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ORIGINAL ARTICLE

# Stratified restricted universals

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**Abstract**
Jc Beall has made several contributions to the theory of restricted quantification in relevant logics. This paper examines these contributions and proposes an alternative account of restricted universals. The alternative is not, however, a theory of relevant restricted universals in any real sense. It is, however, a theory of restricted universals phrased in the most plausible general quantificational theory for relevant logics—Kit Fine's stratified semantics. The motivation both for choosing this semantic framework and for choosing the particular theory of restricted quantification we use is because they are the best way of dealing with these topics from Beall's theory-building theory picture of logic, establishing a second point of contact with Beall's work.

**Keywords** Relevance logic · Restricted quantification · Stratified semantics

## Introduction

Here is the basic problem we will be concerned with in this paper: suppose $\forall x[Ax, Bx]$ is some formula that interprets 'all $A$s are $B$s' in a relevant setting. Then one might hope for the following behavior:

**hope:** for all constants $c$, $Bc$ follows validly from $\forall x[Ax, Bx]$ and $Ac$.

The goal of our paper is to provide a **hope**ful account of restricted universal quantification that, from the point of view of Fine's stratified semantics for unrestricted relevant universals, is particularly natural. Note here that we have carefully avoided saying that what we are providing is in fact an account of relevant restricted quantification. This is not an accident—the account we give is, we think, natural to the point of being

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difficult to reject. It is also disastrously irrelevant. We will have a bit to say at the end of the paper about this. But there is clearly a good deal more philosophical reflection demanded here.

Before getting to the technical details, let us pause to set the scene. To begin, we remind the reader why the 'obvious' candidates for interpreting restricted quantification—viz. the formulas $\forall x(Ax \supset Bx)$ and $\forall x(Ax \to Bx)$ where '$\supset$' is the material conditional and '$\to$' is a relevant conditional—do not work in the sorts of semantic theories that relevant logicians tend to work in.

We begin with the easier case: $\forall x(Ax \supset Bx)$. This fails as an interpretation of 'all $A$s are $B$s' in any paraconsistent (and thus in any relevant) setting. This is for a fairly simple reason: supposing that both $Ac$ and $\neg Ac$ are true, $Ac \supset Bc$ is true regardless of whether $Bc$ is true. Thus, the existence of a $c$ for which $Ac$ is true but $Bc$ is not does not ensure that $\forall x(Ax \supset Bx)$ is false, from which we conclude that $\forall x(Ax \supset Bx)$ is not a good regimentation of 'all $A$s are $B$s' in paraconsistent settings.

The other case is a bit harder. The problem, roughly, is that 'all $A$s are $B$s' ought not say anything about the $B$-hood of non-$A$s. But $\forall x(Ax \to Bx)$ does. This is most visible in the semantics for relevance logics, which we must for the moment assume familiarity with—readers not meeting this supposition might want to return to this discussion after finishing the remainder of the paper. Given the familiarity, though, the rough problem is this: given a point $p$ where $\forall x(Ax \to Bx)$ holds and (say) $Ac$ does not, it is still the case that if we apply $p$ to any point $q$ at which $Ac$ *does* hold, we arrive at a point where $Bc$ holds. Regarding this seeming 'overreach', Ed Mares says the following:

> The material conditional is too weak, but relevant implication is too strong. When one says, for example, "Everyone in this room owns a dog", she does not mean that it follows from being in this room that people own a dog. Rather, it just happens that every person in this room owns a dog. It is this connection, that lies somewhere between material and relevant implication that the restricted quantificational conditional is supposed to capture Mares (2022).

So much for the obvious accounts. The extant literature contains in addition three other non-obvious accounts we will now survey. The first is the proof-theoretic account in Brady (2003). As our interest is in semantic accounts, we put this aside.

