varphi$*,* $\vdash$ *is called* non-trivial*.*

Sometimes, in order to define a logic it is required that $\vdash$ be *finitary* (or *compact*).[4] However, here we consider a logic as it is established in Definition 2.2.

**Definition 2.2** *A* logic *over* $\Theta$ *is a pair* $\mathbf{L} = \langle For(\Theta), \vdash_{\mathbf{L}}\rangle$*, where* $\vdash_{\mathbf{L}}$ *is a structural and non-trivial Tarskian consequence relation such that if its signature* $\Theta$ *contains a binary connective* $\to$*, Modus Ponens (MP) must be satisfied, that is:* $\varphi \to \psi, \varphi \vdash_{\mathbf{L}} \psi$ *for any formulas* $\varphi$ *and* $\psi$*.*

As usual, the fact that $\varphi$ can be inferred from $\Gamma$ in $\mathbf{L}$ will be denoted by $\Gamma \vdash_{\mathbf{L}} \varphi$. The subscript $\mathbf{L}$ will be dropped whenever the logic is clear from the context.

The expressiveness of a logic depends on the available connectives in its signature: thus, as we have pointed out in the introduction, for talking about paracompleteness we need a negation and a disjunction satisfying particular conditions. In order to obtain more expressive logics, we are going to complete the signature with an appropriate conjunction and an appropriate implication. In Definition 2.3 we establish some conditions on connectives so they can be considered as conjunction, disjunction, and implication.

**Definition 2.3** *Let* $\mathbf{L}$ *be a logic over the signature* $\Theta$ *with binary connectives* $\wedge$*,* $\vee$ *and* $\to$*. Then:*

1. $\wedge$ *is a* conjunction *for* $\mathbf{L}$ *when:* $\Gamma \vdash \varphi \wedge \psi$ *iff* $\Gamma \vdash \varphi$ *and* $\Gamma \vdash \psi$*.*
2. $\vee$ *is a* disjunction *for* $\mathbf{L}$ *when:* $\Gamma, \varphi \vee \psi \vdash \sigma$ *iff* $\Gamma, \varphi \vdash \sigma$ *and* $\Gamma, \psi \vdash \sigma$*.*
3. $\to$ *is an* implication *for* $\mathbf{L}$ *when:* $\Gamma, \varphi \vdash \psi$ *iff* $\Gamma \vdash \varphi \to \psi$*.*

Observe that the notions of conjunction and disjunction are the usual ones considered in abstract logic, see for instance [23]. In order to find a suitable implication for the genuine paraconsistent logics **L3A** and **L3B** investigated

[3] That is, for every endomorphism $\rho$ over the algebra $For(\Theta)$ of formulas. From now on, we will write $f[X]$ to denote the set $\{f(x) : x \in X\}$ for any function $f$ and set $X$ contained in its domain.

[4] Informally speaking, it means that every deduction can be obtained from a finite number of hypothesis. In formal terms: if $\Gamma \vdash \varphi$ then $\Gamma_0 \vdash \varphi$ for some finite $\Gamma_0$ contained in $\Gamma$.

in [13], the authors define the concept of classical implication. It is not difficult to prove in the context of Tarskian consequence relations that the notion of implication of Definition 2.3 implies the notion of classical implication given in [13]. The usual manner to define many-valued logics is by means of a logical matrix.

**Definition 2.4** *A* (logical) matrix *over the signature* $\Theta$ *is a structure* $\mathcal{M} = \langle V, F, D \rangle$*, where:*

> $V$ *is a non-empty set of truth values (domain);*
>
> $F := \{f_c \ : \ c \in \Theta\}$ *is a family of truth functions, such that* $f_c : D^n \to D$ *if* $c$ *is a logical connective in* $\Theta$ *with arity* $n$*;*
>
> $D$ *is a subset of* $V$ *(set of designated values).*

Observe that $A = \langle V, F \rangle$ is an algebra for the signature $\Theta$.

**Definition 2.5** *Given a matrix* $\mathcal{M}$ *over* $\Theta$*, a function* $v : \mathcal{V} \longrightarrow V$ *that maps atoms into elements of the domain is called a* valuation *over* $\mathcal{M}$*.*

Any valuation $v$ can be uniquely extended, as usual, to a homomorphism $v : For(\Theta) \longrightarrow V$ such that $v(c(\alpha_1, \ldots, \alpha_n)) = f_c(v(\alpha_1), \ldots, v(\alpha_n))$. Now we can define the notion of model:

**Definition 2.6** *A valuation* $v$ *over* $\mathcal{M}$ *is a* model *of the formula* $\varphi$ *if* $v(\varphi) \in D$*. A model of a set of formulas is a model of each of its elements. A formula* $\varphi$ *is a* tautology *in* $\mathcal{M}$*, denoted by* $\models_{\mathcal{M}} \varphi$*, if every valuation is a model of* $\varphi$*.*

Whenever the matrix is clear from the context, the subscript will be dropped. It is also possible to define a consequence relation by means of a matrix.

**Definition 2.7** *Given a matrix* $\mathcal{M}$*, its* induced consequence relation*, denoted by* $\vdash_{\mathcal{M}}$*, is defined by:* $\Gamma \vdash_{\mathcal{M}} \varphi$ *if every model of* $\Gamma$ *is a model of* $\varphi$*. We denote by* $\mathbf{L}_{\mathcal{M}} = \langle For(\Theta), \vdash_{\mathcal{M}} \rangle$ *the logic obtained from this consequence relation.*

In case a logic is defined via the induced consequence relation of a matrix $\mathcal{M}$ and the cardinality of the set of truth values of $\mathcal{M}$ is $n < \omega$ then the logic is called an $n$-valued logic.

Now we define *neoclassical* connectives. This name can be easily understood if we identify the ‘True’ value with ‘designated value’ and the ‘False’ value with ‘non-designated value’. These conditions are generalizations of those that satisfy and in some way define the nature of the connectives in classical logic. The next definition is common in the literature [13, 2].

**Definition 2.8** *Let* $\mathcal{M} = \langle V, F, D \rangle$ *be a matrix,* $\overline{D}$ *the set of non-designated values (i.e.* $\overline{D} = V \setminus D$*), and* $v$ *any valuation over* $\mathcal{M}$*. Then:*

1. $\wedge$ *is a* neoclassical conjunction*, if it holds that:*
$$v(\varphi \wedge \psi) \in D \text{ iff } v(\varphi) \in D \text{ and } v(\varphi) \in D.$$

2. $\vee$ *is a* neoclassical disjunction*, if it holds that:*
$$v(\varphi \vee \psi) \in \overline{D} \text{ iff } v(\varphi) \in \overline{D} \text{ and } v(\varphi) \in \overline{D}.$$

3. $\rightarrow$ *is a* neoclassical implication*, if it holds that:*
$$v(\varphi \rightarrow \psi) \in D \text{ iff either } v(\varphi) \in \overline{D} \text{ or } v(\psi) \in D.$$

Observe that conditions of neoclassicality of Definition 2.8 are more restrictive than those on Definition 2.3. Specifically, we have that items 2 and 3 on Definition 2.8 imply items 2 and 3 on Definition 2.3. Moreover, item 1 on Definition 2.8 is equivalent to item 1 on Definition 2.3.

**Definition 2.9** *(see [2]) Let $V$ and $V_1$ be sets with $n$ and $m$ elements, respectively, such that $V_1 \subsetneq V$. We say that a function $\circledast : V^k \longrightarrow V$ is an* enlargement *of a function $\odot : V_1^k \longrightarrow V_1$, if the restriction of $\circledast$ to $V_1$ coincides with $\odot$ (i.e. $\circledast|_{V_1^k} = \odot$).*

**Definition 2.10** *Let $\mathbf{L_1}$ be a $n$-valued logic whose set of truth values is $V_1$ and $\mathbf{L_2}$ a $m$-valued logic whose set of truth values is $V_2$, such that $V_1 \subsetneq V_2$ and the set of connectives of $\mathbf{L_1}$ is a subset of the connectives of $\mathbf{L_2}$. $\mathbf{L_2}$ is called a* enlargement *of $\mathbf{L_1}$ if all the truth functions in $\mathbf{L_2}$ are enlargements of the truth functions in $\mathbf{L_1}$.*

The last definition can be recast in algebraic terms as follows: given $\mathbf{L_1}$ and $\mathbf{L_2}$ as above, let $A_i$ be the algebra underlying the matrix $\mathcal{M}_i$ of $\mathbf{L_i}$, for $i = 1, 2$. Then, $\mathbf{L_2}$ is an enlargement of $\mathbf{L_1}$ if and only if $A_1$ is a proper subalgebra of the reduct of $A_2$ to the signature of $A_1$.

By adapting the proof of Lemma 2.3 in [9] we get the following result:

**Proposition 2.11** *Let $\mathbf{L_2}$ be an enlargement of $\mathbf{L_1}$ such that, if $D_i$ is the set of designated values of $\mathbf{L_i}$ (for $i = 1, 2$), then $D_1 = D_2 \cap V_1$. Then, the reduct of $\mathbf{L_2}$ to the signature of $\mathbf{L_1}$ is a sublogic of $\mathbf{L_1}$, that is: for every set of formulas $\Gamma \cup \{\varphi\}$ over the signature of $\mathbf{L_1}$, $\Gamma \vdash_{\mathbf{L_2}} \varphi$ implies that $\Gamma \vdash_{\mathbf{L_1}} \varphi$. In particular, if both logics have the same signature then $\mathbf{L_2}$ is a sublogic of $\mathbf{L_1}$.*

# 3 Three-valued genuine paracomplete logics

In this section we study the notion of genuine paracompleteness among logics $L_{\mathcal{M}} = \langle For(\Theta), \vdash_{\mathcal{M}} \rangle$, where $\mathcal{M} = \langle \{0, \frac{1}{2}, 1\}, F, D \rangle$, i.e. three-valued logics defined over a suitable signature $\Theta$. Particularly 0 and 1 are identified with *False* and *True* respectively. This implies that $1 \in D$ and $0 \notin D$. The goal is to make a detailed analysis to find all the three-valued genuine paracomplete logics such that are enlargements of classical logic, apart from some nice features like being neoclassical, classical, and symmetric,[5] whenever it is possible. As a consequence of Proposition 2.11, all of them will be sublogics of classical logic.

[5]A binary connective $\circledast$ is called symmetric if $a \circledast b = b \circledast a$.

| $\varphi$ | $\neg\varphi$ |
|---|---|
| $0$ | $\boxed{1}$ |
| $\frac{1}{2}$ | $n$ |
| $1$ | $\boxed{0}$ |

(a)

| $\vee$ | $0$ | $\frac{1}{2}$ | $1$ |
|---|---|---|---|
| $0$ | $\boxed{0}$ | $d_1$ | $\boxed{1}$ |
| $\frac{1}{2}$ | $d_1$ | $d_2$ | $d_3$ |
| $1$ | $\boxed{1}$ | $d_3$ | $\boxed{1}$ |

(b)

Table 1: Partial tables for negation and disjunction

### 3.1 Minimal Paracomplete logics

There are two connectives involved in Definition 1.1, namely $\neg$ and $\vee$ i.e. negation and disjunction. In other words, to get a three-valued genuine paracomplete logic we need to consider a signature containing at least $\neg$ and $\vee$ and define a suitable set $D$ of designated values, as well as the interpretation $F$ (the truth tables) for $\neg$ and $\vee$ in the logical matrix $\mathcal{M} = \langle\{0, \frac{1}{2}, 1\}, F, D\rangle$.

First of all, since we are considering connectives that are enlargements of the 2-valued truth functions of classical logic, we have already fixed some of the values of the truth tables, namely those that are boxed in Table 1. As a result of this, only the second row in the table of negation and five entries in the table of disjunction should be analyzed in order to fix their values. However we want our connectives to be well behaved, therefore we add the following restrictions to the connectives:

| Connective | Enlargement | Neoclassical | Symmetric |
|---|---|---|---|
| $\neg$ | ✓ | N/A | N/A |
| $\vee$ | ✓ | ✓ | ✓ |

These conditions reduce the set of unknown values to $n$, $d_1$, $d_2$ and $d_3$ as shown in Table 1.

Let us start by analyzing the characteristics that must obey this pair of connectives in order to satisfy condition $\mathbf{GP1}_D$ of Definition 1.1. In other words it is necessary to set the values in such a way that $\varphi \vee \neg\varphi$ is not a tautology.

