Chapter 16
Inferential Quantification and the ω-Rule

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Abstract Logical inferentialism maintains that the formal rules of inference fix the meanings of the logical terms. The categoricity problem points out to the fact that the standard formalizations of classical logic do not uniquely determine the intended meanings of its logical terms, i.e., these formalizations are not categorical. This means that there are different interpretations of the logical terms that are consistent with the relation of logical derivability in a logical calculus. In the case of the quantificational logic, the categoricity problem is generated by the finite nature of the standard calculi and one direction in which it can be solved is to strengthen the deductive systems by adding infinitary rules (such as the ω-rule), i.e., to construct a *full formalization*. Another main direction is to provide a *natural semantics* for the standard rules of inference, i.e., a semantics for which these rules are categorical. My aim in this paper is to analyze some recent approaches for solving the categoricity problem and to argue that a logical inferentialist should accept the infinitary rules of inference for the first order quantifiers, since our use of the expressions "all" and "there is" leads us beyond the concrete and finite reasoning, and human beings do sometimes employ infinitary rules of inference in their reasoning.

Keywords Quantification · Logical inferentialism · Categoricity · Natural semantics · Infinitary rules

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16.1 Full Formalization, Natural Semantics, and MT-Inferentialism

To *fully formalize* a logical theory, that is already defined on the basis of a semantical system, by a formal calculus means to show that (i) every logical truth is a theorem in the calculus, (ii) every relation of logical consequence is represented by a relation of logical derivability in the calculus, and (iii) all logical terms have the intended semantical meanings in all the permissible interpretations of the calculus.

To provide a *natural semantics* for a logical theory that is already defined by a formal calculus means to uniquely read off the semantical meanings of the logical terms from the deduction rules or axioms that govern their use in the calculus.

(Carnap, 1943) aimed to provide a full formalization of classical logic while (Garson, 2013) aimed to determine the natural semantics for the axiomatic, natural deduction, and multiple-conclusions calculi of classical logic. Both of them formally converge in the results that (i) the multiple conclusions calculi are full formalizations of propositional logic, as defined by the normal truth-tables, and that this semantics is their natural semantics, and (ii) that the transfinite multiple conclusions calculi of first-order logic are full formalizations for the substitutional semantics of the first order quantifiers and that it is their natural semantics. Still, Garson shows that the natural semantics for the standard single conclusion natural deduction formalizations of classical logic is the intuitionistic one, in the case of propositional logic, and the sentential one (as he defines it), in the case of first order quantificational logic.

Model-theoretic logical inferentialism maintains that the meanings of the logical terms are determined by the rules of inference that govern their use and that these meanings can be characterized in model-theoretic terms (reference, truth, validity etc).[1] In the case of the first-order quantifiers, the root of the categoricity problem was initially identified by (Carnap, 1937, 1943) in the finite nature of the standard calculi and one direction in which this problem can be solved is to construct a *full formalization* by strengthening the deductive systems, for instance by adding infinite rules (such as the ω-rule). Another main direction for solving this problem is to provide a *natural semantics* for the standard rules of inference, i.e., a semantics for which these rules are categorical. My aim in this paper is to analyze some recent approaches for solving the categoricity problem and to argue that a model-theoretic logical inferentialist should accept the infinitary rules of inference for the first order quantifiers. Briefly, my argument is that since logical inferentialism maintains that our use of the logical expressions in inferences is what determines their meanings, our use of the expressions "all" and "there is" in ordinary mathematical reasoning leads us beyond the intuitive and finite reasoning, and human beings do sometimes

[1] Proof-theoretic logical inferentialism accepts the idea that the meanings of the logical terms are given by their proof theoretic roles, but maintains that these meanings should be characterized only by using proof-theoretic concepts.

employ infinitary rules in their reasoning, then a logical inferentialist should accept the infinite rules for the quantifiers.

The paper is structured as follows: I shall introduce in the second section (Carnap, 1937, 1943)'s approach for providing a full formalization of classical logic and in the third section I shall present (Garson, 2001, 2013)'s approach for finding out the natural semantics for classical propositional and first-order logical calculi. In the fourth section I shall critically discuss some other recent approaches for obtaining categoricity (McGee, 2000, 2006, 2015; Bonnay & Westerståhl, 2016; Warren, 2020; Murzi & Topey, 2021), and in the fifth section I shall explore Carnap's view on the legitimacy of using infinitary rules of inference in a logical calculus, arguing at the same time that a logical inferentialist should accept the infinitary rules of inference for the first order quantifiers in his logical framework.

16.2 Carnap's Full Formalization of Classical Logic

In *Logical Syntax of Language*, Rudolf Carnap suggested a new standpoint, called by him *the principle of tolerance*, which has a constitutive inferentialist flavour:

> Let any postulates and any rules of inference be chosen arbitrarily; then this choice, whatever it may be, will determine what meaning is to be assigned to the fundamental logical symbols. (Carnap, 1937: xv)

Later on, once he was acquainted with the developments in semantics, Carnap became interested in the relation between logical syntax and semantics and, at this point, the principle of tolerance encountered a first restriction:

> While in constructing a calculus we may choose the rules arbitrarily, in constructing a calculus K in accordance with a given semantical system S we are not entirely free. In some essential respects the features of S determine those of K, although, on the other hand, there is still a freedom of choice left with respect to some features. Thus logic -if taken as a system of formal deduction, in other words, a calculus –is in one way conventional, in another not. (Carnap, 1942: 218–19)

For describing the process of constructing a calculus for a system of logic which is previously semantically defined, (Carnap, 1943: viii, 95–96) introduced the concept of *full formalization*. As we mentioned above, to *fully formalize* a logical theory by a formal calculus means to show that every logical truth is a theorem in the calculus, every relation of logical consequence is represented by a relation of logical derivability in the calculus, and all logical terms have the intended semantical meanings in all the permissible interpretations of the calculus.

(Carnap, 1943) wanted to see whether the standard formalizations of logic are indeed full formalizations. His discoveries were negative, since he found out that the standard (i.e., single conclusion and finite) formalizations of propositional and first-order logic allow for what he called *non-normal interpretations*, i.e., interpretations or valuations for which the logical calculi preserve their soundness, but in which most of the logical signs have different meanings than the standard ones, as provided by their standard semantics (the normal truth tables and the substitutional semantics).[2]

For the propositional calculus, Carnap discovered two kinds on non-normal interpretations: one in which all the sentences are true (and thus both a sentence and its negation are true) and one in which a sentence and its negation are both false (and thus their disjunction is true). These non-normal interpretations arise because the semantic principles of excluded middle and of non-contradiction are not syntactically represented by the standard single-conclusion propositional calculi. More precisely, the semantical concepts of L-exclusive (i.e., a sentence and its negation cannot both be true) and L-disjunct (i.e., a sentence and its negation cannot both be false) are not formalized by these propositional calculi.

To eliminate these non-normal interpretations, Carnap introduced new syntactical concepts based on the notion of *junctive*. A junctive is a potentially infinite sentential class that can be constructed either conjunctively (as it is usually done when we consider the class of premises of an argument) or disjunctively (when we consider the class of conclusions of an argument). For obtaining a fully formalized propositional logic, Carnap introduced two new rules of deduction:

1. $A_i \vee A_j \vdash \{A_i, A_j\}^{\vee}$
2. $V^{\&} \vdash \Lambda^{\vee}$

The first rule fixes the fourth line of the normal truth table for disjunction by requiring that at least one disjunct is true when the disjunction is true, and thus eliminates the second kind of non-normal interpretations.[3] The second rule is a rule of refutation that forbids having all the sentences true in a logical system. ⌜$V^{\&}$⌝ is the universal conjunctive, which is semantically defined as being true when all sentences are true, and ⌜$\wedge^{\vee}$⌝ is the null disjunctive, which is semantically by definition false.[4] Thus, if we consider an interpretation in which all sentences are true, then rule 2) becomes unsound, since this interpretation will make the premise of the rule true and the conclusion is by definition always false. Consequently, this interpretation will not count as a permissible one.

The possibility of non-normal interpretations in predicate logic arises because, in the standard formalizations of first-order logic, a universally quantified sentence is

[2] (Carnap, 1943) *reads off* the meanings of the logical terms from the formal calculi by using, what (Garson, 2013: 13) calls, a deductive model, i.e., a set of valuations V such that every provable sequent in the calculi is satisfied by every valuation in V (see Sect. 16.3 below).

[3] For a recent criticism of using multiple conclusions in a proof-theoretic framework see (Steinberger, 2011) and for a recent defence see (Dicher, 2020).

