Uniformly convex Banach spaces are reflexive—constructively

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Abstract

We propose a natural definition of what it means in a constructive context for a Banach space to be reflexive, and then prove a constructive counterpart of the Milman-Pettis theorem that uniformly convex Banach spaces are reflexive.

Our aim in this note is to present a fully constructive[1] analysis of the Milman-Pettis theorem [11, 12, 9, 13]: a uniformly convex Banach space is reflexive. First, though, we need to outline the constructive context of the statement and proof of this theorem, and to clarify the terms we will use.

The context of our work is that of a quasinormed space: that is, a linear space X equipped with a family $(\| \|_i)_{i\in I}$ of seminorms on X such that the subset $\{\|x\|_i : i \in I\}$ of $\mathbf{R}$ is bounded; that family is called the ***quasinorm*** on X. For all x, x' in the quasinormed space X we define the inequality and equality relations by:

- $(x \neq x') \equiv \exists_{i\in I} (\|x - x'\|_i > 0)$; and
- $(x = x') \equiv \forall_{i\in I} (\|x - x'\|_i = 0)$.

Quasinorms were introduced by Johns and Gibson[2] [8] in their study of Orlicz spaces (such as L_∞), and are important in the constructive analysis of duality; see [4] (Chapter 7, Section 5).

An element x of the quasinormed space X is ***normable***, or ***normed***, if its ***norm***,

$$\|x\| \equiv \sup_{i\in I} \|x\|_i \,, \tag{1}$$

exists. The ***unit ball*** of a quasinormed space $\left(X, (\| \|_i)_{i\in I}\right)$ is the set

$$B_X \equiv \{x \in X : \forall_{i\in I} (\|x\|_i \leqslant 1)\} .$$

If every element of a quasinormed space $\left(X, (\| \|_i)_{i\in I}\right)$ is normable, then we can regard X as a normed space relative to the norm defined at (1).

[1] Our view of *constructive mathematics* is that of Bishop [3, 4, 5]: namely and roughly, it is mathematics with intuitionistic logic together with some appropriate foundation such as Aczel's constructive (ZF) set theory [1, 2]. Note, for the record, that we accept the axiom of dependent choice.

[2] Those authors used the name 'pseudonorm', rather than our 'quasinorm'.

If $\left(Y, \|\ \|_j\right)_{j\in J}$ is also a quasinormed space, then a linear mapping $u : X \to Y$ is said to be ***bounded*** if there exists $c > 0$ (a ***bound*** for u) such that for each $x \in X$,

$$\forall_{j\in J}\forall_{\varepsilon>0}\exists_{i\in I}\left[c\,\|x\|_i > \|u(x)\|_j - \varepsilon\right]$$

For a normable element x of X, this last condition holds if and only if $c\,\|x\| \geqslant \|u(x)\|_j$ for each $j \in J$. In particular, if X, Y are normed spaces, then our definition of 'bounded linear mapping', given above, is equivalent to the usual one in the functional analytic literature.

Let X be a quasinormed space and let Y be a normed space. We define the natural quasinorm on the set $L(X,Y)$ of bounded linear mappings from X into Y as follows:

$$\|u\|_x \equiv \|u(x)\| \qquad (x \in B_X)\,.$$

When we regard $L(X,Y)$ as a quasinormed space, it is this quasinorm that we have in mind.

The particular situation that interests us in the present paper involves a quasinormed linear space $\left(X, (\|\ \|_i)_{i\in I}\right)$, the quasinormed dual space $X^* \equiv L(X, \mathbf{R})$, and the quasinormed second dual $X^{**} \equiv (X^*)^*$ of X. For each $x \in X$ we denote by $\widehat{x}$ the mapping $f \rightsquigarrow f(x)$, which is an element of X^{**}; also, for each $S \subset X$, we denote by $\widehat{S}$ the image of S under the ***natural embedding*** $x \rightsquigarrow \widehat{x}$ of X into X^{**}. Since each element of B_{X^*} has bound 1, we see that

$$\forall_{x\in X}\forall_{f\in B_{X^*}}\forall_{\varepsilon>0}\exists_{i\in I}\left(\|x\|_i > \|\widehat{x}\|_f - \varepsilon\right).$$

So if $\|x\|_i \leqslant c$ for all i, then c is a bound for $\widehat{x}$; in particular, if x is normable, then $\|x\|$ is a bound for $\widehat{x}$. We say that the natural mapping is ***norm-preserving*** if for each normable element x of X, the image $\widehat{x}$ is normable and $\|\widehat{x}\| = \|x\|$.

