Infinite Sets and Hyperoperations

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Abstract

The purpose of this paper is to explore infinite sets and classes by mean hyperoperations. With ideal notion, the idea of extending infinite sets is as large as those objects. In this paper, extensions with hyperoperations are realized, like factorial, derivative, integral and operations between vector spaces. The ideas about infinite and count are enlarged.

Keywords: Set theory, Hyperoperations, Infinity

Contents

1 Introduction

Here, we remember concepts about hyperoperations, which are a generalization of basic arithmetic operations, and a new perspective on counting elements in sets. After these hyperoperations are used to analyse, explore and study some concepts and to extend infinity sets and classes. Hence, the comprehension of some mathematical concepts and tools is improved.

Let be $a,b,n \in \mathbb{N}$, hyperoperations $[\,]$ are ternary operations defined by [1, 2]:

$$\begin{aligned} \cdot[\cdot]\cdot : \mathbb{N}\times\mathbb{N}\times\mathbb{N} &\rightarrow \mathbb{N} \\ (a,b,n) &\mapsto H_n(a,b) = a[n]b, \end{aligned}$$

in which

$$a[n]b = \begin{cases} b+1 & \text{if } n = 0; \\ a, & \text{if } n = 1 \text{ and } b = 0; \\ 0, & \text{if } n = 2 \text{ and } b = 0; \\ 1, & \text{if } n \geq 3 \text{ and } b = 0; \\ a[n-1](a[n](b-1)), & \text{otherwise.} \end{cases}$$

Finite sets do not take into account how they are counted. Infinite sets behave differently: a set can be counted by order or by bijection. They are called ordinals and cardinals [3].

Definition 1.1 Neutral Element *The neutral element by right of an n-hyperope--ration is defined like:*

$$x\textcircled{n}e_n = x, \quad \forall x \in \mathbb{R} \tag{1}$$

Then e_n assumes the values:

$$e_n = \begin{cases} 0, \text{ if } n{=}1 \\ 1, \text{ if } n{>}1 \end{cases} \tag{2}$$

It is seen easy that does not exist a neutral element by left for $n > 2$. However, a question can be done: There is a finite solution for the equations:

$$a\textcircled{n}2 = a \tag{3}$$

$$2\textcircled{n}b = b. \tag{4}$$

In the equation (3), there are solutions only for $n > 1$ and it is exactly the neutral element e_{n-1}. In the (4), there is a solution only for $n = 2$ and it is equal to zero.

Further up, we consider solutions that are infinite numbers, where "=" denotes cardinality.

1.1 Euler relation in terms of hyperoperations

The Euler relation is very known and relates five interesting numbers [4]: the Euler number e, the imaginary number i, the relation in the circle π, the neutral element of multiplication and addition, like shown in the eq. (5). It is written in notation of hyperoperations like:

$$e^{i\pi} + 1 = 0 \Rightarrow [e\,③(i\,②\pi)]\,①e_2 = e_1 \tag{5}$$

His beauty is seen in a more symmetrical way.

1.2 H-factorial

The standard factorial [5, 6] is defined from the first hyperoperations like:

$$a! := a * [(a-1)!] = a②[(a-1e_2)!], \tag{6}$$

which can be extended for other orders of hyperoperations in the following way:

$$a!^n := a\bar{n}[(a-1e_{\bar{n}})!^n]. \tag{7}$$

Or more generally:

$$a!^n_m := an(a-1e_n)!^n_m. \tag{8}$$

1.2.1 Some properties:

i. $n = 1:$ $\quad a!^1 = a!$

ii. $m = n-1:$ $\quad a!^n_m = a!^{n-1}$

1.2.2 Some examples:

i. $n = 2:$ $\quad a!^2 = a^{(a\cdot 1)!^2} = a^{\nearrow\omega}$

ii. $n = 3:$ $\quad a!^3 = a^{\nearrow a!^3} = a⑤\omega$

v. $m = 1:$ $\quad a!^n_1 = an(a-1)!^n_1$

1.3 Hyperoperations' Properties

1. $a③b = a^b \neq b^a = b③a$ - it is not commutative.
2. $(a③a)③a = (a^a)^a = a^{a^2} \neq a^{(a^a)} = a③(a③a)$- it is not associative, neither 2-power associative.
3. $(a⓷b)⓷c \neq (a⓷c)⓷b$