[4] When sentential classes are considered, (Carnap, 1942: 38–39, 1943: 107–108) introduced the following special cases: the universal class that comprises all sentences could be constructed either conjunctively (⌜$V^{\&}$⌝ is the universal conjunctive, which is true if all sentences are true) or disjunctively (⌜$V^{\vee}$⌝ is the universal disjunctive, which is true if at least one sentence is true), and the null class that comprises no sentence could also be taken either conjunctively (⌜$\Lambda^{\&}$⌝ is the null conjunctive, which by definition is true, since it contains no false statement) or disjunctively (⌜$\Lambda^{\vee}$⌝ is the null disjunctive, which by definition is false, since it contains no true statement).

not deductively equivalent (C-equivalent, in Carnap's terms) with the class formed by the conjunction of all the instances of the operand and an existentially quantified sentence is not C-equivalent with the disjunctive class of all the instances of the operand. The deductive implication from the universal sentence to the whole conjunction of its instances is guaranteed by the universal instantiation rule and conjunction introduction, but the other standard rules or axioms do not guarantee the converse. It should be noted, however, that this is not only a limitation of the substitutional semantics[5], since Carnap worked with a denumerable domain in which every object is named by an individual constant from the language, but rather is a consequence of the finite nature of the calculus, since we have no rule in the standard formalizations which licences the deductive transition from an infinite number of premises to a conclusion (when the conclusion does not already follow from a finite subset of the initial set of premises).

Carnap introduces the interpretations for quantificational logic in terms of translations. He provides two translations for the language of predicate logic such that all atomic sentences, all molecular sentences, and all sentential junctions are correlated with themselves by the translations, but one translation correlates ⌜(∀x)Fx⌝ with a proper universal quantifier, i.e., the universal quantifier with its standard meaning (⌜every object is F⌝), while the other correlates it with an improper one that is richer in content, i.e., with ⌜every object is F, and b is G⌝. Of course, the two translations are not equipollent, i.e., they do not have the same content, but they are still consistent with the relation of logical derivability. This means that the formal deductive system remains sound under these interpretations.

Let us consider a model with a denumerable domain of objects whose elements are all named by individual constants from the language. We can interpret, i.e., assign truth-conditions to, the sentence ⌜(∀x)Fx⌝ in the standard way and the calculus will be sound, but we can also interpret it as having the same truth-conditions as the sentence ⌜every object is F, and b is G⌝ has and the calculus will also remain sound. In the latter case, whatever is derivable from ⌜(∀x)Fx⌝ when it is standardly interpreted, will also be derivable from it when it is interpreted as ⌜every object is F, and b is G⌝, and thus the universal elimination rule remains valid. In addition, if ⌜Fa⌝ follows from Γ for an arbitrary a, and we take ⌜Gb⌝ to be true, then the universal introduction rule will also preserve its validity. In order to preserve the duality of the quantifiers, i.e., (∀x)Fx⊣⊢~(∃x)~Fx, in this non-normal interpretation the existential quantifier will be interpreted as ⌜at least one individual is F, or b is not G⌝.[6]

[5] In addition, non-normal interpretations are possible even if we consider the objectual interpretation of the quantifiers (see Garson's Theorem 14.3 below).

[6] These interpretations are discussed from a broader perspective in (Carnap, 1937: 231–32). See also (Carnap, 1943: 140, 148–150).

In order to block the possibility of these non-normal interpretations, Carnap introduced two new rules of inference[7]:

1. $\{A_i\ (^{i}{}_{k})\}^{\&} \vdash A_i$
2. $(\exists i_k)A_i \vdash \{A_i\ (^{i}{}_{k})\}^{\vee}$, where i_k is the only free variable in A_i.

The first rule stipulates that a sentence A_i containing a free variable i_k is directly derivable from the infinite conjunctive set of all its instances. In the next step, the rules of (Carnap, 1943)'s formalism (T28-4b) license the derivation of the sentence ⌜$(\forall i_k)A_i$⌝ from A_i, where i_k is the only free variable in A_i (T30-2 in (Carnap, 1943: 146)). In this way, by transitivity, we can derive a universal sentence from the potentially infinite class of all its instances and, thus, the deductive equivalence between a universal sentence and the class of all its instances is established. Due to the fact that a universal sentence is derived from an infinite number of premises, we have an implicit use of the ω-rule, i.e., the rule that licences the transition from Fa, Fb, Fc … (for all individual constants in a denumerable language) to (∀x)Fx. The second rule stipulates that we can pass from an existentially quantified sentence to the disjunctive class of all its instances. By adding at least one of these two rules to the standard formalizations, the deductive equivalence between the universal (and existential) sentences and all their conjunctive (disjunctive, respectively) instances is obtained, and thus the possibility of non-normal interpretations for the quantifiers disappears. Being syntactically equivalent, the quantified sentences and their classes of instances will also be semantically equivalent.

Certainly, since it allows the transition from an infinite set of sentences to a new sentence, this full formalization of logic employs non-effective syntactical rules. Both (Church, 1944: 498) and (Fitch, 1944: 454) raised scepticism regarding the use of transfinite rules in the construction of a logical calculus and Church argued that these rules should be excluded from elementary syntax, i.e. from that part of syntax that must be known at least implicitly by a speaker in order to use the language correctly, since "it is clearly not possible for the users of a language to systematically follow a non-effective rule in practice". He agreed however that these rules are important in theoretical syntax, i.e., the mathematical theory of the object language.[8] Anticipating Church's criticism, (Carnap, 1943: 113–114) maintained, however, that there is no fundamental change in method, since the metalanguage is necessary even in the standard construction of calculi when we state a rule of inference (a rule being basically a meta-sentence which states that a sentence of a certain form is derivable from sentences of other given forms). We shall return to the

[7] Since the existential quantifier can be defined as ⌜~∀~⌝, one rule of inference is sufficient. This remark applies to all the approaches for obtaining categoricity discussed below.

[8] Likewise, at that time (Tarski, 1933/1983: 295) believed that the use of infinitary rules like the ω-rule "cannot easily be brought into harmony with the current view of the deductive method, and finally that the possibility of its practical application in the construction of deductive systems seems to be problematic in the highest degree". Later on, however, Tarski became more flexible on this matter and even investigated calculi with infinite long expressions; see (Scott & Tarski, 1958) and (Tarski, 1958).

possibility of following infinitary rules of inference in the last section of this paper, when Carnap's view on the legitimacy of using infinitary rules of inference will be analyzed.

16.3 Garson's *Natural Semantics*

Natural semantics is defined by (Garson, 2001:114–15, 2013: 49–50) as a method of providing possible semantic values and *reading off* the semantic properties of the logical terms from the deductive rules that govern their uses –his investigations being in the spirit of a model-theoretic inferentialism. More precisely, the natural semantics is that semantics for which the deductive rules are categorical, i.e., the rules are sound and complete for that semantics and the logical terms are provided with unique meanings.

The usual axiomatic and natural deduction calculi for propositional and first-order logic are sound and complete with respect to the normal truth tables and, respectively, to the substitutional and objectual interpretations, but most of the propositional operators and both the first order quantifiers are provided by these semantics also with non-normal meanings. As we mentioned above, (Carnap, 1943) saw this problem as an asymmetry between syntax and semantics and since he took the standard semantics as unproblematic, his interest was to strengthen the calculi by introducing new syntactical concepts. Garson modifies the standard semantics of the classical logical terms in order to make the classical natural deduction calculi categorical. More precisely, his interest is in finding out what the natural deduction rules actually say about the meanings of the logical terms whose use is govern by them.

To read off the meanings of the logical terms from the logical calculi, (Garson, 2013) distinguishes between three kinds of models, i.e., sets of valuations, in terms of which the expressive power of a set of rules can be formulated, namely, deductive, local, and global models:

Deductive Model: V is a deductive model of a set of rules S iff all the provable sequents of S are all V-valid.

Local Model: V is a local model *of a rule R* iff R preserves V-satisfaction; where a rule R preserves V-satisfaction iff for each member v of V, v satisfies R. A valuation v satisfies R iff whenever v satisfies the inputs of R, it also satisfies the output of R.

Global Model: V is a global model of a set of rules S iff each rule of S preserves V-validity; where a rule R preserves V-validity iff whenever all inputs of R are V-valid, then so is R's output.

The trouble with the deductive models is that they are insensitive to the way in which a logical calculus is formulated. For instance, if negation has a normal meaning, then all the other propositional operators also have only normal meanings.[9] But in this way we know almost nothing about what the rules for each connective say about the meaning of *that* connective. The local models are more adequate for expressing what a set of rules actually say about the operators whose use is governed by them because they can be used relative to a certain set of rules. Garson argues that the classical natural deduction rules for the propositional operators indeed uniquely determine the standard meanings of these terms if the local models are used, but he identifies two main problems with these models[10], and this is why his option is for using the global models.