Note that if we assign 1 to $n$ in Table 1(a) then, in every row, either $\varphi$ or $\neg\varphi$ are designated and, since disjunction is neoclassical, $\varphi\vee\neg\varphi$ is always designated. Therefore, $n$ must be in $\{0, \frac{1}{2}\}$. Up to this point, we know that $1 \in D$ and $0 \notin D$ but, if we set $\frac{1}{2}$ as designated, once again in every row either $\varphi$ or $\neg\varphi$ will be designated, thus $\varphi \vee \neg\varphi$ becomes a tautology. Hence, in a three-valued genuine paracomplete logic we must have $D = \{1\}$. This last result in combination with neoclassicality of $\vee$ forces to have $d_1, d_2 \in \{0, \frac{1}{2}\}$ and $d_3 = 1$ in Table 1(b). As a result, all the variables considered now are $n$, $d_1$ and $d_2$, and all of them take values in $\{0, \frac{1}{2}\}$.

On the other hand, condition $\mathbf{GP2}_D$ of Definition 1.1 corresponds to the condition of $\neg(\varphi \vee \neg\varphi)$ being satisfiable. Let us analyze it by cases:

| $\varphi$ | $\neg\varphi$ | $\varphi \vee \neg\varphi$ | $\neg(\varphi \vee \neg\varphi)$ |
|---|---|---|---|
| 0 | 1 | 1 | 0 |
| $\frac{1}{2}$ | 0 | 0 | 1 |
| 1 | 0 | 1 | 0 |

Table 2: Truth table case $n = 0$ and $d_1 = 0$

| $\varphi$ | $\neg\varphi$ | $\varphi \vee \neg\varphi$ | $\neg(\varphi \vee \neg\varphi)$ |
|---|---|---|---|
| 0 | 1 | 1 | 0 |
| $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 1 |
| 1 | 0 | 1 | 0 |

Table 3: Truth table case $n = \frac{1}{2}$ and $d_2 = 0$

**Case** $n = 0$

Considering the negation whose table takes the value 0 for $n$, we have the following sub-cases:

1. If $d_1 = 0$, $\mathbf{GP1}_D$ and $\mathbf{GP2}_D$ hold and Definition 1.1 is satisfied, regardless of the value of $d_2$, as Table 2 shows. $\mathbf{GP1}_D$ holds since, in the third column, $\varphi \vee \neg\varphi$ is not a tautology due to the value 0 in the second row. On the other hand, $\mathbf{GP2}_D$ holds since, in the fourth column, $\neg(\varphi \vee \neg\varphi)$ has a model due to the value 1 in the second row.

2. If $d_1 = \frac{1}{2}$ then, for any valuation, $v(\neg(\varphi \vee \neg\varphi)) = 0$ and so $\mathbf{GP2}_D$ does not hold.

Therefore, if $n = 0$ the only feasible value for $d_1$ is 0. Thus we have two possible combinations $d_1 = 0$, $d_2 = \frac{1}{2}$, and $d_1 = d_2 = 0$. See Table 4(a) and Table 4(c) respectively.

**Case** $n = \frac{1}{2}$

When $n = \frac{1}{2}$, we have the following sub-cases:

1. If $d_2 = 0$, then $\mathbf{GP1}_D$ as well as $\mathbf{GP2}_D$ hold as desired, without considering $d_1$, as we can see in Table 3 analogously to Table 2.

2. If $d_2 = 1$, then $v(\neg(\varphi \vee \neg\varphi)) \in \overline{D}$ and $\mathbf{GP2}_D$ does not hold.

Analogously to the case $n = 0$ we have only two choices to get a genuine paracomplete disjunction, namely $d_1 = \frac{1}{2}$ and $d_2 = 0$, and $d_1 = d_2 = 0$. See Table 4(b) and Table 4(d) respectively.

The previous analysis leads us to four different three-valued genuine paracomplete logics in the signature $\{\neg, \vee\}$ as shown in the following definition.

| $\varphi$ | $\neg_0\varphi$ |
|---|---|
| $0$ | $1$ |
| $\frac{1}{2}$ | $0$ |
| $1$ | $0$ |

| $\vee_0$ | $0$ | $\frac{1}{2}$ | $1$ |
|---|---|---|---|
| $0$ | $0$ | $0$ | $1$ |
| $\frac{1}{2}$ | $0$ | $\frac{1}{2}$ | $1$ |
| $1$ | $1$ | $1$ | $1$ |

(a)

| $\varphi$ | $\neg_1\varphi$ |
|---|---|
| $0$ | $1$ |
| $\frac{1}{2}$ | $\frac{1}{2}$ |
| $1$ | $0$ |

| $\vee_1$ | $0$ | $\frac{1}{2}$ | $1$ |
|---|---|---|---|
| $0$ | $0$ | $\frac{1}{2}$ | $1$ |
| $\frac{1}{2}$ | $\frac{1}{2}$ | $0$ | $1$ |
| $1$ | $1$ | $1$ | $1$ |

(b)

| $\varphi$ | $\neg_0\varphi$ |
|---|---|
| $0$ | $1$ |
| $\frac{1}{2}$ | $0$ |
| $1$ | $0$ |

| $\vee_2$ | $0$ | $\frac{1}{2}$ | $1$ |
|---|---|---|---|
| $0$ | $0$ | $0$ | $1$ |
| $\frac{1}{2}$ | $0$ | $0$ | $1$ |
| $1$ | $1$ | $1$ | $1$ |

(c)

| $\varphi$ | $\neg_1\varphi$ |
|---|---|
| $0$ | $1$ |
| $\frac{1}{2}$ | $\frac{1}{2}$ |
| $1$ | $0$ |

| $\vee_2$ | $0$ | $\frac{1}{2}$ | $1$ |
|---|---|---|---|
| $0$ | $0$ | $0$ | $1$ |
| $\frac{1}{2}$ | $0$ | $0$ | $1$ |
| $1$ | $1$ | $1$ | $1$ |

(d)

Table 4: Possible truth tables for $\neg$ and $\vee$ in a genuine paracomplete three-valued logic.

| $\wedge$ | $0$ | $\frac{1}{2}$ | $1$ |
|---|---|---|---|
| $0$ | $\boxed{0}$ | $c_1$ | $\boxed{0}$ |
| $\frac{1}{2}$ | $c_1$ | $c_2$ | $c_3$ |
| $1$ | $\boxed{0}$ | $c_3$ | $\boxed{1}$ |

Table 5: Partial table for conjunction

**Definition 3.1** *A three-valued logic* $\mathbf{L}_{\mathcal{M}} = \langle For(\Theta_0), \vdash_{\mathcal{M}} \rangle$ *over the signature* $\Theta_0 = \{\neg, \vee\}$*, where* $\mathcal{M}$ *is the matrix with set of values* $\{0, \frac{1}{2}, 1\}$ *and* $1$ *as the only designated value will be called:*

1. $\mathbf{L3A}^D$ *if the truth tables for* $\neg$ *and* $\vee$ *are those from Table 4(a).*

2. $\mathbf{L3B}^D$ *if the truth tables for* $\neg$ *and* $\vee$ *are those from Table 4(b).*

3. $\mathbf{L3C}^D$ *if the truth tables for* $\neg$ *and* $\vee$ *are those from Table 4(c).*

4. $\mathbf{L3D}^D$ *if the truth tables for* $\neg$ *and* $\vee$ *are those from Table 4(d).*

**Remark 3.2** *As we can see in any of the logics in Definition 3.1 both,* $\perp$ *and* $\top$ *are definable. For the case of* $\mathbf{L3A}^D$ *and* $\mathbf{L3C}^D$ *we have that* $\perp \stackrel{def}{=} \neg(\neg\alpha \vee \neg\neg\alpha)$ *and* $\top \stackrel{def}{=} \neg\alpha \vee \neg\neg\alpha$*. Finally for* $\mathbf{L3B}^D$ *and* $\mathbf{L3D}^D$ $\perp \stackrel{def}{=} \neg((\alpha \vee \neg\alpha) \vee \neg(\alpha \vee \alpha))$ *and* $\top \stackrel{def}{=} (\alpha \vee \neg\alpha) \vee \neg(\alpha \vee \alpha)$*.*

### 3.2 Adequate Conjunctions

Since the definition of genuine paracompleteness does not impose conditions over the conjunction we can choose any of the definable conjunctions in a three-valued logic. Considering restrictions imposed to disjunction we ask the same for conjunction, namely: enlargement of classical conjunction, neoclassical and symmetric.

These restrictions allow us to have a partial table for the conjunction like the one in Table 5 where $c_1$, $c_2$ and $c_3 \in \overline{D}$. Then, there are 8 different conjunctions satisfying all these restrictions as shown in Table 6. Choosing one of these conjunctions to extend each of the logics in Definition 3.1 lead us to 32 different genuine paracomplete logics in the signature $\{\neg, \vee, \wedge\}$.

### 3.3 Adequate Implications

Now we search for suitable implications for $\mathbf{L3A}^D$, $\mathbf{L3B}^D$, $\mathbf{L3C}^D$ and $\mathbf{L3D}^D$. One can consider the same restrictions imposed to disjunction and conjunction except for symmetry, and then we request implication to be an enlargement of classical implication and neoclassical.

The condition of being an enlargement fix four values, see boxes in Table 7. If we ask for neoclassicality to be satisfied too, see Definition 2.8, then the

| $\wedge_0$ | $0$ | $\frac{1}{2}$ | $1$ |
|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ |
| $\frac{1}{2}$ | $0$ | $0$ | $0$ |
| $1$ | $0$ | $0$ | $1$ |

| $\wedge_1$ | $0$ | $\frac{1}{2}$ | $1$ |
|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ |
| $\frac{1}{2}$ | $0$ | $0$ | $\frac{1}{2}$ |
| $1$ | $0$ | $\frac{1}{2}$ | $1$ |

| $\wedge_2$ | $0$ | $\frac{1}{2}$ | $1$ |
|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ |
| $\frac{1}{2}$ | $0$ | $\frac{1}{2}$ | $0$ |
| $1$ | $0$ | $0$ | $1$ |

| $\wedge_3$ | $0$ | $\frac{1}{2}$ | $1$ |
|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ |
| $\frac{1}{2}$ | $0$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $1$ | $0$ | $\frac{1}{2}$ | $1$ |

| $\wedge_4$ | $0$ | $\frac{1}{2}$ | $1$ |
|---|---|---|---|
| $0$ | $0$ | $\frac{1}{2}$ | $0$ |
| $\frac{1}{2}$ | $\frac{1}{2}$ | $0$ | $0$ |
| $1$ | $0$ | $0$ | $1$ |

| $\wedge_5$ | $0$ | $\frac{1}{2}$ | $1$ |
|---|---|---|---|
| $0$ | $0$ | $\frac{1}{2}$ | $0$ |
| $\frac{1}{2}$ | $\frac{1}{2}$ | $0$ | $\frac{1}{2}$ |
| $1$ | $0$ | $\frac{1}{2}$ | $1$ |

| $\wedge_6$ | $0$ | $\frac{1}{2}$ | $1$ |
|---|---|---|---|
| $0$ | $0$ | $\frac{1}{2}$ | $0$ |
| $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $0$ |
| $1$ | $0$ | $0$ | $1$ |

| $\wedge_7$ | $0$ | $\frac{1}{2}$ | $1$ |
|---|---|---|---|
| $0$ | $0$ | $\frac{1}{2}$ | $0$ |
| $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $1$ | $0$ | $\frac{1}{2}$ | $1$ |

Table 6: Possible conjunctions

| $\to$ | $0$ | $\frac{1}{2}$ | $1$ |
|---|---|---|---|
| $0$ | $\boxed{1}$ | $i_1$ | $\boxed{1}$ |
| $\frac{1}{2}$ | $i_2$ | $i_3$ | $i_4$ |
| $1$ | $\boxed{0}$ | $i_5$ | $\boxed{1}$ |

Table 7: Partial table for implication

values of $i_1, i_2, i_3$ and $i_4$ are designated and $i_5$ is not designated. Therefore there are only two different implications that fulfill the conditions, namely $\to_0$ and $\to_1$ defined in Table 8. In fact $\to_0$ and $\to_1$ are implications according to Definition 2.3.

