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THE STACCATO RUN: A CONTEMPORARY ISSUE IN THE ZENONIAN TRADITION

A QUESTION that has risen to prominence in contemporary discussions of Zeno's paradoxes of motion is the possibility of "superfeats" (or "supertasks"), meaning feats that entail the completion in a finite time of an infinite sequence of distinct, physically individuated acts. One such feat, often viewed as *prototypical* of the class of superfeats, is the "staccato run," in which the runner runs from one point in space to another, but makes infinitely many stops along the way. In a number of publications in the late 1960s and early 1970s, Adolf Grünbaum argued that the staccato run, in at least one of its forms, is compatible with the requirements of classical mechanics. And he contended that a demonstration of the *mechanical* feasibility of the run can reasonably be taken as a demonstration of its *logical* feasibility. Grünbaum's position on the staccato run, and on other, comparable superfeats, has come to be widely accepted. Indeed, it can fairly be described as the reigning orthodoxy.[1] In what follows, I offer no argument against the *logical* possibility of the staccato run. But I argue that the run is impossible dynamically. That is, I argue that the run is excluded by Newton's three laws of motion, at least when those laws are supplemented with a certain defensible philosophical judgment.

1. THE STACCATO RUN

As initially described by Grünbaum,[2] the staccato runner, here to be called Achilles, is to start from rest at time T_0 at point 0, to finish at rest at time T at point 1, and to stop en route at each member of the infinite sequence of "Z-points" (Zeno-points): $<1/2, 3/4, 7/8, \ldots>$. At each successive Z-point, Achilles is to rest for half as long as at the preceding Z-point. And each of the subruns (i.e., each of the runs from one Z-point to the next) is to take half the time taken by the preceding subrun.

Two remarks: (1) Grünbaum is, of course, assuming the continuity of space and time. But as Grünbaum notes elsewhere,[3] that assumption is standard even within quantum mechanics. (2) Although Achilles is three-dimensional, it is to be understood that he is located at a given point when and only when his center of *mass* is located at (or above) that point.

In Grünbaum's initial version of the staccato run, Achilles reaches the same peak velocity, and maintains the same average velocity, during each of the subruns. As Grünbaum shows, there is no discontinuity in Achilles' position, not even at the climactic moment, T, when Achilles completes his run. (His position at T, namely 1, is identical to the limit of his positions as $t \to T$.) And although Achilles' *velocity* approaches no limit as $t \to T$, his velocity function is bounded in every earlier neighborhood[4] of T, and so its discontinuity at T is finite.

Grünbaum acknowledges that the run does involve an infinite discontinuity in acceleration. Referring to that discontinuity, and to the finite discontinuity in velocity, he writes in a later publication:

> ... in view of the ... discontinuities in ... velocity and acceleration, our arithmetically simple kind of *staccato* ... [run] may well be kinematically problematic. As far as I know, books on classical or prequantum mechanics do not spell out whether motion involving these particular discontinuities are kinematically possible or not.[5]

Happily. Grünbaum need not be concerned about the kinematic status of the two discontinuities. He had learned from Richard Friedberg, a professor of physics at Columbia University, how those discontinuities can be eliminated. Friedberg had provided Grünbaum with a modified position function for the staccato run, one on which the peak velocities and accelerations attained by Achilles during the subruns converge to zero as $t \to T$. Also converging to zero are the peak values of all of the *higher* time-derivatives of his position. In a Friedberg-style staccato run, there is no discontinuity, infinite or finite, in *any* of the time-derivatives of the runner's position!

[1]Support for Grünbaum's view is found in Philip Morrison, Review of *Zeno's Paradoxes*, *Scientific American* 224 (1971), p. 123; Wesley Salmon, *Space, Time, and Motion* (Minneapolis: University of Minnesota Press, 1980), p. 48; Victor Allis and Teunis Koetsier, "On Some Paradoxes of the Infinite," *British Journal for the Philosophy of Science* 42 (1991), p. 193; Mark Sainsbury, *Paradoxes*, second edition (Cambridge: Cambridge University Press, 1995), pp. 14–15; Craig Harrison, "The Three Arrows of Zeno," *Synthese* 107 (1996), pp. 271–92; and Joseph Alper and Mark Bridger, "Mathematics, Models and Zeno's Paradoxes," *Synthese* 110 (1997), p. 148.

[2]Adolph Grünbaum, *Modern Science and Zeno's Paradoxes of Motion* (Middletown: Connecticut: Wesleyan University Press, 1967), p. 75.

[3]Grünbaum, pp. 109–11.

[4]A *neighborhood* of time *t* is a (temporal) interval containing *t*, either as an endpoint or as an interior point. An *earlier/later neighborhood* of *t* is an interval containing *t* as its later/earlier endpoint.