We call a quasinormed space $\left(X, (\|\ \|_i)_{i\in I}\right)$

- ***uniformly convex*** if for each $\varepsilon > 0$, there exists $\delta \in (0,1)$ such that if $x, y \in B_X$ and there exists $i \in I$ such that $\|x - y\|_i > \varepsilon$, then $\left\|\frac{1}{2}(x+y)\right\|_i < 1 - \delta$ for all $i \in I$;

- ***reflexive***[3] if for each normable element F of X^{**}, there exists a (perforce unique) element x of X such that $F = \widehat{x}$;

- ***pliant***[4] if for each $x \in X$, each $i \in I$, and each $\varepsilon > 0$, there exists $f \in B_{X^*}$ such that $f(x) > \|x\|_i - \varepsilon$.

The pliancy condition is equivalent to:

$$\forall_{x\in X}\forall_{i\in I}\forall_{\varepsilon>0}\exists_{f\in B_{X^*}}\left(\|\widehat{x}\|_f > \|x\|_i - \varepsilon\right) \tag{2}$$

—that is, the boundedness of the identity mapping $\mathfrak{i}$ between the quasinormed spaces $\left(X, \left(\|\ \|_f\right)_{f\in B_{X^*}}\right)$ and $\left(X, (\|\ \|_i)_{i\in I}\right)$; since $\mathfrak{i}^{-1}$ is bounded (by the definition of *bounded linear mapping*), pliancy is actually equivalent to the bicontinuity of $\mathfrak{i}$.

[3]Our notion of 'reflexive' differs from that originally given by Bishop, who also posed in a different way the problem addressed by our main result (Theorem 3 below): he conjectured that the normable linear functionals on a uniformly convex Banach space X form a linear space, and that the normable elements of the dual of the latter space had the form $f \rightsquigarrow f(x)$ with $x \in X$; see [3] (page 295, Problem 8). Ishihara [6] has shown that the second part of Bishop's conjecture is provable if the first part is.

[4]*Pliancy* is a new notion, not found in the earlier literature.

It is easy to show that every Hilbert space is pliant. Classically, every normed space is pliant. Constructively, a normed space is pliant if it is either separable (this is a special case of Proposition 1 below) or else has Gâteaux differentiable norm ([7], Lemma 1; [5], Prop. 5.3.6).

We define a quasinormed space $\left(X, (\| \ \|_i)_{i \in I}\right)$ to be

- ***separable*** if it contains a sequence $(x_n)_{n \geqslant 1}$ that is ***dense***, in the sense that

$$\forall_{x \in X} \forall_{\varepsilon > 0} \exists_n \forall_{i \in I} \left(\|x - x_n\|_i < \varepsilon \right);$$

  and

- ***quasi-separable*** if for each $i \in I$ there exists a sequence $(x_n)_{n \geqslant 1}$ in X such that

$$\forall_{x \in X} \forall_{\varepsilon > 0} \exists_n \left(\|x - x_n\|_i < \varepsilon \right). \tag{3}$$

Of course, separable implies quasi-separable.

Proposition 1. *Every quasi-separable quasinormed space is pliant.*

Proof. Given a quasi-separable quasinormed space $\left(X, (\| \ \|_i)_{i \in I}\right)$, fix $i \in I$ and choose a sequence $(x_n)_{n \geqslant 1}$ in X such that (3) holds. Let X_i denote the space X taken with the seminorm $\| \ \|_i$, and with the equality and inequality defined by

$$\begin{aligned} x =_i y \quad &\Leftrightarrow \quad \|x - y\|_i = 0, \\ x \neq_i y \quad &\Leftrightarrow \quad \|x - y\|_i > 0. \end{aligned}$$

These data turn X_i into a separable *normed* linear space. Given $a \in X_i$ and $\varepsilon > 0$, we apply [5] (Prop. 5.3.1), to obtain a normable linear functional f on X_i with

$$\|f\|_i \equiv \sup \left\{ |f(x)| : x \in X, \|x\|_i \leqslant 1 \right\} = 1$$

such that $f(a) > \|a\|_i - \varepsilon$. If $x = y$ in the quasinormed space X, then $\|x - y\|_i = 0$; whence $x = y$ in the normed space X_i, and therefore $f(x) = f(y)$. Thus f is a linear functional on X. Moreover, since $\|f\|_i = 1$, for each $x \in X$ and each $\alpha > 0$ we have $|f(x)| - \alpha < |f(x)| \leqslant \|x\|_i$; hence $f \in B_{X^*}$. Moreover, $\widehat{a}(f) = f(a) > \|a\|_i - \varepsilon$. □

