Why are all the sets all the sets?

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This is a draft; feel free to get in touch if you have questions or feedback.

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Consider all the sets; why are *they* all the sets?

This question is widely used to motivate *potentialism* about set theory. Potentialism takes many forms, but the point of this paper is to argue that the move to potentialism is at best unmotivated and at worst a mistake.

After fixing notation (§0), I articulate (set-)necessitism (§1). This is the view that the pure sets exist, and are the way they are, as a matter of necessity.[1] I then articulate my target question in detail, and explain why it allegedly constitutes an objection—indeed, the *Ur-Objection*—against necessitism (§2).

The Ur-Objection sometimes motivates a focus on contingency. According to *contingentists*, any things *could* form a set, but it is a contingent matter which pure sets *actually* exist (§3). Contingentists may focus on circumstantial-possibility, but they typically focus on interpretational-possibility (§§4–5). Either way, contingentists must treat *us* as the source of mathematical contingency. This leaves them unable to provide us with a sufficiently rich (potential) hierarchy of sets (§6). Moreover, there are specific difficulties with how we should make sense of interpretational-possibility and interpretational-actuality (§§7–8). All told: contingentism must be rejected.

The Ur-Objection also sometimes motivates a focus on (hyperintensional) priority. On this view, members are ontologically prior to sets; sets metaphysically depend upon their members. I will neither defend nor attack such prioritism; instead, I will show that the notion of priority is simply orthogonal to the Ur-Objection (§§9–10).

Finally: the Ur-Objection sometimes motivates *structural-potentialism*. Again, the motivation is spurious: there is no clear reason to favour structural-potentialism over the structuralist analogue of necessitism (§11).

0 Technical preliminaries

I keep technicalities to a minimum in the bulk of this paper. Still, I must start by fixing some notation.

[1] Berry (2022) and Soysal (2024) use "actualism" for what I specifically call *necessitism*. But other authors use "actualism" as the contrary of *potentialism*. And potentialism can be many things, not all of which are incompatible with necessitism (see §9.2).

I use italic lowercase for singular variables (e.g. x, y). I use bold lowercase for plural variables (e.g. $\mathbf{u}$, $\mathbf{v}$). I write $x \prec \mathbf{u}$ to indicate that x is among $\mathbf{u}$. I use E as an existence predicate

Throughout, I use a positive free modal plural logic, based on S5, which embodies a *nothing-over-and-above* conception of pluralities.[2] Here are the main ideas (for axioms, see §A):

— if everything which could be among $\mathbf{u}$ could be among $\mathbf{v}$, then $\mathbf{u} = \mathbf{v}$
— $\mathbf{u}$ coexist *iff* everything which could be among $\mathbf{u}$ exists
— if x exists and x could be among $\mathbf{u}$, then x is among $\mathbf{u}$
— we have an scheme of unrestricted plural comprehension, $\text{Comp}_{\prec}$

Turning to set theory: I write $\{\mathbf{u}\}$ for the set formed by $\mathbf{u}$, and I write $\mathsf{E}\{\mathbf{u}\}$ to indicate that this exists, i.e. that $\mathsf{E}\mathbf{u} \wedge \exists x \forall z(z \in x \leftrightarrow z \prec \mathbf{u})$. I take no stance on whether we treat the membership relation, $\in$, or the set-formation operation, $\{\cdot\}$, as a theoretical primitive, since it makes no difference (see §B).

Throughout, I will assume the ordinary cumulative iterative conception of pure set. So our sets will be arranged in a (perhaps *potential*) hierarchy, with no urelements in sight. Usually, this would be achieved by deploying some plural version of ZF. Instead, I will deploy an austerely strict sub-theory of ZF: Level Theory, or LT for short (for axioms, see §C). This demonstrably axiomatizes the idea of an *arbitrary* stage in the pure set-hierarchy.[3] And LT's perfect silence on the height of the hierarchy gives it a helpfully neutral status. It can serve as the basis for Potentialist Level Theory, or PLT for short.

PLT demonstrably axiomatizes the very idea of a potential hierarchy of sets,[4] and, thoughout this paper, I will take it as *the* axiomatization of that idea. (Alternative axiomatizations are available,[5] but PLT is pleasingly simple, and it aids readability if I stick to a single formalism.) I provide details of PLT in §C, but here are the key ideas. We treat LT as axiomatic; this ensures that, necessarily, everything is arranged in a set-hierarchy (of utterly unspecified height). We then add some further principles:

$$\begin{aligned} \textbf{Ext}^{\Diamond}_{\in} \quad & \Box\forall z(\Diamond z \in x \leftrightarrow \Diamond z \in y) \rightarrow x = y \\ \textbf{Rigid}_{\in} \quad & x \in y \rightarrow \mathsf{E}x \wedge \mathsf{E}y \wedge \Box(\mathsf{E}y \rightarrow x \in y) \\ \textbf{Vary} \quad & \forall \mathbf{u}\big(\Diamond\mathsf{E}\{\mathbf{u}\} \wedge \Diamond(\mathsf{E}\mathbf{u} \wedge \neg\mathsf{E}\{\mathbf{u}\})\big) \end{aligned}$$

[2] Roberts (2022) provides an excellent discussion of the conception and the associated formalism. I have added Roberts's (ex): this allows me to ask certain questions which would be automatically blocked in a negative free logic (see §5.3, §11.3, and footnote 88).

[3] For the demonstration, see Button (2021a).

[4] Specifically, PLT axiomatizes the position Hamkins and Linnebo (2022: 12ff) call *rank-potentialism*. To obtain the demonstration: see Button's (2021b) discussion of the bi-modal theory LPST; note that LPST significantly simplifies Studd's (2013, 2019) bimodal system; Button proves that LPST is synonymous with a uni-modal theory, MLT; and PLT is a slight modification of MLT.

The slight modification can be thought of as follows. According to the bimodal theory LPST: $\mathbf{u}$ form a set iff *each* $x \prec \mathbf{u}$ existed earlier. In effect, PLT instead says: $\mathbf{u}$ form a set iff $\mathbf{u}$ *co*existed earlier.

[5] See e.g. Parsons (1977, 1983b), Linnebo (2013), and Hamkins and Linnebo (2022) for axiomatizations based on S4.2.

$\text{Ext}_{\in}^{\Diamond}$ says that sets which share all possible members are identical. $\text{Rigid}_{\in}$ says that membership is *rigid*. And, in this context, Vary says that any possible hierarchy can be either enriched with an extra level, or pruned down to some arbitrary level.

This concludes the technicalities; let philosophizing commence.

1 Introducing necessitism

Positions in the philosophy of set theory are either *objectual* or *structural*.[6]

Structuralists approach set theory by considering "set-like" systems: arrangements of objects into a hierarchy. Some structuralists confine themselves to systems of actually-existing objects; modal structuralists entertain possible systems of objects. But structuralists are united in denying that there is any elite or privileged notion of *set* or *membership*. For them: *any* objects can make up a set-like hierarchy.

Objectualists demure. They believe that there is some privileged notion of *set* and *membership*. They insist, for example, that exactly one (possible) object could be *the* empty set, and it *must* be empty. Objectualism is my main focus in this paper; I set aside structuralism until §11.

Traditional objects-platonism is the paradigm of objectualism. Its hallmark commitments are that set theory constitutes a body of truths, and that mathematical truth requires the existence of mathematical objects. From here, it is a small step (but still *a* step) to the claim that the best account of necessary (pure) mathematical truth invokes necessarily-existent (pure) mathematical objects. Set-necessitists, henceforth simply *necessitists*, are those who take this further step.

Necessitists, then, tell us that the pure sets are what they are as a matter of necessity. They therefore endorse the following scheme, for all ϕ in a suitable language for discussing pure sets (for more, see §C):

Empty-Box $\phi \leftrightarrow \Box\phi$

Since we have a normal modal logic, this scheme entails that there are all the sets there could be: any set which could exist does exist and does so necessarily. Hence the mnemonic: for necessitists, $\Box$ is "empty"; adding or dropping it does nothing.

2 The Ur-Objection against necessitism

In his discussions of indefinite extensibility, Michael Dummett posed a problem for necessitists, which we can summarize via this question:[7]

[6] Scambler (draft) and Sutto (2024) suggest this nomenclature. *Caution*: it conflicts with Shapiro's (1997) terminology: Shapiro's "*in re* structuralism" is my *structuralism*; Shapiro's own "*ante rem* structuralism" is a version of *objectualism*.

[7] I am especially thinking about Dummett (1991: 315–16) on "wield[ing] the big stick". However, this section is not Dummett-exegesis; I am not sure that Dummett ever had the *de dicto*/*de rebus* distinction sharply in mind. For fuller discussion of Dummett, see Studd (2017: 84–8, 2019: 55–60).

Dummett's Question. Why is there no set of all the sets?

We can read this question *de dicto* or *de rebus* (§2.1). Necessitists are unable to answer the *de rebus* reading of Dummett's Question, and this leads to the *Ur-Objection* against necessitism (§2.2).

2.1 Two takes on Dummett's Question

We start with a *de dicto* reading of Dummett's Question. Since our quantifiers are restricted to pure sets, on a *de dicto* reading we are simply being asked to explain why this scheme holds:

(1) $(\forall z\, z \prec \mathbf{u}) \rightarrow \neg \mathsf{E}\{\mathbf{u}\}$

Necessitists can answer this question easily: (1) is provable from solid assumptions.[8]

But we can instead read Dummett's Question *de rebus*. To force that reading, stipulate that **s** are all the sets, i.e. that:[9]

(2) $\forall z\, z \prec \mathbf{s}$

Then the *de rebus* version of Dummett's Question asks us why **s**, those very things, do not form a set; why, that is,

(3) $\neg \mathsf{E}\{\mathbf{s}\}$

We might be tempted to reply: "(1) is provable; (2) is true by stipulation; (3) follows; question answered!"[10] But this is wrong-headed. We stipulated the meaning of "**s**", but we are not asking a question about "**s**". We are asking a question about **s**—about those very things—and no stipulations are "baked into" those things. Indeed: to explain why $\neg \mathsf{E}\{\mathbf{s}\}$, we must explain why **s**, those very things, *are* all the sets; we must explain why $\forall z\, z \prec \mathbf{s}$.

(If the point is still unclear, it might help to consider a more concrete question. Let **c** be all the capybaras. Now: why are **c** all the capybaras? A complete answer to that question will involve a detailed history of the capybaras: stories of near-death experiences; tales of missed encounters; the rich tapestry of capybara-life.)

2.2 Explanatory bedrock and the Ur-Objection

Necessitists might reply to the *de rebus* question by reminding us that sets are arranged in stages. So: for there to be a set which is not among **s**, there would have to be a stage which comes after everything among **s**. There is no such stage; ergo, there is no such set.

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[8] We only need Separation (which holds in LT) and elementary logical manipulation which can be carried out even in Tennant's (2017: 127–8) Core Logic.

[9] $\text{Comp}_{\prec}$ ensures that **s** (co)exist; I revisit this in §10.

[10] This is Soysal's (2024: 159, 169–70) envisaged "textbook explanation".

This answer is fine so far as it goes, but it does not go far. We will want to know why no stage comes after everything among **s** (those very things). At this point, necessitists will invoke Empty-Box: the hierarchy could not be otherwise, so that there simply *could not* be other stages, and *that's that*. But this would not be an *answer* to the question;[11] it would be an admission that we have run out of answers. It would be to say: "Shut up; that's why!"

In sum: confronted with the *de rebus* version of Dummett's Question, necessitists hit explanatory-bedrock. This constitutes a prominent objection against necessitism.[12] Henceforth, I call it the *Ur-Objection*.

If the Ur-Objection is to motivate the rejection of necessitism, the point cannot simply be that necessitists leave *something* unexplained. After all: every philosophical position will have to give up on explanations at *some* point. The point must rather be that it is specifically intolerable to leave the *de rebus* version of Dummett's Question unanswered. But is it *really* intolerable?[13]

Rather than answering that question directly, I will consider three families of potentialism which draw their motivation from the Ur-Objection, and see how *they* fare. In thumbnail form, the families are as follows:

— *Contingentism*: some things actually fail to form a set, but (necessarily) any things could form a set (see §§3–8).
— *Prioritism*: things are metaphysically prior to the set they form (see §§9–10).
— *Structural-potentialism*: any things can comprise a set-hierarchy (if there are enough of them), and (necessarily) any set-hierarchy can be extended (see §11).

3 Contingentism

My focus from now through §8 is *contingentism*. In this section, I introduce contingentism; in later sections, I develop a particular version of contingentism in detail, but go on to explain why I reject all versions of contingentism.

[11] It might, though, be a good answer to the (related but distinct) complaint that it is *arbitrary* which sets exist: necessitists will reply that it is not arbitrary but *necessary*, and it is not clear where to go next (see Roberts 2016: 32; cf. Soysal 2020: 579–81, 2024: 165). I am employing "*de rebus*" terminology precisely because I think it cuts through those thickets; it yields the cleanest and strongest challenge to necessitism.

[12] There are many versions of this argument; see Berry (2018: 208, 2022: 3–4, 15–16), Dummett (1991: 315–6), Fine (2006: 23–5), Hellman (1989: 54–5, 2006: 81–2), Lavine (2006: 145), Linnebo (2010: 152–4, 2013: 206–7, 2016: 671–2, 2018c: 56–7), Linnebo and Shapiro (2019: 179), Florio and Linnebo (2021a: 187–91, 2021b: 270–275), Menzel (2021: 298–9), Simmons (2000: 111), and Studd (2013: 699–700, 723, 2019: 190–5). Different authors take the argument to motivate different positions: Fine and Studd advocate contingentism (see §§3–8); Linnebo advocates priority-necessitism with a restriction on $\text{Comp}_{\prec}$ (see §§9–10); Berry and Hellman advocate structural-potentialism (see §11).

[13] It is alleged that it unacceptably curtails mathematical freedom; I rebut this in §10.3.

3.1 *Actuality and objectualism*

Contingentists think that any things *could* form a set, even if they *actually* fail to.

