Inconsistency of $\mathbb{N}$ with the set union operation

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21 January 2024

Abstract

Considering the axiom of infinity, then $\mathbb{N}$ and Peano axioms, together a list of $\mathbb{N}$ subsets, inclusion relation and union operation, a contradiction is obtained.

1 Introduction

The issue of infinity, in particular the actual infinity, leads us to write this article, as other previous ones [1] [2].

We consider the axiom of infinity [4] [5] [8] , then the existence of $\mathbb{N}$ and Peano axioms [3]. Sets are considered with the usual graphical-symbolic notation $\{0, 1, 2, ...n\}$ (see also [6] [7]). We starting with the sets-list $\{x|x \leq y\} \quad \forall y \in \mathbb{N} \quad \textit{with all y taken together}$. Each set is shown to be finite, then using inclusion relation and union operation, we obtain a contradiction about the presence of $\mathbb{N}$ in the list.

2 Iconsistency of $\mathbb{N}$

We consider an infinite list of sets defined by:

$$\{x|x \leq y\} \quad \forall y \in \mathbb{N} \quad \textit{with all y taken together} \tag{1}$$

They are subsets of $\mathbb{N}$. In agreement with the axiom of infinity, the set of all natural numbers $\mathbb{N}$ exists, together its subsets.

We highlight the greatest number of each set, y, and there is the list of these numbers, which are all numbers of $\mathbb{N}$.

$$
\begin{array}{ll}
\mathbf{\{\ 0\ \}} & 0 \\
\mathbf{\{\ 0,\ 1\ \}} & 1 \\
\mathbf{\{\ 0,\ 1,\ 2\ \}} & 2 \\
\mathbf{\{\ 0,\ 1,\ 2,\ 3\ \}} & 3 \\
\mathbf{\{\ 0,\ 1,\ 2,\ 3,\ 4\ \}} & 4 \\
\mathbf{.} & . \\
\mathbf{.} & . \\
\mathbf{.} & . \\
\mathbf{\{\ 0,\ 1,\ 2,\ 3,\ 4,\ ...\ ?\ \}} & ?
\end{array}
$$

Immediately we deduce that there isn't a set equal to $\mathbb{N}$ in the sets-list, because each set is a finite set (but in the list there are all natural numbers). In fact each y is a natural number (a finite number) that does have a successor for Peano axioms; so each set doesn't have all numbers, unlike $\mathbb{N}$. So we have:

$$\{0\} \subset \{0,1\} \subset \{0,1,2\} \subset \{0,1,2,3\} \subset \{0,1,2,3,4\} \subset ... \subset \mathbb{N} \tag{2}$$

Each set is a proper subset of other sets that follow. Then, Given $I_y = \{0, 1, 2, ...y\}$, the union of all I_y is a subset of $\mathbb{N}$, the union of all sets being the set including all numbers of all sets:

$$\bigcup_{y \in \mathbb{N}} I_y \subset \mathbb{N} \tag{3}$$

$\mathbb{N}$ being an excluded limit "value" (set). We also note that (by proof of induction): $I_0 \cup I_1 \cup I_2 \cup ... \cup I_y \cup I_{y+1} = I_{y+1} \ \forall y \in \mathbb{N}$ (obviously with $y + 1 \in \mathbb{N}$) and because in $\bigcup_{y \in \mathbb{N}} I_y$ there are only $y \in \mathbb{N}$ (there aren't terms like I_∞), the previous inductive union applies simultaneously on all terms in $\bigcup_{y \in \mathbb{N}} I_y$, justifying $\bigcup_{y \in \mathbb{N}} I_y \subset \mathbb{N}$.

But all numbers of the list (1) form the set $\mathbb{N}$ and it is given by $\bigcup_{y \in \mathbb{N}} I_y$ (the collection of all numbers of sets), that is:

$$\bigcup_{y \in \mathbb{N}} I_y = \mathbb{N} \tag{4}$$

So, we have a contradiction with (3) and (4): $\bigcup_{y\in\mathbb{N}} I_y \subset \mathbb{N}$ and $\bigcup_{y\in\mathbb{N}} I_y = \mathbb{N}$.

We want to clarify the relation (3) further. The infinite union of a set A, finite or infinite, is A again: $A \cup A \cup A \cup A \cup ... = A$ (infinite "$\cup$" and A objects). But if we have a subset of A, $B \subset A$, then: $B \cup B \cup B \cup B \cup ... \subset A$. So, considering subsets of A $B_y \quad \forall y \in \mathbb{N}$, with $B_y \subseteq B$, all closely included in A as in (2), then: $B_1 \cup B_2 \cup B_3 \cup B_4 \cup ... \subset A$.

We can summarize and visualize what was said above about (3) by considering the set-representation of $\mathbb{N}$ and its all proper subsets, all included between them, as in (2). Then:

$$\{\ \{\ \ \mathbf{0}\}, \mathbf{1}\}, \mathbf{2}\}, \mathbf{3}\}, \mathbf{4}\}, \mathbf{...}\ \ \mathbf{?}\ \ \}\ \ \mathbf{,\ ...\ \ ?,\ ...}\ \ \}_{\mathbb{N}} \tag{5}$$

Blue brackets represent the sets of the list (1) (for simplicity we have only considered a blue left bracket).

So, to describe the fact that **all sets of the list are proper subsets of $\mathbb{N}$ and simultaneously they observe the relation (2)**, in the points "$\}, ...?, ...\}_{\mathbb{N}}$" of (5) there are some numbers without blue brackets; that is there are numbers of $\mathbb{N}$ which don't belong to any set of the list. This is a a symbolic-graphical model for (2) and (3).

3 Conclusion

This proof of inconsistency leads us to ask some questions.

To what extent is this proof valid?

What consequences might this have on other theories that include $\mathbb{N}$?

Without the axiom of infinity, are Peano axioms coherent?

The axiom of infinity implies a time-independent approach (unlike the ambiguous potential infinity). So, is the concept of time necessary to achieve coherence? Does this have a physical meaning?

Is this inconsistency a demonstration of a concrete, finite physical reality?

References

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