Inspection of the proof of Theorem (4.6), the Hahn-Banach theorem for separable spaces, on page 342 of [4] shows that the only place where separability is used is in the application of Corollary (4.5) preceding it, which asserts the pliancy of the space. On the other hand, a well-known application of the Hahn-Banach theorem to the functional $\lambda x \rightsquigarrow \lambda \|x\|$ on the space $\mathbf{R}x$ shows that if that theorem holds, then the space is pliant. Thus we see that the pliancy of arbitrary normed spaces is (constructively) equivalent to the Hahn-Banach theorem holding without the assumption of separability. The status of that version of the Hahn-Banach theorem remains open, as therefore does the pliancy of arbitrary normed spaces.

Lemma 2. *Let $\left(X, (\| \ \|_i)_{i \in I}\right)$ be a pliant quasinormed space, and x an element of X. Then x is normable if and only if $\widehat{x}$ is normable, in which case $\|x\| = \|\widehat{x}\|$.*

Proof. The desired result readily follows from the statement

$$\forall_{f\in B_{X^*}}\forall_{\varepsilon>0}\exists_{i\in I}\left(\|x\|_i > \|\widehat{x}\|_f - \varepsilon\right)$$

together with (2). □

A sequence $(x_n)_{n\geqslant 1}$ in a quasinormed space $\left(X, (\|\ \|_i)_{i\in I}\right)$

▷ ***converges to the limit*** $x \in X$ if

$$\forall_{\varepsilon>0}\exists_N\forall_{n\geqslant N}\forall_{i\in I}\left(\|x-x_n\|_i < \varepsilon\right);$$

▷ is a ***Cauchy sequence*** if

$$\forall_{\varepsilon>0}\exists_N\forall_{m,n\geqslant N}\forall_{i\in I}\left(\|x_m-x_n\|_i < \varepsilon\right).$$

Naturally, if every Cauchy sequence in X converges to a limit in X, then we call X ***complete***.

We now state our constructive version of the Milman-Pettis theorem:

Theorem 3. *Every complete, pliant, uniformly convex quasinormed space is reflexive, and the natural embedding into its second dual is a norm-preserving bijection of the set of normable elements of X onto the set of normable elements of X^{**}.*

Our proof of Theorem 3 (in particular, Lemma 5), is based on ideas of Ringrose [13]. We have the following preliminary lemmas.

Lemma 4. *Let X be a quasinormed space, F an element of $B_{X^{**}}$ with $\|F\| = 1$, and $\delta > 0$. Then there exist $x \in B_X$ and $f \in B_{X^*}$ such that $|F(f) + f(x)| > 2 - \delta$.*

Proof. Choose $f \in B_{X^*}$ such that $F(f) > 1 - \delta/3$. Since 1 is a bound for F, there exists $x \in B_X$ such that $F(f) - \delta/3 < \|f\|_x = f(x)$. Then

$$\begin{aligned} |F(f)+f(x)| &\geqslant F(f)+f(x) \\ &\geqslant 2F(f) - (F(f)-f(x)) \\ &> 2\left(1-\frac{\delta}{3}\right) - \frac{\delta}{3} = 2-\delta. \end{aligned}$$

□

Lemma 5. *If X is a uniformly convex quasinormed space, then the quasinormed space X^{**} is uniformly convex.*

Proof. Let $0 < \varepsilon < 1$, and pick $t \in (0,1)$ such that if $x, y \in B_X$ and there exists $i \in I$ such that $\|x-y\|_i > \varepsilon/4$, then $\left\|\frac{1}{2}(x+y)\right\|_i < 1 - 2t$ for all $i \in I$. Let $\delta \equiv t\varepsilon/4$. Consider $F, G \in B_{X^{**}}$ and $u \in B_{X^*}$ such that $F(u) - G(u) > \varepsilon$. Given f in B_{X^*}, we have

$$F(f) + G(f) = F(f+tu) + G(f-tu) - t\left(F(u) - G(u)\right).$$

Since F and G have bound 1, given $\alpha > 0$, we can pick $x, y \in B_X$ such that $F(f+tu) - \alpha < (f+tu)(x)$ and $G(f-tu) - \alpha < (f-tu)(y)$. Then

$$\begin{aligned} F(f)+G(f) &< (f+tu)(x) + \alpha + (f-tu)(y) + \alpha - t\varepsilon \\ &= f(x+y) + tu(x-y) - t\varepsilon + 2\alpha. \end{aligned} \tag{4}$$