The two other extant accounts are both Beall-related accounts—justifying at last our choice of topic. The first is the Beall, Brady, Hazen, Priest, and Restall (BBHPR) account found in Beall et al. (2006). The second extant account we will examine is Beall's 'simplified' account from Beall (2011). Here, in brief, are the details in the two accounts:

- The BBHPR account first defines a new connective '$\mapsto$' with (essentially) the following semantic clause: $A \mapsto B$ is true at a point $p$ just if for all points $q$ at which $A$ is true, $B$ is true at all points that extend both $p$ and the application of $p$ to $q$. Using this, they propose interpreting 'all $A$s are $B$s' as '$\forall x(Ax \mapsto Bx)$'.
- The simplified account is (as you'd expect) much simpler: it interprets 'all $A$s are $B$s' by the formula '$\forall x[(Ax \to Bx) \vee Bx]$' where $\to$ is a relevant conditional.

Both accounts enjoy a range of virtues that are adumbrated in the relevant papers. Unfortunately, both accounts are also hoplessly **hope**less. In both cases, this is for roughly the same reason as in the relevant conditional account—and with that hint, we leave it to the reader to puzzle out the details. The point, in the end is this: as **hope**less accounts, they are no good for our purposes.

Philosophical and motivational matters in hand, we now turn to honest toil. The plan of the paper is as follows. In the section that immediately follows this one, we lay down the language of restricted quantification theory. Then we give an overview of the basics of stratified semantics essentially along the lines of what is in Fine (1988) followed by our account of stratified restricted universals. We end with a discussion of its merits and problems. An Apendix containing a rather annoying construction that plays a role at a few points follows.

## 1 Linguistic matters

For the most part, the construction of the language of restricted quantification theory $\mathsf{L}$ is entirely routine. But there are some finer points that require great care. We start with the vocabulary of $\mathsf{L}$:

*Logical*
Punctuation: (, ), [, ].
Sentential Connectives: $\neg$, $\rightarrow$, $\vee$, and $\wedge$.
Quantifiers: $\forall$.
Variables: $v_1, v_2, v_3, \ldots \in V$.
*Non-logical*
For each $k \in \mathbb{Z}^+$, $k$-place predicate symbols: $P_1^k, P_2^k, P_3^k \ldots \in Pred^k$.
Ordinary names (or constants): $c_1, c_2, c_3, \ldots \in N$.
Additional names: $\omega_1, \omega_2, \omega_3, \ldots \in \Omega$.

An *expression* is any non-empty finite sequence of these symbols. Let $E$ be the set of such sequences. We identify $(N \cup \Omega) \cup V = Terms$ as the set of *terms* of $\mathsf{L}$. These symbols will serve as the pronouns and nouns of $\mathsf{L}$.

An *atomic formula* is any expression of the form

$$Pn_1 \ldots n_k,$$

where each $n_i \in Terms$ and $P \in Pred^k$. Let $Atomic$ be the set of all such expressions.

Our language $\mathsf{L}$ is then the set of expressions that can be generated from $Atomic$ by applying some finite number of times (possibly zero) the *formula-building operations*:

$$\begin{aligned} F_{\neg}(A) &= \neg A, \\ F_{\square}(A, B) &= (A \,\square\, B), \\ F_{\forall^1_{v_i}}(A) &= \forall v_i\, A, \\ F_{\forall^2_{v_i}}(A, B) &= \forall v_i [A, B], \end{aligned}$$

where $\Box \in \{\rightarrow, \vee, \wedge\}$ and $A, B \in E$. Both $F_{\neg}$ and $F_{\Box}$ are total functions. The remaining formula-building operations are not. Their domains are restricted to sequences of expressions with some special properties: Let $e = \langle e_1 \dots e_j \rangle \in E$; then

$e$ satisfies $(v_i)$ the $v_i$*-open condition* iff for some $e_k$, $e_k = v_i$, and
$e$ satisfies $(\forall v_i)$ the $v_i$*-quantifier-free condition* iff it is not the case that for some $e_k$, $e_k = \forall$ and $e_{k+1} = v_i$.

