Inconsistency of $\mathbb{N}$ from a not-finitist point of view

Enrico Pier Giorgio Cadeddu
(Master of Science, Oristano, Sardinia - Italy)

ABSTRACT: Considering the set of natural numbers $\mathbb{N}$, then in the context of Peano axioms, starting from inequalities between finite sets, we find a fundamental contradiction, about the existence of $\mathbb{N}$, from a not-finitist point of view.

KEYWORDS - Inconsistency, Peano axioms, Natural numbers set, Not-finitist

I. INTRODUCTION

A formal system, together with an interpretation, constituted of an alphabet, grammar, inference rules, axioms, and a reference set, can produce formalized propositions and deductions (theorems) through a finite number of steps, that is a finitist approach [1, 2].

A system is consistent whether a proposition and its negation are not deduced. Godel’s incompleteness theorems [3], developed on the basis of the system of Principia Mathematica including the axiom of infinity, represent a fortress of logic and consistency against inconsistency. But at the same time, they represent a prelude of inconsistency. They give us necessary conditions of consistency, not sufficient ones (undecidable propositions and internal not-demonstrable coherence are these necessary conditions).

Considering the successor function $S(x)$ and the existence of all natural numbers, in concordance with Peano axioms and the axiom of infinity, we show a contradiction in $\mathbb{N}$, in a not-finitist way, that is thinking to take all natural numbers simultaneously.

II. NATURAL NUMBERS SET

The existence of $\mathbb{N}$ is granted by the axiom of infinity [4, 5, 6]. This existence imply that one of each element of the set, also in an actual sense, so taken all together. A finite set wouldn’t admit the Peano axiom: $\forall x(S(x))$, with $S(x) \in \mathbb{N}$, because the greatest number doesn’t have a successor into the finite set. All numbers of $\mathbb{N}$ are defined by Peano axioms [7, 8, 9], together their proprieties thanks to the axiom of induction.

III. A FUNDAMENTAL CONTRADICTION

The two sets: $\{0, \{S(x) \mid x \in \mathbb{N}\}\}$ (with $S(x) \in \mathbb{N}$) and $\mathbb{N}$, are the same set, that is:

$\{0, \{S(x) \mid x \in \mathbb{N}\}\} = \{x \mid x \in \mathbb{N}\} = \mathbb{N}$ (1)

We know, as it is demonstrable, that: $(x \in \mathbb{N}) \Rightarrow \forall x(x < S(x))$. That is $0 < 1, 1 < 2, \ldots, n < n+1$.

At the same time we have:

$\{x \mid x \leq y\} \neq \{x \mid x \leq y + 1\} \forall y$ (2)

with $y+1 = S(y) \in \mathbb{N}$.

That is $\{0, 1, 2, 3\} \neq \{0, 1, 2, 3, 4\}$ and so on, for all $y$.

But necessary condition to have all $y$ (that is $\forall y$) is that at least one of all these sets in (2) exists equal to $\mathbb{N}$, otherwise all $y$ are not taken; the absence of $\mathbb{N}$ (all numbers) in (2) would imply that we could add numbers not present in each set in (2) (so, many numbers would be absent in each set). Then, considering all $y$, then all $x$, and equation (1), we are considering in (2) a set equal to $\mathbb{N}$. So we have $\mathbb{N} \neq \mathbb{N}$, a contradiction.

It is to notice a question: is it necessary to pass through a necessary condition or, directly, do all $y$ imply a set equal to $\mathbb{N}$? At first sight the answer seems no and yes respectively.

CONCLUSION

This proof of inconsistency is not-finitist because it involves infinite totalities. But this is natural considering the set theory with the axiom of infinity (all elements of $\mathbb{N}$). On the other hand, a finitist proof would imply the end of mathematics as we know it.

Anyway, refusing a precise definition of $\mathbb{N}$, then refusing the axiom of infinity (and Peano
axioms?), could be a view to avoid this inconsistency. So the axiom of infinity would seem to have a similar role to coherence. It is not demonstrable, but also it cannot be taken as an axiom if one doesn't want a system to be inconsistent. This proof supports finitist approach in a not arbitrary manner and all theories implying $\mathbb{N}$ with the axiom of infinity could be revisited (including Godel’s theorems).

REFERENCES