One nice additional feature of connectives $\to_0$ and $\to_1$ is that any of the logics obtained by extending $\mathbf{L3A^D}$, $\mathbf{L3B^D}$, $\mathbf{L3C^D}$ or $\mathbf{L3D^D}$ with any of the connectives $\to_0$ or $\to_1$, satisfies the positive fragment of classical logic.

There is a criterion for the construction of a 'paraconsistent' logic, but in terms of implication and negation. It is due to Jaśkowski and consists on the rejection of $\vdash \varphi \to (\neg\varphi \to \psi)$ (the law of Duns Scotus). Clearly, this definition is not equivalent to the definition of paraconsistency in terms of (**NC**) but provides another approach to study paraconsistency in case the language has an implication connective. By analogy, for the paracompleteness case there is also a criterion in terms of implication and negation. According to Karpenko and

| $\to_0$ | $0$ | $\frac{1}{2}$ | $1$ |
|---|---|---|---|
| $0$ | $1$ | $1$ | $1$ |
| $\frac{1}{2}$ | $1$ | $1$ | $1$ |
| $1$ | $0$ | $0$ | $1$ |

| $\to_1$ | $0$ | $\frac{1}{2}$ | $1$ |
|---|---|---|---|
| $0$ | $1$ | $1$ | $1$ |
| $\frac{1}{2}$ | $1$ | $1$ | $1$ |
| $1$ | $0$ | $\frac{1}{2}$ | $1$ |

Table 8: Possible implications

Tomova [17], a logic is ‘paracomplete’ iff $\vdash (\neg\varphi \to \varphi) \to \varphi$ (the law of Clavius), is not valid in it. This definition does not correspond with the definition of paracompleteness in terms of (**EM**) but it gives an intuitionistic flavor to the implication connective. This kind of logics are called *weakly-intuitionistic logics*, see [20] and [8]. Observe that the logics $\mathbf{L3A^D}$, $\mathbf{L3B^D}$, $\mathbf{L3C^D}$ and $\mathbf{L3D^D}$ extended with the implications $\to_0$ or $\to_1$ from Table 8 reject the Clavius law, therefore they are paracomplete in Karpenko and Tomova’s sense.

### 3.4 The family of logics $\mathbf{GP3^D}$

It is possible to extend the genuine paracomplete logics from Definition 3.1 with any of the conjunctions and implications from Sections 3.2 and 3.3.

**Definition 3.3** *The logics* $\mathbf{L3A^D}$, $\mathbf{L3B^D}$, $\mathbf{L3C^D}$ *and* $\mathbf{L3D^D}$ *can be extended to the signature* $\Sigma_0 = \{\wedge, \vee, \to, \neg\}$ *in the following way. Let* $X \in \{A, B, C, D\}$*, let* $i \in \{0, 1, \ldots, 7\}$ *and* $j \in \{0, 1\}$*. The logic* $\mathbf{L3}X^{\mathbf{D}}_{\mathbf{i,j}}$ *over* $\Sigma_0$ *is defined as being the logic* $\mathbf{L3}X^{\mathbf{D}}$ *extended with conjunction* $\wedge_i$ *from Table 6 and with the implication* $\to_j$ *from Table 8.*

**Definition 3.4** *The family of logics* $\mathbf{GP3^D}$ *is the family of 64 genuine paracomplete three-valued logics:* $\mathbf{L3A^D_{i,j}}$, $\mathbf{L3B^D_{i,j}}$, $\mathbf{L3C^D_{i,j}}$ *and* $\mathbf{L3D^D_{i,j}}$ *where* $i \in \{0, 1, \ldots, 7\}$ *and* $j \in \{0, 1\}$*.*

**Remark 3.5** *The logic obtained by extending* $\mathbf{L3B^D}$ *with* $\wedge_3$ *and* $\to_0$*, namely* $\mathbf{L3B^D_{3,0}}$*, coincides with the logic* **MH** *introduced in [3], where a Hilbert system for it was presented. A new axiomatization for this logic, as well as for every logic in Definition 3.4, will be presented in Section 6.*

It is usual to compare new logics with some important well-known systems in order to have a better understanding of them. Let us start by comparing the family $\mathbf{GP3^D}$ with **CPL**. To do this we need the following definition:

**Definition 3.6** *Let* $L_1$ *and* $L_2$ *be two propositional logics defined over the same signature such that* $L_1$ *is a proper sublogic of* $L_2$*, i.e. such that* $\vdash_{L_1} \subsetneq \vdash_{L_2}$*, where* $\vdash_{L_i}$ *denotes the consequence relation of* $L_i$ *(for* $i = 1, 2$*). Then,* $L_1$ *is said to be maximal w.r.t.* $L_2$ *if, for every formula* $\varphi$ *such that* $\vdash_{L_2} \varphi$ *but* $\nvdash_{L_1} \varphi$*, the logic* $L_1^+$ *obtained from* $L_1$ *by adding* $\varphi$ *as a theorem coincides with* $L_2$*.*

According to this definition we can see that any of the logics in the family $\mathbf{GP3^D}$ is a proper sublogic of **CPL**, by virtue of Proposition 2.11 and by the fact that they do not validate (**EM**). Moreover, the following result holds:

**Proposition 3.7** *Let* $L \in \mathbf{GP3^D}$*, then* $L$ *is maximal with respect to* **CPL**.

**Proof:** In Corollary 2.6 of [9] it is shown that if $\mathbf{L_1}$ and $\mathbf{L_2}$ are two matrix logics defined over the same signature $\Theta$ such that:

1. $\mathbf{L_1}$ is the logic defined by the matrix $\mathcal{M}_1 = \langle \{0, \frac{1}{2}, 1\}, F, D^* \rangle$ where $0 \notin D^*$ and $1 \in D^*$.

2. $\mathbf{L_2}$ is the logic defined by the matrix $\mathcal{M}_2$ which is the restriction of $\mathcal{M}_1$ to the domain $\{0,1\}$ (which constitutes a subalgebra over $\Theta$ of $\langle\{0,\frac{1}{2},1\},F\rangle$), and $\mathbf{L_2}$ is a presentation of classical propositional logic $\mathbf{CPL}$ over the signature $\Theta$.

3. $\mathbf{L_1} \neq \mathbf{L_2}$.

4. There are formulas $\top(p)$ and $\bot(p)$ on one variable $p$ such that $v(\top(p)) = 1$ and $v(\bot(p)) = 0$, for every valuation $v$ for $\mathbf{L_1}$.

Then, $\mathbf{L_1}$ is maximal with respect to $\mathbf{CPL}$ (presented as $\mathbf{L_2}$).

First of all we have that, given $L \in \mathbf{GP3^D}$, all the connectives in $L$ are enlargements of the classical ones, hence hypothesis 1 and 2 are satisfied. Clearly for any $L \in \mathbf{GP3^D}$ we have that $L \neq \mathbf{CPL}$ since $\varphi \vee \neg\varphi$ fails to be a valid schema; and finally, it is possible to define the logical constants $\bot(p)$ and $\top(p)$ for each one of them as pointed out in Remark 3.2. ∎

In addition, it is possible to define a determinedness operator $\star$ for each logic $L \in \mathbf{GP3^D}$ in order to recover the law of excluded middle in a controlled way, as it was proposed in [6]. Indeed, this operator can be defined as $\star\,\alpha \stackrel{\text{def}}{=} \alpha \vee \neg\alpha$. Even more, the following Derivability Adjustment Theorem can be proved:

**Proposition 3.8** *Let $L \in \mathbf{GP3^D}$, $\Gamma$ be a set of formulas, and $p_1,\ldots,p_n$ be the set of atomic formulas occurring in $\Gamma \cup \{\varphi\}$. Then, it holds that:*

$$\Gamma \vdash_{\mathbf{CPL}} \varphi \quad \textit{iff} \quad \Gamma, \star\, p_1, \ldots, \star\, p_n \vdash_L \varphi$$

**Proof:** From left to right: it is an immediate consequence of the fact that the connectives of $L$ are enlargements of the classical ones and the fact that, for any valuation $v$ for $L$, $v(\star\, p) = 1$ iff $v(p) \in \{0,1\}$, for any atomic formula $p$. Hence, assuming that $\Gamma \vdash_{\mathbf{CPL}} \varphi$, let $v$ be a valuation for $L$ such that $v(\star\, p_i) = 1$ for any $i = 1,\ldots,n$, and $v(\gamma) = 1$ for every $\gamma \in \Gamma$. Then, $v$ can be seen as a classical valuation over $\Gamma \cup \{\varphi\}$ such that $v(\gamma) = 1$ for every $\gamma \in \Gamma$. By hypothesis, $v(\varphi) = 1$ as well.

From right to left: assume that $\Gamma, \star\, p_1, \ldots, \star\, p_n \vdash_L \varphi$, and let $v$ be a classical valuation such that $v(\gamma) = 1$ for every $\gamma \in \Gamma$. Then, $v$ is a valuation for $L$ such that $v(\star\, p_i) = 1$ for any $i = 1,\ldots,n$. Hence, by hypothesis, $v(\varphi) = 1$. ∎

Intuitively, the last result states that every classical derivation can be recovered in $L$ by 'adjusting' the premises by requiring that every atomic formula involved in the reasoning in $L$ must have a 'classical' behavior.

# 4 Twist structures semantics for the logic $\mathbf{L3A^D_{3,0}}$

In this section we will show how to extend the given three-valued semantics of the logics defined in the previous section to a class of algebras known as *twist* *strutures*. To fix ideas, we first analyze in detail the case of the logic $\mathbf{L3A^{D}_{3,0}}$ (recall Definition 3.4). The general case will be analyzed in Section 6.

Twist structures were independently introduced by M. Fidel [11] and D. Vakarelov [22] in order to semantically characterize Nelson's logic **N4** (recall Example 1.2(3)). The basic idea of twist structures is considering ordered pairs $(a, b)$ (called *snapshots*, following the terminology for swap structures introduced in [4, Chapter 6]) of elements over a certain class of lattices which represent truth-values for formulas $\varphi$ and $\neg\varphi$, respectively. More information about twist structures semantics for **N4** can be found in [19]. As observed in [7], Fidel and Vakalerov's ideas were anticipated by a very general construction over distributive lattices due to Kalman (see [15]). Similar ideas can be found in J. M. Dunn's PhD thesis [10], where he found a representation of De Morgan lattices by means of pairs of sets called *proposition surrogates*. The operations proposed by Dunn coincide with the ones proposed by Kalman, and by Fidel and Vakarelov, namely:

$$\tilde{\neg}\,(a, b) = (b, a);$$

$$(a, b)\,\tilde{\wedge}\,(c, d) = (a \wedge c, b \vee d);$$

$$(a, b)\,\tilde{\vee}\,(c, d) = (a \vee c, b \wedge d).$$

Observe that the negation $\tilde{\neg}$ is involutive: if the negation of $a$ is $b$, the negation of the negation of $b$ must be precisely $a$. On the other hand, conjunction $\tilde{\wedge}$ and disjunction $\tilde{\vee}$ are defined in order to satisfy the De Morgan laws. It is interesting to observe that, starting with a distributive lattice $A$, a new operator $\tilde{\neg}$ over $A \times A$ is obtained, giving origin to a De Morgan lattice, which is a structure richer than the original one. This duality between the first coordinate and the second coordinate of the pairs justifies the name of the structures: both coordinates are logically 'twisted'. On the other hand, the coordinates of the pairs obtained by means of the operations over pairs are obtained from the coordinates of the pairs by using the operations of the original lattice. Twist structures semantics have been afterwards generalized in the literature to several classes of logics, including modal logics.