Thus, $F_{\forall^1_{v_i}}$ is defined on $A$ just in case $A$ satisfies both conditions. And $F_{\forall^2_{v_i}}$ is defined on $\langle A, B\rangle$ just in case $A$ and $B$ satisfy $(\forall v_i)$ and either $A$ or $B$ (or both) satisfies $(v_i)$.

Any $A \in \mathsf{L}$ is known as a *well-formed formula* of $\mathsf{L}$, or *formula* for short.

**Remark** *Comprehending the Conditions: Free variables* will be introduced next. With this notion in hand, the stipulation regarding $F_{\forall^1_{v_i}}$ amounts to saying that it is defined only on those expressions in which $v_i$ occurs free and there is no occurrence of $\forall v_i$; the stipulation regarding $F_{\forall^2_{v_i}}$ amounts to saying that it is defined only on those pairs of expressions in which $v_i$ occurs free in at least one of those expressions and there is no occurrence of $\forall v_i$ in either expression.

Moreover, these two stipulations guarantee two desired features of $\mathsf{L}$: First, no two quantifiers that appear in the same formula will share the same variable. This protects us from ambiguity when we have overlapping quantification. Second, vacuous quantification is not permitted in our language.

Given our definition of $\mathsf{L}$, a typical induction principle for this set immediately follows: for any $S \subseteq \mathsf{L}$, if $Atomic \subseteq S$ and $S$ is closed under the formula-building operations, then $S = \mathsf{L}$.

With the syntactic groundwork in place, we move on to some important concepts and functions. But first, a word on some metalingustic notation: $\ulcorner e_1 @ e_2 \urcorner$ abbreviates $\ulcorner e_1$ occurs in $e_2 \urcorner$ and $\ulcorner e_1 \not@ e_2 \urcorner$ abbreviates its negation, where $e_1, e_2 \in E$.

Let $x \in V$ and $A \in \mathsf{L}$. We define $x @_f A$ - i.e, what it means for $x$ to *occur free* in $A$ - recursively:

$x @_f Pn_1 \dots n_k$ iff $x @ Pn_1 \dots n_k$;
$x @_f \neg B$ iff $x @_f B$;
$x @_f (A \Box B)$ iff $x @_f A$ or $x @_f B$;
$x @_f \forall v_i A$ iff $x @ A$ and $x \neq v_i$.
$x @_f \forall v_i[A, B]$ iff $x @_f A$ or $x @_f B$ and $x \neq v_i$.

Hence, $x @_B A$ - $x$ *occurs bound* in $A$ - if $x @ A$ and $x \not@_f A$. As usual, we say that $A$ is a $\mathsf{L}$-*sentence* when and only when for any $y \in V$, $y \not@_f A$. The set $\mathsf{L}^S$ is the collection of all the sentences of $\mathsf{L}$.

**Remark** *The Salience of Sentences:* $\mathsf{L}^S$ will play an important role in our theory. Only its members will receive a semantic interpretation. Later, $\mathsf{L}^S$ will be carved up into a collection of sentence-based languages. Each sentence-based language $\mathsf{L}^S_Y$ will be defined with respect to some finite $Y \subseteq \Omega$ such that $\mathsf{L}^S_Y \subseteq \mathsf{L}^S$. These sentence-based languages will be evaluated in different strata of a stratified model. Thus, we may assign $\mathsf{L}^S_X$ to a particular zero-order model $M_X$, endowing its members with mathematical meaning. The theorems we will prove concern *these* sets of sentences. But we will have to do this in a rather roundabout way, since our semantic clauses are formulated using elements of distinct sentence-based languages in the pile. Hence, we must show that $\mathsf{L}$ is identical to any subset of itself $\mathsf{L}'$ meeting a special condition and generated from different formula-building operations that make use of certain other subsets of $\mathsf{L}$. We will then carve out its set of sentences, $\mathsf{L}'^S$. All inductive proofs will be performed directly on *this* collection. So, since $\mathsf{L}_Y^S \subseteq \mathsf{L}^S \subseteq \mathsf{L}'^S$, any property that holds of all $A \in \mathsf{L}'^S$ will hold of all $A \in \mathsf{L}_Y^S$.