There is some polysemy here: there are different (literal) notions of *could*, and these give rise to different versions of contingentism. To covers all versions of contingentism, I will retain the polysemy for now. But I must emphasize that I always use "could" to flag some notion of *possibility* with a concomitant notion of *actuality*. So: when a contingentist says that something is *possible* (in their preferred sense), I will always assume it makes sense for us to ask them what they think is *actual*. (The emphasis on "actuality" allows me distinguish contingentism from *prioritism*, which is superficially similar but importantly distinct; I revisit this in §9.)

Contingentists are also *objectualists*, and their objectualism can be fully explicated via two claims. First, sets are *modally-extensional*: sets which share all possible members are identical (this is $\text{Ext}_{\in}^{\Diamond}$ of §0). Second: set membership is *rigid*: a set must always have the same members whenever it exists (this is $\text{Rigid}_{\in}$ of §0.)

Notably, necessitists are also objectualists. (Indeed, necessitists endorse both $\text{Ext}_{\in}^{\Diamond}$ and $\text{Rigid}_{\in}$, via Empty-Box.) So necessitists and contingentists agree that exactly one (possible) object could be *the* empty set, $\emptyset$, and it *must* be empty. Moreover, this agreement percolates up the hierarchy: exactly one (possible) object could be *the* set whose only member is $\emptyset$, and (if it exists) it *must* have exactly $\emptyset$ as member…and so it goes. Indeed, necessitists and contingentists only disagree about whether the "and so it goes" is potential or actual. But here there is real disagreement. Necessitists posit necessarily existent sets, arranged in an *actual hierarchy*. By contrast, contingentists posit a *potential hierarchy* of sets, every possible stage of which can be surpassed; this idea is captured formally by PLT (see §0).

3.2 *Dummett's Question and the need to Explain Contingency*

Since contingentists are objectualists who embrace a notion of actuality, we can actually ask them Dummett's Question. That is, we can ask: *Why is there (actually) no set of all the (actual) sets?*[14]

Like necessitists, contingentists can read Dummett's Question *de dicto* or *de rebus*. Read *de dicto*: contingentists can reply with a simple proof. Read *de rebus*: where $\mathbf{s}$ are all the (actual) sets, contingentists must explain why $\mathbf{s}$, those very things, are all the (actual) sets, i.e. why $\forall z\, z \prec \mathbf{s}$. Indeed, the discussion of §2.1 only presupposes *objectualism*, and necessitists and contingentists agree on objectualism.

Differences between necessitism and contingentism emerge because necessitists insist that $\neg\Diamond \mathsf{E}\{\mathbf{s}\}$ and $\Box\forall z\, z \prec \mathbf{s}$ (via Empty-Box), whilst contingentists insist that $\Diamond \mathsf{E}\{\mathbf{s}\}$ and hence that it is contingent that $\forall z\, z \prec \mathbf{s}$ (via Vary). So, contingentists must Explain Contingency: *they must tell us why $\forall z\, z \prec \mathbf{s}$, given its supposed contingency*. If they cannot do this, then their Ur-Objection has been turned against them.

[14] Burgess (2022: last ¶), Menzel (2021: 303), Roberts (draft: fn.20), Scambler (draft: 17), and Soysal (2024: 159) pose versions of this question.

3.3 *Copernicanism is needed to Explain Contingency*

Given the polysemy of "could" (see §3.1) and hence "contingent", different contingentists may try to Explain Contingency in different ways. However, given the need to Explain Contingency and the dearth of plausible explanations, we can rapidly prune down the space of plausible contingentist positions.

Very likely, the existence of *impure* sets is contingent upon the existence of their supports; the existence of $\{\emptyset, \{\text{Hypatia}\}\}$ is contingent on the existence of Hypatia. But we are only considering pure sets in this paper (see §0). So let me repeat: **s** are all pure sets. They have no supports, on which their existence could be contingent.

Indeed, the *purity* of pure sets removes almost every conceivable source of contingency. Pure sets seem unaffected by: the flapping of a bird's wings; the ebb and flow of the oceans' tides; variations in the initial conditions of the physical universe; changes to the laws of physics. *Perhaps* the pure sets could be contingent upon other pure mathematical objects; but that would only defer the issue, for we would need to explain the contingency of *those* objects…

Only one conceivable source of contingency remains: *us*, *we*, *ourselves*. To Explain Contingency, contingentists must make a Copernican Turn. They must embrace this principle:

> *Copernicanism.* Necessarily: our mathematical behaviour determines which sets exist.

This principle can be fleshed out in a few different ways. The point is that any contingentist must embrace *some* version of Copernicanism.

3.4 *Circumstantial and interpretational modality*

One obvious way to flesh out Copernicanism is to embrace constructivism. Such *constructivism-cum-contingentism* would claim that we determine which sets exist by literally *creating* them. This would also give a way to Explain Contingency: we *happened* to create some sets, but could have created others.

We should pause to consider the *modality* our constructivism-cum-contingentism is invoking. When they consider a claim like $\Diamond \mathsf{E}\{\mathbf{s}\}$, they think that some non-actual set, $\{\mathbf{s}\}$, literally comes into existence. So they are considering variations in our *circumstances*. They use a *circumstantial* modality.[15]

In fact, the converse holds: contingentists who use a circumstantial modality must be constructivists. This follows from their need Explain Contingency via Copernicanism.[16] Given a circumstantial notion of possibility, the generic contingentist claim,

[15] What follows invokes Fine's (2006: 33–4) distinction between circumstantial and interpretational modalities.

[16] *Pace* Scambler (draft: 17), contingentists who invoke a circumstantial possibility have a particular need to Explain Contingency (beyond the need to escape their own Ur-Objection). Suppose that which sets exist is a *brute* circumstantial-contingency. Then there is a world which is exactly like ours

that {**s**} actually does not exist but could, becomes the specific claim that circumstances could vary so that {**s**} would exist. Given Copernicanism, *we* would bring about this variation in circumstance. That is to say: we could bring {**s**} into existence. That is constructivism.

Given the literature, it is safe to assume that contingentists want to avoid constructivism (I revisit this in §6.4). It follows, though, that contingentists must invoke some *non*-circumstantial modality.

The alternative notion of modality is *interpretational*: we consider variations in interpretation, rather than circumstances. *Interpretationalists* are contingentists whose notion of contingency is interpretational, and interpretationalism will be my focus from now through §8.

4 Interpretationalism introduced

In this section, I will outline interpretationalism (§4.1), show where it disagrees with necessitism (§4.2), and describe how it tries to Explain Contingency (§4.3).

4.1 James Studd's interpretational modality

Interpretationalists adopt an interpretational notion of possibility. There is room for variation in exactly what this comes to, but I will take Studd's approach as canonical. For him, interpretational-possibility involves

> …a shift in interpretation understood…as a shift in semantic content.… Like logical necessity, [interpretational-necessity] concerns possible shifts in interpretation rather than circumstance. But…the shifts in interpretation that are admissible are more closely constrained: not every logically-possible interpretation need be counted admissible.[17]

So, when interpretationalists use $\Diamond$ in their modal theory set, they might roughly gloss it as follows:[18]

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except, perhaps, that it contains no sets. Indeed, that world might well *be* our actual world since—on this view—physics, mathematics, and everything else works just as well in set-free worlds as in set-containing worlds. But if the *actual* existence of sets is irrelevant to actual physics, mathematics, etc., it is hard to see how we would benefit from the merely *possible* existence of sets. (Cf. Balaguer's (1998: 132) "makes no difference" argument.)

[17] Studd (2019: 148; see also 105, 144, 147). Linnebo (2018c: 61–3, 205) offers an alternative notion of interpretational-possibility: $\Diamond\phi$ says roughly "we can abstract and thereby shift the meaning of the language so that ϕ". (NB: I *mostly* do not read Linnebo as an interpretationalist; see §9.2.) Fine (2006: 33–41) favours a notion of *postulational-possibility*, which he describes as "a change of interpretation…that is intermediate, as it were, between a change in content and a change in circumstance". Studd's (2019: 104, 146–8) discussion of these alternatives is extremely helpful. The objections I raise against interpretationalism in §§6–8 will apply to interpretationalists who use these other glosses; moreover, use of the other glosses makes it hard to Avoid Revenge (see footnote 32).

[18] See Studd (2019: 148, 172); this is not offered as a reductive definition, but as an explication of a theoretical primitive.

$\Diamond\phi \approx$ it is possible to vary the interpretation (admissibly) so that ϕ

The constraints on admissibility are imposed by interpretationalism's *objectualism*; as in §3.1, all admissible interpretations must agree on what *the* empty set is. Likewise, as in §3.1: interpretationalists countenance a notion of interpretational-actuality: they think that there is a fact of the matter concerning which sets we (actually) discuss.[19]

4.2 Necessitists disagree with interpretationalists

I will start by explaining how and why necessitists disagree with interpretationalism.

Necessitists claim "some things could not form a set". Interpretationalists say "any things could form a set". But we cannot simply take this disagreement at face value. When I described necessitism in §1, I used a fairly *naïve* modality. By contrast, interpretationalists want to use a specifically *interpretational* modality. With different notions of "could" under consideration, disagreement may be superficial. Moreover, interpretationalists (typically) *endorse* the necessitists' Empty-Box, when "$\Diamond$" is read in terms of metaphysical-possibility;[20] and that may well be the necessitist's preferred modality!

For all that, there is real disagreement between necessitists and interpretationalists. As outlined in §1, necessitists infer the necessary existence of pure mathematical objects from the necessity of pure mathematical truth. Necessitists will therefore treat *interpretations*, in this context, as (necessarily existing) entities. So, when interpretationalists gloss a claim as "it is *possible* to vary the interpretation (admissibly) so that ϕ", necessitists will regard this as saying just "there *is* an (admissible) interpretation such that ϕ".[21]

This allows us to formulate a sharp point of disagreement.[22] According to interpretationalists, no things are interpretationally-necessarily *all* the things. Interpretationalists will be happy to gloss this as follows:

(a) given any $\mathbf{u}$, it is *possible* to vary the interpretation so that $\mathsf{E}\mathbf{u} \land \exists z\, z \not\prec \mathbf{u}$

As above, however, necessitists will regard (a) as saying:

(b) given any $\mathbf{u}$, there *is* an interpretation such that $\mathsf{E}\mathbf{u} \land \exists z\, z \not\prec \mathbf{u}$

But this is bad news. Let $\mathcal{I}$ be any interpretation; let $\mathbf{d}$ be the denizens of $\mathcal{I}$. By (b), there is some interpretation, $\mathcal{J}$, with a denizen not among $\mathbf{d}$, i.e. which is not a denizen of $\mathcal{I}$. Generalizing on $\mathcal{I}$, we obtain:

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[19] For contrast, consider the view that our set-theoretic language is always "systematically ambiguous" between different interpretations (cf. Parsons 1977: 289).

[20] See Fine (2006: 31), Parsons (1983b: 328), and Studd (2013: 706–7, 2019: 35, 49, 103, 108, 146).

[21] For necessitists, *admissible interpretations* are those where "$\in$" means $\in$. Consequently, necessitists should not say that interpretations are set-theoretic *objects* (unless they want to restrict $\text{Comp}_{\prec}$; see §10.4 and Williamson 2003).

[22] Cf. Fine (2006: 27), Florio and Linnebo (2021b: 246–8), Studd (2019: 13, 80, 120–2, 144–5, 172), and Williamson (2003: 424–35).

(c) no interpretation includes everything.

and there is no good way to interpret "everything", in (c), so that (c) says something both sensical and true. The moral is that (b) should be rejected; but there are two ways to go. Necessitists infer that (a) is wrong, and hence reject interpretationalism; interpretationalists insist on the ineliminably modal nature of interpretation and hence reject necessitism. This is a real disagreement.

4.3 Interpreting language in use, and Copernicanism

The disagreement we have just encountered allows us to deepen our understanding of the interpretationalists' notion of interpretation, as concerned with language in *use*.

Interpretation is sometimes treated as a mere matter of *abstract-tagging*. To illustrate, consider the notion of an $\mathscr{L}$-structure in model theory. To say that $\mathscr{L}$-constant c names object a in $\mathscr{L}$-structure $\mathcal{M}$ is not to insist that someone somewhere carried out some naming ceremony; it is just to say that $\mathcal{M}$ abstractly tags a with c.[23] Similarly: we tag some (set of) things as $\mathcal{M}$'s domain. On this image: a language is an object of study; just a bundle of tags to apply.

Interpretationalists are not concerned with interpretation as *mere* abstract-tagging.[24] That image is not especially modal, where the interpretationalists' notion of interpretation needs to be (see §4.2). We could try to bolt modality onto the image, by allowing us to tag not just actual objects but merely *possible* objects. But then we would need to hear more about the sense of possibility in question. Claiming that it is *interpretational*-possibility would be viciously circular, given we are still in the process of explicating that notion.

We can go further. Invoking the image of abstract-tagging will make the entire notion of "interpretation" an idle wheel. To the claim that some things, **u**, satisfy (some extension of) LT, when given suitable *admissible* abstract-tags, is just to say that **u** are some sets arranged in a hierarchy.

The moral is clear: interpretationalists are not concerned merely with tagging mathematical entities with terms from abstract formal languages. Their notion of "interpretation" concerns mathematical language *in use*, rather than language as a mere object of study. And modality—indeed, contingency—is an ineliminable part of language in use, because language *use* is inevitably subject to various contingencies.

This observation meshes neatly with the fact that interpretationalists, being contingentists, endorse Copernicanism (see §3.3). Specifically, for interpretationalists, a community's behaviour determines what is interpretationally-actual for that community.[25] And this also tells us how interpretationalists will (try to) Explain Contingency. According to interpretationalists: the interpretationally-actual sets are the sets which

[23] See Lavine (2000: 18–26); contrast Linnebo (2018c: 25, 93).

[24] They need not eschew that notion of "interpretation" forever; I am concerned with "interpretation" as it figures in their gloss on $\Diamond$.

