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The Meaning of "Darn it!"

Luc Bovens and Wlodek Rabinowicz
L.Bovens@LSE.ac.uk and Wlodek.Rabinowicz@fil.lu.se

When vain desire at last and vain regret
Go hand in hand to death, and all is vain,
What shall assuage the unforgotten pain,
And teach the forgetful to forget?

Dante Gabriel Rossetti
The One Hope

1. The *einmal-ist-nicht-keinmal* Game

A bookie picks a random natural number x from 0 to 1,000,000 under a uniform distribution and constructs an urn that has precisely x white balls and $(1,000,000 - x)$ black balls. You are allowed to pick a random ball from this urn. If it's a white ball you get \$1. If it's a black ball you get \$0.

Suppose that you are a risk-neutral person. It is easy to calculate that you would be willing to pay 50 cents for this game. It is also easy to calculate that your credence that there are less than 50% white balls in the urn is just slightly lower than .50.

Now suppose that, before drawing the fateful ball from the urn which will determine your winnings, you get a chance to sample the urn. You draw a ball out of the urn and it is black. You put it back in the urn. How much would you be willing to pay now to play this game? And what is your credence that there are less than 50% white balls in the urn?

There is a German saying that once is never—*einmal ist keinmal*. There is a wisdom embodied in this saying which suggests that sampling a single black ball from a 1,000,000 ball urn should not make much difference to how much a risk neutral person is willing to bet and to her credence that there are less than 50% white balls in the urn.

But in probability theory, appearances may deceive. Indeed, some simple calculations show that a risk-neutral person should now only be willing to pay about 33 cents for this game and that her credence that there are less than 50% white balls in the urn is close to .75. (See appendix.)

Where does one go wrong in underestimating the import of a single observation? One doesn't realize that the minimal information provided by a single draw of one black ball radically changes our credence distribution over the objective chances of winning the game. Obviously, the credence that the urn contains no black balls now goes to zero, but this is by no means the only thing that happens. Posterior credences of all the possible cases in which the urn contains fewer black balls than white balls decrease, whereas posterior credences of all the cases in which it contains more black balls than

white balls increase. It is a matter of a global all-pervasive change in credences. Consequently, drawing a single black ball makes a huge difference to our expectation and to our credence about the number of balls in the urn. Considering this drastic effect of a single draw we call this the *einmal-ist-nicht-keinmal* game—the once-is-not-never game.

Now let us return to the game without sampling—let us call this *the urn game*. Suppose that you can play either the urn game or a game of coin toss, a game which involves tossing a fair coin and which pays \$1 if Heads and \$0 if Tails. Suppose you choose to play the urn game and you lose. "Darn it!" you exclaim. What do you mean by this? That is our subject in this paper.

The relevance of this issue is that we would like to understand the nature of regret after making uncertain choices. We understand "uncertain choices" in the following way. These are choices whose outcomes teach us something about the chances of a win or a loss. Hence, there is a difference between our prior credence that things may turn out well or may turn out poorly and our posterior credence that things might have turned out well or might have turned out poorly. Such choices are different from risky decisions, for example choosing the game of coin toss with a fair coin—whether we win or lose, the chance of winning the game of coin toss is known beforehand. There is no difference between our prior and posterior credence.

2. Twelve Types of Regret

When I ask you for a gloss on your expletive ("Darn it!") after losing the urn game there are two things you might say. Either you might say, "I should not have played the urn game; I should have gone for the coin toss instead!" or you might say, "I could have done so much better!" That is, your regret may focus on your action of choosing the urn game rather than the coin-toss game. Let us call this *action regret*. Or it may focus on the poor outcome that you got. Let us call this *outcome regret*.

There are many terms in the neighbourhood of "regret" that are not quite synonymous. For example, one might say that one *regrets* not having done otherwise, but one is *disappointed* that things did not turn out otherwise. This may be so, but what we are after is something more generic: the negative retrospective emotion that is the root cause and the root expressive content of the expletive "Darn it!" This expletive is fitting both when we realise that we made the wrong choice as well when a coin did not come up the desired way. To start off, we will use 'regret' as a blanket term to cover this emotion. We will draw some further distinctions when the need arises. But first we are going to provide a taxonomy of different kinds of comparisons that might elicit the generic emotion in question.

a. Three Types of Action Regret. As to action regret, I could still push you a bit further to expand on your gloss. Again, there are two things you might say. First, you may point to the fact that, knowing what you know now, it is clear that the coin toss game had a higher expectation than the urn game, viz. 50 cents rather than 33 cents. Let us call this *expectation action regret*. Second, you might say that, knowing what you know now, your credence is high that the chance of winning the coin-toss game was higher than the chance of winning the urn game. Indeed, you have .75

credence that there are less than 50% white balls in the urn. Let us call this *probabilistic action regret*.