In order to define twist structures semantics for $\mathbf{L3A^{D}_{3,0}}$, consider the signatures $\Sigma_+ = \{\wedge, \vee, \to\}$ and $\Sigma = \{\wedge, \vee, \to, \neg, \bot\}$. From now on we will write $\alpha \leftrightarrow \beta$ as an abbreviation of the formula $(\alpha \to \beta) \wedge (\beta \to \alpha)$. The formula ${\sim}\alpha$ will be an abbreviation for $\alpha \to \bot$, expressing a classical negation definable in $\mathbf{L3A^{D}_{3,0}}$. We recall below the truth-tables for $\mathbf{L3A^{D}_{3,0}}$, and describe the truth-table of the derived connective $\sim$.

| | $\neg$ |
|---|---|
| $0$ | $1$ |
| $\frac{1}{2}$ | $0$ |
| $1$ | $0$ |

| $\vee$ | $0$ | $\frac{1}{2}$ | $1$ |
|---|---|---|---|
| $0$ | $0$ | $0$ | $1$ |
| $\frac{1}{2}$ | $0$ | $\frac{1}{2}$ | $1$ |
| $1$ | $1$ | $1$ | $1$ |

| $\wedge$ | $0$ | $\frac{1}{2}$ | $1$ |
|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ |
| $\frac{1}{2}$ | $0$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $1$ | $0$ | $\frac{1}{2}$ | $1$ |

| $\to$ | $0$ | $\frac{1}{2}$ | $1$ |
|---|---|---|---|
| $0$ | $1$ | $1$ | $1$ |
| $\frac{1}{2}$ | $1$ | $1$ | $1$ |
| $1$ | $0$ | $0$ | $1$ |

| | $\sim$ |
|---|---|
| $0$ | $1$ |
| $\frac{1}{2}$ | $1$ |
| $1$ | $0$ |

Let $\Sigma_\perp = \{\wedge, \vee, \to, \perp\}$ be the $\neg$-less fragment of $\Sigma$. It is easy to prove that classical propositional logic **CPL**, presented in the signature $\Sigma_\perp$ (where ${\sim}\alpha \stackrel{\text{def}}{=} \alpha \to \perp$ denotes the negation), coincides with the $\Sigma_\perp$-fragment of $\mathbf{L3A^D_{3,0}}$.

**Proposition 4.1** *Let* **CPL** *presented over the signature* $\Sigma_\perp$ *by means of the usual 2-valued truth-tables, such that* $v(\perp) = 0$ *for every valuation* $v$ *for* **CPL**. *Then, the* $\Sigma_\perp$*-fragment of* $\mathbf{L3A^D_{3,0}}$ *coincides with* **CPL**, *that is: for every* $\Gamma \cup \{\varphi\} \subseteq For(\Sigma_\perp)$, $\Gamma \models_{\mathbf{CPL}} \varphi$ *iff* $\Gamma \models_{\mathbf{L3A^D_{3,0}}} \varphi$.

**Proof:** By Proposition 2.11, $\Gamma \models_{\mathbf{L3A^D_{3,0}}} \varphi$ implies that $\Gamma \models_{\mathbf{CPL}} \varphi$. Conversely, suppose that $\Gamma \models_{\mathbf{CPL}} \varphi$, and let $v$ be a valuation for $\mathbf{L3A^D_{3,0}}$ such that $v(\psi) = 1$ for every $\psi \in \Gamma$. Let $\bar{v}_0 : \mathcal{V} \to \{0,1\}$ be a function such that $\bar{v}_0(p) = {\sim\sim}v(p)$, for every $p \in \mathcal{V}$. Let $\bar{v}$ be the unique homomorphic extension of $\bar{v}_0$ to $For(\Sigma_\perp)$ by using the operators of $\mathbf{L3A^D_{3,0}}$. Since $\mathbf{L3A^D_{3,0}}$ is an enlargement of **CPL** over $\Sigma_\perp$, it follows that the image of $\bar{v}$ is contained in $\{0,1\}$, and so $\bar{v} : For(\Sigma_\perp) \to \{0,1\}$ is a valuation for **CPL**. Moreover, by induction on the complexity of the formula $\psi$ it is easy to prove that $v(\psi) = 1$ iff $\bar{v}(\psi) = 1$. From this, $\bar{v}(\psi) = 1$ for every $\psi \in \Gamma$. By hypothesis, $\bar{v}(\varphi) = 1$ and so $v(\psi) = 1$. This shows that $\Gamma \models_{\mathbf{L3A^D_{3,0}}} \varphi$. ■

The last proposition shows that $\mathbf{L3A^D_{3,0}}$, which is defined over $\Sigma$, is a conservative expansion of **CPL** (defined over $\Sigma_\perp$) by adding a (paracomplete) negation $\neg$. Because of this, in order to define twist structures for $\mathbf{L3A^D_{3,0}}$, the snapshots $(a, b)$ can be defined over an arbitrary Boolean algebra instead of using just a distributive lattice. As in the case of the original Kalman construction, the only condition required to the pairs $(a, b)$ is that $a \wedge b = 0$, by virtue of the validity of the explosion law in $\mathbf{L3A^D_{3,0}}$ w.r.t. $\neg$ (and recalling that $a$ and $b$ represent the truth values assigned to the formulas $\varphi$ and $\neg\varphi$, respectively). The designated elements should be the snapshots $(a, b)$ such that $a = 1$. Since the first coordinate of a snapshot represents the truth-value of a given formula, it makes sense to define any binary operation $\# \in \{\wedge, \vee, \to\}$ over snapshots by computing the corresponding value of $\#$ between the first coordinates of the two given snapshots over the underlying Boolean algebra, and declare this result as being the first component of the resulting snapshot. This is entirely analogous to Kalman and Fidel-Vakarelov construction. The second component of the resulting snapshot should represent the value of the negation of such operation. Namely, if $\tilde{\#}$ represents the operation $\#$ over pairs then

$$(a, b)\tilde{\#}(c, d) = (a\#c, F_\#(a, b, c, d))$$

where $F_\#(a, b, c, d)$ is a Boolean combination of $a, b, c, d$ to be determined. To be more precise, $F_\#(a, b, c, d)$ is a term in the signature $\Sigma_\perp$ for Boolean algebras depending on the variables $a, b, c, d$, which represents, in an analytical way, the expression of the second coordinate of the snapshot obtained by applying the twist operator $\tilde{\#}$ to the generic snapshots $(a,b)$ and $(c,d)$. In order to obtain such operations from the given three-valued truth-tables of $\mathbf{L3A^D_{3,0}}$ described above, it makes sense to consider the two-element Boolean algebra $\mathbb{A}_2$ with domain $A_2 = \{0,1\}$. This is justified by the fact that the variety of Boolean algebras is generated by $\mathbb{A}_2$, which implies that an equation is valid in *every* Boolean algebra iff it is valid in $\mathbb{A}_2$. Now, the only possible snapshots over $\mathbb{A}_2$ are $\{\top, f, \bot\}$ where $\top = (1,0)$, $f = (0,0)$ and $\bot = (0,1)$, with $\top$ being the unique designated element. From now on we will identify $\top$, $f$ and $\bot$ with 1, $\frac{1}{2}$ and 0, respectively. Consider the case $\# = \vee$. With the identification between truth-values we made, the following equations should be satisfied, from the truth-table of $\vee$:

$F_\vee(0,1,0,1) = F_\vee(0,1,0,0) = F_\vee(0,0,0,1) = 1$,
since $0 \vee 0 = 0 \vee \frac{1}{2} = \frac{1}{2} \vee 0 = 0 \simeq (0,1)$;

$F_\vee(0,1,1,0) = F_\vee(0,0,1,0) = F_\vee(1,0,c,d) = 0$,
since $0 \vee 1 = \frac{1}{2} \vee 1 = 1 \vee z = 1 \simeq (1,0)$;

$F_\vee(0,0,0,0) = 0$,
since $\frac{1}{2} \vee \frac{1}{2} = \frac{1}{2} \simeq (0,0)$.

By using Karnaugh maps [16] it is easy to see that the solution of such equations can be expressed by the truth-function $F_\vee(a,b,c,d) = ({\sim}a \wedge d) \vee (b \wedge {\sim}c)$ over $\{0,1\}$. The same technique can be used for determining the second coordinate of the operators $\tilde{\wedge}$ and $\tilde{\to}$:

$F_\wedge(a,b,c,d) = b \vee d$;

$F_\to(a,b,c,d) = a \wedge {\sim}c$.

Concerning the negation $\tilde{\neg}(a,b) = (c,d)$, it is clear that $c$ should coincide with $b$ (since $b$ represents the negation of $a$), while $d$ should represent the negation of $b$, that is, the double negation of $a$ (note that not necessarily $d = a$, since $\neg$ is not involutive in $\mathbf{L3A^D_{3,0}}$). By observing the table for $\neg$ we found the following:

$F_\neg(0,1) = 0$,
since $\neg 0 = 1 \simeq (1,0)$;

$F_\neg(0,0) = F_\neg(1,0) = 1$,
since $\neg\frac{1}{2} = \neg 1 = 0 \simeq (0,1)$.

From this, it is easy to see that the solution of these equations can be expressed by the truth-function $F_\neg(a,b) = {\sim}b$ over $\{0,1\}$. This lead us to the following class of twist structures, when moving from $\mathbb{A}_2$ to any non-trivial Boolean algebra $\mathcal{A}$:

**Definition 4.2** *Let $\mathcal{A}$ be a non-trivial Boolean algebra (that is, $0 \neq 1$) with domain $A$, and let $B_A = \{(a,b) \in A \times A \ : \ a \wedge b = 0\}$. The* twist structure for $\mathbf{L3A^D_{3,0}}$ generated by $\mathcal{A}$ *is the algebra $T_{\mathcal{A}} \stackrel{def}{=} \langle B_A, \tilde{\wedge}, \tilde{\vee}, \tilde{\to}, \tilde{\neg}, \tilde{\bot}\rangle$ over $\Sigma$ such that*

*(i)* $\tilde{\bot} = (0,1)$;

*(ii)* $\tilde{\neg}\,(a,b) = (b, {\sim}b)$;

*(iii)* $(a,b)\,\tilde{\wedge}\,(c,d) = (a \wedge c, b \vee d)$;

*(iv)* $(a,b)\,\tilde{\vee}\,(c,d) = (a \vee c, ({\sim}a \wedge d) \vee (b \wedge {\sim}c))$;

*(v)* $(a,b)\,\tilde{\to}\,(c,d) = (a \to c, a \wedge {\sim}c)$.

**Definition 4.3** *Given $(a,b) \in B_A$ let $\bot_{(a,b)} \stackrel{def}{=} (a,b)\,\tilde{\wedge}\,\tilde{\neg}\,(a,b)$. Observe that $\bot_{(a,b)} = \tilde{\bot}$ for every $(a,b) \in B_A$. Let $\tilde{\top} \stackrel{def}{=} \tilde{\neg}\,\tilde{\bot}$; then $\tilde{\top} = (1,0)$. Let $\tilde{\sim}\,(a,b) \stackrel{def}{=} (a,b)\,\tilde{\to}\,\tilde{\bot}$. Then $\tilde{\sim}\,(a,b) = ({\sim}a, a)$ for every $(a,b) \in B_A$. Finally, let $(a,b)\,\tilde{\leftrightarrow}\,(c,d) \stackrel{def}{=} \big((a,b)\,\tilde{\to}\,(c,d)\big)\,\tilde{\wedge}\,\big((c,d)\,\tilde{\to}\,(a,b)\big)$. Then $(a,b)\,\tilde{\leftrightarrow}\,(c,d) = (a \leftrightarrow c, {\sim}(a \leftrightarrow c))$ where, for any $a, c \in \mathcal{A}$, $a \leftrightarrow c \stackrel{def}{=} (a \to c) \wedge (c \to a)$.*

Observe that, for any $(a,b) \in B_A$, $a = 1$ implies that $b = 0$. Thus, $\{(a,b) \in B_A \ : \ a = 1\} = \{\tilde{\top}\}$.