[25] Studd (2019: ch.8) outlines a metasemantics for Copernicanism.

we *happen* to be talking about. Exactly which sets these are is settled by specific historical contingencies: if mathematicians had done otherwise, we might have talked about other sets than **s**. As it is, our contingent behaviour makes it interpretationally-actually the case that $\forall z\, z \prec \mathbf{s}$. *Contingency explained!*

Or so interpretationalists will insist; I contest this putative explanation in §7. For now, I will continue to develop interpretationalism sympathetically.

5 Interpretationalism can Avoid Revenge

Let us take stock. We saw in §3.2 that an actuality-focussed version of Dummett's Question generates a demand for contingentists: they must Explain Contingency (and we saw in §4.3 how interpretationalists try to do this). In this section, I will explain how a possibility-focussed variant of Dummett's Question generates a second demand for contingentists: they must Avoid Revenge. Whilst Studd's version of interpretationalism fails to meet this demand, interpretationalists can Avoid Revenge by breaking company with Studd to Avoid Revenge. (So this section suggests a "friendly amendment" to Studd's interpretationalism.)

5.1 The threat of revenge

Let us pose the following question to interpretationalists:

Dummettian Revenge. Why couldn't there be a set of all possible things?

If we read this *de dicto*, then the task is to explain the general validity of the scheme:

(4) $(\Box\forall z\, z \prec \mathbf{u}) \rightarrow \neg\Diamond \mathsf{E}\{\mathbf{u}\}$

But this is exactly as easy to explain as (1) of §2.1.[26] Matters become more interesting when we (try to) read Dummettian Revenge *de rebus*. So, we (try to) stipulate that **p** are exactly the possible things:

(5) $\Box\forall z\, z \prec \mathbf{p}$

I have used the caveat "try to" a few times; in §5.3, I will suggest that this attempt fails. For now, suppose it succeeds. So (5) holds. In that case, we shall demand an explanation of the following fact:

(6) $\neg\Diamond \mathsf{E}\{\mathbf{p}\}$

For necessitists, this explanatory-demand is nothing new; necessitists hold that $\mathbf{s} = \mathbf{p}$. But contingentists in general, and interpretationalists in particular, believe that $\Diamond \mathsf{E}\{\mathbf{s}\}$. So interpretationalists are in new territory.

[26] With (1) as a theorem, (4) follows in a normal modal logic.

To meet the explanatory-demand, interpretationalists may first note that Vary and (6) together entail that **p** cannot coexist, i.e. that:

(7) $\neg\Diamond \mathsf{E}\mathbf{p}$

But interpretationalists hold that any things which form a set coexist.[27] So, interpretationalists may say that **p** cannot form a set because they cannot coexist, and they cannot coexist because $\Box\forall z\, z \prec \mathbf{p}$.[28]

This answer to the *de rebus* version of Dummettian Revenge is fine so far as it goes, but it does not go far. We will want interpretationalists to explain why $\Box\forall z\, z \prec \mathbf{p}$; that is where the action is (compare §2.2). So we come to a second demand for interpretationalists. They must Avoid Revenge: *they must explain why* $\Box\forall z\, z \prec \mathbf{p}$ *(or: explain why we failed to stipulate the meaning of "*$\mathbf{p}$*").*

As I will show: Studd's version of interpretationalism does not Avoid Revenge (§5.2), but a nearby version of interpretationalism does (§5.3).

5.2 Studd does not Avoid Revenge

Studd thinks that the impulses which drive interpretationalists to embrace a $\Diamond$-potential hierarchy of sets should push them ever onwards.[29] In more detail: we start with some interpretational modality, $\Diamond$, which we use to characterize the $\Diamond$-potential hierarchy. We then consider whether the "$\Diamond$-potential hierarchy could be extended beyond the stages [that $\Diamond$] generalizes about".[30] Studd thinks that we should conclude that it *could* be and hence—by parity of reasoning with the move from necessitism to interpretationalism—*should* be. We therefore posit a strictly richer kind of interpretational modality, $\Diamond_1$, and use this to characterize the $\Diamond_1$-hierarchy, which strictly extends the $\Diamond$-hierarchy.

Since Studd countenances this $\Diamond_1$-hierarchy, he will say that **p** (co)exist at the first stage of the $\Diamond_1$-hierarchy which comes after the $\Diamond$-hierarchy. (So: we *succeeded* when we tried to stipulate the meaning of "$\mathbf{p}$".) To Avoid Revenge, Studd must therefore explain why $\Box\forall z\, z \prec \mathbf{p}$.

I maintain that he cannot do this. In this context, we can re-phrase the explanatory demand as follows: *Explain why the divide between* $\Diamond$ *and* $\Diamond_1$ *is exactly where it is, and not somewhere else.* Put thus, though, it is obvious that no explanation is possible; we have nothing but a *syntactic* grasp on the supposed distinction between $\Diamond$ and $\Diamond_1$. So Studd cannot Avoid Revenge. In effect, the Ur-Objection has been turned against him.

Matters get worse. We can re-run Dummettian Revenge in terms of $\Diamond_1$. So: we stipulate that $\mathbf{p}_1$ are all the $\Diamond_1$-potential things, i.e. that $\Box_1\forall z\, z \prec \mathbf{p}_1$; we ask why $\neg\Diamond_1\mathsf{E}\{\mathbf{p}_1\}$; ultimately this becomes a demand to explain why $\Box_1\forall z\, z \prec \mathbf{p}_1$. The same

[27] This is Rigid_{β}; given NOaA it follows from $\text{Rigid}_{\in}$.
[28] Cf. Parsons (1977: 281–2) on Cantor.
[29] See Studd (2019: esp.201–13; also 155–6, 158, 161–2, 176–7, 243–4). Studd's interpretationalism is (in his terms) $\forall\Box$-relativist; I think interpretationalists should be $\forall\Box$-absolutist. My $\Diamond_1$ is Studd's $\oplus$.
[30] Studd (2019: 212, 213).

structural pressures which lead Studd from $\Diamond$ to $\Diamond_1$ will lead him to posit a yet-richer modality, $\Diamond_2$... and so it goes. Studd will be pushed to embrace an indefinite sequence of strictly richer modalities, $\Diamond_i$, for various indexes i, and, he will face an indefinite sequence of unanswerable questions concerning why $\neg\Diamond_i \mathsf{E}\{\mathbf{p}_i\}$.

The phrase "indefinite sequence" itself masks further trouble. How should we characterize the schematic indexes (our various i's)? A moment's reflection will convince us that no answer will ever suffice. And this is a sadly familiar point. Indeed, much of the promise of introducing modal operators was to obtain a kind of generality concerning sets *without* resorting to schemas.[31]

For all these reasons, *pace* Studd, interpretationalists *should* not countenance $\Diamond_1$. Fortunately, they do not *need* to.

5.3 *A transcendental way to Avoid Revenge*

I will offer a (transcendental) argument, on behalf of interpretationalists, which allows them to eschew $\Diamond_1$ and Avoid Revenge. The argument starts with a conceptual platitude about *de rebus* representation:

(i) when we represent some things *de rebus*, we adopt an interpretation which includes all of those things.

Hence:

(ii) if we can represent some things *de rebus*, then it is possible to vary the interpretation to include those very things.

Given our gloss on interpretational-possibility (see §4.1), this becomes:

(iii) if we can represent some things *de rebus*, they interpretationally-could coexist.

The next step in the argument involves considering the role of variables. In first-order logic, variables are devices of *de re* representation; their role is just to present some *thing* (on a value-assignment). Analogously: in plural logic, plural-variables are devices of *de rebus* representation. So:

(iv) if some things can be the values of a plural-variable, then we can represent those things *de rebus*

Combining (iii) and (iv), and recalling that $\Diamond$ expresses interpretational-possibility, we obtain this open scheme:

PossCo $\Diamond \mathsf{E}\mathbf{u}$

[31] Cf. Fine 2006: 29–30; Studd 2019: 144.

Indeed, PossCo just expresses our initial conceptual platitude about *de rebus* representation.

Armed with this insight, let us revisit the attempt to raise a *de rebus* version of Dummettian Revenge. Via (4), all possible things cannot coexist. But then, via PossCo, the phrase "all possible things" cannot present some things *de rebus*. So when we tried to stipulate "**p** are exactly the possible things", that attempt *misfired*. More generally: since PossCo captures a conceptual platitude about *de rebus* representation, we cannot so much as formulate a *de rebus* version of Dummettian Revenge. We Avoid Revenge because there is, literally, *nothing* to explain.[32]

To close the section, note that the transcendental argument for PossCo also allows interpretationalists to deny (*pace* Studd) that the same impulses which drove them to embrace a $\Diamond$-potential hierarchy should drive them to countenance a $\Diamond_1$-hierarchy. Interpretationalists are (as I frame things) motivated by the Ur-Objection; they are driven to embrace a $\Diamond$-hierarchy because they are embarrassed by necessitism's inability to explain why **s**, *de rebus*, do not and cannot form a set. But, given PossCo, you just *misfire* if you try to embarrass someone by saying "you can't explain why **p** cannot form a set". So: no Ur-Objection-shaped reasoning pushes interpretationalists into (disastrously) countenancing $\Diamond_1$. Interpretationalists really can, straightforwardly, Avoid Revenge.

This concludes my sympathetic development of interpretationalism. But I am not an interpretationalist. Over the next three sections, I will explain why.

6 Interpretationalism cannot Regain Paradise

David Hilbert famously insisted that "No one shall drive us out of the Paradise that Cantor created for us".[33] Hilbert's insistence is perfectly reasonable, and it generates a third demand. Interpretationalists must be able to Regain Paradise: *they must ensure that their potential hierarchy is richly populated with possible sets.*

In this section, I show that interpretationalists cannot Regain Paradise. Their commitment to Copernicanism keeps them out of Eden.

6.1 The problem, starkly presented

To raise a problem for interpretationalists, let me (temporarily) treat interpretationalism's $\Diamond$-operator in terms of quantification over interpretational-possibilities. (This

[32] Linnebo (2018c: 66–9) offers a similar argument from PossCo against revenge. However, he simply *assumes* PossCo, rather than offering an argument for it on the basis of his preferred understanding of $\Diamond$. My argument (i)–(iv) uses Studd's gloss on $\Diamond$, and Linnebo's gloss does not seem to license the same argument (see footnote 17).

[33] Hilbert (1926: 170).

is unfaithful to the spirit of interpretationalism, but it affords a very clean statement of the problem; I revisit the infidelity in §6.2.) Given this treatment of $\Diamond$, PLT entails:[34]

(1) there are at least as many interpretational-possibilities as there are levels in the potential hierarchy

Next, Copernicanism entails:

(2) there are at least as many distinct possible ways for a community to behave as there are interpretational-possibilities

Combining these:

(3) there are at least as many distinct possible ways for a community to behave as there are levels in the potential hierarchy.

But now we have a problem. To Regain Paradise, interpretationalists must—at the very least—be able to offer a potentialist treatment of ZF.[35] Standard models of ZF have inaccessibly many levels. So the potential hierarchy must have inaccessibly many possible levels. By (3), there must be inaccessibly many distinct possible ways for a community to behave. And that is implausible, on any notion of "behaviour".

6.2 *Eliminating quantification over possibilities*

That is the problem, starkly presented. I will now explain how to present the problem without quantifying over interpretational-possibilities.

First, note that PLT proves that any possible level could exhaust all there is, i.e. we have this theorem scheme (Lemma 3 of §C):

$$Lev(s) \rightarrow \Diamond(\mathsf{E}s \land \forall x\, x \subseteq s)$$

Given $\text{Rigid}_{\in}$, this is a good surrogate for premise (1) of the above argument.

It is less obvious what surrogate to offer for (2). However, this is the interpretationalists' problem, not mine. To Explain Contingency, interpretationalists must endorse Copernicanism (see and §3.3 and §4.3). So they need some way, which they find satisfactory, to *express* the connection between possible behaviour and their $\Diamond$-operator. We can simply use whatever they provide us with as a surrogate for (2), to obtain a surrogate for (3).

[34] Suppose s and t are possible levels. Using Lemma 3 of §C, let $\boldsymbol{w} \Vdash \mathsf{E}s \land \forall z\, z \subseteq s$ and let $\boldsymbol{x} \Vdash \mathsf{E}t \land \forall z\, z \subseteq t$. So if $\boldsymbol{w} = \boldsymbol{x}$, then $s = t$ by $\text{Rigid}_{\in}$ and $\text{Ext}^{\Diamond}_{\in}$.

[35] Presumably via (de)modalization in §9.2.

6.3 *Shoring up the argument*

I have set out my argument. To shore it up, I will consider the (only) two strategies that an interpretationalist might deploy in an attempt to rebut my argument, and show that neither works.

Strategy 1: idealize the notion of behaviour. The first strategy is to agree with each of (1)–(3), but to deny that there is a problem, because there *are* inaccessibly many possible ways for a community to behave. Granted, these would not be possible *human* behaviours, since we are bound by spacetime; but we can (according to this strategy) countenance an *idealized* notion of behaviour. For example: for any arbitrary ordinal, α, we can posit an idealized community who perform "α-super-tasks"; by such means, we Regain Paradise.

I can deal with this strategy briskly. Interpretationalists want to give an account of set theory in terms of interpretational-possibility, and hence (by Copernicanism) in terms of *behaviour*. This strategy instead seeks to give an account of the required notion of behaviour in terms of *set theory* (e.g. via the notion of arbitrary ordinals).[36] So this strategy involves abandoning the interpretationalist project.

Strategy 2: allow supra-behavioural interpretations. The second strategy concedes that possible behaviours are limited, and so attempts to avoid the problem by denying premise (2). We would still hope to explain the actual contingency, that $\forall z\, z \prec \mathbf{s}$, by insisting that our actual language-use determines what is interpretationally-actual. But we would allow that some (indeed, most) interpretations are *supra*-behavioural; they are not the correct interpretation *of* any possible community's language-use.

Again: this strategy involves abandoning the interpretationalist project. A supra-behavioural interpretation is precisely *not* an interpretation of language in (any possible) *use*. So to countenance "supra-behavioural interpretation" is to countenance a notion of "interpretation" which is anathema to interpretationalist, for the reasons given in §4.3.[37]

[36] An interpretationalist might try to avoid circularity, by insisting that they are engaged in virtuous "boot-strapping". In thumbnail: we account for a small potential hierarchy in terms of possible behaviour; that potential hierarchy supplies us with notions (e.g. some ordinals) with which we enrich our notion of behaviour; we use the enriched notion of behaviour to account for a taller potential hierarchy; etc. Interesting as this is, we cannot bootstrap our way to Paradise; as Button and Trueman (2022: §3.3), hereditary-points impose hard limits.