By adopting the global models, Garson is committed to the assumption that the rules of inference are validity preserving relative to a given semantics, i.e., if the premises are semantically valid (V-valid), then the conclusions obtained from them should also be semantically valid.[11]

The general idea of Garson's approach is that every system of rules S expresses a condition [S] on the model V. This condition [S] determines the canonical model for S, i.e., the set of all valuations that satisfy S, and (Garson, 2013: 53) proves that each system of rules S is adequate (sound and complete) with respect to its canonical model [S]:

[S] Adequacy Theorem. $H \vdash_S C$ iff $H\ _{[S]}\ C$

If the condition [S] is equivalent to the intended semantics of the logical terms ($\|S\|$), then [S] is a natural semantics for S, i.e., $[S] = \|S\|$.

In the case of propositional logic, Garson acknowledges the fact that if we use the global models, then the standard natural deduction rules for the classical propositional operators do not uniquely determine their standard meanings and he shows that the natural semantics for the classical propositional natural deductive rules is the intuitionistic one.[12]

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[9] If negation is normal, then disjunction is also normal, otherwise the Disjunctive Syllogism Rule (AvB, ~ A ⊢ B) would become unsound (i.e., if both "A" and "B" are false and negation is normal (thus, "~A" is true), then "AvB" cannot be true). However, since negation and disjunction form a functionally complete set of connectives, then all the other connectives will be normal.

[10] The problems with the local models that (Garson, 2013: 42–43) identifies are that of incompleteness in the case of classical propositional logic and that the local models do not generalize properly at the quantificational level. We shall discuss these problems in Sect. 16.4.4) below when the approach of obtaining categoricity *by convention* will be explored.

[11] Certainly, a rule of inference in the sequential format says that if a sequent is provable, then another sequent is also provable. Thus, if a proved sequent is taken to be valid, then the requirement of validity preserving seems reasonable. Some authors, as (Bonnay & Westerståhl, 2016: 724), consider it to be too strong since it involves a complex grasp of logical consequence, in particular that it requires an understanding of validity preserving mechanisms.

[12] More generally, Garson shows that: (1) if we consider the axiomatic formalizations of propositional logic, then they fail to uniquely determine the standard meanings of the propositional operators no matter what kind of models (deductive, local, global) we use; (2) the multiple conclusions sequent formalizations determine the classical meanings no matter what models of the three we use; (3) if we consider the natural deduction format and the global models, then the

For the first order quantifiers, (Garson, 2013: 237) proves that neither the substitutional (‖s∀‖), nor the objectual (‖d∀‖) semantics are uniquely determined by the standard natural deduction rules for the quantifiers and he introduces a new semantics that is supposed to count as their natural semantics. His system of natural deduction for the universal quantifier, i.e. (S∀), consists of three rules that make no use of individual constants: the introduction and elimination rules, and a structural rule that allows the substitution of a variable with another. His proof of non-categoricity consists in providing a classical valuation v* that satisfies the rules, but violates both these semantics:

Theorem 14.3 S∀ does not express ‖s∀‖, nor does it express ‖d∀‖.

Proof Consider [S∀] the canonical model for S∀, i.e. the set of all valuations over wffs of a language L that satisfy S∀. By the [S] Adequacy Theorem, [S∀] is a model of S∀. The set {A^y/x: y is a variable of L} ∪ {~∀xA} is consistent in S∀, and the set *e* of all wffs B such that {A^y/x: y is a variable of L} ∪ {~∀xA} ⊢ B is deductively closed and so a member of [S∀]. The set *e* however, although it contains A^y/x for each variable y, it does not contain ∀xA on pain of inconsistency. The representing function for set e is the classical valuation v*, which is a member of [S∀], but violates ‖s∀‖. Likewise an objectual model <D,V> that violates ‖d∀‖ is obtained by taking D as the set of all variables of L and v* will deliver the same result.

The idea that the set ⌜{A^y/x: y is a variable of L} ∪ {~∀xA}⌝ is consistent in the natural deduction system S∀ is analogous with (Carnap, 1943: 149)'s observation that there is no axiom or rule in the standard formalizations of predicate logic that legitimates the deductive equivalence between a universal sentence and the class of all its instances. Garson's proof strategy is to show that ‖s∀‖ is not forced by S∀. To show ‖s∀‖ from left to right (i.e. if v(∀xA) is true, then for all variables y of L, v(A^y/x) is true), we assume that v(∀xA) is true, and by the soundness of ∀-elimination rule, each instance will be true. The problem is to show ‖s∀‖ from right to left. By contraposition, if we assume that v(∀xA) is false, it should follow that v(A^y/x) is false. However, this is precisely what v* shows, that v*(∀xA) is false, although v*(A^y/x) is true for each variable y.[13]

meanings of the operators is the intuitionistic one. If we use the local models however, then the meanings of the operators are the classical ones.

[13] If we consider the ∀-introduction rule and the set Γ, which does not contain x free, the validity of this rule tells us that: if $\Gamma \models_{[S\forall]}$ A, then $\Gamma \models_{[S\forall]}$ (∀xA). Since v*(A^y/x) is true for each variable y, and v*(∀xA) is false, for preserving the validity of the rule, v*(Γ) has to be false. Hence, the global validity of the rules from S∀ is consistent with this valuation v*, but v* provides the universal quantifiers with a different meaning than those defined by the standard substitutional (‖s∀‖) or objectual (‖d∀‖) semantics. Therefore, these semantics cannot be the natural semantics for S∀. Moreover, for the same reason, this valuation v* seems to do the same thing even if we consider the local validity of S∀. The valuation v* preserves the sequent satisfaction of the (meta)rule of ∀-introduction (provided that v*(Γ) is false), and thus the rule is locally valid, while the universal quantifier is false even though its instances are true.

(Garson, 2013: 217) introduces a new semantics which is supposed to be the natural semantics for his natural deduction rules for the universal quantifier. This semantics provides, what Garson calls, *the sentential interpretation* of the quantifiers and has an intensional flavor:

$\|\forall\|$ v(∀xA) = t iff every v' in V, if v $\leq_x$ v', then v'(A) = t, where v $\leq_x$ v' holds exactly when v' is an extension of v save for the formulas containing x free.

$\|\leq_x\|$ v $\leq_x$ v' iff for every wffs A which does not contain x free, if v(A) = t, then v' (A) = t.

According to this sentential interpretation, in order to calculate the value of v(∀xA), one has to check all the extensions v' of v from V. (Garson, 2013: 238, Theorem 14.4.1) proves that the system S∀-, i.e. the introduction and elimination rules for the quantifiers without the structural rule for substituting variables, expresses $\|\forall\|$. However, S∀- does not preserve preserve $\|\forall\|$-validity, because there are models V which contain a single valuation v such that v(Ax) = true, v(Ay) = false, and in this case the only valuation v' such that v $\leq_x$ v' is v itself, which will make v(∀xAx) true. Thus, the structural rule of S∀, which allows the substitution of a variable with another (Sub), does not preserve $\|\forall\|$-validity. If the condition $\|$Sub$\|$ defined by the structural rule Sub is added to the condition $\|\forall\|$ previously defined, then the system S∀ expresses $\|$S∀$\|$, i.e., the conjunction of $\|\forall\|$ and $\|$Sub$\|$. $\|$Sub$\|$ guarantees the fact that if v(A^y/x) is false, then for some extension v' in V, v $\leq_x$ v' and v'(A) is false. Consequently, $\|$S∀$\|$ is the natural semantics for S∀.

As Garson remarks, however, the semantical definition $\|\forall\|$ works as a condition for a modal operator, where $\leq_x$ plays the role of the accessibility relation. More precisely, $\leq$ is an analogue of the accessibility relation $\subseteq$ from Kripke's models for intuitionistic logic. This is why the sentential interpretation has an intensional character. For this reason, a classical logical inferentialist, who is interested in determining the classical meanings of the quantifiers, should be reluctant in accepting an intensional semantics for what is taken to be the benchmark of extensionality, i.e., first order logic.

(Garson, 2013: section 14.9)'s results converge with Carnap's in the idea that if we add the ω-rule in the deductive systems of first-order logic, then the substitutional semantics is their natural semantics. Moreover, the ω-rule can be formulated in an axiomatic manner, i.e.{A^y/x: y is a variable of L} ⊢ ∀xA, and, thus, its use does not seem to be so problematic, since it can be introduced in a sequent proof in a single line. One immediate consequence of having the ω-rule is that the relation of logical consequence will no longer be compact and, thus, the notions of inference and proof will go beyond what is finite. However, as I will argue in Sect. 16.5 below, if the meanings of the quantifiers lead us beyond what is intuitive and finite, then a logical inferentialist should develop and accept formal logical tools that are able to represent all the semantical properties of the quantifiers. Hence, I think that a fully fledged logical inferentialist should be disposed to renounce to compactness, as (Carnap, 1943) did, if the symmetry between syntactical and semantical methods is to be attained.