**Definition 4.4** *The* logical matrix generated by $T_{\mathcal{A}}$ *is $\mathcal{M}_{\mathcal{A}} \stackrel{def}{=} \langle T_{\mathcal{A}}, \{\tilde{\top}\}\rangle$.*

**Remark 4.5** *Consider once again the two-element Boolean algebra $\mathbb{A}_2$ with domain $A_2 = \{0,1\}$. As observed above, $T_{\mathbb{A}_2}$ has domain $B_{A_2} = \{\top, f, \bot\}$ where $\top = (1,0)$, $f = (0,0)$ and $\bot = (0,1)$, and $\top$ is the unique designated element. It is immediate to see that the operations on the 3-valued algebra $T_{\mathbb{A}_2}$ coincide with the ones defined for the 3-valued characteristic matrix for $\mathbf{L3A^D_{3,0}}$ by identifying $\bot$, $f$ and $\top$ with $0$, $\frac{1}{2}$ and $1$, respectively. Thus, up to names, $\mathcal{M}_{\mathbb{A}_2}$ is the 3-valued matrix which defines $\mathbf{L3A^D_{3,0}}$. Hence, $\Gamma \models_{\mathcal{M}_{\mathbb{A}_2}} \varphi$ iff $\Gamma \models_{\mathbf{L3A^D_{3,0}}} \varphi$.*

**Definition 4.6** *Let $\Gamma \cup \{\varphi\}$ be a set of formulas in $For(\Sigma)$. Then $\varphi$ is said to be a* consequence of $\Gamma$ w.r.t. twist-structures semantics for $\mathbf{L3A^D_{3,0}}$, *denoted by $\Gamma \models^{T}_{\mathbf{L3A^D_{3,0}}} \varphi$, if $\Gamma \models_{\mathcal{M}_{\mathcal{A}}} \varphi$ for every matrix $\mathcal{M}_{\mathcal{A}}$ and every Boolean algebra $\mathcal{A}$.*

**Remark 4.7** *By Definitions 4.3 and 4.4 we have the following, for every valuation $v$ over $\mathcal{M}_{\mathcal{A}}$: $v(\varphi \wedge \neg\varphi) = \tilde{\bot}$; $v(\varphi \to \varphi) = \tilde{\top}$; $v({\sim}\varphi) = \tilde{\sim}\,v(\varphi)$. Finally, suppose that $v(\varphi) = (a,b)$ and $v(\psi) = (c,d)$; then, $v(\varphi \leftrightarrow \psi) = \tilde{\top}$ iff $a = c$.*

# 5 The Hilbert calculus $\mathbf{L3A^D_{3,0h}}$ for $\mathbf{L3A^D_{3,0}}$

Now, inspired by the definition of twist structures for $\mathbf{L3A^D_{3,0}}$ given in Definition 4.2, a Hilbert calculus adequate to such semantics will be obtained in a natural way. Recall that, for $\# \in \{\wedge, \vee, \rightarrow\}$, $(a,b)\tilde{\#}(c,d) = (a\#c, F_{\#}(a,b,c,d))$ and $\tilde{\neg}(a,b) = (b, F_{\neg}(a,b))$. Here, $F_{\#}(a,b,c,d)$ and $F_{\neg}(a,b)$ are terms in the signature $\Sigma_{\bot}$ for Boolean algebras. This means that they can be also interpreted as formulas in $For(\Sigma_{\bot})$ depending on the propositional variables $a,b,c,d$. On the other hand, the intended meaning of a snapshot $(a,b)$ is that it represents a pair of formulas of the form $(\alpha, \neg\alpha)$, that is, the first coordinate represents $\alpha$ while the second one represents $\neg\alpha$. Thus, at the logical level, the following formulas should be valid, for $\# \in \{\wedge, \vee, \rightarrow\}$:

$$\neg(\alpha\#\beta) \leftrightarrow F_{\#}(\alpha, \neg\alpha, \beta, \neg\beta)$$

$$\neg\neg\alpha \leftrightarrow F_{\neg}(\alpha, \neg\alpha).$$

For instance, assuming that $(a,b)$ represents the pair of formulas $(\alpha, \neg\alpha)$, the formula $\neg\neg\alpha \leftrightarrow {\sim}\neg\alpha$ describes in the logic that the second coordinate of $\tilde{\neg}\,(a,b)$ (corresponding to $\neg\neg\alpha$) is ${\sim}b$, according to Definition 4.2, where ${\sim}b$ represents ${\sim}\neg\alpha$. On the other hand, assuming that $(a,b)$ and $(c,d)$ represent, respectively, the pairs of formulas $(\alpha, \neg\alpha)$ and $(\beta, \neg\beta)$, the formula $\neg(\alpha \rightarrow \beta) \leftrightarrow (\alpha \wedge {\sim}\beta)$ describes the fact that the second coordinate of $(a,b) \mathrel{\tilde{\rightarrow}} (c,d)$ (corresponding to the formula $\neg(\alpha \rightarrow \beta)$) is $a \wedge {\sim}c$ (corresponding the formula $(\alpha \wedge {\sim}\beta)$).

Taking this into consideration, and by recalling that the logic $\mathbf{L3A^{D}_{3,0}}$ contains classical logic, as stated in Proposition 4.1, we arrive to the Hilbert calculus for $\mathbf{L3A^{D}_{3,0}}$ to be described in Definition 5.2 below. We start by recalling the standard Hilbert calculus for classical positive logic.

**Definition 5.1 (Classical Positive Logic)** *The* classical positive logic $\mathbf{CPL}^{+}$ *is defined over the language $For(\Sigma_{+})$ by the following Hilbert calculus:*

**Axiom schemas:**

$$\alpha \rightarrow \big(\beta \rightarrow \alpha\big) \tag{\textbf{Ax1}}$$

$$\Big(\alpha \rightarrow \big(\beta \rightarrow \gamma\big)\Big) \rightarrow \Big(\big(\alpha \rightarrow \beta\big) \rightarrow \big(\alpha \rightarrow \gamma\big)\Big) \tag{\textbf{Ax2}}$$

$$\alpha \rightarrow \Big(\beta \rightarrow \big(\alpha \wedge \beta\big)\Big) \tag{\textbf{Ax3}}$$

$$\big(\alpha \wedge \beta\big) \rightarrow \alpha \tag{\textbf{Ax4}}$$

$$\big(\alpha \wedge \beta\big) \rightarrow \beta \tag{\textbf{Ax5}}$$

$$\alpha \rightarrow \big(\alpha \vee \beta\big) \tag{\textbf{Ax6}}$$

$$\beta \rightarrow \big(\alpha \vee \beta\big) \tag{\textbf{Ax7}}$$

$$\Big(\alpha \rightarrow \gamma\Big) \rightarrow \Big((\beta \rightarrow \gamma) \rightarrow \big((\alpha \vee \beta) \rightarrow \gamma\big)\Big) \tag{\textbf{Ax8}}$$

$$\big(\alpha \rightarrow \beta\big) \vee \alpha \tag{\textbf{Ax9}}$$

**Inference rule:**

$$\frac{\alpha \quad\quad \alpha \rightarrow \beta}{\beta} \tag{\textbf{MP}}$$

**Definition 5.2** *The logic* $\mathbf{L3A^D_{3,0h}}$*, defined over signature* $\Sigma$*, is given by the Hilbert calculus obtained from* $\mathbf{CPL}^+$ *by adding the following axiom schemas:*

$$\bot \to \alpha \qquad (\mathbf{bot})$$

$$\neg\bot \qquad (\mathbf{top})$$

$$\alpha \to \big(\neg\alpha \to \beta\big) \qquad (\mathbf{exp})$$

$$\neg\neg\alpha \leftrightarrow {\sim}\neg\alpha \qquad (\mathbf{NN})$$

$$\neg\big(\alpha \wedge \beta\big) \leftrightarrow \big(\neg\alpha \vee \neg\beta\big) \qquad (\mathbf{NC})$$

$$\neg\big(\alpha \vee \beta\big) \leftrightarrow \big(({\sim}\alpha \wedge \neg\beta) \vee (\neg\alpha \wedge {\sim}\beta)\big) \qquad (\mathbf{ND})$$

$$\neg\big(\alpha \to \beta\big) \leftrightarrow \big(\alpha \wedge {\sim}\beta\big) \qquad (\mathbf{NI})$$

Now, soundness and completeness of $\mathbf{L3A^D_{3,0h}}$ with respect to the corresponding twist structures semantics $\models^T_{\mathbf{L3A^D_{3,0}}}$ will be proved. From this, soundness and completeness of $\mathbf{L3A^D_{3,0h}}$ w.r.t. the 3-valued matrix logic $\mathbf{L3A^D_{3,0}}$ will be obtained by using general results on Boolean algebras, as we shall see in Theorem 5.9 below. Firstly, some previous definitions and results are needed.

**Definition 5.3** *Let* $\mathbf{L}$ *be a logic. A set of formulas* $\Gamma$ *is* closed in $\mathbf{L}$ *if, for every formula* $\psi$*:* $\Gamma \vdash \psi$ *iff* $\psi \in \Gamma$*.*

**Definition 5.4** *Let* $\mathbf{L}$ *be a logic, and let* $\Gamma \cup \{\varphi\}$ *be a set of formulas. The set* $\Gamma$ *is* maximal non-trivial w.r.t. $\varphi$ *in* $\mathbf{L}$*, or* $\varphi$-saturated *in* $\mathbf{L}$*, if* $\Gamma \nvdash \varphi$ *but* $\Gamma, \psi \vdash \varphi$ *for any* $\psi \notin \Gamma$*.*

It is easy to prove that any $\varphi$-saturated set of formulas in a logic is closed. Recall now the following classical result (see [23, Theorem 22.2]):

**Theorem 5.5 (Lindenbaum-Łos)** *Let* $\mathbf{L}$ *be a finitary logic, and let* $\Gamma \cup \{\varphi\}$ *be a set of formulas such that* $\Gamma \nvdash \varphi$*. Then, there exists a set of formulas* $\Delta$ *such that* $\Delta$ *is* $\varphi$*-saturated in* $\mathbf{L}$ *and* $\Gamma \subseteq \Delta$*.*

**Remark 5.6** *Clearly* $\mathbf{L3A^D_{3,0h}}$ *is a finitary logic, then Theorem 5.5 applies to it.*

**Theorem 5.7 (Soundness and completeness of $\mathbf{L3A^D_{3,0h}}$ w.r.t. twist structures)** *Let* $\Gamma \cup \{\varphi\}$ *be a set of formulas in* $For(\Sigma)$*. Then,* $\Gamma \vdash_{\mathbf{L3A^D_{3,0h}}} \varphi$ *iff* $\Gamma \models^T_{\mathbf{L3A^D_{3,0}}} \varphi$*.*

**Proof:** 'Only if' part (Soundness): It is easy to see that every axiom in $\mathbf{L3A^D_{3,0h}}$ is valid in any twist structure, that is: for every $\mathcal{M}_{\mathcal{A}}$ and for every valuation $v$ over $\mathcal{M}_{\mathcal{A}}$, $v(\varphi) = \tilde{\top}$ for every instance $\varphi$ of every axiom of $\mathbf{L3A^D_{3,0h}}$. Indeed, the axioms of $\mathbf{CPL}^+$ are obviously valid in every $\mathcal{M}_{\mathcal{A}}$, by Definitions 4.2 and 2.7. Axioms (**bot**) and (**top**) are clearly valid in every $\mathcal{M}_{\mathcal{A}}$. For (**exp**), let $v$ be a valuation over $\mathcal{M}_{\mathcal{A}}$ and let $v(\alpha) = (a, b)$ and $v(\beta) = (c, d)$. Then $v(\alpha \to (\neg\alpha \to \beta)) = (a, b) \tilde{\to} (\tilde{\neg}\, (a, b) \tilde{\to} (c, d)) = (a, b) \tilde{\to} ((b, {\sim}b) \tilde{\to} (c, d)) = (a, b) \tilde{\to} (b \to c, b \wedge {\sim}c) = (a \to (b \to c), a \wedge {\sim}(b \to c)) = ((a \wedge b) \to$

$c, (a \wedge b) \wedge {\sim}c) = (1, 0) = \tilde{\top}$ since $a \wedge b = 0$. The validity of the other axioms follows easily by Definition 4.2 and the last observation in Remark 4.7. In addition, satisfaction is preserved by (**MP**). Indeed, suppose that $v$ is a valuation over $\mathcal{M}_{\mathcal{A}}$ such that $v(\varphi) = v(\varphi \to \psi) = \tilde{\top}$. Let $v(\psi) = (a, b)$. Then $\tilde{\top} = v(\varphi) \tilde{\to} v(\psi) = \tilde{\top} \tilde{\to} (a, b) = (1 \to a, 1 \wedge {\sim}a) = (a, {\sim}a)$. Hence $a = 1$ and so $b = 0$. That is, $v(\psi) = \tilde{\top}$. Using this, the result follows by induction on the length of derivations.