There is one more way of assessing *action regret* which our set up does not permit. In our set up, actions and states are not independent. As a matter of fact, if we chose to play the urn game, the coin will not be tossed and thus neither the state of Heads nor the state of Tails in the coin-toss game will materialise. (And vice versa: If we play the coin-toss game, neither the state of drawing a white ball from the urn nor the state of drawing the black ball will materialize.) When actions and states are independent, then it makes sense to compare the outcome that you did get to the outcome that you would have got if you had chosen differently, assuming that the state which is actualised is kept fixed. For instance, suppose that my actions are predicting Heads or predicting Tails, whereupon a fair coin is being tossed and I win a \$1 if and only if I correctly called the outcome. If I predict Heads, and Tails comes up, then I can compare my \$0 with the \$1 if I had predicted Tails. Let us call this *outcome action regret*.

We measure expectation action regret by means of the posterior expectation of the urn game minus the expectation of the coin toss game. We measure probabilistic action regret by means of the posterior credence that the chance of winning the coin-toss game was higher than the chance of winning the urn game.

b. Nine Types of Outcome Regret. While there is no room for outcome action regret in the urn game, due to the dependence of states on actions, it is still possible to talk about pure outcome regret in this case: When you lose the urn game, you may well be disappointed with the outcome. As to such outcome regret, indeed, you could have done better—but in what way could you have done better? Well, there are six possibilities: You may point to your loss and compare it to:

- i. The win that you could have had doing what you did, i.e. playing the urn game;
- ii. The win that you could have had had you chosen to do differently, i.e. playing the coin-toss game;
- iii. The prior expectation of what you did, i.e. of playing the urn game;
- iv. The posterior expectation of what you did, i.e. of playing the urn game;
- v. The prior expectation of what you could have done, i.e. of playing the coin-toss game.
- vi. The posterior expectation of what you could have done, i.e. of playing the coin-toss game.

(v) and (vi) are not extensionally distinct from each other, since the prior and the posterior expectation of what you could have done are the same: you did not learn anything about the unchosen action. Indeed, even if you had played the coin-toss game, your prior and posterior expectation for that game would have been the same: a win or a loss would not have affected your expectation of this game.

Furthermore, in comparing your loss to the win that you did not get, it might also be natural to compare this loss to the *best* win that you could have had, be it from the chosen or the unchosen action. Correspondingly, in measuring your loss by

comparing it to the expectation, it might be natural to compare this loss to the best prior or posterior expectation, be it from the chosen or the unchosen action. In our case, the best posterior expectation is the expectation of the unchosen action, i.e. the coin-toss game, since the posterior expectation of the urn game has gone down to .33.

The following table summarizes all these different types of outcome regret.

| | Win | Prior Expectation | Posterior Expectation |
|-----------------|---------------|-------------------|-----------------------|
| Chosen Action | $(1 - 0) = 1$ | $(.50 - 0) = .50$ | $(.33 - 0) = .33$ |
| Unchosen Action | $(1 - 0) = 1$ | $(.50 - 0) = .50$ | $(.50 - 0) = .50$ |
| Best | $(1 - 0) = 1$ | $(.50 - 0) = .50$ | $(.50 - 0) = .50$ |

Table 1. Nine Types of Outcome Regret

We will name these types outcome regret by concatenating row names and column names, e.g. *chosen-action/win outcome regret etc.*

3. Uncertainty Aversion and Uncertainty Proneness

Krähmer and Stone (2013) provide an explanation for uncertainty aversion that can be framed in terms of our analysis of regret. Suppose that you are offered a choice between 50 cents in hand or playing the urn game. You are uncertainty averse and take the 50 cents. What might explain your refusal to play?