16.4 Some Recent Logico-Philosophical Approaches for Obtaining Categoricity

In this section I shall discuss four recent approaches for obtaining categoricity, namely, (i) the open-endedness approach, (ii) the topic-neutrality approach, (iii) the open-ended unrestricted inferentialism, and (iv) the categoricity by convention approach.[14] The main idea that this section will argue for is that these four approaches succeed in showing that the universal quantifier ranges over the entire domain or that it has an unrestricted interpretation, but they let untouched the non-normal interpretation that Carnap pointed out to, namely, an interpretation in which the universal quantifier ranges over the entire domain, but it is provided by this interpretation with an improper meaning.

16.4.1 McGee's Open-Endedness Approach

Vann McGee argued in a couple of papers (2000, 2006, 2015) that if we take the natural deduction rules for the propositional operators and for the first-order quantifiers to be open-ended, i.e. if they are sound not only within a certain language, but they remain sound in any mathematically possible extension of that language, then they uniquely determine the 'semantic role' of these logical terms.

In the case of the propositional logic, McGee assumes that at least one sentence is false, and thus excludes by a semantic assumption the first kind of non-normal interpretations. In order to exclude the second type of non-normal interpretations, he introduces the semantic assumption that *for any class of models there is a sentence in a mathematically possible language such that that sentence is true only in these models.* However, strictly taken, this assumption is false, because if we consider the non-normal class of models in which a sentence and its negation are both false, only the theorems of propositional logic will be true in these models. But these theorems are also true in the normal models. Therefore, since there is no sentence which is true *only* in the class of non-normal models, it follows, contrary to McGee's assumption, that there is a class of models such that there is no sentence true only in them.[15]

In the case of the first order quantifiers, (McGee, 2000: 71) introduces a particular form of the semantic assumption mentioned above in order to show that the natural deduction rules for the quantifiers uniquely determine their meanings:

> If the constant c does not occur in ψ, then the class of models in which ψ is true is closed under c-variants, and, conversely, that, if a class of models is closed under c-variants, then there is, in some mathematically permissible language, a sentence not containing c that is true in all and only the members of the class.

[14] The latter two approaches are discussed in details in Brîncuș (2024).

[15] See (Brîncuș, 2021) for a discussion of this second assumption of McGee's approach.

The semantics for the quantifiers is formulated in (Mates, 1972: 60)'s manner, without using the notion of satisfaction. The central notion in this semantics is that of *c-variant interpretation*. Two interpretations I and I' are c-variants if and only if they are the same or differ only in what they assign to c. The truth conditions for the universal quantifier in terms of *c-variant interpretations* are the following:

$\ulcorner(\forall x)\phi x\urcorner$ is true under the interpretation I if and only if ϕc is true under every c-variant of I.

McGee's arguments meant to show that the rules for the universal quantifier do indeed fix its standard truth-conditions go as follows:

Sufficiency:

(1) If c does not occur in $\ulcorner(\forall x)\phi x\urcorner$, and $\ulcorner(\forall x)\phi x\urcorner$ is true, then it is true in every c-variant of I.
(2) $\{(\forall x)\phi x\}\vdash \phi c$.
(3) Therefore, ϕc is true in every c-variant of I.

The sufficiency direction in McGee's argument is unproblematic, since the validity of the $\forall$E-rule guarantees that each instance is true in all the models in which the universal sentence is true. Since the rule is taken to be open-ended, then for every new individual constant from an extension of the initial language, the instance formed with it will be a consequence of the universal sentence and, thus, true in every c-variant of I.

Necessity:

(1) Let ϕc be true in every c-variant of I and let ψ be a sentence, in which c does not occur, that is true in all and only the c-variants of I.
(2) Thus, ϕc is true in all the models of ψ and, consequently, $\{\psi\} \models \phi c$.
(3) From (2), by the $\forall$I-rule, $\{\psi\} \vdash (\forall x)\phi x$.
(4) Thus, since ψ is true in all the c-variants of I, $\ulcorner(\forall x)\phi x\urcorner$ is also true in the c-variants of I.

The necessity direction, however, is problematic. If we accept in step (1) the assumption formulated above, and ϕc is true in all c-variants of I, then there is no model in which ψ is true but ϕc is false, i.e. $\{\psi\} \vdash \phi c$. Step (3) of McGee's argument assumes, however, that if $\{\psi\} \vdash \phi c$, then $\{\psi\} \vdash \phi c$. Since the argument talks about domains or classes of models, it involves higher-order logic and, therefore, completeness should not be taken for granted. Nevertheless, if we assume completeness, the $\forall$I-rule and transitivity allow us next to derive $\ulcorner(\forall x)\phi x\urcorner$ from ψ, and the validity of these derivations guarantees that $\ulcorner(\forall x)\phi x\urcorner$ is true in every c-variant of I, i.e., in the class of models in which ψ is true. However, does the conclusion of this argument guarantee that the universal quantifier is provided by the formal deductive rules with a *unique* meaning?

Let us take a look at the universal introduction rule. If the domain of quantification is finite and every object is named by an individual constant, a universal sentence is equivalent with the finite conjunction of its instances and we can easily infer a universal sentence from the conjunction of all its instances. However, when we have an infinite domain and there are objects in that domain which are not named in the language, a universal sentence is not a logical consequence of, and it is not logically derivable from, any finite conjunction of its instances.

Someone may think that Gentzen's formulation of the (∀I)-rule solves this problem. The problem of the universal introduction is how to derive a universal sentence ⌜(∀x)ϕx⌝ from a collection of premises Γ about a domain of individuals D. Gentzen's rules provide an answer to this problem through an analogy to a problem about sentential reasoning. If ϕ is a formula with one free variable, then we may label by $D_{\&}\phi$ the ϕ-conjunction over D, i.e., the conjunction formed such that for each object α in D there is a conjunct ϕa in $D_{\&}\phi$. At this point, a universal sentence can be introduced if we succeed to derive all its instances from the set of premises Γ and then apply the conjunction introduction rule. Certainly, as we may have an infinite domain, the reduction of the universal to conjunction cannot be fruitfully conducted. Gentzen's formulation of the rules, however, solves this problem by suggesting that, instead of deriving all the instances, it is enough to derive only one instance, but "in a way which would allow for the derivation *in the same way of any other* conjunct" (Forbes, 1993: 24). The conditions imposed on the instantial individual constant in the formulation of the introduction rule for the universal quantifies guarantee this *instance-invariant* derivation of one conjunct instead of any other.

We should immediately note, however, that the ∀I-rule does not establish the deductive implication from $D_{\&}\phi$ to ⌜(∀x)ϕx⌝. The rule simply says that if ⌜Γ ⊢ $D_{\&}\phi$⌝, then ⌜Γ ⊢ (∀x)ϕx⌝. But this does not imply that ⌜$D_{\&}\phi$ ⊢ (∀x)ϕx⌝. For instance, we may interpret Γ as being the universally quantified sentence when its meaning is provided by a non-normal interpretation, namely, ⌜$D_{\&}\phi$ & ψb⌝. In this case, ⌜($D_{\&}\phi$ & ψb) ⊢ $D_{\&}\phi$⌝, and , ⌜($D_{\&}\phi$ & ψb) ⊢ (∀x)ϕx⌝, but $D_{\&}\phi$ does not imply ⌜(∀x)ϕx⌝, i.e. ⌜$D_{\&}\phi$ & ψb⌝.

Hence, since the introduction rule for the universal quantifier fails to establish the deductive implication from $D_{\&}\phi$ to the universal sentence ⌜(∀x)ϕx⌝, the meaning of the universal quantifier is not uniquely determined by the rules. McGee's arguments for the necessity and sufficiency of the rules in uniquely determining the meaning of the universal quantifier (in terms of truth-conditions) remain insensitive in regard to a normal interpretation of ⌜(∀x)ϕx⌝ as ⌜ϕa_1 & ϕa_2 & ...⌝ (where a_1, a_2, ... are all the individual constants of the language that completely denote all the objects in the domain) and a non-normal interpretation of ⌜(∀x)ϕx⌝ as ⌜ϕa_1 & ϕa_2 & ... & ψb⌝. The validity of the ∀I-rule is consistent with both types of interpretations. The open-endedness requirement seems to establish that the interpretation of the universal quantifier is unrestricted (provided that everything is nameable and assuming the truth of his semantic assumption, which is already problematic for the propositional case), but it fails to eliminate the non-normal interpretations that Carnap pointed out. We shall return to the adequacy of open-endedness for solving Carnap's Problem in Sects. 16.4.3 and 16.4.4) below. [16]

[16] (McGee, 2015: 179) suggests a very interesting new approach for obtaining categoricity at the quantificational level through an open-ended application of Hilbert's rule for the ε-operator.