[37] There is another problem with this strategy. In brief: even if it allows us to explain the interpretationally-*actual* contingency that $\forall z\, z \prec \mathbf{s}$, it cannot to handle the interpretational-*necessity* of contingency. In detail: interpretationalists hold that $\Box\exists\mathbf{u}(\forall z\, z \prec \mathbf{u} \land \Diamond\exists z\, z \not\prec \mathbf{u})$. That is: interpretationally-necessarily, some things are all the sets but only contingently all the sets. So: interpretationally-necessarily, some things, $\mathbf{u}$, are such that there is a question of *why* $\mathbf{u}$ are contingently all the sets. Copernicanism addresses all such questions. It say: "*necessarily*: our mathematical behaviour determines which sets exist"; the "*necessarily*" prefix is there to ensure that, if $\mathbf{u}$ were all the sets, they would be *all* the sets due to the contingent behaviour of some community. But this entails (2). So: if interpretationalists want to reject (2), as on this strategy, they will re-open these explanatory-gaps.

6.4 A generalization of a venerable objection

The problem I have raised in this section is easy to state: interpretationalism is undone by a basic tension, between the needs to Explain Contingency and to Regain Paradise. In fact, this is a generalization of a venerable objection against constructivism.

In §3.4, I noted that interpretationalism is not the only version of contingentism; we can consider versions of contingentism which use some kind of *circumstantial* modality (rather than interpretational modality). But is easy to see that circumstantially-formulated contingentism succumbs to exactly the same argument as I have levelled against interpretationalism: just re-read §§6.1–6.3, replacing every occurrence of “interpretational-” with “circumstantial-”.

This failure of circumstantially-formulated contingentism should come as no surprise. After all, in §3.4, I showed that circumstantially-formulated contingentism is committed to constructivism (via Copernicanism). And the inability to Regain Paradise, in all its glory, is a venerable objection against constructivism.[38]

The upshot of this section is that the objection is more general than has been noted. It does not just affect constructivists, but all copernicans, and hence all contingentists, including interpretationalists.

7 Interpretationalism is monstrously unexplanatory

The inability to Regain Paradise is a sufficient reason to reject any version of contingentism. However, I have two further complaints against interpretationalism (specifically). In this section I will argue that, contrary to §4.3 interpretationalism cannot Explain Contingency, because their notion of interpretational-*possibility* is monstrous.

Towards the end of §4.3, I sketched how interpretationalists might try to Explain Contingency. That discussion can be condensed down into a short dialogue:

— *Michael:* Why is every set among **s**?
— *Interpretationalist:* Because our contingent behaviour determines what counts as *everything*, since our behaviour determines what is interpretationally-actual.

For comparison, imagine a conversation between two children at a museum.[39]

— *Abby:* Why did stegosaurs have four legs?
— *Ben:* Because our contingent behaviour determines what counts as a *leg*.

Abby should protest that Ben is being silly. My objection is that the interpretationalist’s attempt to Explain Contingency is dangerously close to Ben’s silliness.

Playing with meanings cannot affect the legs of long-extinct animals. So Abby might reply to Ben, very soberly:

[38] See e.g. Parsons (1977: 271–80), Potter (2004: 37), and Studd (2013: 706, 2019: 49). Compare also Hewitt’s (n.d.: 325) argument that Linnebo (2010: 158–9) lacks sufficient expressive resources.

[39] This is a riff on an adage attributed to Abraham Lincoln.

(a) however we vary the meaning of *leg*, stegosaurs would have had four legs.

Even more pedantically: in evaluating "stegosaurs would have had four legs" in (a), we use our existing meaning of *leg*, not the hypothesized varied meaning. That is, the prefix "however we vary the meaning of *leg*" is not a Kaplanian monster. And Ben is being silly because he has explained (in part) why "stegosaurs have four legs" is a *true sentence*, without offering any explanation of the *fact that* stegosaurs have four legs.

Now an interpretationalist who was *also* a constructivist would say that, when we vary the (admissible) interpretation of our set-theoretic language, we literally construct new sets. But every interpretationalist I know of rejects constructivism.[40] So they should agree that we cannot affect the pure sets in any way, *including* by varying interpretations. By parity with (a), we should expect them to say:

(b) however we vary the interpretation, everything would be among **s** i.e. $\forall z\, z \prec \mathbf{s}$.

That is: "however we vary the interpretation" is not a Kaplanian monster.

This causes a snag. Interpretationalists are committed to $\Diamond\exists z\, z \not\prec \mathbf{s}$. In §4.1, we *roughly* glossed $\Diamond\phi$ as "it is possible to vary the interpretation (admissibly) so that ϕ". If we deploy that rough gloss on $\Diamond\exists z\, z \not\prec \mathbf{s}$, we seem to contradict (b).

This is only a small snag, not an outright contradiction; the gloss was, indeed, *rough*. To be a bit less rough, interpretationalists might say this: $\Diamond$ is a semi-technical notion; we can approximate its meaning with some English phrases that might not (naturally) be read monstrously; but $\Diamond$ is a Kaplanian monster by design.[41] Perhaps we could have made this clearer, by writing something like this:

$$\Diamond\phi \approx^{\text{but read this}}_{\text{monstrously}} \text{it is possible to vary the interpretation (admissibly) so that } \phi$$

In any case: I happily grant interpretationalists their monster. My concern is that monsters do not lend themselves to good explanations of the facts we care about (note: the *facts*, not the truth values of *sentences*). To see why, revisit Abby and Ben, in the museum, and now suppose we furnish them with an explicitly monstrous operator, via this gloss:

$$\blacklozenge\phi \approx^{\text{but read this}}_{\text{monstrously}} \text{it is possible to vary the meaning of } leg \text{ so that } \phi$$

Using this operator, it would be *correct* to say:

(a_*) stegosaurs had four legs $\wedge \blacklozenge$(stegosaurs had five legs)

Comparably, interpretationalist think it is correct to say:

(b_*) $\forall z\, z \prec \mathbf{s} \wedge \Diamond\exists z\, z \not\prec \mathbf{s}$

But now (b_*) stands to the interpretationalist's attempt to explain the fact that $\forall z\, z \prec \mathbf{s}$ as (a_*) stands to Ben's attempt to explain the fact that stegosaurs had four legs: neither *explains*. So, contrary to §4.3, interpretationalists cannot Explain Contingency after all, even by invoking Copernicanism.

[40] See footnote 20.

[41] This is Studd's (2019: 107–8, 174–6) approach.

8 Interpretationalism struggles with actuality

I turn to my third and final reason for rejecting interpretationalism: we need to reject their notion of interpretational-actuality.

Modal locutions are peppered throughout semi-formal mathematics. One set theorist may say:

(c) every set has a choice set

Another may say:

(d) for any set there might be, it would have a choice set

Whatever we ultimately make of mathematical modality, when we consider these claims as bits of mathematics, they are surely intended to have the same content. It would be bizarre to think that (c) is only concerned with the *actual* sets, whilst (d) concerns all *possible* sets.

Bearing this in mind, consider interpretationalism's commitment to Copernicanism. So far, I have said almost nothing about *how* our behaviour is supposed to determine which sets are interpretationally-actual. But the outline of the picture is clear enough: the interpretationally-actual set-hierarchy is exactly as tall as required to furnish a "natural" interpretation of our actual mathematical claims.[42] (And, as in the previous paragraph, it does not matter whether those mathematical claims use any modal locutions.) It follows that mathematical claims *only* concern interpretationally-actual objects.

Suppose, now, that we hear (c) in the mathematics classroom. On the present picture, this claim only concerns the interpretationally-*actual* hierarchy. It tells us only that every interpretationally-actual set has a choice set. It says nothing about whether there interpretationally-could be sets which interpretationally-necessarily lack choice sets. The same point holds even if we hear (d), rather than (c), in the mathematics classroom. Indeed, quite generally: no mathematical claim could even *express* the proposition that there interpretationally-could be sets which interpretationally-necessarily lack choice sets, for that is a claim about the entire *potential* hierarchy.

The problem is obvious: here *we* are, apparently expressing the proposition in question! Since it cannot be expressed by any mathematical claim, we must insist that we (philosophers, philosophizing) are doing something *supra*-mathematical.[43] This avoids self-stultification, but at terrible cost. Our subject matter is (choice) sets; is it so much as conceivable that this is *supra*-mathematical?

[42] For ideas in this ballpark, see Parsons (1977: 289–9), Linnebo (2010: 159n21, 2013: 207–8), Menzel (2021: 303), and Studd (2019: 238–40). Berry (2022: 73–6) and Soysal (2024: 160–2) raise excellent worries about the tenability of this metasemantics.

[43] It is not enough to suggest that our discussion is mathematical and that, in the course of the discussion, we thereby expand the interpretation. That would shift interpretational-actuality a few stages up the potential hierarchy, as it were; it would not still not allow us to discuss the entire potential hierarchy.

Potentialists of any stripe must therefore reject the picture I just sketched. Instead, they must say that mathematical claims concern the whole potential hierarchy (by default, whether or not the claim uses modal locutions).[44] So if we hear (c) in the mathematics classroom, potentialists should take it to say: *necessarily*, every set *could* have a choice set.[45]

Unfortunately for interpretationalists, interpretational-*actuality* has no role to play on this account. Any claim about the whole potential hierarchy is compatible with assigning any interpretationally-possible level as interpretationally-actual.[46] So, on this alternative picture, where all mathematical claims concern the whole potential hierarchy (by default),[47] none of our mathematical claims can conceivably constrain which sets are interpretationally-actual. But then we are left without any notion of "interpretational-actuality".

9 Prioritism

Let us take stock. We have seen that contingentists cannot simultaneously Explain Contingency and Regain Paradise (§6). We also saw that interpretationalists can neither Explain Contingency (§7) nor offer a workable notion of (interpretational-)actuality (§8). So we must reject contingentism.

In §2.2, though, I explained that the Ur-Objection has been taken to motivate two other broad positions: *prioritism* and *structural-potentialism*. In this section and the next, I will consider prioritism. My claim is that it simply does not connect with the Ur-Objection.

9.1 Disentangling contingency from priority

Contingentists formalize their position using normal modal logics. But normal modal logics can be used for a rather different purpose in this debate. Specifically: *prioritists* hold that $\mathbf{u}$ are always prior to $\{\mathbf{u}\}$, in some metaphysically substantial, hyperintensional, sense of *priority*. (In this paper, I use "priority" as a colourless label for any posited hyperintensional relation, to cover ground, essence, building, etc.)

The notion of priority is obviously time-like, and we can easily regiment time-like structures using normal modal logics.[48] For example, to say that a is prior to b—

[44] See Linnebo (2010: 155–6, 2013: 225) and Studd (2019: 156–7). I include the caveat "by default" since context or decision can restrict our claims to some fragment.

[45] More generally, we can use (de)modalization (see §9.2).

[46] Consider §9.2, and the question of "which level is actual-under-demodalization".

[47] Mathematical claims from outside set theory will not help to restore a notion of interpretational-actuality, since they are entangled with set theory. For example, each of these claims is equivalent to (c): every vector space has a basis; products of compact topological spaces are compact; every connected graph has a spanning tree.

[48] Contemporary prioritists may, however, prefer to use a more fine-grained logic of essence; e.g. Fine (1995, 2000) and Ditter (2020a,b).

that a comes earlier than b in some priority-ordering—we might use PLT and write: $\Diamond(\mathsf{E}a \land \neg\mathsf{E}b)$.

If we do not attend to the differences between contingency and priority, we will quickly get confused. So let me introduce a crucial litmus test. *To figure out whether someone has contingency or priority in mind, when they use ◇, ask about "actuality"*. If they are concerned with *contingency*, it will make sense to contrast $\Diamond\phi$ with $@\phi$. (And all my arguments against contingentism in §§6–8 presupposed that we can ask about "actuality".) But if they are concerned with *priority*, the very idea of contrasting $\Diamond\phi$ with $@\phi$ will be nonsense. After all: only a deep conceptual confusion could lead someone to say "I know that sets depend upon their members, so that the world is arranged into levels by priority; but which level is *priority-actual*?"

Armed with this litmus test, we might suspect that priority and contingency are *orthogonal* notions. Indeed so: any attitude towards priority is compatible with both necessitism and contingentism. That is the point of the following matrix, which outlines the four available positions:

	Necessitism	*Contingentism*
Prority	Priority relations account for the well-foundness of the (necessary) hierarchy.	Priority relations account for the possible connections between (contingent) sets.
Minimal	Priority-talk is just a loose gloss on the (necessary) well-foundedness of the hierarchy.	Priority-talk is just a loose gloss on possible connections between (contingent) sets.

These are not merely hypothetical options; all four positions have been defended. Michael Potter is a priority-necessitist.[49] Luca Incurvati outlines the definitive version of minimal-necessitism.[50] Studd is a minimal-contingentist.[51] Kit Fine[52] and Charles Parsons[53] are priority-contingentists.

The objections to contingentism go through whether we consider minimal- or priority-contingentism. But it is worth spending a little time—indeed, this section and the next—discussing *priority-necessitism*.

[49] Potter (2004: 36–40).

[50] Incurvati (2012, 2020: 51–69).

[51] Studd (2013, 2019).

[52] Fine (2006: 34–5) evidently countenances interpretational-actuality. And Fine (1994: 4–9) treats the relationship between a set and its members as paradigmatically hyperintensional.

[53] Parsons (1977: 270, 293, 1983b) explicitly endorses prioritism. He commits himself to contingentism by explicitly contrasting possible and *actual* existence for sets (1977: 274, 293–5, 1983b: 315–7). So, on my reading, Parsons uses ◇ in dual duty, for both priority and contingency. Linnebo (2018b: 250, 264) offers a similar reading of Parsons.

