Inconsistency of $\mathbb{N}$ with the set union operation

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Abstract

A contradiction is obtained, considering the axiom of infinity, then $\mathbb{N}$ and Peano axioms, together a list of $\mathbb{N}$ subsets and with inclusion relation and union operation. Natural numbers constitute an infinite set, $\mathbb{N}$, but we show the union of its proper subsets, with a specific form, isn't an infinite set. Also we get a simpler explanation and a symbolic representation. Lastly, inconsistency of Peano successor axiom is a consequence of rejecting infinity.

Introduction

The issue of infinity, in particular the actual infinity, leads us to write this article, as other previous ones [1] [2]. The purpose has always been to get a proof of inconsistency rather than hypothesize it in a new system, or arbitrarily deny infinity.

We consider the axiom of infinity [4] [5] [8] , then the existence of $\mathbb{N}$ and Peano axioms [3]. Sets are considered with the usual graphical-symbolic notation $\{0, 1, 2, ...n\}$ (see also [6] [7]). We start with the sets-list $\{x|x \leq y\} \quad \forall y \in \mathbb{N}$ *with all y taken together*. Each set is shown to be finite, then using inclusion relation and union operation, we obtain a contradiction.

We only consider actual infinity as the true infinity. Potential infinity should just be viewed as a growing finite.

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1 Subsets, inclusion relation, set union operation and inconsistency of $\mathbb{N}$

We consider an infinite list of sets defined by:

$$\{x|x \leq y\} \quad \forall y \in \mathbb{N} \quad \textit{with all y taken together} \tag{1}$$

They are proper subsets of $\mathbb{N}$. In agreement with the axiom of infinity and the axiom of separation, the set of all Natural numbers $\mathbb{N}$ exists, together its subsets.

We highlight the greatest number of each set, y, and there is the list of these numbers, which are all numbers of $\mathbb{N}$.

{ 0 }	0
{ 0, 1 }	1
{ 0, 1, 2 }	2
{ 0, 1, 2, 3 }	3
{ 0, 1, 2, 3, 4 }	4
.	.
.	.
.	.
{ 0, 1, 2, 3, 4, ... ? }	?

Also we could associate a different number to each set, for example as for Von Neumann number definition, but this is irrelevant about all considerations that follow; all numbers having to belong to the list in any case. Immediately we deduce that there isn't a set equal to $\mathbb{N}$ in the sets-list, because each set is a finite set, but in the list there are all natural numbers. In fact each y is a natural number (a finite number, as it is possible to demonstrate) that does have a successor for Peano axioms; so each set doesn't have all numbers, unlike $\mathbb{N}$. So we have the chain: $\{0\} \subset \{0,1\} \subset \{0,1,2\} \subset \{0,1,2,3\} \subset \{0,1,2,3,4\} \subset ... \subset \mathbb{N}$. Or in a more compact form:

$$I_0 \subset I_1 \subset I_2 \subset ... \subset I_y \subset ... \subset \mathbb{N} \tag{2}$$

Each set is a proper subset of other sets that follow and obviously of $\mathbb{N}$.

At finite (y finite) it is: $I_0 \cup I_1 \cup I_2 \cup ... \cup I_y = I_y = I_{union_{(y)}} \subset \mathbb{N} \ \ \forall y \in \mathbb{N}$. In fact I_y includes all numbers of its subsets.

But considering infinite terms I_i, like those would seem on the left of "$\subset \mathbb{N}$" in (2), at first sight "$I_{union} \subset \mathbb{N}$" (with $I_{union} = \bigcup_{i \in \mathbb{N}} I_i$) isn't demonstrable, because we can't take a terminal I_y that includes all I_i (I_y has a successor), and then that couldn't be equal to I_{union}.

However, in any case, set union operation gives a set including all sets involved. With (2) I_{union} includes all I_i, which are included one in the other and this being also valid until infinity. The same I_{union} has to belong to the chain (2) on the left of "$\subset \mathbb{N}$", otherwise I_{union} would contains numbers not contained in any other set I_i. $I_0 \subset I_1 \subset I_2 \subset ... \subset I_y \subset ... \subset I_{union}$.

On the left of "$\subset \mathbb{N}$" we don't have an infinite I_ω including a I_i and so on. Then $I_{union} \neq I_\omega$, it isn't infinite.