16.4.2 Bonnay and Westerståhl's Semantic Strategy

As we mentioned din Sect. 16.2 above, (Church, 1944) raised scepticism regarding the possibility of a *full formalization* of classical logic, criticizing Carnap's formalizations as embedding "a concealed use of semantics", and argued that no purely syntactic solution would work. (Bonnay & Westerståhl, 2016) followed Church's suggestion and adopted a 'semantic strategy' for solving the categoricity problem, by imposing some general semantic constrains on the permissible class of interpretations.

In the case of propositional logic, they show that if we impose the principles of non-triviality, i.e., at least one sentence is false, and that of compositionality on the class of interpretations, then both the first kind and the second kind of non-normal interpretations are blocked. The problem with these two assumptions from an inferential perspective is that, as (Murzi & Topey, 2021: section 2.2) remarked, it is not so clear whether they can be justified only by appeal to inferential practice – although compositionality seems to be a very reasonable assumption if we think to the way in which we learn and extend our languages.

In the case of quantificational logic, the semantic principle invoked is that of topic-neutrality, in the specific form of the invariance under permutations. It is important to note that Bonnay and Westerståhl take the quantifiers as instances of generalized quantifiers, i.e. properties of sets of objects, and *Carnap's problem* becomes whether the rules of deduction allow any other generalized quantifiers beside the standard ones (∀,∃). Carnap's treatment of the quantifiers as infinite conjunctions and disjunctions and his use of infinitary rules are considered by them as a procrustean strategy, because the quantificational case in basically reduced to the propositional one. Let us briefly see how their approach goes for the quantificational case.

When L is a language of FOL interpreted over a domain D, ⊢ is the deducibility relation in FOL, and an interpretation I is a pair of the form (M, Q), where M is an L-structure based on D, which interprets the non-logical terms of L, and Q is the set of subsets of D, which interprets the quantifiers, then the truth conditions for the universal quantifier are the following:

$$(M, Q) \models (\forall x)\varphi\ \sigma \text{ if and only if } \mathrm{Ext}_{(M,Q)}\varphi \in Q$$

The interpretation (M, Q) is standard if and only if Q = {D}. However, the authors prove that if Q is a principal filter closed under the interpretation of terms in M, then the interpretation (M,Q) will be consistent with ⊢, but the standard interpretation for ∀ in which Q = {D} is only one among many other possible interpretations. For instance, if Q is generated by a subset A ⊆ D, then the interpretation will be non-standard. In a non-standard interpretation, the objects from the domain are

(Carnap, 1937: 197) referred to Hilbert's version of the omega rule as being sufficient for eliminating the non-normal interpretations, but made no remarks in this respect on the rules for the epsilon operator. For some brief remarks on the epsilon operator see Sect. 16.5 below.

not treated on the same par, but only the objects from the subset A generating the principal filter, and named in L, are in the range of the quantifiers. Bonnay and Westerståhl's solution is to impose the model-theoretic requirement of invariance under permutations on the principal filter Q. As they prove, a principal filter Q on D is invariant under permutation if and only if Q = {D}. With this semantic assumption at work, the non-standard interpretations, as they define them, are excluded.

Although this analysis of the non-standard interpretations that are consistent with the relation of logical derivability in FOL is very interesting and illuminating, we should immediately note that the non-standard interpretations described by them are in fact different from the non-normal interpretations of the quantifiers that Carnap referred to –and, thus, their solution does not directly solve Carnap's original problem and, *ipso facto*, the categoricity problem for logical inferentialism. In the non-standard interpretations described by Bonnay and Westerståhl, what differs essentially from one interpretation to another is the range of the quantifiers, while in the non-normal interpretations described by Carnap the quantifiers range over the entire domain (due to the fact that (Carnap, 1943: 136) works with a denumerable domain and each object from this domain is named by an individual constant in the language), but they still have different *meanings* provided by different interpretations for which the formal deductive system remains sound. In Carnap's non-normal interpretations, the universal quantifier has an improper meaning, i.e., although it ranges over the entire domain, its content is richer. The interpretation of the universal quantifier as "every object is F, and b is G" is an interpretation in which the range of the universal quantifier is the entire domain, but its meaning is richer, since it says in addition that the object denoted by b, and which is also F, has the property G.

In addition, I think that a full-fledged logical inferentialist should not be entirely satisfied with this solution due to the fact that categoricity is obtained by imposing semantic constraints that are not justified only on the basis of the inferential practice. We shall discuss in Sect. 16.4.4 below an inferential justification of the invariance-permutation assumption.

16.4.3 Warren's Open-Ended Unrestricted Inferentialism

(Warren, 2020) proposed a revitalization of conventionalism about logic and mathematics, i.e., of the idea that logical and mathematical properties in any given language are fully explained by its linguistic conventions, and argued that if the linguistic conventions are taken to be meaning determining rules of inference, then logical conventionalism follows from logical inferentialism. (Warren, 2020: 63–64) is committed to a particular form of logical inferentialism, i.e., unrestricted logical inferentialism, according to which *any* set of rules of inference that can be used for an expression can be meaning constituting for it.

For solving the categoricity problem in the case of propositional logic, (Warren, 2020, 78–84) moves to a natural deduction bilateralist formalization which has two force indicators as primitive signs ("+" for acceptance and "-" for rejection) and a structural rule for *reductio* (If $\Gamma, \alpha \vdash \beta$ and $\Gamma, \alpha \vdash \beta^*$, then $\Gamma \vdash \alpha^*$, where the '*' reverses the primitive signs '+' and '−', i.e., if $\alpha = -\varphi$, then $\alpha^* = +\varphi$). The notion of validity generalizes as follows: an inference is valid if and only if every valuation which makes all the plus-signed premises true and all the minus-signed premises false also make the conclusion true if it is plus signed, and false if it is minus-signed. In this bilateralist system, the elimination rule for negation instructs us to reject φ if not-φ is accepted, i.e. $+\sim\varphi \vdash -\varphi$. Thus, a valuation which makes both φ and not-φ true will make the elimination of negation rule invalid. Likewise, a valuation which makes both φ and not-φ false will make the bilateralist introduction rule for negation invalid, i.e., $-\varphi \vdash +\sim\varphi$. (Warren, 2020: 81) considers that the use of the force indicators is acceptable from an inferential perspective, although he notes that the most important advantage of bilateralism is "the ability to cleanly solve Carnap's problem".[17]

In the case of quantificational logic, (Warren, 2020: 85–86) appeals to open-endedness for solving (a version) of Carnap's categoricity problem. He argues that if we take the standard natural deduction rules to be open-ended, then the standard semantic values of the first order quantifiers are forced by the standard natural deduction rules. In the line of (Bonnay & Westerståhl, 2016), Warren considers the generalized version of the quantifiers, i.e., as properties of properties, and the problem is to show that the open-endedness of the natural deduction rules guarantees that the meanings of the quantifiers are the standard ones. If we consider a non-empty domain D and take "ext(x)" denote the extension A of "x" in D, by *standard* Warren means that the extension of the universal quantifier is the entire domain ($A = D$ iff $A \in \text{ext}(\forall)$) and that the extension of the existential quantifier is a non-empty subset of the domain (A is a non-empty subset of D iff $A \in \text{ext}(\exists)$). Let us consider (Warren, 2020: 85–86)'s proof for the universal quantifier:

Theorem $A = D$ iff $A \in \text{ext}(\forall)$

(Necessity): Let us assume that $A \in \text{ext}(\forall)$. We add a monadic predicate 'F' to our initial language L such that $\text{ext}(F) = A$ and let us assume for *reductio* that $A \neq D$. Since $A \neq D$, it follows that there is an object o in D\A. Open-endedness allows us to add an individual constant c such that $\text{ext}(c) = o$. Now we are in the expanded language where "(∀x)Fx" is true, but "Fc" is false. This contradicts the open-ended validity of the ∀-elimination rule. Thus, our assumption that $A \neq D$ is false.

(Sufficiency): Let us assume that $A = D$ and for *reductio* that $A \notin \text{ext}(\forall)$. We add the predicate 'F' to our language such that $\text{ext}(F) = A$. Let c be an individual constant such that $\text{ext}(c) = o$, for some member o of D. We are now in an expanded language where "Fc" is true for some arbitrary 'c', but "(∀x)Fx" is false. This contradicts the validity of open-ended validity of the ∀-introduction rule. Hence, $A \in \text{ext}(\forall)$.

[17] The original bilateralist proposal to solve Carnap's problem for propositional logic is due to (Smiley, 1996). For a discussion of the adequacy of the bilateralist framework for solving the categoricity problem see for instance (Murzi & Hjortland, 2009; Incurvati & Smith, 2010).