9.2 Reading Øystein Linnebo as a priority-necessitist

In this subsection, I will argue that Linnebo, a prominent (self-identifying) potentialist, is best read as a priority-necessitist.

Linnebo is certainly a prioritist: he variously describes sets as "prior to", "constituted by", or "metaphysically derived from" their members.[54] He also explicitly rejects circumstantialism. The subtle question is whether he is a priority-*interpretationalist*, like Parsons and Fine, or whether Linnebo is a priority-*necessitist*, like Potter.[55]

Most evidence for an interpretationalist-reading of Linnebo is by association: he often explicitly associates himself with Parsons and Studd,[56] who are both interpretationalists. The only direct textual data which support an interpretationalist-reading of Linnebo are a couple of passages, where he outlines a role for *interpretational-actuality*;[57] via my litmus test, this commits him to interpretationalist (§9.1). I propose to read these passages as a brief lapse. I maintain that, overall, Linnebo is best read as a priority-*necessitist* who uses $\Diamond$ to flag *priority*, rather than contingency.

The necessitist-reading is required, given Linnebo's reliance upon certain equivalence results. These equivalence results rely upon simple translations, between modal and non-modal set theories, with these key clauses:[58]

$$\textit{Modalization:} \qquad \exists x\phi \rightsquigarrow \Diamond\exists x\phi^{\Diamond}$$

$$\textit{Demodalization:} \qquad \Diamond\phi \rightsquigarrow \exists s(Lev(s) \land \phi^{s})$$

If we modalize then demodalize, or vice versa, we get back where we started. Invoking such translations, Linnebo claims that his modal set theory and non-modal set theories are two different viewpoints "for studying the same subject matter".[59] This is exactly what you would expect from a necessitist who is using $\Diamond$ (solely) to indicate *priority*; on this view, translating away the $\Diamond$-operators by demodalization simply leaves us with a "flatter" view of the same subject matter. By contrast, a contingentist would need to tell us how to demodalize their actuality-operator. Linnebo evidently (rightly) feels under no obligation to say which level is actual-under-demodalization, so he should be read as a (priority-)necessitist.

[54] Linnebo (2013: 214–7, 2018c: 211–2) and Florio and Linnebo (2021a: 183, 2021b: 63); see also Linnebo (2008: 71–4). Much of Linnebo (2018c: esp. 11–19, 45–6, 189–95) and all of deRosset and Linnebo (2024) connects with metaphysical *ground*. And Florio and Linnebo (2021a: 200)[2–3, 63, 88, 289]FlorioLinnebo:MO posit both pluralities and sets on the grounds that pluralities *explain* sets.

[55] Linnebo (2010: 158, 2013: 207–8, 2016: 672, 2018a: 202, 2018b: 265, 2018c: 72, 189–90) and Linnebo and Shapiro (2021: 288) endorse the necessitist's Empty-Box, when $\Box$ is read as metaphysical-necessity; but so do Fine, Parsons, and Studd (see footnote 20).

[56] See Linnebo (2013: 225–6, 2018c: 205) and Linnebo and Shapiro (2019: 171, 2021: 288).

[57] Linnebo (2018c: 62, 189–90). He had (2010: 159fn21) explicitly described a position which countenanced interpretational-actuality as "[a]n interesting *alternative*" (my emphasis) to his own position.

[58] Linnebo speaks of "mirroring theorems"; see Button (2021b) on near-synonymy.

[59] Linnebo (2013: 206).

9.3 *The use of modal logics*

I have cast Linnebo and Potter, alike, as priority-necessitists. But Linnebo, unlike Potter, uses a (normal) modal logic to explicate priority. In this section, I will suggest the non-modal approach is slightly preferable.

As mentioned in §9.1: the notion of "priority" is obviously time-like and hence amenable to treatment via a normal modal logic. For example, in PLT, we can say that a arises earlier than b by writing: $\Diamond(\mathsf{E}a \wedge \neg\mathsf{E}b)$.[60] But we can equally well regiment time-like structures non-modally, just by quantifying over "time-like stages". In that case, we can say that a arises earlier than b by writing: there is a stage, s, such that a is found at s and b is not found at s.

At this point, a priority-necessitist may think that they face a choice: to regiment their notion of priority, they must either enrich their primitive *ideology* with a $\Diamond$-operator, or enrich their primitive *ontology* with stages. In fact, as Potter helped to popularize, no enrichment is required.[61] We can explicitly define *levels* as particular sets which go proxy for "time-like stages". So we can then say that a arises earlier than b by writing: there is a level, s, such that $a \subseteq s$ but $b \nsubseteq s$. That is a claim in pure (non-modal) set theory. .

The modal theory (PLT), the stage theory, and the pure set theory (LT) are perfectly equivalent in straightforward ways.[62] Indeed, the equivalence between PLT and LT is just given by demodalization and modalization, and $\exists s(Lev(s) \wedge a \subseteq s \wedge b \nsubseteq s)$ is exactly the demodalization of $\Diamond(\mathsf{E}a \wedge \neg\mathsf{E}b)$; both formalisms equally well explicate the claim that a arises earlier than b.

At one point, Linnebo favoured the use of a modal theory because he claimed that something is *lost* in demodalization; that "modal theories provide powerful instruments for studying the same subject matter under a finer resolution" than non-modal theories allow.[63] My reply to this worry is simple: since the modal and non-modal theories are *equivalent*,[64] they study the same subject matter at the *same* level of resolution. Nothing is lost, nor even *losable*.

The equivalence cuts both ways of course: I can hardly *complain* if someone prefers one formalism over another. But I can issue two cautions. First: modal theories are undoubtedly more technically complicated than non-modal theories, for no technical

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60 Linnebo (2013: 216, 2018c: 212–13) actually formulates "the principle that the elements of a set are prior to the set itself" by adopting the *non-modal* axiom of Foundation. This is because Linnebo's own S4.2-based system (unlike PLT) is expressively impoverished; it cannot speak about "earlier worlds". Compare expressive concerns raised by Button (2021b: §10.1), Roberts (2016: 31–2), and Studd (2013: 700–1, 723–4, 2019: 169–71).

61 The definition of "level" I offer for LT is just a simplification of Potter's (2004: ch.3).

62 See Button (2021a,b). A *fourth* approach is to use arbitrary ordinals as syntactic type-markers for stages (see Linnebo and Rayo 2012; Florio and Linnebo 2021b: 249–61); the resulting type theory is *not* equivalent to the other three approaches (see Button and Trueman 2022: §2).

63 Linnebo (2013: 206); see also Linnebo (2013: 221, 2018c: 215).

64 Admittedly, Linnebo's (2013) own S4.2-based approach (unlike PLT) is *not* exactly equivalent to a non-modal theory. But this is because it suffers from a *defect* of resolution: it cannot speak about "earlier worlds" (see footnote 60).

gain (given the equivalence). Second: using modal operators to express *priority*, rather than *contingency*, can lead to philosophical confusion; I take up this theme in the next section.

10 Priority is unrelated to the Ur-Objection

Having outlined priority-necessitism, the obvious next step is to compare the merits of priority- and minimal-necessitism. I will not do that.[65] Instead, I will return to my focus: the Ur-Objection.

The Ur-Objection starts by asking a necessitist to explain why all the sets, **s**, do not form a set. This quickly becomes the task of explaining why $\forall z\, z \prec \mathbf{s}$ (see §2). Now, our necessitist might be a minimalist, insisting that priority-talk is just a loose gloss on the well-foundedness of the hierarchy. Or they might be a prioritist, insisting that priority relations ground the well-foundedness of the hierarchy. But, pretty clearly, going one way rather than the other will not help them to explain why $\forall z\, z \prec \mathbf{s}$.

At first glance, then, there is no connection between priority and the Ur-Objection. But Linnebo is led to his priority-necessitism *via* the Ur-Objection. In this section, I will explain his motivations (§10.1). I will then argue that Linnebo is mistaken: our first glance verdict was correct, and there is no connection between priority and the Ur-Objection (§§10.2–10.3). I will close with a diagnosis: appearance to the contrary arises by equivocating between contingency and priority (§10.4).

10.1 Linnebo on Collapse and priority

Whilst Linnebo wields the Ur-Objection as fiercely as anyone,[66] he never presents it as a criticism of *necessitism*. Instead, Linnebo holds that necessitists (himself included) should respond to the Ur-Objection by restricting $\text{Comp}_{\prec}$. To explain this, I need to say a bit more about his notion of (what I have simply glossed as) *priority*.

Linnebo holds that abstract(ed) objects are, in a word, *thin*. In a few more words:[67]

(a) Sets are obtained by abstraction, using a plural version of Frege's Basic Law V.
(b) When the set $\{\mathbf{u}\}$ exists, it makes "demands on the world that go beyond" the demands imposed by **u**'s coexistence.
(c) However, "the former demands do not *substantially* exceed the latter", for the latter "ground" the former.
(d) This grounding occurs because sets depend on their members via *definition*; and the demand is not substantial, because *any* permissible definition succeeds.

[65] I recommend Incurvati's (2012, 2020: 51–69) discussion.
[66] See references in footnote 12.
[67] All quotes from Linnebo (2018c: 5); see also Linnebo (2010: 156–8, 2016: 672, 2018c: 61, 191–2, 205) and deRosset and Linnebo (2024: 358–60, 378–81, 386–8).

Linnebo then attempts to extract a controversial mathematical conclusion from the claim that sets are thin. He writes: "given some objects $\mathbf{u}$, we can make good mathematical and philosophical sense of the associated set $\{\mathbf{u}\}$; so this is a permissible definition."[68] By (d), it follows that $\{\mathbf{u}\}$ exists. But $\mathbf{u}$ were arbitrary; so we obtain the following axiom:

Collapse $\forall \mathbf{u}\, \mathsf{E}\{\mathbf{u}\}$

This is controversial, because Collapse is inconsistent with unrestricted plural comprehension, i.e. $\text{Comp}_{\prec}$.[69] So Linnebo concludes that we must reject $\text{Comp}_{\prec}$.

Having rejected $\text{Comp}_{\prec}$, though, Linnebo has a neat response to the Ur-Objection. When we try to stipulate that $\mathbf{s}$ are such that $\forall z\, z \prec \mathbf{s}$, our success essentially relies upon the $\text{Comp}_{\prec}$-instance $\exists \mathbf{u} \forall x(x \prec \mathbf{u} \leftrightarrow x = x)$. Absent $\text{Comp}_{\prec}$, we can say that the stipulation simply *fails*; that "$\mathbf{s}$" is simply referentless. In that case, though, it is nonsense to ask us to "explain why $\forall z\, z \prec \mathbf{s}$": there is literally nothing to explain. More generally, without $\text{Comp}_{\prec}$, there just is no intelligible *de rebus* version of Dummett's Question; there is only the (easily-answered) *de dicto* version. The Ur-Objection dissolves.[70]

10.2 *Disconnecting Collapse from priority*

I have just outlined two Linnebovian ideas—thinness and Collapse—and explained how Linnebo thinks they are connected. I think the ideas are individually and jointly coherent; but they are simply *orthogonal*. As I will now show, we can easily consider necessitists who accept one but reject the other.

Meet Lynda. She is a minimal-necessitist. She dismisses all of Linnebo's metametaphysics; she cannot make head nor tail of the highfalutin notion of *grounding* that figured in (a)–(d) of §10.1. Still, Lynda fully agrees that we can always obtain $\{\mathbf{u}\}$ from $\mathbf{u}$ by a permissible definition, so she accepts Linnebo's argument for Collapse.

Next, meet Steinvör. She is a priority-necessitist. She heartily agrees with Linnebo that sets are thin, in the sense of (a)–(d). But she rejects Linnebo's argument for Collapse. As she sees it: given some condition ϕ, we can always make good mathematical and philosophical sense of the $\mathbf{v}$ such that $\forall z(z \prec \mathbf{v} \leftrightarrow \phi)$. That is, we obtain $\mathbf{v}$ by a permissible definition; so, by (d), $\mathbf{v}$ exists. More generally: Steinvör thinks that (d) mandates unrestricted $\text{Comp}_{\prec}$![71]

Linnebo and Steinvör disagree over the import of (d): Linnebo insists that it supports Collapse; Steinvör insists that it supports $\text{Comp}_{\prec}$. My point is that nothing in

[68] Linnebo (2018c: 191–2). See also Florio and Linnebo (2021b: 276). Notation changed since I tend to use $\mathbf{u}$ for arbitrary plural variables.

[69] Tennant's (2017: 127–8) Core Logic suffices to prove the inconsistency.

[70] In effect, this is the demodalization of the use of PossCo to Avoid Revenge (see §5.3). Note that contingentists cannot attempt to avoid the need to Explain Contingency on similar lines: $\text{Comp}_{\prec}$ is an axiom scheme of PLT, and Vary draws its strength from $\text{Comp}_{\prec}$.

[71] Steinvör defends what deRosset and Linnebo (2024: 387) call a *moderate* attitude towards abstraction and grounding, where Linnebo (2018c) defended the *radical* attitude.

the core notion of *thinness* can help to resolve this disagreement. In particular, note that principles (b)–(c) only establish a *conditional*: *if* $\{\mathbf{u}\}$ exists, then its existence is grounded in the coexistence of $\mathbf{u}$; it is not immediate that the coexistence of $\mathbf{u}$ *does* ground the existence of $\{\mathbf{u}\}$, as Collapse would require. On the contrary: (b) tells us that the existence of $\{\mathbf{u}\}$ demands *more* of the world than the mere coexistence of $\mathbf{u}$. This allows Steinvör to insist that, whilst sets are thin, sometimes the *extra* demand is a demand too far; in those cases, Collapse fails.[72]

10.3 Mathematical freedom

The apparent coherence of Lynda's and Steinvör's positions seems to indicate that we can keep thinness and Collapse completely separate. But, before we draw that conclusion, there is one last point to consider. Linnebo holds that his notion of "permissible mathematical definitions", which would vindicate Collapse, is needed "to safeguard the freedom of mathematics".[73] This claim could be accepted by a minimal-necessitist—indeed, it may well motivate Lynda—but it is especially interesting for our understanding of thinness. For, if Linnebo is right about freedom, then (*pace* Steinvör) (d) should indeed entail Collapse, so that thinness connects to the Ur-Objection after all.