[5]Adolph Grünbaum, "Modern Science and Zeno's Paradoxes of Motion," in Wesley Salmon, ed., *Zeno's Paradoxes* (Indianapolis: Bobbs-Merrill, 1970), p. 213.

[6]Grünbaum, pp. 215–16.

[7]Actually, in the reversed run *a la* Friedberg, the stopping points differ somewhat from the traditional ones. But since the difference is irrelevant to the arguments ahead, and since Friedberg's stopping points take many more digits to express, the difference will be ignored.

[8]Physicist Paul Davies writes, "Newtonian mechanics is symmetric in time. What this means is that any motion of atoms which proceeds in accordance with the Newtonian laws of motion has a *reverse* pattern of motion which is equally in accordance with those laws" (*Space and Time in the Modern Universe*, Cambridge: Cambridge University Press, 1977, p. 70).

Grünbaum concludes:

> ... Friedberg's version of the ... *staccato* run obviates all of the kinematically and dynamically problematic features of the arithmetically simple example... if we wish to call the staccato runner's execution of the aleph null *separate* motions "doing infinitely many things," then his performance shows that infinitely many things can be done in a finite time.[6]

2. THE REVERSED STACCATO RUN

I will dispute the just-quoted claim that the (Friedberg version of the) staccato run is dynamically unproblematic. But to enhance the clarity and vividness of my arguments, I will focus on a version of the staccato run that we'll call the *reversed* run.

The reversed run is exactly like the (Friedberg version of the) regular staccato run, except that the aleph null distinct subruns are to be made in reverse order. Instead of concentrating the subruns at the *end* of his run, Achilles is to concentrate them at the *beginning*. As in the regular staccato run, Achilles is to start from rest at point 0 at time T_0, to finish at rest at point 1 at time T, and to run in a straight line, and always in the same direction, between those two points. But along the way he is to stop for a period of rest at each of the *reverse* Z-points of 01. That is, he is to stop at each member of this infinite sequence of spatial points: <... 1/8, 1/4, 1/2>.[7]

Presumably, if the regular staccato run is feasible *logically*, the same is true of the reversed run. After all, we could film the regular run and then play the film back in reverse. That would provide a completely detailed *depiction* of the reversed run. And if the reversed run can be thus depicted, there should be no logical obstacle to its occurrence. In general, it seems safe to say that if an itinerary is logically feasible, so is its reverse. Furthermore, the laws of (classical) mechanics are invariant under time reversal.[8] So the reverse of a dynamically feasible itinerary is itself dynamically feasible. Accordingly, I will assume that a demonstration of the dynamic infeasibility of the reversed staccato run is tantamount to a demonstration of the dynamic infeasibility of the regular staccato run. (If this assumption turned out to be questionable, that would be an interesting and important development.)

3. THE LOGICAL POSSIBILITY OF THE REVERSED RUN

It may well appear that the reversed run (and therefore the regular staccato run as well) is a logical impossibility. For one thing, it is hard to understand how Achilles might *begin* the run. But it is widely held that the impossibility of the run is *not* a logical one. Let me explain why.

First, it would be hard to deny that it's logically possible, if only through divine agency, for Achilles to *be* at all of the places he is required to be, at all of the times he is required to be at them. (After all, he is never required to be at more than one place at a time.) Furthermore, if he *does* occupy all of those places at all of those times, he will occupy, beginning with 0 and ending with 1, a *continuous* series of places at a *continuous* series of times. Thus, by the usual definition, Achilles will indeed "move" from 0 to 1.

Of course, there is more to *running* from 0 to 1 than *moving* from 0 to 1. In order for Achilles to run from 0 to 1, he must use his legs to *propel* himself from 0 to 1. But presumably there are possible worlds in which Achilles not only is located at all of the prescribed places at all of the prescribed times, and thus *moves* from 0 to 1, but is indeed propelling himself with his legs at all of the times at which he is moving, and thus *runs* from 0 to 1.

If there *aren't* such worlds, it is for a reason that has yet to emerge in the literature on this topic. Accordingly, I will allow that staccato runs are possible logically (even though I doubt that they are). I will be content to dispute the widely accepted view of Adolf Grünbaum: that (Friedberg style) staccato runs are possible even mechanically. I will undertake to show that the reversed staccato run is excluded by the basic laws of Newtonian dynamics.[9]

4. THE DYNAMIC IMPOSSIBILITY OF THE REVERSED STACCATO RUN

In the reversed run, Achilles is to start from rest at point 0 and to run eastward to point 1, last occupying 0 at time T_0 and first occupying 1 at time T. Along the way Achilles is to stop for a period of rest at each of the (reverse)

[9]For an attempt to show the *logical* impossibility of superfeats, see my "The Impossibility of Superfeats," *The Southern Journal of Philosophy* 38 (2000), pp. 207–220.