'If' part (Completeness): Suppose that $\Gamma \nvdash_{\mathbf{L3A^D_{3,0h}}} \varphi$. By Theorem 5.5 and Remark 5.6 there exists a set $\Delta$ which is $\varphi$-saturated in $\mathbf{L3A^D_{3,0h}}$ such that $\Gamma \subseteq \Delta$. Define the following relation in $For(\Sigma)$: $\beta \equiv_\Delta \gamma$ iff $\Delta \vdash_{\mathbf{L3A^D_{3,0h}}} \beta \leftrightarrow \gamma$. By the properties of $\mathbf{CPL}^+$ it is easy to prove that $\equiv_\Delta$ is a congruence over $For(\Sigma)$ with respect to the connectives $\wedge$, $\vee$ and $\to$. Moreover, $\mathcal{A}_\Delta \stackrel{\text{def}}{=} For(\Sigma)/_{\equiv_\Delta}$ is a Boolean algebra such that $1 = [\beta \to \beta]_\Delta$ and $0 = [\bot]_\Delta$ for any formula $\beta$, where $[\gamma]_\Delta$ denotes the equivalence class of the formula $\gamma$ w.r.t. $\equiv_\Delta$. From this $[{\sim}\beta]_\Delta = {\sim}[\beta]_\Delta$ in $\mathcal{A}_\Delta$. Consider the twist structure $T_{\mathcal{A}_\Delta}$ generated by $\mathcal{A}_\Delta$ as in Definition 4.2 and let $\mathcal{M}_{\mathcal{A}_\Delta}$ be the corresponding logical matrix. It is worth noting that $[\gamma]_\Delta = 1$ iff $\Delta \vdash_{\mathbf{L3A^D_{3,0h}}} \gamma$ iff $\gamma \in \Delta$. Observe that $[\beta]_\Delta \wedge [\neg\beta]_\Delta = [\beta \wedge \neg\beta]_\Delta = 0$ for every $\beta$, by axiom (**exp**). Then, $([\beta]_\Delta, [\neg\beta]_\Delta) \in B_{A_\Delta}$ for every $\beta$, where $B_{A_\Delta}$ is the domain of $T_{\mathcal{A}_\Delta}$. Therefore, there is a function $v_\Delta : For(\Sigma) \to B_{\mathcal{A}_\Delta}$ given by $v_\Delta(\gamma) = ([\gamma]_\Delta, [\neg\gamma]_\Delta)$. Moreover, $v_\Delta(\gamma) = (1, 0) = \tilde{\top}$ iff $[\gamma]_\Delta = 1$ iff $\gamma \in \Delta$. It is easy to see that $v_\Delta$ is a valuation over $\mathcal{M}_{\mathcal{A}_\Delta}$. Indeed, it is clear that $v_\Delta(\bot) = (0, 1) = \tilde{\bot}$, by axioms (**bot**) and (**top**). On the other hand, $v_\Delta(\neg\gamma) = ([\neg\gamma]_\Delta, [\neg\neg\gamma]_\Delta) = ([\neg\gamma]_\Delta, [{\sim}\neg\gamma]_\Delta)$, by axiom (**NN**). But $[{\sim}\neg\gamma]_\Delta = {\sim}[\neg\gamma]_\Delta$. Hence $v_\Delta(\neg\gamma) = \tilde{\neg}\, v_\Delta(\gamma)$. Let $\# \in \{\wedge, \vee, \to\}$. Then $v_\Delta(\beta\#\gamma) = v_\Delta(\beta) \,\tilde{\#}\, v_\Delta(\gamma)$, by axioms (**NC**), (**ND**) and (**NI**) and the definition of the operations in the Boolean algebra $\mathcal{A}_\Delta$. Therefore, $v_\Delta$ is a valuation over $\mathcal{M}_{\mathcal{A}_\Delta}$ such that $v_\Delta(\gamma) = \tilde{\top}$ for every $\gamma \in \Gamma$ but $v_\Delta(\varphi) \neq \tilde{\top}$, since $\varphi \notin \Delta$. This shows that $\Gamma \nvDash^T_{\mathbf{L3A^D_{3,0}}} \varphi$. ■

From Remark 4.5 and from soundness and completeness of $\mathbf{L3A^D_{3,0h}}$ w.r.t. twist structures semantics, it can be obtained soundness and completeness of $\mathbf{L3A^D_{3,0h}}$ w.r.t. the 3-valued matrix logic $\mathbf{L3A^D_{3,0}}$. In order to prove this, some well-known results from Boolean algebras will be recalled (see, for instance, [12]):

**Proposition 5.8** *Let $\mathcal{A}$ be a Boolean algebra with domain $A$.*

*(1) If $a$ is an element of $A$ different from 1, then there exists an ultrafilter $F$ over $\mathcal{A}$ such that $a \notin F$.*

*(2) If $F$ is an ultrafilter over $\mathcal{A}$, then the characteristic funcion $h_F : A \to \{0, 1\}$ of $F$ (that is, $h_F(x) = 1$ iff $x \in F$) is a homomorphism of Boolean algebras between $\mathcal{A}$ and the two-element Boolean algebra $\mathbb{A}_2$.*

**Theorem 5.9 (Soundness and completeness of $\mathbf{L3A^D_{3,0h}}$ w.r.t. $\mathbf{L3A^D_{3,0}}$)**
*Let $\Gamma \cup \{\varphi\}$ be a set of formulas in $For(\Sigma)$. Then, $\Gamma \vdash_{\mathbf{L3A^D_{3,0h}}} \varphi$ iff $\Gamma \models_{\mathbf{L3A^D_{3,0}}} \varphi$.*

**Proof:** 'Only if' part (Soundness): It follows from Theorem 5.7 (Soundness) and from the last observation in Remark 4.5.

'If' part (Completeness): Suppose that $\Gamma \nvdash_{\mathbf{L3A^D_{3,0h}}} \varphi$. By completeness of $\mathbf{L3A^D_{3,0h}}$ w.r.t. twist structures semantics (Theorem 5.7), $\Gamma \not\models^T_{\mathbf{L3A^D_{3,0}}} \varphi$. Then, there exists a Boolean algebra $\mathcal{A}$ and a valuation $v$ over $\mathcal{M}_{\mathcal{A}}$ such that $v(\gamma) = \tilde{\top}$ for every $\gamma \in \Gamma$, but $v(\varphi) \neq \tilde{\top}$. Consider the following notation: for every formula $\alpha$ we will write $v(\alpha) = (v(\alpha)_1, v(\alpha)_2)$. Then, $v(\gamma)_1 = 1$ for every $\gamma \in \Gamma$, but $v(\varphi)_1 \neq 1$. By Proposition 5.8(1), there is an ultrafilter $F$ over $\mathcal{A}$ such that $v(\varphi)_1 \notin F$. Let $h_F : A \to \{0, 1\}$ be the characteristic funcion of $F$. By Proposition 5.8(2), $h_F$ is a homomorphism of Boolean algebras between $\mathcal{A}$ and the two-element Boolean algebra $\mathbb{A}_2$. Let $\bar{v} : For(\Sigma) \to B_{A_2}$ given by $\bar{v}(\alpha) = (h_F(v(\alpha)_1), h_F(v(\alpha)_2))$. Since $h_F$ is a homomorphism of Boolean algebras, it follows that $\bar{v}(\alpha)$ indeed belongs to $B_{A_2}$ for every $\alpha$. Morever, $\bar{v}$ is a valuation over the matrix $\mathcal{M}_{\mathbb{A}_2}$. For instance, we can prove that $\bar{v}(\alpha \to \beta) = \bar{v}(\alpha) \tilde{\to} \bar{v}(\beta)$ as follows:

$$
\begin{aligned}
\bar{v}(\alpha \to \beta) &= (h_F(v(\alpha \to \beta)_1), h_F(v(\alpha \to \beta)_2)) \\
&= (h_F(v(\alpha)_1 \to v(\beta)_1), h_F(v(\alpha)_1 \wedge {\sim} v(\beta)_2)) \\
&= (h_F(v(\alpha)_1) \to h_F(v(\beta)_1), h_F(v(\alpha)_1) \wedge {\sim} h_F(v(\beta)_2)) \\
&= (h_F(v(\alpha)_1), h_F(v(\alpha)_2)) \tilde{\to} (h_F(v(\beta)_1), h_F(v(\beta)_2)) \\
&= \bar{v}(\alpha) \tilde{\to} \bar{v}(\beta).
\end{aligned}
$$

In addition, the valuation $\bar{v}$ is such that $\bar{v}(\gamma) = \top$ for every $\gamma \in \Gamma$ (since $v(\gamma)_1 = 1$, hence $h_F(v(\gamma)_1) = 1$), but $\bar{v}(\varphi) \neq \top$ (since $v(\varphi)_1 \notin F$, hence $h_F(v(\varphi)_1) = 0$). This shows that $\Gamma \not\models_{\mathcal{M}_{\mathbb{A}_2}} \varphi$. By Remark 4.5, $\Gamma \not\models_{\mathbf{L3A^D_{3,0}}} \varphi$. ■

# 6 Twist structures and Hilbert calculus for the family $\mathbf{GP3^D}$

As observed above, the logics in the family of three-valued genuine paracomplete logics $\mathbf{GP3^D}$ are pretty similar. In most of the cases they just change one entry of the truth table for some of their connectives. These small changes can be easily captured by the twist structures, allowing us to define them in a modular way. Hence, by following the steps shown in Sections 4 and 5, we can easily obtain twist stuctures and Hilbert systems for each of the logics in the family. Let us start by generalizing Definition 4.2.

**Definition 6.1** *Let $\mathcal{A}$ be a non-trivial Boolean algebra with domain $A$, and let $B_A = \{(a, b) \in A \times A \ : \ a \wedge b = 0\}$. For $X \in \{A, B, C, D\}$, $i \in \{0, 1, \ldots, 7\}$ and $j \in \{0, 1\}$, the* twist structure for $\mathbf{L3}X^{\mathbf{D}}_{\mathbf{i,j}}$ generated by $\mathcal{A}$ *is the algebra $T_{\mathcal{A}} \overset{def}{=} \langle B_A, \tilde{\wedge}, \tilde{\vee}, \tilde{\to}, \tilde{\neg}, \tilde{\bot} \rangle$ over $\Sigma$ such that $\tilde{\bot}$, $\tilde{\neg}$, $\tilde{\wedge}$, $\tilde{\vee}$ and $\tilde{\to}$ are defined as follows:*

| Connective | Operation on snapshots |
|---|---|
| $\neg_0$ | $\tilde{\neg}\,(a,b) = (b, {\sim}b)$ |
| $\neg_1$ | $\tilde{\neg}\,(a,b) = (b, a)$ |
| $\vee_0$ | $(a,b)\,\tilde{\vee}\,(c,d) = (a \vee c, ({\sim}a \wedge d) \vee (b \wedge {\sim}c))$ |
| $\vee_1$ | $(a,b)\,\tilde{\vee}\,(c,d) = (a \vee c, {\sim}((a \vee b) \vee (c \vee d)) \vee (b \wedge d))$ |
| $\vee_2$ | $(a,b)\,\tilde{\vee}\,(c,d) = (a \vee c, {\sim}a \wedge {\sim}c)$ |
| $\wedge_0$ | $(a,b)\,\tilde{\wedge}\,(c,d) = (a \wedge c, {\sim}a \vee {\sim}c)$ |
| $\wedge_1$ | $(a,b)\,\tilde{\wedge}\,(c,d) = (a \wedge c, ({\sim}a \wedge {\sim}c) \vee b \vee d)$ |
| $\wedge_2$ | $(a,b)\,\tilde{\wedge}\,(c,d) = (a \wedge c, (a \wedge {\sim}c) \vee ({\sim}a \wedge c) \vee b \vee d)$ |
| $\wedge_3$ | $(a,b)\,\tilde{\wedge}\,(c,d) = (a \wedge c, b \vee d)$ |
| $\wedge_4$ | $(a,b)\,\tilde{\wedge}\,(c,d) = (a \wedge c, (a \wedge {\sim}c) \vee ({\sim}a \wedge c) \vee (b \wedge d) \vee ({\sim}b \wedge {\sim}c \wedge {\sim}d))$ |
| $\wedge_5$ | $(a,b)\,\tilde{\wedge}\,(c,d) = (a \wedge c, (a \wedge d) \vee (b \wedge c) \vee (b \wedge d) \vee ({\sim}a \wedge {\sim}b \wedge {\sim}c \wedge {\sim}d))$ |
| $\wedge_6$ | $(a,b)\,\tilde{\wedge}\,(c,d) = (a \wedge c, (a \wedge {\sim}c) \vee ({\sim}a \wedge c) \vee (b \wedge d))$ |
| $\wedge_7$ | $(a,b)\,\tilde{\wedge}\,(c,d) = (a \wedge c, (a \wedge d) \vee (b \wedge c) \vee (b \wedge d))$ |
| $\to_0$ | $(a,b)\,\tilde{\to}\,(c,d) = (a \to c, a \wedge {\sim}c)$ |
| $\to_1$ | $(a,b)\,\tilde{\to}\,(c,d) = (a \to c, a \wedge d)$ |