   Ex: $(2④3)④2 = \left(2^{\nearrow^3}\right)^{\nearrow^2} \neq \left(2^{\nearrow^2}\right)^{\nearrow^3} = (2④2)④3$
4. $a⓷(b+1) = a⓶(a⓷b)$, $m = n-1$
5. $a \hat{\ominus} b = a⓷(-b)$

1.3.1 Inverse hyperoperation $\hat{\ominus}$

In the same way that equivalence relation are used to construct inverse basic operation as difference and quotient, we construct inverse of hyperoperation and its formal aspects. First, take the exponentiation case:

$$a^b = a③b = c③d = c^d \Leftrightarrow a \hat{\ominus}_3 d = c \hat{\ominus}_3 b \tag{9}$$

$$\Leftrightarrow (x,1) \sim (a,d) \sim (c,b) \Rightarrow x^d = a^1 \tag{10}$$

Then, equivalence relation can be constructed for the general case:

$$a⓷b = c⓷d \Leftrightarrow a \hat{\ominus}_3 d = c \hat{\ominus}_3 b \tag{11}$$

$$\Leftrightarrow (x,1) \sim (a,d) \sim (c,b) \Rightarrow x⓷d = a \tag{12}$$

1.4 Ordinal Algebra

We know the succession operation is fundamental to order relations and defined by $\bar{n} = n+1$. For finite numbers, sum operation commutes, which does not occur with infinite numbers [7, 8]. In a short way: $1+\omega = \omega$, however $\omega + 1 = \bar{\omega} \neq \omega$.

Writing the known numbers by order, we have:

$$0, 1, 2, \ldots, n, \ldots, \omega, \omega+1, \omega+2, \ldots, \omega+\omega = \omega \cdot 2, \ldots, \omega \cdot 3, \ldots, \omega \cdot n, \ldots$$

$$\ldots, \omega \cdot \omega = \omega^2, \ldots, \omega^n, \ldots, \omega^\omega, \ldots, \omega^{\omega^\omega}, \ldots$$

Like $\omega + \omega^2 = \omega(1+\omega) = \omega^2$, the identity $\omega + a = a$ can be used to characterizing the square ω^2 and greater numbers. In a analogous way, ω^ω is the first ordinal a that satisfies $\omega \cdot a = a$. In the next step, we found the first ordinal that satisfies $\omega^a = a$. Because it misses a finite representation with the known operation, it is named ε_0 [3]. In the notation of hyperoperations, it would be denoted by

$$\varepsilon_0 = \omega^{\nearrow^\omega} = \omega ④ \omega = \omega ⑤ 2. \tag{13}$$

Extending this idea arbitrary to hyperoperations $\textcircled{m}$, we have:

$$\lim_{n\to\omega} (\omega \textcircled{m} n) = \omega \textcircled{m} \omega = \omega \textcircled{\widehat{m}} 2.$$

Analogous characterizations of $\omega \textcircled{\widehat{m}} 2$ with ideals can be stated also[9].

Proposition: Order of hyperoperations are useful to order ideal ordinals. $a = \omega \textcircled{\widehat{m}} \omega$ is the first ordinal that satisfies

$$\omega \textcircled{m} a = a. \tag{14}$$

1.5 Cardinal Algebra

Another form of counting can be considered by bijection. The first cardinals are represented like [10]:

$$\aleph_0, \aleph_1, \aleph_2, \ldots, \aleph_\omega, \aleph_{\omega+1}, \ldots, \aleph_{\omega^\omega}, \ldots, \aleph_{\aleph_1}, \ldots, \aleph_{\aleph_\omega}, \ldots, \theta, \ldots, \tag{15}$$

where θ satisfies $\theta = \aleph_\theta$. After this, we have many others numbers, like $\aleph_{\theta+1}$, $\aleph_\theta + \aleph_\omega, \aleph_{\theta+\omega}$. This sequence never ends. Absolute Infinity, which is beyond all the ordinals, is denoted by the symbol Ω, which is not an ordinal number because all ordinal numbers are before it [3].