You might reason as follows. There is a 50-50 probability of winning or losing. Suppose I lose. Then I will look back on my choice and say: I chose to play a game that has a posterior expectation of 33 cents rather than doing the smart thing and opting for the 50 cents for sure. I now have zero cents and so my loss, as compared to the expected value of the smart action, is 50 cents. Let this be a measure of my regret. Suppose I win. Then I will look back on my choice and say: I chose to play a game that has a posterior expectation of 67 cents. That was a smart thing to do. In addition, Lady Luck smiled on me. I have \$1 in my hands and so my gain, as compared to the posterior expectation of the smart action, is 33 cents. Let this be a measure of my joy. If I choose to play the urn game, then (i) the chances that I will experience regret or joy are 50-50 and (ii) my anticipated regret is greater than my anticipated joy. Hence I refuse to play the urn game.¹

This is a clever suggestion, but it requires a very particular way of framing our anticipation of feelings of regret and joy. Regret is measured as Best/Posterior-Expectation Outcome Regret. I ask myself: How much worse did I do than the posterior expectation of the action that has the best posterior expectations? The answer is: I have \$0 and that is 50 cents less than the 50 cents I could have had by

¹ Krähmer and Stone apply this kind of reasoning to the famous Ellsberg's paradox. They use their model to explain why agents tend to prefer risky choices to the uncertain ones. Unlike in uncertain choices, in risky choices the posterior expectation doesn't increase in case of a win, which explains why the expected joy from a win is higher in these cases; so much higher that it offsets the expected regret from a loss.

taking the 50 cents in hand rather than playing a game with a posterior expectation of 33 cents. Correspondingly, joy is measured as Best/Posterior Expectation Outcome Joy. In the case of joy, I ask myself: How much better did I do than the posterior expectation of the action that has the best posterior expectation? The answer is: I have \$1 and that is 33 cents more than the posterior expectation of 67 cents of playing the urn game.

But if we measure regret relative to the *best* thing that I could have done from an ex post perspective, then should we not measure joy relative to the *worst* thing that I could have done from an ex post perspective? Should I not say: I am so joyful that I did not take the 50 cents in hand – that would have been a stupid thing to do since the urn game has a posterior expectation of 66 cents? And hence I should frame my joy in terms of a gain from 50 cents to \$1. And then my anticipated regret and anticipated joy are equal and the argument for abstaining from the urn game crumbles.

Krähmer and Stone may respond that this way of framing the choice problem may well be *reasonable*, but the fact of the matter is that some people are uncertainty averse. They are merely providing a behavioural explanation of uncertainty aversion by pointing to some feature of the psychology of uncertainty averse people. But then, could we extend this behavioural explanation for uncertainty proneness? Building on our analysis, we can indeed provide an explanation of both uncertainty attitudes.

Note the terms “joy” and “regret”. They are not quite antonyms. So far we have let “regret” cover the widest range of emotions and the term is indeed used broadly in the English language. But we could distinguish between two negative emotions: An emotion of sorrow (sadness, disappointment...) which is responsive to bad luck in the chosen action and an emotion of regret proper, so to speak, which is responsive to the realisation that things might have been better with unchosen actions. Similarly, we could distinguish between two positive emotions: An emotion of joy (gladness, being thrilled...) which is responsive to good luck in the chosen action and an emotion of relief which is responsive to the realisation that things might have been much worse with unchosen actions. “Sorrow” and related expressions (“sadness”, “disappointment”...) are then antonyms of “joy” (“gladness”, “being thrilled”...), whereas “regret” in this more restricted sense is an antonym of “relief”. Much more can be said in the way of conceptual analysis of terms denoting retrospective emotions, but this will do for our purposes.

Sorrow and joy are responsive to Lady Luck’s frowning or smiling upon us. Now, from an ex post perspective, to what extent did Lady Luck frown upon us in the urn game when we lose? She presented us with \$0 in a game with a posterior expectation of 33 cents. This difference of 33 cents is a measure of our sorrow, which equals Chosen-Action/ Posterior-Expectation Outcome Regret in our scheme. Similarly, the extent to which Lady Luck smiled upon us when we have a win is the difference between the \$1 that we won and the posterior expectation of 67 cents, i.e. 33 cents.

Regret and relief are responsive to features of the unchosen actions. A measure of my regret is the difference between the \$0 resulting from my loss compared to the expectation of 50 cents, had I chosen for the money in hand, i.e. (50 cents – \$0) = 50 cents. This is Unchosen Action/Expectation Outcome Regret in our scheme. A measure of my relief is the difference between the \$1 resulting from my win

compared to the expectation of 50 cents had I chosen for the money in hand, that is $(\$1 - 50 \text{ cents}) = 50 \text{ cents}$.