Two functions are now introduced. With them, we rigorously codify the intuitive notion of taking a formula of $\mathsf{L}$ and replacing every occurrence of some term $a$ in it with a term $b$. The purpose: to facilitate easy syntactic manipulation of formulas in later proofs and to allow for clearer formulation of our semantic clauses. Thus, for each $a, b \in Terms$, we define $(a/b) : Terms \longrightarrow Terms$ as follows:

$$n(a/b) = \begin{cases} b & \text{if } n = a; \\ n & \text{if } n \neq a. \end{cases}$$

Two simple claims follow from this definition. We invite the reader to verify them on their own.

**Fact 1** $n(a/a) = n$.

**Fact 2** Whenever $n \neq b$, it follows that $n(a/b)(b/c) = n(a/c)$.

The function $\overline{(a/b)} : \mathsf{L} \longrightarrow \mathsf{L}$ - the extension of $(a/b)$ - is defined like so: $[A]\,\overline{(a/b)} = A$ if $d \in \{a, b\}$ and $d \ @_B \ A$ or $a \not\!@ A$; otherwise,

$$\begin{aligned}
[Pn_1 \ldots n_k]\overline{(a/b)} &= Pn_1(a/b) \ldots n_k(a/b); \\
[\neg A]\overline{(a/b)} &= \neg[A]\overline{(a/b)}; \\
[(A \,\Box\, B)]\overline{(a/b)} &= ([A]\overline{(a/b)} \,\Box\, [B]\overline{(a/b)}); \\
[\forall x A]\overline{(a/b)} &= \forall x[A]\overline{(a/b)}; \\
[\forall x[A, B]]\overline{(a/b)} &= \forall x[[A]\overline{(a/b)},\ [B]\overline{(a/b)}].
\end{aligned}$$

If $a, b \in N \cup \Omega$, we say that $[A]\overline{(a/b)}$ is the *a,b-variant* of $A \in \mathsf{L}$.

As would be expected, two claims follow from the definition of $\overline{(a/b)}$ as well. The reader may verify these claims by a simple induction on $A \in \mathsf{L}$.

**Fact 3** $[A]\overline{(a/a)} = A$.

**Fact 4** Whenever $b \not\!@ A$, $[[A]\overline{(a/b)}]\overline{(b/c)} = [A]\overline{(a/c)}$.

Recall that the purpose of introducing $(a/b)$ and its extension $\overline{(a/b)}$ is to rigorously codify in our language the intuitive notion of replacing all those instances of $a$ in a formula $A$ with $b$. More importantly, when we compose $\overline{(a/b)}$ and $\overline{(b/c)}$, we want to switch out all and *only* those instances of $a$ in $A$ with $c$. But what we have so far does not do the job. Let $Pab \in Atomic$ and assume $a \neq b$. Notice the result of these computations:

$$\begin{aligned}[[Pab]\overline{(a/b)}]\overline{(b/c)} &= [Pa(a/b)b(a/b)]\overline{(b/c)};\\ &= Pa(a/b)(b/c)b(a/b)(b/c);\\ &= Pb(b/c)b(b/c);\\ &= Pcc.\end{aligned}$$

Thus, composing these two functions will not always result in replacing all and only those occurrences of $a$ with $c$ in $A$. The problem: it is possible that $b$ @ $A$, making it subject to replacement. Luckily, this is easily fixed by a simple stipulation: Whenever $b$ @ $A$,

$$[[A]\overline{(a/b)}]\overline{(b/c)} = [[A]\overline{(a/c_k)}]\overline{(c_k/c)}$$

where $c_k \in N$ and $k$ is the least number greater than any $i$, for all $c_i$ @ $A$. Since $A$ is a finite sequence and $N$ is infinite, we know that $c_k$ exists. Consequently, this stipulation and Fact 4, guarantee that

$$[[A]\overline{(a/b)}]\overline{(b/c)} = [A]\overline{(a/c)},$$

for any $a, b, c \in Terms$ and any $A \in \mathsf{L}$.