Table 9: Twist operations

| Connective | Axiom | |
|---|---|---|
| $\neg_0$ | $\mathbf{NN}_0$ | $\neg\neg\alpha \leftrightarrow {\sim}\neg\alpha$ |
| $\neg_1$ | $\mathbf{NN}_1$ | $\neg\neg\alpha \leftrightarrow \alpha$ |
| $\vee_0$ | $\mathbf{ND}_0$ | $\neg\big(\alpha \vee \beta\big) \leftrightarrow \big(({\sim}\alpha \wedge \neg\beta) \vee (\neg\alpha \wedge {\sim}\beta)\big)$ |
| $\vee_1$ | $\mathbf{ND}_1$ | $\neg\big(\alpha \vee \beta\big) \leftrightarrow \big(((\alpha \vee \neg\alpha) \vee (\beta \vee \neg\beta)) \to (\neg\alpha \wedge \neg\beta)\big)$ |
| $\vee_2$ | $\mathbf{ND}_2$ | $\neg\big(\alpha \vee \beta\big) \leftrightarrow \big({\sim}\alpha \wedge {\sim}\beta\big)$ |
| $\wedge_0$ | $\mathbf{NC}_0$ | $\neg\big(\alpha \wedge \beta\big) \leftrightarrow ({\sim}\alpha \vee {\sim}\beta)$ |
| $\wedge_1$ | $\mathbf{NC}_1$ | $\neg\big(\alpha \wedge \beta\big) \leftrightarrow ({\sim}\alpha \wedge {\sim}\beta) \vee \neg\alpha \vee \neg\beta$ |
| $\wedge_2$ | $\mathbf{NC}_2$ | $\neg\big(\alpha \wedge \beta\big) \leftrightarrow (\alpha \wedge {\sim}\beta) \vee ({\sim}\alpha \wedge \beta) \vee \neg\alpha \vee \neg\beta$ |
| $\wedge_3$ | $\mathbf{NC}_3$ | $\neg\big(\alpha \wedge \beta\big) \leftrightarrow \big(\neg\alpha \vee \neg\beta\big)$ |
| $\wedge_4$ | $\mathbf{NC}_4$ | $\neg\big(\alpha \wedge \beta\big) \leftrightarrow (\alpha \wedge {\sim}\beta) \vee ({\sim}\alpha \wedge \beta) \vee (\neg\alpha \wedge \neg\beta) \vee ({\sim}\neg\alpha \wedge {\sim}\beta \wedge {\sim}\neg\beta)$ |
| $\wedge_5$ | $\mathbf{NC}_5$ | $\neg\big(\alpha \wedge \beta\big) \leftrightarrow (\alpha \wedge \neg\beta) \vee (\neg\alpha \wedge \beta) \vee (\neg\alpha \wedge \neg\beta) \vee ({\sim}\alpha \wedge {\sim}\neg\alpha \wedge {\sim}\beta \wedge {\sim}\neg\beta)$ |
| $\wedge_6$ | $\mathbf{NC}_6$ | $\neg\big(\alpha \wedge \beta\big) \leftrightarrow (\alpha \wedge {\sim}\beta) \vee ({\sim}\alpha \wedge \beta) \vee (\neg\alpha \wedge \neg\beta)$ |
| $\wedge_7$ | $\mathbf{NC}_7$ | $\neg\big(\alpha \wedge \beta\big) \leftrightarrow (\alpha \wedge \neg\beta) \vee (\neg\alpha \wedge \beta) \vee (\neg\alpha \wedge \neg\beta)$ |
| $\to_0$ | $\mathbf{NI}_0$ | $\neg\big(\alpha \to \beta\big) \leftrightarrow \big(\alpha \wedge {\sim}\beta\big)$ |
| $\to_1$ | $\mathbf{NI}_1$ | $\neg\big(\alpha \to \beta\big) \leftrightarrow \big(\alpha \wedge \neg\beta\big)$ |

Table 10: Axiom schemes

*(i)* $\tilde{\bot} = (0, 1)$;

*(ii)* $\tilde{\neg}\,(a, b)$ *corresponds with the operation for* $\neg_0$ *in Table 9 if* $X \in \{A, C\}$ *or corresponds with the operation for* $\neg_1$ *in Table 9 if* $X \in \{B, D\}$;

*(iii)* $(a, b)\,\tilde{\wedge}\,(c, d)$ *corresponds with the operation for* $\wedge_i$ *in Table 9.*

*(iv)* $(a, b)\,\tilde{\vee}\,(c, d)$ *corresponds with the operation for* $\vee_0$ *in Table 9 if* $X = A$, *corresponds with the operation for* $\vee_1$ *in Table 9 if* $X = B$, *or corresponds with the operation for* $\vee_2$ *in Table 9 if* $X \in \{C, D\}$;

*(v)* $(a, b)\,\tilde{\to}\,(c, d)$ *corresponds with the operation for* $\to_j$ *in Table 9.*

**Definition 6.2** *The logic* $\mathbf{L3}X^{\mathbf{D}}_{\mathbf{i,jh}}$, *defined over signature* $\Sigma$, *is given by the Hilbert calculus obtained from* $\mathbf{L3A}^{\mathbf{D}}_{\mathbf{3,0h}}$ *(recall Definition 5.2) by changing axiom schemas* ***(NN)***, ***(NC)***, ***(ND)*** *and* ***(NI)*** *respectively by the following axiom schemes from Table 10:*

- ***NN***$_0$ *if* $X \in \{A, C\}$ *or* ***NN***$_1$ *if* $X \in \{B, D\}$;
- ***NC***$_i$;
- ***ND***$_0$ *if* $X = A$, ***ND***$_1$ *if* $X = B$ *or* ***ND***$_2$ *if* $X \in \{C, D\}$;
- ***NI***$_j$.

For $\mathbf{L} \in \{\mathbf{L3}X^{\mathbf{D}}_{\mathbf{i,j}} \ : \ X \in \{A, B, C, D\},\ i \in \{0, 1, \ldots, 7\}$ *and* $j \in \{0, 1\}\}$, let $\mathbf{L_h}$ be the corresponding Hilbert calculus. The consequence relation w.r.t. $\mathbf{L_h}$, twist structures semantics and the original 3-valued matrices will be denoted by $\vdash_{\mathbf{L_h}}$, $\models^T_{\mathbf{L}}$ and $\models_{\mathbf{L}}$, respectively. The following results can be proven by a straightforward adaptation of the proofs for $\mathbf{L3A}^{\mathbf{D}}_{\mathbf{3,0}}$:

**Theorem 6.3 (Soundness and completeness of $\mathbf{L_h}$ w.r.t. twist structures)** *Let* $\Gamma \cup \{\varphi\}$ *be a set of formulas in* $For(\Sigma)$. *Then,* $\Gamma \vdash_{\mathbf{L_h}} \varphi$ *iff* $\Gamma \models^T_{\mathbf{L}} \varphi$.

**Theorem 6.4 (Soundness and completeness of $\mathbf{L_h}$ w.r.t. L)** *Let* $\Gamma \cup \{\varphi\}$ *be a set of formulas in* $For(\Sigma)$. *Then,* $\Gamma \vdash_{\mathbf{L_h}} \varphi$ *iff* $\Gamma \models_{\mathbf{L}} \varphi$.

# 7 LGP: a genuine paracomplete, non-finite-valued logic

Observe that every example of genuine paracomplete (not paraconsistent) logic given up to now consists of three-valued logics. In this section we will introduce a logic, called **LGP**, which is genuine paracomplete (but not paraconsistent), on the one hand, and it cannot be semantically characterized by a single finite-valued logical matrix, on the other. The logic **LGP** will be defined as an axiomatic extension of $\mathbf{CPL}^+$ by adding a (genuine) paracomplete negation $\neg$ satisfying basic properties of classical negation (including explosion).

Recall from Definition 3.3 the propositional signature $\Sigma_0 = \{\wedge, \vee, \to, \neg\}$, and let $For(\Sigma_0)$ be the algebra of formulas generated over $\Sigma_0$ by the set $\mathcal{V} = \{p_n \ : n \in \mathbb{N}\}$ of propositional variables.

**Definition 7.1** *The logic* **LGP**, *defined over signature* $\Sigma_0$, *is obtained from* $\mathbf{CPL}^+$ *(recall Definition 5.1) by adding the following axiom schemas:*

$$\alpha \to \big(\neg\alpha \to \beta\big) \tag{\textbf{exp}}$$

$$\alpha \leftrightarrow \neg\neg\alpha \tag{\textbf{dn}}$$

$$\neg\alpha \leftrightarrow \neg\beta \quad \textit{if } \vdash_{\mathbf{CPL}^+} \alpha \leftrightarrow \beta \tag{\textbf{eq}}$$

**Remark 7.2** *In the axiom schema* (**eq**), $\vdash_{\mathbf{CPL}^+}$ *represents the consequence relation of* $\mathbf{CPL}^+$ *expanded to the signature* $\Sigma_0$ *(which is a conservative expansion of the original logic* $\mathbf{CPL}^+$ *defined over the signature* $\Sigma_+$*). An alternative, equivalent (and maybe more precise) formulation of* (**eq**) *is as follows:* $\neg\alpha \leftrightarrow \neg\beta$ *if there exist formulas* $\alpha', \beta' \in For(\Sigma_+)$ *and a homomorphism* $\rho : For(\Sigma_+) \to For(\Sigma_0)$ *of* $\Sigma_+$*-algebras such that* $\rho(\alpha') = \alpha$, $\rho(\beta') = \beta$, *and* $\vdash_{\mathbf{CPL}^+} \alpha' \leftrightarrow \beta'$. *This is the formulation adopted in Definition 7.4 below.*

*Observe that, since* $\mathbf{CPL}^+$ *is decidable (by the usual two-valued truth-tables), it is effective decidable whether a given formula in* $For(\Sigma_0)$ *is an instance of axiom* (**eq**) *or not. Obviously* (**eq**) *is stable under substitutions: if* $\varphi = \neg\alpha \leftrightarrow \neg\beta$ *is an instance of* (**eq**) *and* $\rho : For(\Sigma_0) \to For(\Sigma_0)$ *is a substitution then, given that* $\alpha \leftrightarrow \beta$ *is a tautology in* $\mathbf{CPL}^+$ *over* $\Sigma_0$, *so is* $\rho(\alpha) \leftrightarrow \rho(\beta)$. *By* (**eq**), *the formula* $\neg\rho(\alpha) \leftrightarrow \neg\rho(\beta)$ *is an axiom of* **LGP**. *That is,* $\rho(\varphi)$ *is an axiom of* **LGP**. *For example,* $\vdash_{\mathbf{CPL}^+} (p_1 \wedge \neg p_2) \leftrightarrow (\neg p_2 \wedge p_1)$. *Hence,* $\neg(p_1 \wedge \neg p_2) \leftrightarrow \neg(\neg p_2 \wedge p_1)$ *is an instance of* (**eq**). *By substitution,* $\neg(\alpha \wedge \neg\beta) \leftrightarrow \neg(\neg\beta \wedge \alpha)$ *is an instance of* (**eq**), *for every* $\alpha, \beta \in For(\Sigma_0)$. *Analogously,* $\neg(\alpha \wedge \beta) \leftrightarrow \neg(\beta \wedge \alpha)$ *is an instance of* (**eq**), *for every* $\alpha, \beta \in For(\Sigma_0)$.[6]

**Proposition 7.3** *The logic* **LGP** *satisfies the* Deduction Metatheorem*: for every set of formulas* $\Gamma \cup \{\varphi, \psi\} \subseteq For(\Sigma_0)$, $\Gamma, \varphi \vdash_{\mathbf{LGP}} \psi$ *iff* $\Gamma \vdash_{\mathbf{LGP}} \varphi \to \psi$.