We can now let uncertainty attitudes be determined by our anticipated retrospective emotions in decisions under uncertainty. If an agent anticipates joy (in case of a win) and regret (in case of a loss), then the negative emotion of regret measured at 50 cents trumps the positive emotion of joy measured at 33 cents and hence she will be uncertainty averse.² If an agent instead anticipates relief (in case of a win) and sorrow (in case of a loss), then the positive emotion of relief measured at 50 cents trumps the negative emotion of sorrow measured at 33 cents and she will be uncertainty prone.

Let us try to make this psychology plausible. Suppose that we interview uncertainty averse and uncertainty prone persons. In cases in which a win or a loss affects our posterior expectation of the game, an uncertainty averse person would justify her choice of taking the money in hand as follows: See, if I win, I would not be all that excited – I would feel that my win was very much in the cards. But if I lose, I would kick myself, because I could instead have opted for an action whose posterior expectation was so much better than the outcome I have got now. An uncertainty prone person would say: See, if I lose, then I would not feel that bad – I would just feel that there was not much of a win in the cards anyway. But if I win, I would be pleased that I did not choose the option that would have kept me down at a much lower posterior expectation.

Now, as a behavioural explanation, we do not take issue with the reasonableness of these responses. However, we can do an empirical test of the plausibility of the explanation. To start with, using choice problems like the ones described above, and relying on the suggested explanation of uncertainty aversion and uncertainty proneness, we can identify among our subjects the ones who are uncertainty averse and the ones who are uncertainty prone. Now, consider a one-ball urn version of our game: the bookie picks a random natural number x from 0 to 1 under a uniform distribution and constructs a one-ball urn that has precisely x white balls and $(1 - x)$ black balls. I.e., its contents are one white ball or one black ball, depending on the bookie's random choice. Note that the posterior expectation of this game is 1 upon a win and 0 upon a loss.

Suppose that we offer a subject a choice between the urn game with 1,000,000 balls or the urn game with a single ball. Now if the explanation of uncertainty aversion that has been suggested above is correct, then an uncertainty averse person should pick the 1,000,000 balls urn game: His regret upon a loss is fixed at 50 cents in both games, but his joy upon a win is 0 in the single-ball urn game and 33 cents in the 1,000,000 balls urn game. An uncertainty prone person should pick the single-ball urn game: Her relief upon a win is fixed at 50 cents in both games, but her sorrow upon a loss is 0 in the single-ball urn game and 33 cents in the 1,000,000 balls urn game. We do not

² Our analysis is slightly different from Krähmer and Stone's. They measure joy by comparing the value of a win to the posterior expectation of the action that has the highest posterior expectation. We measure joy by comparing the value of a win to the posterior expectation of the chosen action. These measures are extensionally equivalent in the case at hand, since the chosen action turns out to have the highest posterior expectation if we win, but they are not intensionally equivalent.

know whether this empirical test will hold up. If it won't, then the analysis of uncertainty aversion and uncertainty proneness along Krähmer and Stone lines is questionable.

In case preparing an urn game with 1,000,000 balls in an experimental setting would be too difficult and an urn game with a single ball too contrived, no matter: We can instead offer the subject a choice between an urn game with fewer balls and an urn game with more balls. On the Krähmer-Stone-type modelling, the uncertainty averse person will prefer the more-balls urn game, whereas the uncertainty prone person will choose the fewer-balls urn game: Sensitivity to uncertainty decreases as the number of balls increases. This is a rather strange implication of the model, to say the least, and one might doubt that it will stand up to experimental testing.

There is a temptation to say that uncertainty neutrality is explained by being the anticipation of sorrow and joy or by the anticipation of regret and relief. In both cases the anticipated emotions balance each other out. But there is no need to do that. Relying on Broome's (1991) account of the goodness of an action, we can just provide a rational choice explanation of risk neutral choice. Broome argues that the goodness of an action equals its expected goodness. Goodness is linear in money in the context of this paper. Hence the rational action is the action that maximises the expectation of the game. Since the expectations of the certain choice (i.e. 50 cents), the risky choice (i.e. the single-ball urn game) and the uncertain choice (i.e. the multiple-ball urn game) are all the same, a rational person should be indifferent between any of these options. For Broome, the goodness of an action is insensitive to the distinction between certainty, risk and uncertainty: Only the expected goodness matters. Hence this is a rational choice explanation of uncertainty neutrality. By contrast, the explanations of uncertainty attitudes in terms of anticipated regret, relief, sorrow or joy are behavioural; they carry no normative weight.

4. Regret Avoidance Therapy

What do we say to a person who is overcome by regret? If we cannot convince her to give up on this emotion, we may try to steer them towards a kind of regret that is minimal for the problem at hand by framing the problem in a particular way. Consider the following examples.