1.5.1 Extension of set idea

The generalized continuum hypothesis states that the infinite cardinal number $\mathfrak{a}^+$, defined by the minimal cardinality bigger than another cardinal number $\mathfrak{a}$, is the power set $2^\mathfrak{a}$ [11]. However, let be an object $\mathfrak{B}$ with cardinality $\mathfrak{b}$ that satisfies $\mathfrak{b} = 2^\mathfrak{b} = 2③\mathfrak{b}$, it cannot be a set. This implies that the set of subsets of $\mathfrak{B}$ has the same number of elements as $\mathfrak{B}$. This requires another succession rule in terms of cardinality. Let be the rule $\mathfrak{b}^+ = 2④\mathfrak{b}$. Then, we define a ③-set as an object that satisfies this succession rule.

For extension, define a ⓐ-set as an object $\mathfrak{B}$ that satisfies the identity $\mathfrak{b} = 2ⓐ\mathfrak{b}$ and the succession rule $\mathfrak{b}^+ = 2ⓐ\mathfrak{b}$. In particular, the standard succession $\mathfrak{b}^+ = 2③\mathfrak{b}$ for infinite numbers is satisfied by ②-set, that are standard sets.

Suppose that $\aleph_0 = 2^{\nearrow \aleph_0}$, then

$$\aleph_n = 2^{\nearrow \aleph_0 + n} = 2^{\nearrow \aleph_0} = \aleph_0,$$

which is absurd. However

$$\mathfrak{b} = 2^{2^{\nearrow \aleph_0}} = 2^{\nearrow \aleph_0} \text{ satisfies } \mathfrak{b} = 2^{\mathfrak{b}} = 2③\mathfrak{b},$$

i.e., it does not represent a cardinality of a set.

To summarize, the sequence below is increasing:

$$2 \cdot \aleph_0 \prec 2^{\aleph_0} \prec 2^{\nearrow \aleph_0} = \mathfrak{b} \prec \cdots \prec 2ⓝ\aleph_0 \prec \cdots \tag{16}$$

1.6 Surreal Numbers

Elements from a type of special algebra in the non-standard analysis can rule in [12, 13]:

$$a\left(\frac{1}{b}\right) = c \;\Leftrightarrow\; a = bc \;;\; a,b,c \in \{\mathbb{N}, \zeta_1, \zeta_2, \ldots,\} \tag{17}$$

It does exist $\mu(s)$, where $\mu(s)$ is a measure of the set s and $\mu(s) = \frac{1}{b}$.

$$\zeta_i = \aleph_{i-1}, \quad \text{for } i \in \mathbb{N}^* \tag{18}$$

$$\zeta_3 = \aleph_2 \quad - \quad \text{functions set infinity} \tag{19}$$

$$\zeta_2 = \aleph_1 \quad - \quad \text{continuum infinity} \tag{20}$$

$$\zeta_1 = \aleph_0 \quad - \quad \text{countable} \tag{21}$$

$$\zeta_0 = \begin{matrix} n, \forall n \in \mathbb{N} - \text{ finite numbers} \\ \frac{1}{n} \end{matrix} \tag{22}$$

$$\left.\begin{matrix} \zeta_{-1} = \frac{1}{\aleph_0} \\ \zeta_{-2} = \frac{1}{\aleph_1} \\ \zeta_{-3} = \frac{1}{\aleph_2} \end{matrix}\right] - - \text{infinitesimals} \notin \mathbb{R}. \tag{23}$$

1.7 Infinitesimal calculus with hyperoperations

The standard calculus can be extended by the hyperoperations [14, 13]. Then, for each order, we write the differential and integral correspondents. The next step is to construct the analogous for higher orders.

$n=0$	successor	$^-$			$f(x)$	$_-$	
$n=1$	addition	$+$	$1+1$	$\frac{df}{dx}$	$\int f(x)dx$	$-$	$\mathbb{Z}$
$n=2$	multiplication	$\cdot$	$2\cdot 3$	f_ρ	$\wr f(x)^{dx}$	$\div,:,/$	$\mathbb{Q}$
$n=3$	potentiation	exp	3^2		$\vdots$	$\sqrt[n]{\cdot}$	$\mathbb{R}$
$n=4$		$\nearrow$	$5^{\nearrow 4}$		$\vdots$	$\swarrow$	
$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n		$\textcircled{n}$	$2\textcircled{n}3$		$_n\wr f(x)\bar{\textcircled{n}}dx$	$\textcircled{n}$	,