Alas, Linnebo is wrong; mathematical freedom does not militate for Collapse.

Restricting Collapse certainly seems to stifle our mathematical freedom to form sets. But Steinvör's point is precisely that restricting $\text{Comp}_{\prec}$ also seems to stifle our freedom to represent things *de rebus*. We learned from Russell that we must restrict *some* initially attractive principle. So Linnebo needs to give us an argument that Collapse is *more* important to mathematical freedom than $\text{Comp}_{\prec}$.

Unfortunately, Linnebo gives us no such argument. Moreover, none is possible. What Collapse says is that any things, *de rebus*, could form a set. But when does ordinary mathematics clearly present us with sets *de rebus*? In the first instance, mathematical entities are always given to us under description: we speak about the ϕ sets (maybe *the hereditarily-accessible sets*), or we speak about the ψs (maybe *the groups*) which we decide to treat *as* sets. Fans of Collapse hold that any plurality yields a set, but that substantial work is needed to determine whether ϕ or ψ is plurality-forming (some things are all the hereditarily accessible sets; no things are all the groups). Fans of $\text{Comp}_{\prec}$ hold that any ϕ or ψ is plurality-forming, but that substantial work is needed to determine whether that plurality forms a set (the hereditarily-accessible sets do; the groups do not). The substantial work is shoved around, but the upshot for mathematical freedom seems the same whether we favour Collapse or $\text{Comp}_{\prec}$ (there is a set of all inaccessibles; there is no set of all groups).

[72] Linnebo (2018c: ch.5) considers and rejects the *ultra-thin* conception of objects, according to which for $\mathbf{u}$ to exist *just is* for $\{\mathbf{u}\}$ to exist. The ultra-thin conception leaves no room for any extra demand, so it entails Collapse; but equally, the ultra-thin conception leaves no room for any sense in which $\mathbf{u}$ are *prior* to $\{\mathbf{u}\}$.

[73] Linnebo (2018c: 192) and Florio and Linnebo (2021a: 191, 2021b: 275); see also Berry (2022: 3–4, 16) and Hellman (1989: 54, 2006: 81).

In fact, there is a subtle difference; but it ultimately indicates that (unrestricted) $\text{Comp}_{\prec}$ yields *greater* freedom than (unrestricted) Collapse. Accepting Collapse leads essentially to first-order set theory; since any $\mathbf{x}$ form a set, you can forget about $\mathbf{x}$ and deal with the set. By contrast, accepting $\text{Comp}_{\prec}$ leads essentially to second-order set theory. But second-order set theory is *deductively stronger* than first-order set theory.[74] So: there is a clear sense in which unrestricted $\text{Comp}_{\prec}$ allows us to see and do strictly more than unrestricted Collapse. Insofar as I understand the idea of mathematical freedom, I suspect it looks like this.

This does not, of course, prove that we should accept $\text{Comp}_{\prec}$ and dismiss Collapse; I deliberately leave that question open.[75] But it does refute Linnebo's claim that mathematical freedom requires that we accept Collapse rather than $\text{Comp}_{\prec}$. And it shuts down the hope of drawing any connection between *thinness* (or *priority* more generally), on the other hand, and Collapse (or the Ur-Objection), on the other.

10.4 A diagnosis

I will close the section with a tentative diagnosis. A subtle equivocation, between contingency and priority, led Linnebo to see a connection with Collapse (and the Ur-Objection), where none really exists.[76]

My discussion of Dummettian Revenge in §5.3 suggests the following: if we read $\Diamond$ in terms of certain (specific) kinds of *contingency*, it becomes attractive to regard PossCo, i.e. the scheme $\Diamond \mathsf{E}\mathbf{u}$, as expressing an important conceptual truth (about *de rebus* representation). Suppose we endorse PossCo on some such grounds. Subsequently, we decide to move away from a modal language, via *demodalization* (see §9.2). Under demodalization, PossCo becomes the open scheme $\exists x\big(Lev(x) \land (\forall z \prec \mathbf{u})z \in x\big)$. And this amounts to a restriction of $\text{Comp}_{\prec}$. Specifically, it amounts to insisting that pluralities are level-bound. So: maybe we have an attractive path, from a conceptual truth (about *de rebus* representation), to a restriction on $\text{Comp}_{\prec}$?

No. PossCo is attractive *only* when $\Diamond$ is read as expressing certain (specific) kinds of *contingency*. But priority-necessitists treat $\Diamond$ as expressing *priority*. For example, perhaps $\Diamond \mathsf{E}\mathbf{u}$ expresses something like "$\mathbf{u}$ are jointly available at some time-like dependence-stage". Put like that, though, it is obviously *just* a direct restriction on $\text{Comp}_{\prec}$, with some added hyperintensional bells and whistles.

11 Structuralism

Having investigated contingentism (§§3–8) and priority-necessitism (§§9–10), I hereby conclude my investigation of objectual positions. I will conclude this paper by dis-

[74] See Button (2021b: §C); cf. Williamson (2016: 681).

[75] For what it is worth: I lean towards keeping $\text{Comp}_{\prec}$.

[76] This is my diagnosis of Linnebo (2010: 154–8, 2013: 219–20, 2018c: 66–9, 191–2, 215) and Florio and Linnebo (2021b: 249).

cussing structuralism. There are structuralist analogues of necessitism and contingentism, namely, structural-absolutism and structural-potentialism (§11.1). The point of this section is to show that considerations related to the Ur-Objection give no clear reason to favour one version of structuralism over the other (§§11.2–11.3).

11.1 Structural-absolutism and structural-potentialism

Structuralists consider many possible set-hierarchies. In one hierarchy, "the empty set" might be an aardvark; in another, "the empty set" might be a baboon. Here, structuralists disagree with both contingentists and necessitists (see §3.1). However, we can find an analogue of the contingentist / necessitist distinction within the family of structuralist positions.

Structural-*potentialists* insist that no set-hierarchy is as tall as possible; that any possible hierarchy could be extended.[77] Slightly more formally,[78] structural-potentialists hold: for any possible set-hierarchy, $\mathcal{H}$, there *could* be a hierarchy with an isomorphic copy of $\mathcal{H}$ as a proper initial segment. This position is grounded in the insistence that, given any things, $\mathbf{v}$, we could always find far more things than $\mathbf{v}$; and if $\mathbf{v}$ were arranged in a set-hierarchy, then these more numerous things could be arranged into a taller hierarchy.

Structural-*absolutists*, by contrast, are structuralists who hold that there could be some *maximal* set-hierarchy; that some possible set-hierarchy is at least as tall as any other. Their position is grounded in the insistence that there could be some things, $\mathbf{v}$, which are *absolutely infinite*, so that it is impossible to find more things than $\mathbf{v}$.[79]

Alas: I know of no representatives of structural-absolutism. The best I can do is generate a version of structural-absolutism, by combining two well-known positions: structuralism about set theory *plus* first-order necessitism (i.e. the view that *every object* exists necessarily).[80]

By contrast, Sharon Berry and Geoffrey Hellman have given book-length defences of structural-potentialism. Moreover, both Berry and Hellman present the Ur-Objection as a reason to embrace structural-potentialism.[81] So it may be surprising to

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[77] See e.g. Berry (2022: esp. 28, ch.3, ch.6), Hellman (1989: esp. 54–7, 59(5), 72, 2011: 635), Hellman and Cook (2018: 57), and Putnam (1967: 21–2).

[78] Full formalization is genuinely difficult (cf. Linnebo 2018b: 257–8). Berry (2018: 202ff, 2022: 45–51) uses a novel notion of "conditional logical possibility" (see footnote 81). Roberts (2019: esp. 829, 841, 850) uses a primitive (rigid) notion of ordered-pair.

[79] I have conflated claims about the height of (possible) hierarchies with mere claims about (possible) cardinalities. There is room for slippage here, by denying any version of CH (so that we can always find more things, but not enough to "add another layer of sets"). I set this slippage aside: dealing with it would make my discussion more complicated, without really affecting the key issues.

[80] As popularized by Williamson (2013), who also endorses higher-order necessitism.

[81] See the references of footnote 12. An important difference between Berry and Hellman is worth noting. Hellman (1996) came to favour a plural modal logic. So: the *de rebus*, Ur-Objection-esque questions I formulate below (Putnam's Question and Putnamian Revenge) are roughly questions in Hellman's vernacular.

Berry (2018: esp. 202–7, 2022: esp. ch.4) uses *first-order* logic and bans quantifying-in to modal discover that, when we think through the Ur-Objection in the light of this paper, we find no obvious reason to favour structural-potentialism over structural-absolutism. That is the point of what follows.

Before we consider the Ur-Objection in a structuralist setting, permit me a pre-emptive digression. A structuralist might claim that mathematical freedom requires structural-potentialism, since structural-absolutism cannot *always* find more things.[82] Were they to do so, they would commit the same mistake as a necessitist who claims that mathematical freedom requires Collapse rather than $\text{Comp}_{\prec}$. Rather than repeat the argument of §10.3, transposed into a structuralist key, I will illustrate the point with an example. According to the structural-absolutist, (exemplars of) all possible *groups* belong together in one possibility; according to the structural-potentialist, the possible groups are spread out diffusely through the possible universe. The former view grants at least as much freedom as the latter and, if anything, slightly more: the structural-absolutist has **Grp** as a possible category, where the structural-potentialist only ever has approximations to **Grp**. This concludes my pre-emptive digression.

11.2 Structuralism and the Ur-Objection

In §2.2, we considered Dummett's Question: *Why is there no set of all the sets?* As formulated, the question clearly presupposes objectualism. Structuralists cannot ask whether certain things "form a set" until they have stipulated which set-hierarchy is under consideration. But, having made such a stipulation, Dummett's Question generates no philosophical puzzles. Ultimately, the answer will be: *these things, de rebus, don't form a set in this set-hierarchy, because that's what this set-hierarchy is like.*

Nevertheless, there is an interesting "structuralist transposition" of Dummett's Question. Structural-absolutists believe that some (possible) hierarchy is as tall as possible, because some (possible) things are absolutely infinite. So we can ask:[83]

> *Putnam's Question.* Fix some (possible) absolutely infinite things, $\mathbf{o}$; why can't there be more things than $\mathbf{o}$?

We saw that necessitists quickly hit bedrock when asked Dummett's Question. Structural-absolutists hit bedrock equally rapidly when asked Putnam's Question. To illustrate: suppose our structural-absolutists are first-order-necessitists. They will say that all the things, $\mathbf{o}$, are absolutely infinite because $\mathbf{o}$ are, necessarily, all the things. Okay; but the obvious question is *why* there could not be some $z \nprec \mathbf{o}$, i.e. why $\Box \forall z\, z \prec \mathbf{o}$. First-order-necessitists will have little to say in reply.

contexts. She compensates for this loss in expressive power by positing a new, structure-preserving, modal operator. Now, Berry favours structural-potentialism, but we could in principle combine her new logic with structural-absolutism. In this vernacular, we cannot raise any *de rebus*, Ur-Objection-esque questions, to arbitrate between these alternatives.

[82] Cf. Berry (2022: 3–4, 16) and Hellman (1989: 54, 2006: 81), though note that they are criticizing *necessitists* not structural-absolutists.

[83] The name is a homage to Putnam (1967: 21): "Even God could not make a model for Zermelo set theory that it would be *mathematically* impossible to extend."

Notably, structural-potentialists face no version of Putnam's Question. This is a *very* good thing for them. Contingentists faced Dummett's Question; this forced them to Explain Contingency; this forced them to adopt Copernicanism; and everything went rapidly downhill. No similar problems await structural-potentialists. (So, nothing that follows is meant to impugn the coherence of structural-potentialism.)

11.3 Structuralism and Revenge

Nevertheless, structuralists should pause before rushing to embrace potentialism, simply on the basis that absolutists have hit explanatory rock-bottom. At a minimum, we should first check what rock-bottom looks like for structural-potentialists.

Whilst structural-potentialists do not face Putnam's Question, they *do* face some version of "revenge". The specific formulation of Dummettian Revenge again presupposes objectualism, but we can again "transpose" the question so that it raises issues for structural-potentialists:[84]

Putnamian Revenge. Let **p** be all possible things (so that $\Box\forall z\, z \prec \mathbf{p}$); can as many things as **p** coexist? If not, *why*?

Suppose, first, that structural-potentialists agree that Putnamian Revenge is successfully posed (that the stipulation of "**p**" succeeds). Then they must offer a negative answer to the first part of the question: there cannot be as many things as **p**.[85] So they must explain *why*. Evidently, this becomes the challenge of explaining why $\Box\forall z\, z \prec \mathbf{p}$. And, at this point, I cannot see what structural-potentialists might hope to say in reply, beyond: "**p** *just are* all possible things". They would thereby hit explanatory rock-bottom, much like the structural-absolutist who claimed: "**o** *just are* absolutely infinite".

Structural-potentialists may, though, respond to Putnamian Revenge by insisting that it *misfires* (that the attempted stipulation of "**p**" fails). As in §5.3: the key step in arguing for a *misfire* is to establish PossCo, i.e. the open scheme $\Diamond\mathsf{E}\mathbf{u}$. However, the specific argument for PossCo given in §5.3 deployed a very particular gloss on interpretational-modality; that specific argument is therefore unavailable to structural-potentialists, unless they happen to use a very similar modality. More generally: whether (and how) a structural-potentialist can defend PossCo will depend

[84] It also worth considering how Putnamian Revenge applies to structural-absolutists. First-order-necessitism affirm $\Box\mathsf{E}\mathbf{p}$, and so (trivially) hold that there could be as many things as **p**. More generally, my formulation of structural-absolutism (in §11.1) does not *entail* that there could be as many things as **p**, but the invoked notion of "absolute infinity" strongly *suggests* it.