There are a few more syntactic items that deserve treatment before we get down to business. Let $Y \subseteq X \subseteq \Omega$ and suppose $X$ and $Y$ are finite.

First, fix $Terms_Y = (N \cup Y) \cup V$ and let $Atomic_Y$ be the collection of all $Pn_1 \ldots n_k \in Atomic$ such that $n_i \in Terms_Y$. The language $\mathsf{L}_Y$ is then the set generated from $Atomic_Y$ by applying the formula-building operations in the usual manner. One can easily prove that $\mathsf{L}_Y \subseteq \mathsf{L}$ and $\mathsf{L}_Y \subseteq \mathsf{L}_X$, by induction on $A \in \mathsf{L}_Y$. When $\mathsf{L}_Y \subseteq \mathsf{L}$, we say that $\mathsf{L}_Y$ is a *sublanguage* of $\mathsf{L}$. And when $\mathsf{L}_Y \subseteq \mathsf{L}_X$, we say that $\mathsf{L}_X$ is an *extension of* $\mathsf{L}_Y$. (Observe that it's 'an extension of $\mathsf{L}_Y$,' not 'the extension of $\mathsf{L}_Y$.' This is due to the fact that $\mathsf{L}_Y$ is contained in an infinite number of sublanguages of $\mathsf{L}$, each of which can be generated by adding additional names to $Y$.)

Second, a $\mathsf{L}_Y$-*sentence* is any $A \in \mathsf{L}_Y$ such that for all $y \in V$, $y \not@_f A$. We take $\mathsf{L}_Y^S$ to be the collection of such formulas, and refer to it as a *sentence-based sublanguage* of $\mathsf{L}$, since it plainly follows that $\mathsf{L}_Y^S \subseteq \mathsf{L}$.

Third, we form the collection $\Lambda^S$ of all these sentence-based sublanguages of $\mathsf{L}$. (The class $\Lambda^S$ is, then, constituted by levels of sentence-based sublanguages of $\mathsf{L}$, as described in the first remark.) Clearly, it is partially ordered by inclusion. Our stratified model is partially ordered too with respect to $\leq$, defined like this:

$$M_Z \leq M_W \text{ iff } Z \subseteq W,$$

where $M_Z$ and $M_W$ are zero-order models. Naturally, these two partial orderings coincide with each other structurally. Take the bijection $I$ from $\Lambda^S$ to our stratified model defined like so: $I(\mathsf{L}^S_Z) = M_Z$. And observe that

$$\mathsf{L}^S_Z \subseteq \mathsf{L}^S_W \text{ iff } Z \subseteq W.$$

The mapping $I$, therefore, serves as an *interpretation function* in this setting, assigning “meaning" to every sentence-based sublanguage of L. (Consequently, each level of the sentence based-language pile is uniquely associated with a level in our stratified model by “laying" one partial ordering on top of the other.)

Finally (and as promised), for the purpose of providing clear and rigorous induction proofs using the semantic clauses for the restricted and unrestricted universal quantifiers, both of which require us to utilize elements of distinct sublanguages of L, we define the set L′ and establish that L′ = L.

