Inconsistency of $\mathbb{N}$ with the set union operation

Enrico P. G. Cadeddu *

22 June 2024

Abstract

A contradiction is obtained, considering the axiom of infinity, then $\mathbb{N}$ and Peano axioms, together a list of $\mathbb{N}$ subsets and with inclusion relation and union operation.

Introduction

The issue of infinity, in particular the actual infinity, leads us to write this article, as other previous ones [1] [2]. The purpose has always been to get a proof of inconsistency rather than hypothesize it in a new system, or arbitrarily deny the infinity.

We consider the axiom of infinity [4] [5] [8] , then the existence of $\mathbb{N}$ and Peano axioms [3]. Sets are considered with the usual graphical-symbolic notation $\{0, 1, 2, ...n\}$ (see also [6] [7]). We start with the sets-list $\{x|x \leq y\} \quad \forall y \in \mathbb{N} \quad \textit{with all y taken together}$. Each set is shown to be finite, then using inclusion relation and union operation, we obtain a contradiction.

1 Iconsistency of $\mathbb{N}$

We consider an infinite list of sets defined by:

$$\{x|x \leq y\} \quad \forall y \in \mathbb{N} \quad \textit{with all y taken together} \tag{1}$$

* Email Address cadeddu.e@gmail.com

They are proper subsets of $\mathbb{N}$. In agreement with the axiom of infinity and the axiom of separation, the set of all Natural numbers $\mathbb{N}$ exists, together its subsets.

We highlight the greatest number of each set, y, and there is the list of these numbers, which are all numbers of $\mathbb{N}$.

{ 0 }	0
{ 0, 1 }	1
{ 0, 1, 2 }	2
{ 0, 1, 2, 3 }	3
{ 0, 1, 2, 3, 4 }	4
.	.
.	.
.	.
{ 0, 1, 2, 3, 4, ... ? }	?

Immediately we deduce that there isn't a set equal to $\mathbb{N}$ in the sets-list, because each set is a finite set (but in the list there are all natural numbers). In fact each y is a natural number (a finite number) that does have a successor for Peano axioms; so each set doesn't have all numbers, unlike $\mathbb{N}$. So we have: $\{0\} \subset \{0,1\} \subset \{0,1,2\} \subset \{0,1,2,3\} \subset \{0,1,2,3,4\} \subset ... \subset \mathbb{N}$. Or in a more compact form:

$$I_0 \subset I_1 \subset I_2 \subset ... \subset I_y \subset ... \subset \mathbb{N} \tag{2}$$

Each set is a proper subset of other sets that follow and obviously of $\mathbb{N}$.

At finite (y finite) it is: $I_0 \cup I_1 \cup I_2 \cup ... \cup I_y = I_y = I_{union_{(y)}} \subset \mathbb{N} \ \ \forall y \in \mathbb{N}$. In fact I_y includes all numbers of its subsets.

But considering an infinite quantity of terms I_i, like that would seem on the left of "$\subset \mathbb{N}$" in (2), "$I_{union} \subset \mathbb{N}$" (with $I_{union} = \bigcup_{i \in \mathbb{N}} I_i$) at first sight isn't demonstrable, because we can't take a terminal I_y that includes all I_i (I_y has a successor), and then that couldn't be equal to I_{union}.

However **each I_i in (2) is a proper subset of $\mathbb{N}$ like in (1), it is finite, each "i" is finite. There aren't infinite terms I_i. Then, on the left of "$\subset \mathbb{N}$" in (2) there is a finite number of terms I_i**, although clearly indeterminate (for example for y = n the number of terms I_i is n + 1).

So I_{union} is finite, it contains a finite number (indeterminate) of terms and we have:

$$I_{union} \subset \mathbb{N} \tag{3}$$

But from (1), in which are contained all natural numbers in agreement with $\forall y$, also we have $\bigcup_{i \in \mathbb{N}} I_i = \mathbb{N}$, that is $I_{union} = \mathbb{N}$ and then:

$$\neg(I_{union} \subset \mathbb{N}) \tag{4}$$

So we have a contradiction, being simultaneously (3) and (4).