Suppose that we choose to play the single-ball urn game rather than the coin toss game and we lose. Then all types of regret can kick in. But a therapist will tell us: "What you didn't choose is water under the bridge—forget about that; As to what you did choose, there was no way that you could have won; It simply was not in the cards (or, *in casu*, in the urn). Hence, there is no reason to regret anything."

In this case, the therapist tries to convince us that the relevant way to experience regret is through chosen-action/posterior-expectation outcome regret. And then he points out that there is no regret to be had of this kind. It is the fatalistic antidote to regret. Things went as they went and though we might not have known at the time, it was never in the cards for them to go differently.

Suppose that we have an urn game in which we know that there are an equal number of black and white balls in the urn. We play this urn game rather than the coin toss game and we lose. All kinds of regret can kick in. But now a therapist will tell us: “The action that you did choose was perfectly reasonable. What else could you have done? Play the coin-toss game? That game had exactly the same expectation as the game you chose to play and you can rest assured that it did not have a higher chance of a win. Hence there is no reason to regret anything.”

In this case, the therapist tries to convince us that the relevant way to experience regret is through expectation action regret or probabilistic action regret. There is no regret to be had of this kind. This is the random-world antidote to regret. What we could have done was just as much a crap shoot as what we did do. Hence there is no reason to regret anything.

The therapist teaches the agent how to focus regret assessment so that it disappears in the situation at hand. He teaches regret avoidance. The agent may follow. But the agent may resist and be regret seeking, by turning the therapist’s advice topsy-turvy.

In the one-ball urn case, a regret-seeking agent will focus on expectation action regret or probabilistic action regret—which she does have reason to have. She will say to the therapist: Yes, but it would have been so much better to play the coin-toss game!

In the equal-number-of-black-and-white-balls urn case, a regret-seeking agent will focus on chosen-action/posterior-expectation outcome regret and chosen-action/win outcome regret—which she does have reason to have. She will say to the therapist: Yes, but think of the expectation of my choice and of the win that I could have had in the game that I played!

Now depending on the nature of the chosen and unchosen action, one therapeutic strategy may be more successful than another. But often we don’t know what the true nature of our choices is. Then life is a *Rashomon*—after the 1951 Kurosawa movie in which a crime and its aftermath are recalled from differing points of view—we can frame our actions in a particular way that permits us to deal with regret. There are two such frames.

Either we may frame our choices as choices between actions that have random outcomes—it’s all a crap shoot. This corresponds to Aristotle’s open future. The point of such framing is to steer us away from action regret. If things go wrong, there is no reason for regret: If we had chosen differently, things might have gone equally wrong! The accompanying virtue is epistemic modesty—we should not pretend to know how things might have gone if we had acted otherwise.

Or we may frame our choices as choices between actions that are fated—actions with pre-ordained outcomes. This corresponds to the Stoic conception that the future is pre-determined. The point of such framing is to steer us away from outcome regret relative to the chosen action. If things go wrong, there is no reason for regret: Things couldn’t have gone any better—it just wasn’t in the cards! The virtue in this case is serenity—we should learn to accept what could not have been otherwise.

It is curious that opposing positions on this deep metaphysical question concerning the status of future events can be called in as antidotes to retrospective emotions in response to misfortunes resulting from our past choices.

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Appendix

Let n be the number of balls in the urn, with x white balls and $(n - x)$ black balls for $x = 1, \dots, n$. My credence of drawing a white ball, conditional on the proportion of white balls being x/n equals x/n . My credence of the proportion of white balls being x/n equals $1/(n + 1)$. Let W be the event of drawing a white ball and let $Prop(White)$ be the proportion of white balls in the urn.

The following calculations are based on simple algebra and the formulas for the sum of the first natural numbers and the sum of the squares of the first natural numbers.³

My credence in W equals

$$(1) \quad c(W) =$$

$$\sum_{x=0}^n c\left(W \mid Prop(White) = \frac{x}{n}\right) c\left(Prop(White) = \frac{x}{n}\right) =$$

$$\sum_{x=0}^n \frac{x}{n} \frac{1}{(n+1)} =$$

$$\frac{1}{(n+1)n} \sum_{x=0}^n x =$$

$$\frac{1}{(n+1)n} \frac{(n+1)n}{2} = \frac{1}{2}$$

³ That is, $\sum_{x=1}^n x = \frac{n(n+1)}{2}$ and $\sum_{x=1}^n x^2 = \frac{n(n+1)(2n+1)}{6}$.