[85] Suppose otherwise, i.e. $\Diamond\mathsf{E}\mathbf{u}$, and **u** are as many as **p**. By the structural-potentialists own principles, there could be strictly more things than **u**, and hence strictly more things that **p**. But then **p** are strictly more than **p**; a contradiction.

upon the precise details of their preferred modality.[86] It is therefore worth considering the following counterexample to PossCo, which can be adapted for many different modalities:[87]

> *Handle, Blade, and Bowl are actual objects. Respectively, they are a cutlery-handle, a knife-blade, and a spoon-bowl. We could assemble Blade and Handle; they would then comprise a specific knife, Knifey. We could also assemble Blade and Bowl; they would then comprise a specific spoon, Spoony. But we cannot create both a spoon and a knife using Handle; what Handle is joined to cannot be sundered. So Knifey and Spoony are incompossible: they cannot coexist, though each of them can exist.*

Pause on the last sentence of the example. It uses a natural language plural-term, "Knifey and Spoony"; the anaphora "they" and "them" pick this up, and so apparently allow us to say of two things, *de rebus*, that they cannot coexist. And this just seems like an informal counterexample to PossCo. More formally: let **c** be Knifey and Spoony;[88] by fiat, $\neg\Diamond \mathsf{E}\mathbf{c}$.

No doubt there is more to say. Still, it is extremely unclear that structural-potentialists can brush aside Putnamian Revenge. For now, then, we should doubt whether structural-potentialism enjoys any real advantage over structural-absolutism.

12 Conclusion

Objectual potentialists use possibility-talk. But this can mean many different things. Crucially, we must disentangle *contingency* from *priority*; the crucial litmus test, here, is whether it makes sense to ask about what is *actual*.

The first lesson of this paper is that, whatever we think about the Ur-Objection, contingentism is bad news. All contingentists run into a basic tension. To Explain Contingency, they need Copernicanism (§3), but Copernicanism means they cannot Regain Paradise (§6). Moreover, interpretationalists face further specific problems: their notion of possibility is monstrously unexplanatory, and their notion of actuality must be jettisoned (§§7–8).

Among objectual positions, this leaves only versions of necessitism standing. But a secondary lesson of this paper is that the Ur-Objection is orthogonal to the choice between minimal-necessitism and priority-necessitism (whether modally-formulated or

[86] Hellman (1989: 17–18, 59(4), 2011: 635–6) and Hellman and Cook (2018: 57–8) in effect defend PossCo by invoking Stalnakerian actualism about modal metaphysics. Scambler (2025: §4, draft: 20) also suggests a Stalnakerian actualism.

[87] This is adapted from Williamson (2013: 21, 149–50). Bigger Knifey/Spoony cases can be created, thereby blocking paraphrase-strategies to defuse this apparent counterexample; see Fritz and Goodman (2017: 1073ff) and Fritz (2023) for more.

[88] So our stipulation is that $\Box\forall z\big(z \prec \mathbf{c} \leftrightarrow (z = \text{Knifey} \vee z = \text{Spooney})\big)$. This explains why I have not adopted the scheme of negative free logic, $z \prec \mathbf{u} \rightarrow \mathsf{E}\mathbf{u}$; for that would entail $\neg\Diamond\mathsf{E}\mathbf{u} \rightarrow \Box\forall z\, z \nprec \mathbf{c}$.

not; see §§9–10). The last lesson of this paper is that considerations related to the Ur-Objection have surprisingly little impact on debates between structural-absolutists and structural-potentialists (§11).

All told: the Ur-Objection should not move us one bit: certainly not towards contingentism; not either way, as regards prioritism or minimalism; nor between different versions of structuralism. We can stop asking: *Why are all the sets all the sets?*

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Thanks ANONYMIZED at this stage.

A Background logic

My background modal logic is from Roberts (2022), incorporating Roberts's optional (ex).[89] NOaA stands for "nothing-over-and-above". It is based on positive free quantified S5, and adds the following distinctive axiom schemes for plurals:

$$\begin{aligned}
&\textbf{Rigid}_{\prec} \quad x \prec \mathbf{u} \rightarrow \big(\Box(\mathsf{E}\mathbf{u} \rightarrow \mathsf{E}x) \wedge \Box(\mathsf{E}x \leftrightarrow x \prec \mathbf{u})\big)\\
&\textbf{Ext}^{\Diamond}_{\prec} \quad \Box\forall x(\Diamond x \prec \mathbf{u} \leftrightarrow \Diamond x \prec \mathbf{v}) \rightarrow \mathbf{u} = \mathbf{v}\\
&\textbf{Comp}_{\prec} \quad \exists \mathbf{u}\forall x(x \prec \mathbf{u} \leftrightarrow \phi) \qquad \text{with "}\mathbf{u}\text{" not free in } \phi
\end{aligned}$$

Roberts proves that, given NOaA: some things, **u**, coexist *iff* everything which could ever be among **u** exist, i.e.:

Proof on p.38 **Lemma 1** (NOaA)**:** $w \Vdash \mathsf{E}\mathbf{c}$ iff: $w \Vdash \Diamond a \prec \mathbf{c} \rightarrow \mathsf{E}a$ for each a

B An objectual core for set theory

The next step is to augment NOaA with principles governing sets. First, we need to discuss the choice of *primitives*. There are two options.

Membership-first. We might start with membership as a primitive relation. In that case, we would lay down these schemes:

$$\begin{aligned}
&\textbf{Ext}^{\Diamond}_{\in} \quad \Box\forall z(\Diamond z \in x \leftrightarrow \Diamond z \in y) \rightarrow x = y\\
&\textbf{Rigid}_{\in} \quad x \in y \rightarrow \mathsf{E}x \wedge \big(\mathsf{E}y \wedge \Box(\mathsf{E}y \rightarrow x \in y)\big)\\
&\textbf{Exists}_{\Diamond} \quad \Diamond \mathsf{E}x
\end{aligned}$$

So: $\text{Ext}^{\Diamond}_{\in}$ lays down a modalized version of extensionality; $\text{Rigid}_{\in}$ ensures that membership behaves in a rigid sort of way; then $\text{Exists}_{\Diamond}$ guarantees that our first-order variables range only over possible things.

[89] According to which $(\mathsf{E}x \wedge \Diamond x \prec \mathbf{u}) \rightarrow x \prec \mathbf{u}$.

Formation-first. Alternatively, we might use a primitive which corresponds to set-formation, i.e. to the operation which maps some things, $\mathbf{u}$, to their set, $\{\mathbf{u}\}$.[90] We could take a (partial-)function-symbol as primitive; but it is easier to use a two-place predicate, $ß$, which is plural in the first position and singular in the second. We then lay down these schemes:

$$\begin{array}{rl} \textbf{Inject}_{ß} & \big(\Diamond ß(\mathbf{u}, x) \wedge \Diamond ß(\mathbf{v}, y)\big) \rightarrow (\mathbf{u} = \mathbf{v} \leftrightarrow x = y) \\ \textbf{Rigid}_{ß} & ß(\mathbf{u}, x) \rightarrow \big(\mathsf{E}\mathbf{u} \wedge \mathsf{E}x \wedge \Box(\mathsf{E}x \rightarrow ß(\mathbf{u}, x))\big) \\ \textbf{Pure}_{ß} & \Diamond \exists \mathbf{v}\, ß(\mathbf{v}, x) \end{array}$$

So: Inject$_{ß}$ tells us that $ß$ is in effect an injective (partial) function; Rigid$_{ß}$ ensures it behaves in a rigid sort of way; then Pure$_{ß}$ guarantees, in effect, that our first-order variables range only over possible sets.

The approaches are synonymous. There is no need to chose between these alternatives since, over a background of NOaA, they are synonymous.[91] Specifically: synonymy is witnessed by direct translations with these clauses (all other clauses are verbatim):

$$x \in_I y :\equiv (\exists \mathbf{v} \succ x) ß(\mathbf{v}, y)$$
$$ß_J(\mathbf{u}, x) :\equiv \mathsf{E}x \wedge \mathsf{E}\mathbf{u} \wedge \forall z(z \prec \mathbf{u} \leftrightarrow z \in x)$$

Proof on p.38 I leave the proof of synonymy to the reader, but this means we do not need to decide which of $\in$ or $ß$ is "officially" primitive and which is defined. Either way: call the resulting theory ObjCore (taken to include all of NOaA), since it captures nothing other than objectualism (see §1 and §3.1). So anyone using ObjCore can write $\{\mathbf{u}\}$ for the unique a (if it exists) such that $ß(\mathbf{u}, a)$, i.e. such that $\mathsf{E}\mathbf{u} \wedge \mathsf{E}a \wedge \forall z(z \in a \leftrightarrow z \prec \mathbf{u})$. We write $\mathsf{E}\{\mathbf{u}\}$ to indicate that this a does exist. Note that ObjCore proves the usual Proof on p.39 version of extensionality, i.e. $\forall a \forall b\big(\forall x(x \in a \leftrightarrow x \in b) \rightarrow a = b\big)$.

C Specific set theories

The principles of ObjCore are compatible with many difference conceptions of *set*. But our focus is the ordinary iterative conception. To obtain that we invoke this idea: *we find* $\{\mathbf{u}\}$ *at some stage in the hierarchy iff we found* $\mathbf{u}$ *at earlier stages*. This idea can be axiomatized very directly. Define:

$$Lev(s) :\equiv \exists h \forall z\Big[(\forall a \in h)\big(z \in a \leftrightarrow (\exists c \in a)(z \subseteq c \in h)\big) \wedge \big(z \in s \leftrightarrow \exists c(z \subseteq c \in h)\big)\Big]$$

Whilst it may not be immediately obvious, Button (2021a) shows that the thus-defined levels go proxy for stages. So we can capture our iterative idea using a single scheme:

$$\textbf{Levelling}\quad \mathsf{E}\{\mathbf{u}\} \leftrightarrow \exists s\big(\mathsf{E}\mathbf{u} \wedge Lev(s) \wedge (\forall z \prec \mathbf{u}) z \in s\big)$$

[90] This idea goes back to Frege (1893: §9, §20).
[91] Linnebo (2013: 217) makes a similar observation for his approach, based on S4.2.

This says: *some things form a set iff those things all exist and indeed as members of some level*. It is easy to show that *being a level* is rigid (see Button 2021b: Lemma A.3):

Proof on p.39 **Lemma 2** (ObjCore + Levelling)**:** $Lev(s) \rightarrow \Box(\mathsf{E}s \rightarrow Lev(s))$

We can now define PLT as the theory ObjCore + Levelling + Vary. This is the theory I ascribe to all objectual potentialists. Using possibility-talk, PLT amounts to this: necessarily, sets are arranged in a hierarchy which could be arbitrarily pruned down or strictly augmented. Indeed, we have:

Proof on p.39 **Lemma 3** (PLT)**:** $Lev(s) \rightarrow \Diamond(\mathsf{E}s \wedge \forall x\, x \subseteq s)$

We can also define the theory common to all necessitists, NLT. This will tell us that the pure sets are arranged in a hierarchy which could not have been otherwise. Perhaps the simplest axiomatization of NLT is given laying down a new scheme:

$$\mathbf{Act}_{\beta}\ \ \mathsf{E}x \wedge \exists \mathbf{v} \Box \beta(\mathbf{v}, x)$$

And defining NLT as NOaA + Inject$_{\beta}$ + Act$_{\beta}$ + Levelling. Provably, NLT extends ObjCore and has some further nice properties:

Proof on p.39 **Lemma 4** (NOaA, Inject$_{\beta}$, Act$_{\beta}$)**:** ObjCore holds; we also have the schema **Eu**, and therefore can use classical (non-free) S5; and we have Empty-Box.

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Proof of Lemma 1. *Left-to-right.* By Rigid$_\prec$. *Right-to-left.* Using Comp$_\prec$ at $\boldsymbol{w}$, fix $\mathbf{b}$ such that

$$\boldsymbol{w} \Vdash \mathsf{E}\mathbf{b} \wedge \forall x(x \prec \mathbf{b} \leftrightarrow \Diamond x \prec \mathbf{c})$$

It suffices to show that $\boldsymbol{w} \Vdash \mathbf{b} = \mathbf{c}$; we will show that $\boldsymbol{w} \Vdash \Box\forall x(\Diamond x \prec \mathbf{b} \leftrightarrow \Diamond x \prec \mathbf{c})$ and invoke Ext$^\Diamond_\prec$. So let $\boldsymbol{x} \Vdash \mathsf{E}a$.

First, let $\boldsymbol{y} \Vdash a \prec \mathbf{b}$. By Rigid$_\prec$, we have $\boldsymbol{w} \Vdash \mathsf{E}a \wedge a \prec \mathbf{b}$. So $\boldsymbol{w} \Vdash \Diamond a \prec \mathbf{c}$ and hence $\boldsymbol{x} \Vdash \Diamond a \prec \mathbf{c}$. Conversely, let $\boldsymbol{z} \Vdash a \prec \mathbf{c}$, so that $\boldsymbol{z} \Vdash \mathsf{E}a$ by Rigid$_\prec$. We assumed that if $\boldsymbol{w} \Vdash \Diamond\mathsf{E}a$ then $\boldsymbol{w} \Vdash \mathsf{E}a$; so indeed $\boldsymbol{w} \Vdash \mathsf{E}a$, so that $\boldsymbol{w} \Vdash a \prec \mathbf{b}$, i.e. $\boldsymbol{x} \Vdash \Diamond a \prec \mathbf{b}$. ☐

Lemma 5 (NOaA): $\forall\mathbf{u}\forall\mathbf{v}\big(\forall z(z \prec \mathbf{u} \leftrightarrow z \prec \mathbf{v}) \to \mathbf{u} = \mathbf{v}\big)$

Proof. Let $\boldsymbol{w} \Vdash \mathsf{E}\mathbf{a} \wedge \mathsf{E}\mathbf{b} \wedge \forall z(z \prec \mathbf{a} \leftrightarrow z \prec \mathbf{b})$. I will show $\boldsymbol{w} \Vdash \Box\forall z(\Diamond z \prec \mathbf{a} \leftrightarrow \Diamond z \prec \mathbf{b})$ and then invoke Ext$^\Diamond_\prec$. So let $\boldsymbol{x} \Vdash \mathsf{E}e$ and $\boldsymbol{y} \Vdash e \prec \mathbf{a}$. By Rigid$_\prec$, $\boldsymbol{w} \Vdash e \prec \mathbf{a}$, so $\boldsymbol{w} \Vdash e \prec \mathbf{b}$. The converse is exactly similar. ☐

The next two lemmas establish the synonymy mentioned on p.33 ("$\Rightarrow$" indicates that assuming all scheme instances on the left entails all scheme instances on the right):

Lemma 6 (NOaA): (1) Rigid$_ß$ $\Rightarrow$ Rigid$_\in{}^I$

(2) Pure$_ß$ + Rigid$_ß$ $\Rightarrow$ Exists$_\Diamond{}^I$

(3) Pure$_ß$ + Inject$_ß$ $\Rightarrow$ Ext$^\Diamond_\in{}^I$

(4) Pure$_ß$ + Rigid$_ß$ + Inject$_ß$ $\Rightarrow$ $\big(ß(\mathbf{u}, x) \leftrightarrow (ß_J(\mathbf{u}, x))^I\big)$

Proof. *(1)* Let $\boldsymbol{w} \Vdash a \in_I b$. Fix $\mathbf{c}$ with $\boldsymbol{w} \Vdash \mathsf{E}\mathbf{c} \wedge ß(\mathbf{c}, b) \wedge a \prec \mathbf{c}$; note $\boldsymbol{w} \Vdash \mathsf{E}a$ by Rigid$_\prec$ and $\boldsymbol{w} \Vdash \mathsf{E}b$ by Rigid$_ß$. Next, let $\boldsymbol{x} \Vdash \mathsf{E}b$. By Rigid$_ß$, $\boldsymbol{x} \Vdash ß(\mathbf{c}, b) \wedge \mathsf{E}\mathbf{c}$; now $\boldsymbol{x} \Vdash a \prec \mathbf{c}$ by Rigid$_\prec$, so $\boldsymbol{x} \Vdash a \in_I b$.