2 Inconsistency in symbolic representation of $\mathbb{N}$

We consider a set-representation of $\mathbb{N}$ and its all proper subsets, all included between them, like in (2). Then:

$$\{\ \{\ \ \mathbf{0}\}, \mathbf{1}\}, \mathbf{2}\}, \mathbf{3}\}, \mathbf{4}\}, \ldots\ ?\ \}\ , \ldots\ ?, \ldots\ \}_{\mathbb{N}} \tag{5}$$

Blue brackets represent the sets of the list (1) (for simplicity we have only considered one blue left bracket).

So, to describe the fact that **all sets of the list are proper subsets of $\mathbb{N}$ and simultaneously they observe relation (2)**, in the points "$\}, \ldots ?, \ldots \}_{\mathbb{N}}$" of (5) there are some numbers without blue brackets; that is there are numbers of $\mathbb{N}$ which don't belong to any set of the list. This is a contradiction.

Another approach is the following.

We are referring to (1) and $\bigcup_{i \in \mathbb{N}} I_i = \mathbb{N}$.

$$\{\ \{\ \ \mathbf{0}\}, \mathbf{1}\}, \mathbf{2}\}, \mathbf{3}\}, \mathbf{4}\}, \ldots\ \}\ \ \}_{\mathbb{N}} \tag{6}$$

$\mathbb{N}$ includes all subsets (blue brackets) defined by list (1). These subsets contain each natural number y (as defined in (1)). Then we see a blue bracket including **all natural numbers** (and all subsets). So, there is a proper subset (a blue bracket) including all natural numbers (let's keep in mind that there are all numbers).

But no proper subset on the list (1) includes all numbers; each number having a successor (for Peano axioms) and then each subset having a "successor-set". So there is a contradiction.

In this section we have shown that sets graphic symbolic representation leads to contradictions when some consequences of axioms are applied to that. So, if symbolism was a necessary aspect of Natural Numbers Set, it would imply its inconsistency.

Conclusion

Infinity (actual infinity) is a fundamental part of $\mathbb{N}$, as set, that determines its existence. Inconsistency of one is that of the other and vice versa.

This proof uses second-order logic, because it directly refers to subsets, and a first-order translation, if possible, would be desirable. This proof leads us to ask some questions.

What consequences might this have on other theories that include $\mathbb{N}$?

Without the axiom of infinity, are Peano axioms coherent?

The axiom of infinity implies a time-independent approach (unlike the ambiguous potential infinity). So, is the concept of time necessary to achieve coherence? Does this have a physical meaning?

Is this inconsistency a demonstration of a concrete, finite physical reality? In our opinion the answer to this last question is affirmative.

References

[1] Enrico P G Cadeddu. Inconsistency of N and the question of infinity. *OSF Preprints 10.31219/osf.io/2rs8u 2024 Jan.*

[2] Cadeddu Enrico P. G. Inconsistency of N from a not-finitist point of view. *International Journal of Modern Research in Engineering and Technology (IJMRET)*, 8(10):15–16, 2023.

[3] Giuseppe Peano. *Arithmetices principia: Nova methodo exposita*. Fratres Bocca, 1889.

[4] Jerzy Pogonowski. "mathematics is the logic of the infinite": Zermelo's project of infinitary logic. *Studies in Logic, Grammar and Rhetoric*, 66(3):673–708, 2021.

[5] Bertrand Russell. *Introduction to mathematical philosophy*. Taylor & Francis, 2022.

[6] D Singh and JN Singh. von neumann universe: A perspective. *International Journal of Contemporary Mathematical Sciences*, 2:475–478, 2007.

[7] John Von Neumann. Zur einfuhrung der transfiniten zahlen. *Acta Litterarum ac Scientiarum Regiae Universitatis Hungaricae Francisco-Josephinae, sectio scientiarum mathematicarum*, 1:199–208, 1923.

[8] Ernst Zermelo. Investigations in the foundations of set theory i. *From Frege to Gödel*, pages 199–215, 1908.