Indeed, as we discussed Vann McGee's proofs in Sect. 16.4.1, the open-endedness of the ∀-elimination rule assures us that from a universally quantified sentence we can derive each of its instances and, thus, under the assumption that each individual constant denotes an object from the domain, the extension of the universal quantifier is the entire domain. The open-endedness of the ∀-introduction rule in Warren's proof, however, seems to do essentially the same thing. Conceptually, it would be strange to see how the extension of the quantifier in the sentence "(∀x)Fx" would be different from the entire domain since, by assumption, the extension of the predicate "F" is D itself. Thus, if we accept that the meanings of the quantifiers are exhausted by specifying their extensions, the open-endedness requirement seems to do its job. However, Carnap's original categoricity problem requires more than specifying the extension of the quantifiers. It requires from rules to force a unique interpretation of the quantifiers such that alternative interpretations in which ⌜(∀x)Fx⌝ is interpreted as "all objects are F and b is G" and ⌜(∃x)Fx⌝ is interpreted as "at least one object is F or b is not G" are excluded.

Although Warren does not explicitly acknowledge Carnap's original categoricity problem for the quantifiers[18], it seems to me that he might be sympathetic to Carnap's solution for solving it. (Warren, 2020: 263–79; 2021) argued in favour of the possibility of inferring according to the omega rule when its premises are, in principle, recursively enumerable.

Consider for instance Goldbach conjecture (GB) which asserts that every even number greater than 2 is the sum of two prime numbers. Warren invites us to ideally consider a supertask computer (SC) that is able to perform a countably infinite number of computations in a finite time. The SC is set to verify GB and it checks 0 in half a minute, one in half of half a minute, and n in $1/2^{n+1}$ minutes and, thus, the computation will finish in one minute. The SC either sends a halt signal, if a counterexample is found before one minute, or no counterexample is found and, thus, we receive no signal. If the SC fails to halt, then we accept as observers GB(0), GB(1), GB(2) … by using as evidence the computations. Then we conclude (∀x)GBx. Therefore, on the basis of the computation, we accept each of the infinitely many premises and infer from them, by using the omega rule, the truth of GB. Thus, this would count as a situation in which we, human beings, perform an infinite reasoning, by using infinite inferences.[19]

[18] For instance, (Warren, 2020: 85) wonders whether "a version of Carnap's problem" appears for the standard quantifier rules. Still, as we discussed above, (Carnap, 1943) had a unitary treatment of the non-normal interpretations both for the propositional connectives and for the first-order quantifiers.

[19] (Warren, 2021) approaches the possibility of infinite reasoning from a naturalist view on human cognition and on this view inferences are seen as "causal processes realized by our brains", although they are not exhausted by these processes. The possibility of accepting infinite many premises becomes plausible, he argues, once we accept a dispositionalist account on acceptance and believing. We can have some behavioural dispositions for accepting a sentence without considering it in advance. Hence, as long as the premises of the omega rule are recursively enumerable, we can have the behavioural dispositions to accept them without considering each of them individually in advance.

I think that Carnap's full formalization of quantificational logic by using infinitary rules could be easily accepted by Warren if, in addition, we require for the premise-instances of the quantified sentences to be, in principle, recursively enumerable by the users of the quantificational language. If we consider Carnap's problem as a problem for logical inferentialism, which is very sensible to our ordinary inferential practices, Warren's condition of recursive enumerability seems reasonable. However, if we see Carnap's problem as a mathematical problem, which regards the connection between abstract syntax and semantics, then the condition of the enumerability of the instances is sufficient, because the problem gets solved once the deductive equivalence between a quantified sentence and its conjunctive or disjunctive class of instances is obtained. In a denumerable domain in which every object is named, the transfinite rules of inference will establish this equivalence.

16.4.4 Categoricity by Convention

(Murzi & Topey, 2021) adopted a moderate model-theoretic inferentialist stance, according to which our open-ended syntactical dispositions for inferring with basic rules of inference uniquely determine the meanings of the logical terms. Their account follows with some emendations (Garson, 2013)'s path of reading off the meanings of the logical terms from the rules of inference with the help of the local models and combines McGee's open-endedness constraint with Bonnay and Westerståhl's permutation invariance assumption.

In the case of propositional logic, the main problem with the local models that (Garson, 2013: 42–43) indicated was that of incompleteness. The incompleteness problem refers to the fact that the rules for the material implication fix the classical meaning of this operator, while its introduction and elimination rules cannot prove by themselves all the logical truths formulated only in terms of it (e.g. Peirce's Law needs the rules for negation). This problem is solved in (Murzi, 2020) by developing a calculus in which classical *reductio ad absurdum* is taken as a structural metarule. His system has the proof-theoretic feature of embedding negation in nice clothes at the level of the structural rules, case in which Peirce's law can be proven only by using the operational rules for the material implication.

In the case of quantificational logic, Garson's problem with the local models was that if the meanings of the quantifiers are to be given by using local models, then the introduction rules for the universal quantifier will be unsound, since a valuation may satisfy the premise ⌜Γ⊢Ft⌝ without satisfying *ipso facto* the conclusion ⌜Γ⊢(∀x)Fx⌝, even if t does not occur in Γ or (∀x)Fx. (Murzi & Topey, 2021) argue that the local models could be used, with some emendations of the formalism, even for the quantifiers. They introduce a form of the ∀I-rule with open sentences, in which φ has at most the variable x free in the premise of the metarule:

$$\frac{\Gamma \vdash \phi}{\Gamma \vdash \forall x \phi},$$

where x does not appear free in Γ.

By using this new formulation of the rule and a definition of satisfaction relative to variable assignments,[20] Garson's counterexample to the local validity of ∀I-rule vanishes, since if *every* assignment satisfies ⌜Γ⊢φ⌝, then it also satisfies ⌜Γ⊢∀xφ⌝.[21]

What the authors have to establish next is that the natural deduction rules, in this specific form for the ∀I-rule, are categorical when validity is defined by using the local models. Roughly, what they need to establish is that the universal quantifier ranges over the entire domain. This idea is labelled by the authors *the first order thesis*:

First Order Thesis. The rules of FOL are locally valid with respect to a class of valuations V only if all v ∈ V obey the standard interpretation of ∀ -i.e. are such that, for any φ with at most x free, ∀xφ is true in v iff every object in the domain is in the extension of φ in v (or, more briefly, iff Ext_v (φ) = M, where M is v's domain).

(Murzi & Topey, 2021) justify this thesis by proving a weakened version of it and then enforcing this weakened thesis by using (Bonnay & Westerståhl, 2016)'s lemma which shows that if the universal quantifier is permutation invariant, then it ranges over the entire domain. The weakened thesis is the following:

Weakened First Order Thesis. The rules of FOL are locally valid with respect to a class of valuations V only if all v ∈ V are such that, for any φ, ∀xφ is true in v iff $M_x \subseteq Ext_v(\varphi)$, where M_x is the range of x in v.

This thesis establishes that the range of the universal quantifier is a subset of the domain of objects. In other words, by assuming that the introduction and elimination rules for the universal quantifier are satisfaction preserving, we achieve the result that the range of the variable x is included in the set defined by the formula φ,

[20] A valuation v $satisfies_s$ ⌜Γ⊢ φ⌝, where s is a variable assignment, iff, in v, either s fails to make true some δ∈Γ or s makes true φ. A sequent ⌜Γ⊢ φ⌝ will be thus satisfied by a valuation v iff v $satisfies_s$ ⌜Γ⊢φ⌝ for every variable assignment s.

[21] Garson suggests that a valuation may satisfy ⌜Γ⊢Ft⌝ without satisfying ⌜Γ⊢(∀x)Fx⌝, but the inference from ⌜Γ⊢Ft⌝ to ⌜Γ⊢(∀x)Fx⌝ is not an instance of the ∀I-rule, as Murzi and Topey formulate it, since there is no free variable in the premise. An instance of it would be rather the inference from ⌜Γ⊢ Ft⌝ to ⌜Γ⊢(∀x)Ft⌝, in which the use of ∀ is vacuous. Murzi and Topey do not consider, however, Garson's valuation v* discussed in Sect. 16.3 which, as I suggested in footnote 14 above, may also be used against the determinacy of the ∀-rules when local validity is used.

which has at most x free.[22] The step from this result to the *first order thesis* is justified by the invariance under permutation lemma, which guarantees that, under the assumption that ∀ is permutation invariant, if its range is a subset of the domain, then that range is the domain itself. The assumption of permutation invariance is justified inferentially by (Murzi & Topey, 2021) by assuming that we inherit some syntactic dispositions to follow a logical rule in an open-ended way[23] and then using McGee's reasoning for the sufficiency of the rules in determining the meaning of ∀ (see Sect. 16.4.1 above). In other words, if ∀E-rule is open-ended, then its interpretation is permutation invariant in any model of FOL, i.e., it ranges over the entire domain.