A white ball yields \$1 and a black ball \$0; hence the expectation of the game is 50 cents.

If n is odd, my credence that there are less than 50% white balls in the urn is

$$(2) \quad c(\text{Prop}(\text{White}) < .50) = \sum_{x=0}^{\frac{(n-1)}{2}} c\left(\text{Prop}(W) = \frac{x}{n}\right) = \frac{(n+1)}{2} \frac{1}{(n+1)} = \frac{1}{2}.$$

If n is even,

$$(3) \quad c(\text{Prop}(\text{White}) < .50) = \sum_{x=0}^{\frac{n}{2}-1} c\left(\text{Prop}(W) = \frac{x}{n}\right) = \binom{n}{2} \frac{1}{(n+1)} = \frac{n}{(n+1)} \frac{1}{2},$$

which approaches $\frac{1}{2}$ as $n \rightarrow \infty$.

We calculate the posterior credence (c^*) of the proportion of white balls being x/n , upon having drawn a black ball with replacement (i.e. upon learning that B). By conditionalization and Bayes Formula:

$$(4) \quad c^*(\text{Prop}(\text{White}) = \frac{x}{n}) =$$

$$c\left(\text{Prop}(\text{White}) = \frac{x}{n} \mid B\right) =$$

$$\frac{c\left(B \mid \text{Prop}(\text{White}) = \frac{x}{n}\right) c\left(\text{Prop}(\text{White}) = \frac{x}{n}\right)}{c(B)} =$$

$$\frac{\left(1 - \frac{x}{n}\right) \left(\frac{1}{n+1}\right)}{1/2}$$

We now calculate my posterior credence of drawing a white ball upon learning that B . This is the sum for x ranging from 0 to n of the products of the posterior credences of drawing a white ball conditional on the proportion of white balls being x/n and the posterior credences that there are x white balls in the urn:

$$(5) \quad c^*(W) = \sum_{x=0}^n c^*(W \mid \text{Prop}(\text{White}) = \frac{x}{n}) c^*\left(\text{Prop}(\text{White}) = \frac{x}{n}\right)$$

Clearly

(6)

$$c^* \left(W \mid \text{Prop}(\text{White}) = \frac{x}{n} \right) = \frac{x}{n}$$

Hence, from (4), (5) and (6),

(7) $c^*(W) =$

$$\begin{aligned} & \sum_{x=0}^n \frac{x}{n} \frac{\left(1 - \frac{x}{n}\right) \left(\frac{1}{n+1}\right)}{\frac{1}{2}} = \\ & \frac{2}{n^2(n+1)} \left(n \sum_{x=0}^n x - \sum_{x=0}^n x^2 \right) = \\ & \frac{2}{n^2(n+1)} \left(n \frac{n(n+1)}{2} - \frac{n(n+1)(2n+1)}{6} \right) = \\ & \frac{1}{3} \frac{(n+1)}{n} \end{aligned}$$

which approaches $1/3$ as $n \rightarrow \infty$.

Furthermore, if n is odd,

(8)

$$c^*(P(W) < .50) = \sum_{x=0}^{\frac{(n-1)}{2}} c^* \left(\text{Prop}(\text{White}) = \frac{x}{n} \right) =$$

$$\sum_{x=0}^{\frac{(n-1)}{2}} \frac{\left(1 - \frac{x}{n}\right) \left(\frac{1}{n+1}\right)}{\frac{1}{2}} =$$

$$\frac{2}{(n+1)} \sum_{x=0}^{\frac{(n-1)}{2}} \left(1 - \frac{x}{n}\right) =$$

$$1 - \frac{1}{4} \frac{(n-1)}{n}$$

$$\frac{(3n+1)}{4n}$$

which approaches .75 as $n \rightarrow \infty$.

For n is even,

(9)

$$c^*(P(W) < .50) = \sum_{x=0}^{\binom{n}{2}-1} c^*(\text{Prop(White)} = \frac{x}{n}) =$$

$$\sum_{x=0}^{\binom{n}{2}-1} \frac{(1 - \frac{x}{n})(\frac{1}{n+1})}{1/2} =$$

$$\frac{2}{(n+1)} \sum_{x=0}^{\binom{n}{2}-1} \left(1 - \frac{x}{n}\right) =$$

$$\frac{n}{(n+1)} - \frac{1}{4} \frac{(n-2)}{(n+1)} =$$

$$\frac{(3n+2)}{(4n+4)}$$

which also approaches .75 as $n \rightarrow \infty$.