[10]Suppose, for the purpose of *reductio*, that there is body *b*, a time *t*, and an interval t_1t_2 containing *t* (as an interior point) such that (1) *b* has at *t* a nonzero acceleration and (2) *b* has at every other time belonging to t_1t_2 an acceleration of 0. From (2) it follows that *b* has a constant velocity, *v*, throughout the half-open interval $t_1\underline{t}$ (which contains all points belonging to t_1t except *t*) and a constant velocity, v', throughout the half-open interval $\underline{t}t_2$. But since (1) can be true only if *b has* an acceleration at *t*, *b*'s velocity function must have a *derivative* at *t*, which entails that it's *continuous* at *t*. From (1) and (2) together, then, it follows that the body's velocity at *t* must be equal both to *v* and to v'. But then the limit of the average accelerations of *b* in ever smaller neighborhoods of *t* is 0. That means, by the definition of instantaneous acceleration, that the acceleration of *b* at *t* is 0, which contradicts (1).

[11]At least, he would not have been at rest throughout some *period* ending at T_0. For simplicity, let's stipulate that Achilles *is* to be at rest throughout such a period.

[12]See J. J. C. McKinsey, A. C. Sugar, and Patrick Suppes, "Axiomatic Foundations of Classical Particle Mechanics," *Journal of Rational Mechanics and Analysis* 2 (1953), p. 258.

Z-points of 01, <... 1/8, 1/4, 1/2>, which means he is to stop infinitely often in every later neighborhood of T_0. (At any time later than T_0, Achilles is at some point east of 0. But every point east of 0 is east of infinitely many Z-points, at each of which Achilles is to have stopped.) We will find that Achilles cannot run as required while complying with Newton's laws of motion. In particular, we will find that the force needed to start him on his way will cause him to have an initial motion that is incompatible with his being at rest at each of the Z-points.

Achilles is to start from rest. By Newton's first law, Achilles will remain at rest, and thus will remain at point 0, unless he is caused to move by some external force. Since Achilles is to move by running, the motive force will be a force exerted on him by the ground, a force resulting, in accordance with Newton's third law, from an equal but oppositely directed force exerted on the ground by his legs. But the question that needs to be asked is *when* the motive force is exerted.

Note first that the motive force (i.e., the force or sequence of forces that causes Achilles to cease to be at rest at point 0) cannot be a force that acts only for a moment. There *are* no such forces. By Newton's second law, $F = ma$, a (nonzero) force that acted only for a moment would produce a (nonzero) *acceleration* that lasted only for a moment. But the very definition of acceleration makes that impossible.[10] Therefore, the motive force must be exerted throughout some *interval* of time. (In the present section we will assume that there is a single temporal interval, as opposed to a sequence of separated temporal intervals, that includes all and only the times at which the motive force acts. In section 5 this assumption will be defended.)

Note second that the (longest) temporal interval throughout which the motive force acts must be open at both ends, since (in a Friedberg-style run) Achilles' acceleration varies continuously. Thus there is no *first* time at which the force acts (if there were, the force and acceleration functions would be discontinuous at that time), but rather a last time at which the force has *yet* to act.

Note third that the last time at which the motive force has yet to act cannot be a time earlier than T_0. If the motive force were exerted throughout an interval that contains both T_0 and a time *earlier* than T_0, then, contrary to his instructions, Achilles would already be in motion at T_0.[11] If, on the other hand, the motive force both began and *ended* earlier than T_0, then it could not explain Achilles' departure from point 0, where he is at rest *at* T_0. (The second law is standardly understood to mean that forces produce accelerations only at the times at which they are *exerted*.[12])

Note finally that the last time at which the motive force has yet to act cannot be a time *later* than T_0. For any such time *t*, Achilles is to have departed from point 0 some nonzero amount of time *prior* to *t*. So a force that has yet to begin at *t* cannot explain Achilles' ceasing to be at rest at 0.

Putting these points together, we may conclude that the motive force can only be a (nonzero) force that acts at all and only the times contained in some open temporal interval bounded on the early end by T_0. But now there is a problem. By Newton's second law, any such force will cause Achilles to have a nonzero acceleration throughout that same open interval. And that is incompatible with Achilles' stopping for a period of rest at each Z-point, since the latter entails having an acceleration of 0 infinitely often in every later neighborhood of T_0, no matter how small.

In brief: The force required (by Newton's first law) to explain Achilles' ceasing to be at rest will (by Newton's second law) cause Achilles to accelerate. But (as shown in note 10) there is no such thing as an acceleration that lasts only for a moment. And no matter how short the initial period of acceleration, it will be incompatible with Achilles' stopping for a period of rest, as he is required to do, infinitely often in every later neighborhood of T_0. Thus Achilles' task is dynamically infeasible.