This approach is very nicely conducted, and combines in an elegant manner the ideas of local models, open-endedness and invariance under permutations, but as we have seen in Sect. 16.4.1 above, what McGee's approach succeeds in establishing is that the open-ended deductive rules for the universal quantifier provide it with an unrestricted interpretation (under the assumption that every-thing is nameable). Nevertheless, this result is perfectly compatible with the existence of the non-normal interpretations of the quantifiers that Carnap pointed out. We can have a non-normal interpretation in which the universal quantifier ranges over the entire domain, but still provides it with a non-normal meaning. The criticism raised against McGee's approach remains valid against Bonnay and Westerståhl's, Warren's, as well as against the approach discussed in this section.

In addition, inferentialists like (Murzi & Topey, 2021: section 2.4.4) agree that we have the dispositions "to accept *all* of the infinitely many instances of our logical rules". But if we have dispositions for accepting the *infinitely* many instances of a logical rule, this already opens the way for embracing the infinitely many premises involved in a logical rule, as the ω-rule. Certainly, this is big step forward, but the empirical research on dispositions may validate it in the future.[24] In addition, once

[22] It should be noted that (Murzi & Topey, 2021: 3407) prove this weakened thesis by using a restricted formulation of the ∀-introduction rule: if $\vdash\varphi$, then $\vdash\forall x\varphi$. If we consider the rule in its general form 'if $\Gamma\vdash\varphi$, then $\Gamma\vdash\forall x\varphi$', then Garson's valuation v* could be used to show that the ∀-introduction rule is locally valid, but $\forall x\varphi$ is not satisfied, although φ is satisfied (see also footnote 14 above). If Γ is taken to be empty, then if φ is a theorem, its logical closure will also be a theorem. However, it seems to me that the problem which remains in this case is to describe the relation between φ and its instances, since the logical inferentialist needs individual constants in his language. If φ follows from the set of all its potentially infinite set of sentences, without following from a finite subset of it, then a transfinite rule of inference, as that of Carnap's above, seems to be needed.

[23] (Murzi & Topey, 2021) embrace a naturalist standpoint and assume that we have some general dispositions to infer in accordance to logical rules. These dispositions are supposed to have a syntactic nature and they allow us to infer in an open-ended way, i.e. we can accept instances of the logical rules formed with expressions that are in some extensions of the original language that we use.

[24] The idea is that, as long as we do not have a generally accepted theory of dispositions, we cannot convincingly argue for or against the idea that human beings have dispositions for following infinitary rules. If we have the dispositions to accept a class of sentences without considering individually each of them in advance, as (Warren, 2021) argue, then it is plausible to accept that

we accept the omega rule, we obtain the deductive equivalence between a universal sentence and the class of all its instances, and the possibility of assigning to the universal quantifier an improper meaning is thus blocked. My aim in the next section is to explore Carnap's view on the use of infinitary rules and to argue for the idea that a logical inferentialist should *in principle* accept the infinitary rules of inference for the first order quantifiers.

16.5 The ω-Rule Again

Kurt Gödel showed that most mathematical theories that are formal, in the sense that all reasoning in them can be completely replaced by finite mechanical devices, are incomplete, i.e., there is a sentence in the language of such a theory that is neither demonstrable nor refutable in its formal system. This understanding of formality is an important assumption for the applicability of his theorems, and it requires that all the logical rules of deduction used in the formal system should be finite (or effective).[25]

To overcome the limitations revealed by Gödel's results, (Carnap, 1937) envisaged a new method of deduction that makes essential use of transfinite rules, i.e., non-effective ones (as the ω-rule), and used it to obtain a complete formal criterion of validity for classical mathematics, i.e., to state necessary and sufficient conditions for what counts for a sentence to be true in classical mathematics. The peculiarity of this new method of deduction, i.e., the method of consequence (c-method), is that it operates not with sentences, but with sentential classes, which may be infinite. Both the languages I and II of *Logical Syntax of Language* contain transfinite rules for these sentential classes (§14, §34b):

> In order to attain completeness for our criterion we are thus forced to renounce definiteness, not only for the criterion itself but also for the individual steps of the deduction. A method of deduction which depends upon indefinite individual steps, and in which the number of the premises need not be finite, we call a method of consequence or a *c-method.* In the case of a method of this kind, we operate, not with sentences but with sentential classes, which may also be infinite. (Carnap, 1937: 98–99)

Concerning the transfinite rules, (Carnap, 1937: 173) believed that "there is nothing to prevent the practical application of such a rule". Later on, after his semantic turn, (Carnap, 1942: 247) amended this method of consequence, by taking the transfinite rules as being part of I_t and II_t, two related systems obtained from the finite rules of Language I and Language II by adding the transfinite ones. I_t and II_t define 'provable

we can follow infinitary rules. Beside this, since we can prove that Peano Arithmetic closed under the ω-rule is deductively complete, it is clear that in a certain sense we can fruitfully *use* infinitary rules (see also footnote 30).

[25] Some logicians, as (Curry, 1968: 261), even took recursive effectiveness as a necessary condition for logical formalization.

in I_t,' and 'provable in II_t'. Moreover, 'analytic in I' becomes 'provable in I_t,' and 'analytic in II,' becomes 'provable in II_t'. Thus, the distinction between d-terms and c-terms and between derivation and consequence series is abandoned, as (Carnap, 1942: 248) explicitly recognizes. The main reason for abandoning this distinction is that he acknowledged that the same procedure of constructing a sequence of sentences can be applied with both finite and transfinite rules.

From a more general philosophical perspective, the use of indefinite rules seems to be justified on pragmatic grounds. As (Carnap, 1943: 143) emphasized, "indefinite calculi seem to be admissible and convenient and even necessary for certain purposes". By this he referred to the fact that the transfinite rules of inference are necessary for the purpose of constructing an L-exhaustive calculus[26] for arithmetic and a full formalization of logic. In addition to this general motivation, (Carnap, 1942: 160–161) believes that the transfinite proofs and derivations are legitimate for the reason mentioned above, namely, that the same procedure of constructing a sequence of sentences can be applied also to transfinite proofs:

> Till recently, all rules applied in systems of modern logic have been finite, R_i usually contains one or two sentences. In recent years, however, it has been found that transfinite rules can be applied, and that they are useful and even necessary for certain purposes. [. . .] It will, however, be shown that the application of transfinite rules can also be made in the form of transfinite proofs or derivations. The definitions DA1 to 4 given above are then sufficient to cover the use of transfinite rules also, by a (finite or transfinite) sequence we understand a one-many correlation of sentences with the ordinal numbers of a (finite or transfinite) initial segment of the series of ordinal numbers. Hence a proof or derivation in which no repetition of sentences occurs may be regarded as a well-ordered series of sentences. (Carnap, 1942: 160–161)

The last part of this passage simply points out to the difference between series and sequences. (Carnap, 1942:18–19) describes two different ways of ordering objects in a linear order, namely, by series or by sequences. A series of objects is a transitive, irreflexive and connected relation, while a sequence is simply an enumeration of objects. We can have repetitions in sequences, but not in a series, being irreflexive. Thus, an infinite proof in which there is no repetition of sentences can be represented, in the sense of a one to one correlation, by a series of natural numbers. If there are repetitions, then we will have a sequence, and there will be a correlation one-many, i.e., the same sentence will be associated with different numbers. Consequently, at least from a theoretical perspective, there seems to be no problem in representing an infinite proof by an infinite series of sentences. This representation would still count as a purely formal one.[27]

[26] If S is a semantical system and there is a sentence S_2 and a infinitely class of sentences C_1 in S such that S_2 is a logical consequence of C_1 without being a logical consequence of any finite subclass of C_1, then an L-exhaustive calculus C for S can be constructed only if transfinite rules are admitted. See (Carnap, 1939: 23).

[27] As (Tennant, 2008: 103) points out, if the finite serial structures are to be likened to geometrical points, then an infinite sequence of premises involved in an application of the ω-rule may be likened to a geometrical line, being an infinite sequence of such points. Thus, 'simply being infinitary does not count against being *purely formal*'.

The argument that I want to briefly put forward here for the idea that the logical inferentialists should accept the infinitary rules for the first order quantifiers can be formulated as follows:

As.1 Logical inferentialism maintains that our use of the logical expressions in inferences is what determines their meanings.
As.2 Our use of the expressions “all” and “there is” in mathematical inferences leads us beyond the intuitive and finite reasoning.
P1. If human beings do sometimes use infinitary rules of inference in their reasoning, then a logical inferentialist should in principle accept the infinitary rules for the quantifiers.
P2. Human beings do sometimes use infinitary rules of inference in their reasoning.
C. Therefore, a logical inferentialist should in principle accept the infinitary rules for the quantifiers.