So I_{union}, the union of all subsets of $\mathbb{N}$ as in (1), isn't infinite, unlike $\mathbb{N}$, $\neg(I_{union} = \mathbb{N})$ and we have:

$$I_{union} \subset \mathbb{N} \tag{3}$$

This is also available just considering (2). In fact $\mathbb{N}$ contains numbers not contained in any other finite proper subset, each subset including the union of its previous subsets. Then $I_{union} \subset \mathbb{N}$.

But from (1), in which are contained all natural numbers in agreement with $\forall y$, also we have $\bigcup_{i \in \mathbb{N}} I_i = \mathbb{N}$, that is $I_{union} = \mathbb{N}$ and then:

$$\neg(I_{union} \subset \mathbb{N}) \tag{4}$$

So we have a contradiction, being simultaneously (3) and (4).

1.1 Another approach to a not-infinite I_{union}

Each I_i in (2) is a proper subset of $\mathbb{N}$ as in (1), it is finite, each "i" is finite. There aren't infinite terms I_i. Then, on the left of "$\subset \mathbb{N}$" in (2) there would be a finite number of terms I_i, although indeterminate (for an y = n the number of terms I_i is n + 1).

We further explain. The chain written in this manner: $I_0 \subset I_1 \subset I_2 \subset ....$, containing all Natural numbers, cannot be considered consisting of infinite terms I_i, because all terms are finite, with a finite index "i", and then with a finite number of previous and subsequent terms; the number of terms between two terms is always finite. But if we write this: $I_0 \subset I_1 \subset I_2 \subset ... \subset I_\omega$ ($I_\omega = \mathbb{N}$), now an infinite number of terms can be exist, because the last term is infinite and then includes an infinite number of terms. We could think of an analogy with an open finite interval; for example the segment defined for $[0, 5)$ is not "5" long but is less than "5". So, on the left of "$\subset \mathbb{N}$" in (2), there isn't I_ω, then the number of terms isn't infinite (and each term is finite) and their union isn't infinite.

Even more clearly, if we had an infinite I_{union}, then infinite quantity of numbers and an infinite chain, on the left of "$\subset \mathbb{N}$", we would have infinite terms I_i but all finite, which only admit a finite chain (an I_y is preceded by a finite number of I_i); there isn't an infinite chain. This would be absurd.

$I_0 \subset I_1 \subset I_2 \subset ... \subset I_y \subset ...$

$/ - - - \infty\ I_i - - - - /$ *It would be like locating infinity on a finite I_y, that involves a finite chain, not infinite.*

In conclusion, $\mathbb{N}$ is preceded, considering the ordering relation "$\subset$", by a finite set. A finite set is preceded by a finite number of sets, then a not-infinite chain. But this chain contains all Natural numbers, this is infinite and there is a contradiction.

These considerations wouldn't be valid if we thought to infinity as a finite chain increasing over time; in fact in this case the concept of infinity would be independent of the finiteness of every I_i. But time dependency cannot be taken into account to define actual infinity. We only have to consider time-independent statements.

1.2 A simplified and a first order approach

A second-order logic has been used, because we have quantified on subsets. For a simplified approach and as an attempt at translation into the first order, we consider the chain:

$0 < 1 < 2 < 3 < 4 < .... < \omega$.

Each Natural number n is finite and preceded by a finite number n of numbers. On the left of "$< \omega$" the chain $0 < 1 < 2 < 3 < 4 < ....$ (that we call $C_\mathbb{N}$) isn't infinite. In fact ω is greater than any Natural number and it is preceded, considering the ordering relation "$<$", by a Natural number (the chain only consists of Natural numbers), then a finite number. A finite number is preceded by a finite number of numbers, then the chain $C_\mathbb{N}$ isn't infinite. But it contains all Natural numbers and therefore it is the infinite set $\mathbb{N}$, so we have a contradiction. Considering ω is necessary because $\mathbb{N}$ is unlimited.

$\forall n \in C_{\mathbb{N}}(\omega > n) \longrightarrow^{?} \nexists x(x \in C_{card.\infty})$ and $\forall x(x \in C_{card.\infty})$. Where $C_{card.\infty}$ = a cardinal infinite chain. The difficult is to consider a generic finite number without a specific value n, to maintain Peano successor axiom. $C_{\mathbb{N}}$ isn't limited by a specific n, but it is limited by a finite number. These concepts seems to be outside the usual system of rules, but not wrong. $\forall n \in C_{\mathbb{N}}(\omega > n) \longrightarrow \nexists x(x \leq n)(x \in C_{card.\infty})$ is a solution?