(2) Fix a; using Pure$_ß$, let $\boldsymbol{w} \Vdash ß(\mathbf{b}, a)$; now $\boldsymbol{w} \Vdash \mathsf{E}a$ by Rigid$_ß$.

(3) Let $\boldsymbol{w} \Vdash \Box\forall x(\Diamond x \in_I a \leftrightarrow \Diamond x \in_I b)$. Using Pure$_ß$, let $\boldsymbol{x}_a \Vdash \mathsf{E}\mathbf{c} \wedge ß(\mathbf{c}, a)$ and $\boldsymbol{x}_b \Vdash \mathsf{E}\mathbf{d} \wedge ß(\mathbf{d}, b)$. I claim $\boldsymbol{w} \Vdash \Box\forall x(\Diamond x \prec \mathbf{c} \leftrightarrow \Diamond x \prec \mathbf{d})$. To prove this, let $\boldsymbol{x} \Vdash \mathsf{E}s$; if $\boldsymbol{x} \Vdash \Diamond s \prec \mathbf{c}$ then $\boldsymbol{x}_a \Vdash s \prec \mathbf{c}$ by Rigid$_\prec$, so $\boldsymbol{x}_a \Vdash s \in_I a$ and $\boldsymbol{x} \Vdash \Diamond s \in_I a$, so $\boldsymbol{x} \Vdash \Diamond s \in_I b$; so let $\boldsymbol{y}$ and $\mathbf{e}$ be such that $\boldsymbol{y} \Vdash \mathsf{E}\mathbf{e} \wedge ß(\mathbf{e}, b) \wedge s \prec \mathbf{e}$; now $\boldsymbol{y} \Vdash \mathbf{d} = \mathbf{e}$ by Inject$_ß$, so $\boldsymbol{y} \Vdash s \prec \mathbf{d}$. The converse is similar, establishing the claim. Hence $\boldsymbol{w} \Vdash \mathbf{c} = \mathbf{d}$ by Ext$^\Diamond_\prec$, and $\boldsymbol{x}_a \Vdash a = b$ by Inject$_ß$.

(4) I must prove $ß(\mathbf{b}, a) \leftrightarrow \mathsf{E}a \wedge \mathsf{E}\mathbf{b} \wedge \forall z\big(z \prec \mathbf{b} \leftrightarrow \exists\mathbf{v}(ß(\mathbf{v}, a) \wedge z \prec \mathbf{v})\big)$.

Left-to-right. Let $\boldsymbol{w} \Vdash ß(\mathbf{b}, a)$. So $\boldsymbol{w} \Vdash \mathsf{E}\mathbf{b} \wedge \mathsf{E}a$ by Rigid$_ß$. Let $\boldsymbol{w} \Vdash \mathsf{E}s$: if $\boldsymbol{w} \Vdash s \prec \mathbf{b}$, then $\mathbf{b}$ witnesses the right-hand-side; if $\boldsymbol{w} \Vdash ß(\mathbf{c}, a) \wedge s \prec \mathbf{c}$ then $\boldsymbol{w} \Vdash \mathbf{b} = \mathbf{c}$ by Rigid$_ß$ so that $s \prec \mathbf{b}$.

Right-to-left. Let $\boldsymbol{w} \Vdash \mathsf{E}a \wedge \mathsf{E}\mathbf{b} \wedge \forall z\big(z \prec \mathbf{b} \leftrightarrow \exists\mathbf{v}(ß(\mathbf{v}, a) \wedge z \prec \mathbf{v})\big)$. Using Pure$_ß$ and Rigid$_ß$, let $\boldsymbol{u} \Vdash \mathsf{E}a \wedge \mathsf{E}\mathbf{c} \wedge ß(\mathbf{c}, a)$. By Rigid$_ß$, $\boldsymbol{w} \Vdash ß(\mathbf{c}, a) \wedge \mathsf{E}\mathbf{c}$. I claim that $\boldsymbol{w} \Vdash \forall z(z \prec \mathbf{b} \leftrightarrow z \prec \mathbf{c})$. To prove this claim, let $\boldsymbol{w} \Vdash \mathsf{E}s$. If $\boldsymbol{w} \Vdash s \prec \mathbf{c}$, then $\boldsymbol{w} \Vdash s \prec \mathbf{b}$ by choice of $\boldsymbol{w}$. Conversely, if $\boldsymbol{w} \Vdash s \prec \mathbf{b}$, by choice of $\boldsymbol{w}$ fix $\mathbf{d}$ such that $\boldsymbol{w} \Vdash ß(\mathbf{d}, a) \wedge s \prec \mathbf{d}$; now $\boldsymbol{w} \Vdash \mathbf{c} = \mathbf{d}$ by Inject$_ß$ so $\boldsymbol{w} \Vdash s \prec \mathbf{c}$. This establishes the claim. Now $\boldsymbol{w} \Vdash \mathbf{b} = \mathbf{c}$ by Lemma 5, so that $\boldsymbol{w} \Vdash ß(\mathbf{b}, a)$. ☐

Lemma 7 (NOaA): (1) $\text{Rigid}_{\in} \Rightarrow \text{Rigid}_{ß}{}^{J}$

(2) $\text{Exists}_{\Diamond} \Rightarrow \text{Pure}_{ß}{}^{J}$

(3) $\text{Ext}_{\in}^{\Diamond} + \text{Rigid}_{\in} \Rightarrow \text{Inject}_{ß}{}^{J}$

(4) $\text{Rigid}_{\in} \Rightarrow \big(x \in y \leftrightarrow (x \in_I y)^J\big)$

Proof. (1) Suppose $\boldsymbol{w} \Vdash \mathsf{E}a \wedge \mathsf{E}\mathbf{b} \wedge \forall z(z \prec \mathbf{b} \leftrightarrow z \in a)$. I will show that $\boldsymbol{x} \Vdash \mathsf{E}\mathbf{b}$. Let $\boldsymbol{x} \Vdash \Diamond s \prec \mathbf{b}$; then $\boldsymbol{w} \Vdash s \prec \mathbf{b}$ by $\text{Rigid}_{\prec}$, so $\boldsymbol{w} \Vdash s \in a$ and hence $\boldsymbol{x} \Vdash \mathsf{E}s$ by $\text{Rigid}_{\in}$. Hence $\boldsymbol{x} \Vdash \mathsf{E}\mathbf{b}$ by Lemma 1. Now $\text{Rigid}_{\prec}$ and $\text{Rigid}_{\in}$ together yield that $\boldsymbol{x} \Vdash \forall z(z \prec \mathbf{b} \leftrightarrow z \in a)$.

(2) Fix a; using $\text{Exists}_{\Diamond}$ let $\boldsymbol{w} \Vdash \mathsf{E}a$. Using $\text{Comp}_{\prec}$, $\boldsymbol{w} \Vdash \exists \mathbf{c} \forall x(x \prec \mathbf{c} \leftrightarrow x \in a)$.

(3) Let $\boldsymbol{x}_a \Vdash \mathsf{E}a \wedge \mathsf{E}\mathbf{c} \wedge \forall z(z \prec \mathbf{c} \leftrightarrow z \in a)$ and $\boldsymbol{x}_b \Vdash \mathsf{E}b \wedge \mathsf{E}\mathbf{d} \wedge \forall z(z \prec \mathbf{d} \leftrightarrow z \in b)$. First suppose $\boldsymbol{w} \Vdash \mathbf{c} = \mathbf{d}$ and let $\boldsymbol{y} \Vdash \mathsf{E}s$; now note $\boldsymbol{y} \Vdash \Diamond s \in a$ iff $\boldsymbol{x}_a \Vdash s \in a$ (by $\text{Rigid}_{\in}$) iff $\boldsymbol{x}_a \Vdash s \prec \mathbf{c}$ iff $\boldsymbol{x}_b \Vdash s \prec \mathbf{d}$ (by $\text{Rigid}_{\prec}$) iff $\boldsymbol{x}_b \Vdash s \in b$ iff $\boldsymbol{y} \Vdash \Diamond s \in b$ (by $\text{Rigid}_{\in}$). Hence $\boldsymbol{w} \Vdash \Box \forall z(\Diamond z \in a \leftrightarrow \Diamond z \in b)$, so that $\boldsymbol{w} \Vdash a = b$ by $\text{Ext}_{\in}^{\Diamond}$. A very similar argument shows that if $\boldsymbol{w} \Vdash a = b$ then $\boldsymbol{w} \Vdash \mathbf{c} = \mathbf{d}$.

(4) I must prove $b \in a \leftrightarrow \mathsf{E}a \wedge \exists \mathbf{v}\big(\forall z(z \prec \mathbf{v} \leftrightarrow z \in a) \wedge b \prec \mathbf{v}\big)$. *Right-to-left.* Trivial using $\text{Rigid}_{\prec}$. *Left-to-right.* Let $\boldsymbol{w} \Vdash b \in a$. Then $\boldsymbol{w} \Vdash \mathsf{E}b \wedge \mathsf{E}a$ by $\text{Rigid}_{\in}$. Using $\text{Comp}_{\prec}$ fix $\mathbf{c}$ so that $\boldsymbol{w} \Vdash \mathsf{E}\mathbf{c} \wedge \forall z(z \prec \mathbf{c} \leftrightarrow z \in a)$; now $b \prec \mathbf{c}$. □

Proof of Extensionality in ObjCore. Let $\boldsymbol{w} \Vdash \mathsf{E}a \wedge \mathsf{E}b \wedge \forall x(x \in a \leftrightarrow x \in b)$. Let $\boldsymbol{x} \Vdash \mathsf{E}s$; now $\boldsymbol{x} \Vdash \Diamond s \in a$ iff $\boldsymbol{w} \Vdash s \in a$ (by $\text{Rigid}_{\in}$) iff $\boldsymbol{w} \Vdash s \in b$ iff $\boldsymbol{x} \Vdash \Diamond s \in b$ (by $\text{Rigid}_{\in}$). So $a = b$ by $\text{Ext}_{\in}^{\Diamond}$. □

Proof of Lemma 2. First we show the rigidity of subsethood, i.e.: $(\forall b \subseteq a)\Box(\mathsf{E}a \rightarrow (\mathsf{E}b \wedge b \subseteq a))$. This holds by Separation (which follows from Levelling) and $\text{Rigid}_{\in}$.

Now let $\boldsymbol{w} \Vdash Lev(s)$, with this witnessed by $\boldsymbol{w} \Vdash \mathsf{E}h$. Suppose $\boldsymbol{v} \Vdash \mathsf{E}s$. Note that $\boldsymbol{w} \Vdash h \subseteq s$ hence $\boldsymbol{v} \Vdash \mathsf{E}h$. By the rigidity of membership and the subsethood, $\boldsymbol{v} \Vdash Lev(s)$ with h as witness. □

Proof of Lemma 3. Let $\boldsymbol{w} \Vdash Lev(s)$. Using $\text{Comp}_{\prec}$ and Vary let $\boldsymbol{x} \Vdash \mathsf{E}s \wedge \neg\mathsf{E}\{s\}$. By Lemma 2, $\boldsymbol{x} \Vdash Lev(s)$. Now for reductio suppose $\boldsymbol{x} \Vdash \mathsf{E}a \wedge a \not\subseteq s$. By Levelling there is t with $\boldsymbol{x} \Vdash Lev(t) \wedge a \subseteq t$, so that $s \in t$ by the necessary well-ordering of the levels and hence $\boldsymbol{x} \Vdash \mathsf{E}\{s\}$ by Separation, a contradiction. □

Proof of Lemma 4. *$\text{Pure}_{ß}$.* Trivial.

$\text{Rigid}_{ß}$. Let $\boldsymbol{w} \Vdash ß(\mathbf{b}, a)$. Using $\text{Act}_{ß}$, let $\boldsymbol{w} \Vdash \mathsf{E}a \wedge \mathsf{E}\mathbf{c} \wedge \Box ß(\mathbf{c}, a)$. So $\boldsymbol{w} \Vdash \mathbf{b} = \mathbf{c}$ by $\text{Inject}_{ß}$, hence $\boldsymbol{w} \Vdash \mathsf{E}b \wedge \Box ß(\mathbf{b}, a)$.

The **Eu** *schema.* Via $\text{Act}_{ß}$ and Lemma 1.

Empty-Box. A simple induction on complexity. □