Either $3\varepsilon/8 < u(x-y)$ or $u(x-y) < \varepsilon/2$. In the first case, pick $j \in I$ such that $\|x-y\|_j > u(x-y) - \varepsilon/8 > \varepsilon/4$. Then for each $i \in I$ we have $\left\|\frac{1}{2}(x+y)\right\|_i < 1-2t$. Choosing i such that $\|x+y\|_i > |f(x+y)| - \alpha$, and referring to (4), we now obtain

$$\begin{aligned} F(f)+G(f) &< (\|x+y\|_i + \alpha) + 2t - t\varepsilon + 2\alpha \\ &< 2(1-2t) + 2t - 0 + 3\alpha = 2(1-t) + 3\alpha \end{aligned}$$

and therefore

$$F(f)+G(f) < 2(1-\delta) + 3\alpha. \tag{5}$$

In the case $u(x-y) < \varepsilon/2$ we have, by (4),

$$F(f)+G(f) < 2 + \frac{t\varepsilon}{2} - t\varepsilon + 2\alpha = 2 - \frac{t\varepsilon}{2} + 2\alpha,$$

which, since $t\varepsilon/2 = 2\delta$, implies that (5) holds. Since $\alpha > 0$ is arbitrary, we now see that $|F(f)+G(f)| \leqslant 2(1-\delta)$ for all $f \in B_{X^*}$. □

This brings us to the (constructive) proof of Theorem 3:

Proof. Let $\left(X, (\| \ \|_i)_{i\in I}\right)$ be a complete, pliant, uniformly convex quasinormed space, and F a normed element of X^{**}; we may assume that $\|F\| = 1$. Given $\varepsilon > 0$, and using Lemma 5, compute $\delta \in (0,1)$ such that if $F, G \in B_{X^{**}}$, $u \in B_{X^*}$, and $F(u) - G(u) > \varepsilon$, then $\left|\frac{1}{2}(F(f)+G(f))\right| \leqslant 1-\delta$ for all $f \in B_{X^*}$. By Lemma 4, there exist $x \in B_X$ and $f \in B_{X^*}$ such that $|F(f)+f(x)| > 2(1-\delta)$; whence $|F(g) - \widehat{x}(g)| \leqslant \varepsilon$ for each $g \in B_{X^*}$. Since $\varepsilon > 0$ is arbitrary, we can construct a sequence $(x_n)_{n\geqslant 1}$ in B_X such that

$$\forall_n \forall_{g\in B_{X^*}} \left(|(F - \widehat{x}_n)(g)| < 2^{-n-3}\right).$$

Then for $m \geqslant n$ and all $g \in B_{X^*}$ we have

$$|g(x_m - x_n)| = |\widehat{x}_m(g) - \widehat{x}_n(g)| < 2^{-m-3} + 2^{-n-3} \leqslant 2^{-n-2}.$$

But X is pliant, so for each $i \in I$ we can pick $g \in B_{X^*}$ such that $g(x_m - x_n) > \|x_m - x_n\|_i - 2^{-n-2}$. Hence

$$\|x_m - x_n\|_i < 2^{-n-1} \qquad (m \geqslant n). \tag{6}$$

Since this holds for each $i \in I$, we see that $(x_n)_{n\geqslant 1}$ is a Cauchy sequence in B_X and therefore converges to a limit x in our complete quasinormed space X. For each n, letting $m \to \infty$ in (6), we see that $\|x - x_n\|_i \leqslant 2^{-n-1}$ for all i. Given $g \in B_{X^*}$, we can find i such that $\|x - x_n\|_i > |g(x - x_n)| - 2^{-n-3}$; whence

$$\begin{aligned} |(F - \widehat{x})(g)| &\leqslant |(F - \widehat{x}_n)(g)| + |g(x - x_n)| \\ &\leqslant 2^{-n-3} + \|x - x_n\|_i + 2^{-n-3} \\ &< 2^{-n-2} + 2^{-n-1} < 2^{-n}. \end{aligned}$$

Since g and n are arbitrary, we conclude that $F = \widehat{x}$. It follows from Lemma 2 that x is normable, with $\|x\| = \|F\|$. This leads almost immediately to the second conclusion of the statement of our theorem. □

Since, as we observed earlier, a normed space is pliant if either it is separable or else it has a Gâteaux differentiable norm, we have the following special cases of Theorem 3.

Corollary 6. *A separable, uniformly convex Banach space is reflexive.*

Corollary 7. *A uniformly convex Banach space with Gâteaux differentiable norm is reflexive.*

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