The concepts used are better explained forward. Beside this, $\bar{n}=n+1$ and the inverse of $\textcircled{n}$is denoted by $\textcircled{n}$:

$$(a\textcircled{n}b)\,\textcircled{n}b=a \tag{24}$$

$$(a\,\textcircled{n}b)\textcircled{n}b=a \tag{25}$$

1.7.1 Generalization of derivative by mean hyperoperations

First, the derivative definition is written in terms of hyperoperation [14, 13]

$$\frac{df}{dx}=\frac{f(x+dx)-f(x)}{dx}=[f(x\textcircled{1}dx)\,\textcircled{1}f(x)]\,\textcircled{2}\,dx$$

Then, we generalize by induction

$$\mathfrak{D}_a^x f=[f(x\textcircled{@}dx)\,\textcircled{@}f(x)]\,\bar{\textcircled{@}}dx$$

Put $\Delta x=x'\,\textcircled{@}x$, then

$$\mathfrak{D}_a^x f=\lim_{x'\to x}\left[f(x')\,\textcircled{@}f(x)\right]\bar{\textcircled{@}}\Delta x, \tag{26}$$

1.7.2 Generalization of Integral by mean hyperoperations

Extension of Riemann Integral [14, 13] can be extended by mean hyperoperation. Let be a partition $P=[\gamma=x^0,x^1,\ldots,x^n,\ldots,\delta=x^N]$. Apply $\textcircled{@}\Delta x$ to the equation

(26) on each point of the partition P and do the hyperoperation @among all them.

$$
\begin{aligned}
{}_a\Big)_\gamma^\delta f(x)@dx &= \lim_{min(\Delta x)\to 0} \textcircled{a}_\gamma^\delta f(x)@\Delta x && (27)\\
&= [f(x^1) \mathbin{\textcircled{a}} f(\gamma)]@[f(x^2) \mathbin{\textcircled{a}} f(x^1)]\cdots[f(\delta) \mathbin{\textcircled{a}} f(x^{N-1})] && (28)
\end{aligned}
$$

These kinds of integral can help to comprehend the behaviour of functions at many levels of infinitude.

Examples:

1. ${}_1\Big)_\gamma^\delta f(x)②dx = \int_\gamma^\delta f(x)dx$

   [Riemmann Integral]

2. ${}_2\Big)_\gamma^\delta f(x)③dx = \Big)_\gamma^\delta f(x)^{dx} = \Big)_\gamma^\delta e^{\ln f(x)dx} = exp(\int_\gamma^\delta \ln f(x)dx)$

   [A path in the Feymann integral]

1.8 Extension of tensor space by mean hyperoperations

Let be $\dim V = n$, then the operation of direct sum and tensor product [15] can be extended by mean hyperoperation analogous, like the following recurrence of vectorial space construction.

$$
\begin{aligned}
S_0 = \left[\omega^0, V, m\right] &= V \oplus \mathbb{R}^m\\
S_1 = \left[\omega^1 = \oplus, V, m\right] &= \underbrace{V \oplus \cdots \oplus V}_{m \text{ times}}, \text{ where } \left[\omega^1, V, 2\right] \approx \left[\omega^0, V, n\right]\\
S_2 = \left[\omega^2 = \otimes, V, m\right] &= \underbrace{V \otimes \cdots \otimes V}_{m \text{ times}}, \text{ where } \left[\omega^2, V, 2\right] \approx \left[\omega^1, V, n\right]\\
&\vdots \qquad \vdots \qquad\qquad\qquad \vdots\\
S_p = \left[\omega^p, V, m\right] &= \underbrace{V\omega^{p-1}\cdots\omega^{p-1}V}_{m \text{ times}}, \text{ where } \left[\omega^p, V, 2\right] \approx \left[\omega^{p-1}, V, n\right]
\end{aligned}
$$

Solving the recurrence, the dimensions of those spaces are found:

- $\dim S_0 = n + m$

- $\dim S_1 = n \cdot m$
- $\dim S_2 = n^m$

  $\vdots \qquad \vdots$
- $\dim S_p = n \textcircled{\tilde{p}} m$

To summarize, some functions and concepts are extended with hyperoperation. Those are essential to large numbers comprehension. New larger infinities and their properties arise with the ideal notion and class notion. Derivative and integral concepts from calculus are also extended with hyperoperations. Besides this, operations between vector space, like direct sum and tensor product, are also generalized.

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