Inconsistency of ℕ from a not-finitist point of view

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ABSTRACT: *Considering the set of natural numbers ℕ, then in the context of Peano axioms, starting from inequalities between finite sets, we find a fundamental contradiction, about the existence of ℕ, from a not-finitist point of view.*

KEYWORDS *-Inconsistency, Peano axioms, Natural numbers set, Not -finitist*

I.INTRODUCTION

A formal system together an interpretation, constituted of an alphabet, grammar, inference rules, axioms, and a reference set, can produce formalized propositions and deductions (theorems) through with a finite number of steps, that is a finitist approach [1, 2].

A system is consistent whether a proposition and its negation are not deduced. Godel's incompleteness theorems [3], developed on the basis of the system of Principia Mathematica including the axiom of infinity, represent a fortress of logic and consistency against inconsistency. But at the same time they represent a prelude of inconsistency. They give us necessary conditions of consistency, not sufficient ones (undecidable propositions and internal not-demonstrable coherence are these necessary conditions).

Considering the successor function S(x) and the existence of all natural numbers, in concordance with Peano axioms and the axiom of infinity, we show a contradiction in $\mathbb{N}$, in a not-finitist way, that is thinking to take all natural numbers simultaneously.

II.NATURAL NUMBERS SET

The existence of $\mathbb{N}$ is granted by the axiom of infinity [4, 5, 6]. This existence imply that one of each element of the set, also in an actual sense, so taken all together. A finite set wouldn't admit the Peano axiom: $\forall x(S(x))$, with $S(x) \in \mathbb{N}$, because the greatest number doesn't have a successor into the finite set. All numbers of $\mathbb{N}$ are defined by Peano axioms [7, 8, 9], together their proprieties thanks to the axiom of induction.

III.A FUNDAMENTAL CONTRADICTION

The two sets: $\{0, \{S(x) \mid x \in \mathbb{N}\}\}$ (with $S(x) \in \mathbb{N}$) and $\mathbb{N}$, are the same set, that is:

$$\mathbf{\{0, \{S(x) \mid x \in \mathbb{N}\}\} = \{x \mid x \in \mathbb{N}\} = \mathbb{N}} \quad (1)$$

We know, as it is demonstrable, that: $(x \in \mathbb{N})(\forall x(x < S(x))$. That is $0 < 1$, $1 < 2$, …, $n < n+1$.

At the same time we have:

$$\mathbf{\{x \mid x \leq y\} \neq \{x \mid x \leq y + 1\} \quad \forall y} \quad (2)$$

with $y+1 = S(y) \in \mathbb{N}$.

That is $\{0, 1, 2, 3\} \neq \{0, 1, 2, 3, 4\}$ and so on, for all y.

But necessary condition to have all y (that is $\forall y$) is that at least one of all these sets in (2) exists equal to $\mathbb{N}$, otherwise all y are not taken; the absence of $\mathbb{N}$ (all numbers) in (2) would imply that we could add numbers not present in each set in (2) (so, many numbers would be absent in each set). Then, considering all y, then all x, and equation (1), we are considering in (2) a set equal to $\mathbb{N}$. So we have $\mathbb{N} \neq \mathbb{N}$, a contradiction.

It is to notice a question: is it necessary to pass through a necessary condition or, directly, do all y imply a set equal to $\mathbb{N}$? At first sight the answer seems no and yes respectively.

CONCLUSION

This proof of inconsistency is not-finitist because it involves infinite totalities. But this is natural considering the set theory with the axiom of infinity (all elements of $\mathbb{N}$). On the other hand a finitist proof would imply the end of mathematics as we know it.

Anyway, refusing a precise definition of $\mathbb{N}$, then refusing the axiom of infinity (and Peano axioms?), could be a view to avoid this inconsistency. So the axiom of infinity would seem to have a similar role to coherence. It is not demonstrable, but also it cannot be taken as an axiom if one doesn't want a system to be inconsistent. This proof supports finitist approach in a not arbitrary manner and all theories implying $\mathbb{N}$ with the axiom of infinity could be revisited (including Godel's theorems).

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