A rigid actual meaning (about time, considering actual infinity) implies $C_{\mathbb{N}}$ is simply limited by a specific n ($< \omega$) and all formal deductions, then contradictions, would fall within the normal rules, without the necessity of taking an unknown, undetermined number. Clearly invalidity of Peano successor axiom is an immediate consequence of the inferred finiteness of $C_{\mathbb{N}}$.

$\forall n \in C_{\mathbb{N}}(\omega > n) \longrightarrow \exists M \forall n(n \leq M)$. M would be the finite number preceding ω by "<". This (contradictory) symbolic assertion (like the previous ones) has to be regarded as having a meta-theoretical meaning anyway, concerning the definition of the reference set (used for the theoretical symbolic calculation). Considering a successor of M, we would start a time-depending process, in contrast with the concept of "actuality". In practice $\mathbb{N}$ should be thought of as limited by a specific natural number, great enough to contain all necessary calculations.

2 Inconsistency in symbolic representation of $\mathbb{N}$

We consider a set-representation of $\mathbb{N}$ and its all proper subsets, all included between them, like in (2). Then:

$$\{\ \{\ \mathbf{0}\},\mathbf{1}\},\mathbf{2}\},\mathbf{3}\},\mathbf{4}\},\ldots\ ?\ \}\ ,\ \ldots\ ?,\ \ldots\ \}_{\mathbb{N}} \tag{5}$$

Blue brackets represent the sets of the list (1) (for simplicity we have only considered one blue left bracket).

So, to describe the fact that **all sets of the list are proper subsets of $\mathbb{N}$ and simultaneously they observe relation (2)**, in the points "$\}, \ldots ?, \ldots \}_{\mathbb{N}}$" of (5) there are some numbers without blue brackets; that is there are numbers of $\mathbb{N}$ which don't belong to any set of the list. This is a contradiction; all numbers have to belong to those.

Another approach is the following.

We are referring to (1) and $\bigcup_{i\in\mathbb{N}} I_i = \mathbb{N}$.

$$\{\ \{\ \ \mathbf{0}\},\mathbf{1}\},\mathbf{2}\},\mathbf{3}\},\mathbf{4}\},\ldots\ \}\ \ \}_{\mathbb{N}} \tag{6}$$

$\mathbb{N}$ includes all subsets (blue brackets) defined by list (1). These subsets contain each natural number y (as defined in (1)). Then we see a blue bracket including **all natural numbers** (and all subsets). So, there is a proper subset (a blue bracket) including all natural numbers (let's keep in mind that there are all numbers).

But no proper subset on the list (1) includes all numbers; each number having a successor (for Peano axioms) and then each subset having a "successor-set". So there is a contradiction.

In this section we have shown that sets graphic symbolic representation and evidences from (2) lead to contradictions. But, is sets symbolism so important? Anyway, this representation is in line with what's said above all in the first part of section 1.

3 Inconsistency of Peano successor axiom

Supposing the set of Natural numbers is finite, there is a number, the greatest number, without a successor, in contrast with the Peano axiom. The successor not ever could be "0", for the other Peano axiom: zero is not the successor of any natural number.

So the set of Natural numbers isn't finite.

But this is in contrast to the inconsistency of infinity, then to its nonexistence. Also an infinite time, with its infinite intervals, couldn't exist, and the process involving numbers that follow each other continuously would end. Then there is a contradiction, as already seen in subsection 1.2.

Conclusion

Infinity (actual infinity) is a fundamental part of $\mathbb{N}$, as set, that determines its existence. Inconsistency of one is that of the other and vice versa. We have come to affirm the inconsistency of actual infinity, which can be summarized as follows: $\mathbb{N}$ is an infinite chain consisting of finite numbers (all natural numbers exist), which imply a finite chain. Also in (5) we have a clear representation about the inconsistency, in a different form. Moreover, having obtained inconsistency of Peano successor axiom appears very important.

All this leads us to ask some questions.

What consequences might this inconsistency have on other theories that include $\mathbb{N}$?

Is it possible to speak about the existence of a local coherence, concerning time, with reference to Peano successor axiom?

Is this inconsistency a demonstration of a concrete, finite physical reality? In our opinion the answer to this last question is affirmative.

References

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