5. COUNTERING A POSSIBLE OBJECTION

In section 4 we determined that the force which explains Achilles' ceasing to be at rest at 0 can only be a force that acts at all and only the times contained in some open temporal interval bounded on the early end by T_0. But we made that determination with the aid of a certain assumption. We assumed that the motive force is not a sequence of temporally *separated* forces (or, equivalently, a single, temporally *discontinuous* force). But that assumption is open to question. It might be suggested that Achilles' departure from point 0 is explained, not by any *one* force, but by the infinite sequence of forces that explain Achilles' departures from the infinite sequence of (reverse) Z-points, <... 1/8, 1/4, 1/2>. It may be noted that those forces are applied at points arbitrarily close to 0 and at times arbitrarily close to T_0.

In order to evaluate the idea that Achilles' ceasing to be at rest at 0 might be explained by an infinite sequence of forces, none of which is applied throughout some deleted later neighborhood of T_0, and none of which is applied throughout some deleted upper neighborhood of 0,[13] let's perform a thought experiment. For the purposes of this thought experiment, we'll think of Achilles as a point mass. And we'll imagine that each of the *positive* accelerations of Achilles is to result, not from his own efforts, but from the eastward (1-ward) force exerted on him by a mechanical thruster. Located underground between each pair of successive Z-points is a thrusting device (one that is half as large in its east-west dimension as its successor to its east). As Achilles rests at each Z-point, the first thruster to his west rises from the ground behind him, moves eastward until it makes contact with him, and, at the appointed time, thrusts him forward from that Z-point (with just the accelerative force prescribed by Friedberg). When the thruster completes its thrust, Achilles is slowed by friction (at the rate prescribed by Friedberg) and thus brought to a stop at the next Z-point.

The suggestion to be countered is that Achilles' departure from point 0 (which, of course, is not a Z-point) could be explained by the sequence of forces that cause him to leave the Z-points. So we need to stipulate that there is no thruster that rises behind Achilles while he is at rest at point 0 and thrusts him forward from that point. All of the thrusters are located to the *east* of 0.

Suppose now that Achilles has taken his position at the starting point, 0. Given the hypothesis we wish to test, we remind Achilles not to propel *himself* forward. We then activate the machinery that controls the thrusters. In the minute that follows, each of the thrusters springs into action at the appointed time. But where, at the end of the minute, do we find Achilles? It is obvious, I trust, that Achilles is still at the starting point. After all, no thruster propelled Achilles forward from that point. And Achilles did not propel *himself* forward from it. The infinity of thrusters to the east of Achilles would have propelled him on to point 1 if something had gotten him started. But the eastward thrusting thrusters to his east were powerless, individually and collectively, to exert a force on someone located to their west. So, by Newton's first law,[14] Achilles must still be at rest at point 0.

This thought experiment makes it sufficiently clear, if it wasn't already, that Achilles' ceasing to be at rest at 0 cannot be explained by forces none of which acts on Achilles throughout some deleted upper neighborhood of 0. Thus we have defended an inference drawn in section 4: Achilles' ceasing to be at rest at point 0 can be explained only by a force that acts on him at all and only the times contained in some open temporal interval bounded on the early end by T_0, since only a force of that description would act on him throughout some open spatial interval bounded on the lower side by 0. Since, as we saw, the

[13]A *deleted later neighborhood* of T_0 contains all times in some later neighborhood of T_0 (see note 4) except T_0 itself. A *deleted upper neighborhood* of 0 contains all points in some spatial interval 0X, where X is greater than 0, except 0 itself.

[14]As translated from the Latin by Bernard Cohen and Anne Whitman, Newton's formulation of the first law reads, "Every body perseveres in its state of being at rest or of moving uniformly straightforward, except insofar as it is compelled to change its state by forces impressed" (*The Principia: Mathematical Principles of Natural Philosophy*, Berkeley: University of California Press, 1999).

application of such a force is incompatible, given Newton's second law, with Achilles' running as instructed, we have also defended the overall conclusion of section 4: Achilles' task, the reversed staccato run, is dynamically infeasible.

6. CONCLUDING SUMMARY

We have concluded that the reversed staccato run is dynamically infeasible. Since the laws of classical mechanics are invariant under time reversal (the reader is referred again to note 8), we can conclude, in opposition to the received view, that the regular staccato run is dynamically infeasible as well. In arguing for the infeasibility of the reversed run, we relied not just on Newton's three laws of motion but on a supplementary philosophical judgment: that the forces which explain the reversed runner's departures from the infinite sequence of reverse Z-points fail to provide the explanation required, by Newton's first law, of the runner's ceasing to be at rest at the starting point. The needed supplementary judgment was defended by the thought experiment of section 5.[15]

[15]I thank James Buxton for very helpful suggestions.