The assumptions are not needed for the validity of the argument, but they rather shape the background in which the argument is formulated. From the assumptions it follows that the logical inferentialist implicitly accepts that the full use of the expressions “all” and “there is” goes beyond what is finite, and the premises simply add the information that the infinitary rules which govern the performance of infinite inferences are the needed tools for expressing this use. The first assumption is simply the definition of logical inferentialism, so it cannot be disputed in this context. Some remarks on the second assumption, however, are necessary.

The infinite has always been a challenge for human mind and (Hilbert, 1926: 371) even remarked that the clarification of the nature of infinite is necessary “for *the honor of the human understanding* itself”. (Hilbert, 1923: 1139) emphasized that the first point where we go beyond the concrete, intuitive, and the finite is the application of the concepts ‘all‘ and ‘there is‘. We can deal very easily with these two concepts if we work with a finite domain of objects, case in which the universal and existential quantifications are reducible to finite conjunctions and disjunctions, and their duality is justified by using the DeMorgan rules. Still, this duality is usually assumed in mathematical reasoning to also hold when we deal with an infinite domain of objects.[28]

In order to secure the duality of the quantifiers in an infinite domain, Hilbert introduced a transfinite axiom that implicitly defines the logical choice function ε, a function that assigns a definite object ε(F) to each predicate F. The epsilon axiom $Fx \rightarrow F(\varepsilon xF)$ says that if a predicate F is satisfied at all, then it is satisfied by εxF. “εxF” stands for an arbitrary object for which the proposition Fx certainly holds if it holds of any object at all. The definitions of the quantifiers based on this axiom

[28] (Lewis, 1918: 236), for instance, makes the assumption that “any law of the algebra which holds *whatever finite* number of elements be involved holds for any number of elements whatever.” This assumption is taken by him to be true and is grounded by the convention that the quantifiers are equivalent with (possibly infinite) conjunctions and, respectively, disjunctions.

are the following:

$$(\forall): \quad (\forall x)\,Fx \leftrightarrow F(\varepsilon x(\sim F)) \qquad (\exists): \quad (\exists x)\,Fx \leftrightarrow F(\varepsilon x F)\,.$$

(Weyl, 1929: 259) criticized this axiom on the ground that in order *to construct* the representative object, εF, for the property F, "we must imagine that we have a divine automaton which accomplishes this task". This automaton should produce for every property that we insert in it the representative of that property, i.e., an object such that, if it instantiates the property, then any object instantiates it.[29]

I think that it should be clear at this point that the full uses of the expressions "all" and "there is" lead us beyond the finite reasoning. The transfinite axioms and rules, as epsilon or omega, manage to help us in fully operating with the quantifiers, but their use raises difficulties. As (Weyl, 1929: 259) emphasized:

> If we had a automaton like this at our disposal we would be free from the troubles caused by "all" and "any"; but of course the belief in its existence is pure nonsense. Mathematics, however, behaves as though it did exist.

Of course, we do not have such an automaton, as we do not have a *real machine* that could check in a finite time all the premises of the omega rule. But, nevertheless, as Weyl remarked, classical mathematicians behave as if this automaton would exist. This suggests that *at least* some classical mathematicians are disposed to accept and even to engage in inferences that go beyond the intuitive and finite reasoning. In other words, the use of the quantifiers by the classical mathematicians, the very quantifiers that generate Carnap's original problem, is also governed in inferential practices by infinitary rules. If this is so, however, then I think that logical inferentialists should accept the infinitary rules of inference for the quantifiers.[30]

The approach of a logical inferentialist, it seems to me, should be similar to the one of a linguistic anthropologist. In order to recover and formulate the formal grammar that governs the use of a language in a community, the anthropologist has to carefully study the everyday linguistic behaviour of the members of the community. Likewise, in order to recover and formulate the proof-theoretical framework of quantificational human inferences, a logical inferentialist should take into account all the uses of the quantifiers from the reasoning practice. Since the full use of the quantifiers in classical mathematics involves the implicit reference to infinite totalities and this use is better reconstructed as being governed by infinite

[29] (Carnap, 1961) also acknowledged the indeterminate character of the ε-operator, but, nevertheless, he found it useful not only for logic and mathematics, but also for defining the theoretical concepts of scientific theories.

[30] For a very interesting and elaborate discussion on the ability of human beings to perform infinite inferences see (Warren, 2021). For a criticism, see (Marschall, 2021). Marschall assumes, however, that we do not have dispositions for following infinite rules and, thus, he concludes that only languages with recursive rules are permissible. His conclusion is a radical one, since he also applies this constrain to meta-languages. As we have seen in Sect. 16.2 above, even (Church, 1944) admitted the fruitfulness of the infinitary rules in the meta-theory. After all, we can prove new theorems precisely by *using* these rules (see also footnote 25).

rules, then a logical inferentialist should accept these rules in his proof-theoretical framework.

Certainly, (Church, 1944: 498)'s criticism that it is not possible to *systematically* follow a non-effective rule in practice remains valid, but I do not think that it should be considered an insurmountable obstacle. For instance, as (Warren, 2021) argued, if we consider an infinite inference with recursively enumerable premises, then the possibility of performing infinite inferences is at least plausible from a naturalist perspective. An actual example of performance of an infinite inference is difficult (if not impossible) to be found, but if we abandon our reluctance for accepting these rules, then our dispositions for inferring in accordance with them may get a more substantial shape. As we have seen above, Carnap was even more optimistic on this matter and held the belief that not only infinite inferences are possible, but also infinite derivations and proofs.

One further philosophical remark that I want to make regards the notions of *categoricity* and *infinitary rules of inference* as normative ideas or ideals to be followed. It can be seen as a step back, but I think that the symmetry between the proof-theoretic methods and the model-theoretic ones can be fruitfully seen as an ideal to be attained. Certainly, model-theoretic inferentialism requires categoricity, but from a more general perspective, categoricity could be seen as an ideal to be attained. Likewise, we can think of the idea of following an infinite rule of inference as an ideal, a norm for our reasoning. We do not *systematically* follow in practice an infinite rule of inference, but following such rules is as idea that guides our reasoning. The idea of following an infinite rule could be seen, to quote (Hilbert, 1926: 392), as "a concept of reason that transcends all experience and through which the concrete is completed so as to form a totality".

Regarding the use of infinitary rules in constructing a logical calculus, I totally agree with (Carnap, 1937: 52) that "everyone is at liberty to build up his own logic". The methods for using transfinite calculi are clearly stated and the syntactical rules are precisely formulated. However, since the semantic meanings of the quantifiers involve infinity, then the construction of a logical calculus for them, as (Carnap, 1942: 219) emphasized, is not entirely conventional. We need a logical instrument, i.e., a calculus, which can help us to exhaustively deal with its semantic counterpart. Certainly, this calculus can be seen as a normative instrument. After all, logic is not entirely concerned with the way in which people actually think and infer, but rather with the way in which they ought to think and infer if truth is to be attained. If we abandon our reluctance towards the infinitary rules and exercise our dispositions for performing inferences that go beyond finite reasoning, logic may even gain a better understanding of mathematical practice, where infinity plays a major role.[31]

[31] See (Barwise, 1981) and (Moore, 1990) for a discussion of the development of infinitary logics and its relation with mathematical practice.

16.6 Final Remarks

My aim in this paper was to analyze the importance of the infinitary rules of inference for the first order quantifiers by discussing the main approaches to the categoricity problem for the classical logical inferentialism. Although there are various solutions to this problem, depending on the way in which the rules of inference are formulated and the semantics is specified, I tried to emphasize Carnap's original idea that this problem for the quantifiers originates in the finite nature of the standard formalizations of classical logic. The meanings of the first order quantifiers go beyond what is finite and thus the standard formalizations that use only finitary rules of inference, i.e., rules with a finite number of premises (and conclusions), have problems in fully capturing their meanings.

As we have seen in the fourth section, McGee, Bonnay and Westerståhl, Warren, as well as Murzi and Topey, are interested in finding constraints that force the universal quantifier to range over the entire domain of objects. However, the non-normal interpretations pointed out by Carnap are also interpretations in which the universal quantifier ranges over the entire domain (since the domain is denumerable and all the objects are named by individual constants). The main difference is that its meaning in a non-normal interpretation is richer that in a normal one. This is a consequence of the fact that the standard finite rules of inference do not establish the deductive equivalence between a universally quantified sentence and the class of all its instances.

At the same time, the main idea that I argued for is that a model-theoretic logical inferentialist should in principle accept the infinitary rules of inference for the first order quantifiers since our use of the quantifiers, and thus their meanings, go beyond the intuitive and finite reasoning, and human beings do sometimes employ infinite rules of inference in their reasoning. Certainly, by accepting the infinitary rules of inference we obtain the symmetry between logical syntax and semantics and, thus, the categoricity problem gets solved.

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