**Proof:** It is an immediate consequence of the fact that **LGP** contains axioms (**Ax1**) and (**Ax2**), taking into account that (**MP**) is the only inference rule. ■

**Definition 7.4** *A function* $\mu : For(\Sigma_0) \to \big\{0, 1\big\}$ *is a* bivaluation for **LGP** *if it satisfies the following clauses:*

***(vAnd)*** $\mu(\alpha \wedge \beta) = 1$ *iff* $\mu(\alpha) = 1$ *and* $\mu(\beta) = 1$

***(vOr)*** $\mu(\alpha \vee \beta) = 1$ *iff* $\mu(\alpha) = 1$ *or* $\mu(\beta) = 1$

***(vImp)*** $\mu(\alpha \to \beta) = 1$ *iff* $\mu(\alpha) = 0$ *or* $\mu(\beta) = 1$

***(vNeg)*** $\mu(\alpha) = 1$ *implies* $\mu(\neg\alpha) = 0$

[6] One could consider the possibility of defining (**eq**) as an inference rule instead of an axiom. This is the way taken, for instance, by Lemmon in [18], when defining the (non-normal) modal system **S0.5**, in which the usual *necessitation rule* ($\Box\varphi$ follows if $\varphi$ is provable) is changed to the following: $\Box\varphi$ follows if $\varphi$ is provable in classical logic **CPL** expanded to the modal language. We prefer to keep (**eq**) as a schema axiom, since it guarantees immediately the validity of the Deduction Metatheorem (see Proposition 7.3) and, as discussed above, the resulting Hilbert calculus is still recursively axiomatizable and structural. It should be observed that the same move could be made with **S0.5** and similar systems.

***(vDneg)*** $\mu(\alpha) = 1$ *iff* $\mu(\neg\neg\alpha) = 1$

***(vEq)*** $\mu(\neg\alpha) = \mu(\neg\beta)$ *if* $\alpha = \rho(\alpha')$ *and* $\beta = \rho(\beta')$ *for some homomorphism* $\rho : For(\Sigma_+) \to For(\Sigma_0)$ *of* $\Sigma_+$*-algebras and some* $\alpha', \beta' \in For(\Sigma_+)$ *such that* $\vdash_{\mathbf{CPL}^+} \alpha' \leftrightarrow \beta'$.

*The consequence relation of* **LGP** *w.r.t. bivaluations is defined as follows: for every set of formulas* $\Gamma \cup \{\varphi\} \subseteq For(\Sigma_0)$, $\Gamma \models^2_{\mathbf{LGP}} \varphi$ *iff* $\mu(\varphi) = 1$ *for every bivaluation for* **LGP** *such that* $\mu[\Gamma] \subseteq \{1\}$.

**Remark 7.5** *Observe that, if* $\alpha = \rho(\alpha')$ *and* $\beta = \rho(\beta')$ *for some homomorphism* $\rho : For(\Sigma_+) \to For(\Sigma_0)$ *of* $\Sigma_+$*-algebras and some* $\alpha', \beta' \in For(\Sigma_+)$ *such that* $\vdash_{\mathbf{CPL}^+} \alpha' \leftrightarrow \beta'$, *then* $\mu(\alpha) = \mu(\beta)$ *(besides having* $\mu(\neg\alpha) = \mu(\neg\beta)$ *by (vEq)). Given a bivaluation* $\mu$ *and a formula* $\alpha$ *such that* $\mu(\alpha) = 0$, *then* $\mu(\neg\alpha)$ *can take any value in* $\{0, 1\}$. *Thus, taking into account (vNeg) and (vDneg), there are three possibilities for the sequence of formulas* $\alpha$, $\neg\alpha$, $\neg\neg\alpha$, $\neg\neg\neg\alpha$, $\ldots$*: it takes the values* $1, 0, 1, 0, \ldots$*; or it takes the values* $0, 1, 0, 1, \ldots$*; or it takes the values* $0, 0, 0, 0, \ldots$.

By adapting the proof of completeness of **LFI**s w.r.t. bivaluations (see, for instance, [5] and [4]), the following result can be obtained:

**Theorem 7.6** *For every set of formulas* $\Gamma \cup \{\varphi\} \subseteq For(\Sigma_0)$*:* $\Gamma \vdash_{\mathbf{LGP}} \varphi$ *iff* $\Gamma \models^2_{\mathbf{LGP}} \varphi$.

**Corollary 7.7** *The logic* **LGP** *is genuine paracomplete, and it is not paraconsistent.*

**Proof:** Let $\alpha$ and $\beta$ be formulas. By (**exp**) and *Modus Ponens* it follows that $\alpha, \neg\alpha \vdash_{\mathbf{LGP}} \beta$. Hence, **LGP** is not paraconsistent.

Now, let $p$ be a propositional variable and let $\mu$ be a bivaluation such that $\mu(p) = \mu(\neg p) = 0$. Then $\mu(p \vee \neg p) = 0$, showing that $\not\models^2_{\mathbf{LGP}} p \vee \neg p$. By soundness, $\nvdash_{\mathbf{LGP}} p \vee \neg p$. Morever, assume that $\mu$ additionally satisfies that $\mu(\neg(p \vee \neg p)) = 1$ (this is always possible, by Definition 7.4). Then $\mu$ shows that $\neg(p \vee \neg p) \not\models^2_{\mathbf{LGP}} p$. By soundness, $\neg(p \vee \neg p) \nvdash_{\mathbf{LGP}} p$. From this, **LGP** is genuine paracomplete. ■

## 8 Uncharacterizability of LGP by finite matrices

It is well-known that Dugundji showed in 1940 that no modal system between **S1** and **S5** can be characterized by a single finite logical matrix. By adapting his proof, it was shown in [5] and [4] that several **LFI**s cannot be characterized by a single finite logical matrix. In this section it will be shown that **LGP** is not semantically characterizable by a single finite logical matrix.

To begin with, some notation will be introduced. If $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ is a finite family of $n$ distinct formulas in $For(\Sigma_0)$, with $n \geq 3$, then $\bigvee(\Gamma)$ will denote the formula $(\ldots((\gamma_1 \vee \gamma_2) \vee \gamma_3) \vee \ldots) \vee \gamma_n$. Define $\bigvee(\{\gamma_1\}) = \gamma_1$ and $\bigvee(\{\gamma_1, \gamma_2\}) = \gamma_1 \vee \gamma_2$. Given a natural number $n \geq 1$ and $n$ different propositional variables $p_1, \ldots, p_n$, let $\alpha_n := \bigvee(\{p_j \ : \ 1 \leq j \leq n\})$ and $\beta^i_n := \bigvee(\{p_j \ : \ 1 \leq j \leq n, \ j \neq i\})$ for $1 \leq i \leq n$ (in the latter, $n \geq 2$).

**Definition 8.1** *Let $n \geq 3$ be a natural number, and let $p_1, \ldots, p_n$ be $n$ different propositional variables. We define the following formula schema:*

$$\delta(n) = \bigvee \big(\{(\neg\alpha_n \to \neg\beta^i_n) \ : \ 1 \leq i \leq n\}\big).$$

**Proposition 8.2** *Let $\mathcal{M} = \langle V, F, D\rangle$ be a logical matrix such that $V$ has $n \geq 2$ elements, and $\mathcal{M}$ is a model of* **LGP***. Then, $\mathcal{M}$ validates the formula $\delta(n+1)$.*

**Proof:** Let $\mathcal{M} = \langle V, F, D\rangle$ be an $n$-valued logical matrix which is a model of the logic **LGP**, and consider a valuation $h$ over $\mathcal{M}$. Since $V$ has $n$ values, there exists $1 \leq i < k \leq n+1$ such that $h(p_i) = h(p_k)$, by the Pigeonhole Principle. Let $\alpha'_{n+1}$ be the formula obtained from $\alpha_{n+1}$ by substituting the unique occurrence of $p_i$ by $p_k$. It is clear that $\alpha'_{n+1} \leftrightarrow \beta^i_{n+1}$ is a theorem of $\mathbf{CPL}^+$, since the only difference between $\alpha'_{n+1}$ and $\beta^i_{n+1}$ is that the former has two occurrences of $p_k$ while the latter has just one. By axiom (**eq**), the formula $\neg\alpha'_{n+1} \leftrightarrow \neg\beta^i_{n+1}$ is a theorem of **LGP**. This means that $h(\neg\alpha'_{n+1}) \in D$ iff $h(\neg\beta^i_{n+1}) \in D$. But $h(\alpha_{n+1}) = h(\alpha'_{n+1})$, since $h(p_i) = h(p_k)$. From this $h(\neg\alpha_{n+1}) = h(\neg\alpha'_{n+1})$, since $h$ is a homomorphism. Therefore: $h(\neg\alpha_{n+1}) \in D$ iff $h(\neg\beta^i_{n+1}) \in D$. Thus, $h(\neg\alpha_{n+1} \to \neg\beta^i_{n+1}) \in D$, given that $\mathcal{M}$ is a model of $\mathbf{CPL}^+$. Using again that $\mathcal{M}$ is a model of $\mathbf{CPL}^+$, we infer that $h(\delta(n+1)) \in D$. Since this holds for every valuation $h$, the result follows. ■

**Proposition 8.3** *The formula $\delta(n)$ is not a theorem of* **LGP***, for every $n \geq 3$.*

**Proof:** Let $\mu$ be a bivaluation for **LGP** such that $\mu(p_j) = 0$ for every $1 \leq j \leq n$. Hence, $\mu(\alpha_n) = \mu(\beta^i_n) = 0$, for every $1 \leq i \leq n$. Assume also that $\mu(\neg\alpha_n) = 1$ and $\mu(\neg\beta^i_n) = 0$, for every $1 \leq i \leq n$. It is worth noting that this choice is always possible, from Definition 7.4, taking into consideration that $\alpha_n \neq \beta^i_n$ for every $1 \leq i \leq n$. From this, it follows that $\mu(\neg\alpha_n \to \neg\beta^i_n) = 0$, for every $1 \leq i \leq n$. Therefore $\mu(\delta(n)) = 0$ and so $\not\models^2_{\mathbf{LGP}} \delta(n))$. By soundness of **LGP** w.r.t. bivaluations, $\delta(n)$ is not a theorem of **LGP**. ■

**Theorem 8.4** *The logic* **LGP** *cannot be characterized by a single finite logical matrix.*

**Proof:** Suppose, by absurd, that **LGP** can be semantically characterized by a finite logical matrix $\mathcal{M}$ with, say, $n$ elements. Observe that $n \geq 2$ (otherwise, the logic **LGP** would be trivial, which is not the case). Given that $\mathcal{M}$ is a matrix model of **LGP** with $n \geq 2$ values, it validates $\delta(n+1)$, by Proposition 8.2. By completeness of **LGP** w.r.t. $\mathcal{M}$, the formula $\delta(n+1)$ must be a theorem of **LGP**. But this is an absurd, in view of Proposition 8.3. This means that **LGP** cannot be characterized by a single finite logical matrix. ■

## 9 Conclusions

In this paper the notion of genuine paracomplete logics, which are logics rejecting the dual principles that define genuine paraconsistent logics, where introduced and analyzed. First, in a similar way to the analysis done in [2] for genuine paraconsistent logics, we develop a study among three-valued logics in order to find all the (standard) connectives defining genuine paracomplete logics. This analysis allows us to define 64 different three-valued systems. After this, we found in a systematic way an algebraic semantics for each of these systems by means of twist structures semantics. These algebraic semantics allow us to find, in a natural and direct way, simple and uniform Hilbert-style axiomatizations of all these systems. In particular, the logic **MH** introduced in [3], is one of the 64 systems we found here, and so a new (and shorter) axiomatization of **MH** is presented here. Finally, we show an example of a genuine paracomplete (non paraconsistent) logic called **LGP** which cannot be characterized by a single finite logical matrix.

We believe that genuine paracomplete logics are interesting not only from a conceptual perspective, but also for applications. It should be interesting to observe that the methodology presented here for finding twist structures semantics generalizing a given three-valued matrix semantics, as well as the Hilbert calculus obtained from there, could be successfully adapted to other logics presented by means of a finite logical matrix. This is a topic that deserves future research.

### Acknowledgements

The first and third authors acknowledge financial support from the Teacher's Professional Development Program (PRODEP) of the Secretary of Public Education (SEP), Mexico, under the grants UTMIX-EXB-042 and UTMIX-EXB-050. The second author acknowledges support from the National Council for Scientific and Technological Development (CNPq), Brazil, under research grant 306530/2019-8.

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