So, let $\mathsf{L}' \subseteq \mathsf{L}$ be the set generated from $Atomic$ by applying the formula-building operations (two of which we have seen before) defined on sequences of expressions as you would expect:

$$\begin{aligned}
F_{\neg}(A) &= \neg A,\\
F_{\Box}(A \Box B) &= (A \Box B),\\
F^{\omega}_{\forall^1_x}([A]\overline{(x/\omega)}) &= \forall x[[A]\overline{(x/\omega)}]\overline{(\omega/x)},\\
F^{\omega}_{\forall^2_x}([A]\overline{(x/\omega)},\ [B]\overline{(x/\omega)}) &= \forall x[[[A]\overline{(x/\omega)}]\overline{(\omega/x)},\ [[B]\overline{(x/\omega)}]\overline{(\omega/x)}],
\end{aligned}$$

where $\omega \in \Omega$, $A, B \in \mathsf{L}$ both satisfy $(\forall x)$, and $\omega \not\in\!\!\!@\, A, B$; furthermore, $F^{\omega}_{\forall^1_x}$ is defined on $A$ just in case $A$ satisfies $(x)$, and $F^{\omega}_{\forall^2_x}$ is defined on $\langle [A]\overline{(x/\omega)}, [B]\overline{(x/\omega)}\rangle$ just in case either $A$ or $B$ (or both) satisfy $(x)$. We also stipulate that L′ is closed under all $\overline{(a/b)}$.

This definition yields a corresponding induction principle for L′: for any $S \subseteq \mathsf{L}'$, if $Atomic \subseteq S$ and $S$ is closed under the formula-building operations for L′, then $S = \mathsf{L}'$.

Now, we demonstrate that L′ = L. In order to do this, an auxiliary claim is needed.

**Fact 5** $F^{\omega}_{\forall^1_X}([A]\overline{(x/\omega)}) = F_{\forall^1_x}(A)$ and $F^{\omega}_{\forall^2_x}([A]\overline{(x/\omega)},\ [B]\overline{(x/\omega)}) = F_{\forall^2_x}(A, B)$.

***Proof*** To see that this reasoning works, recall that L is closed under all $\overline{(a/b)}$ and note the conditions that restrict the domains of the old and the new quantifier formula-building operations.

$$\begin{aligned}
F^{\omega}_{\forall^1_x}([A]\overline{(x/\omega)}) &= \forall x[[A]\overline{(x/\omega)}]\overline{(\omega/x)}\\
&= \forall x[A]\overline{(x/x)}, \text{ by } \textbf{Fact 4};\\
&= \forall x A, \text{ by } \textbf{Fact 3};\\
&= F_{\forall^1_x}(A).
\end{aligned}$$

$$F^{\omega}_{\forall^2_x}([A]\overline{(x/\omega)},\ [B]\overline{(x/\omega)}) = \forall x[[[A]\overline{(x/\omega)}]\overline{(\omega/x)},\ [[B]\overline{(x/\omega)}]\overline{(\omega/x)}];$$
$$= \forall x[[A]\overline{(x/x)}, [B]\overline{(x/x)}],\ \text{by } \textbf{Fact 4};$$
$$= \forall x[A, B],\ \text{by } \textbf{Fact 3};$$
$$= F_{\forall^2_x}(A, B).$$

□

**Fact 6** $\mathsf{L}' = \mathsf{L}$

***Proof*** We know that $\mathsf{L}' \subseteq \mathsf{L}$. Hence, it is sufficient to demonstrate that $\mathsf{L} \subseteq \mathsf{L}'$, by induction on $A \in \mathsf{L}$. We prove only the quantifier cases. All other cases are utterly trivial. For the remainder of the proof, let $S = \{A \in \mathsf{L} : A \in \mathsf{L}'\}$.

**Unrestricted Quantifier Case:**

Suppose that $B$ satisfies both $(x)$ and $(\forall x)$, and that $\omega \not@ B$. For the IH, assume further that $B \in S$. We must show that $F_{\forall^1_x}(B) \in S$.

By the IH,

$$B \in \mathsf{L} \text{ and } B \in \mathsf{L}'.$$

The set $\mathsf{L}$ is closed under $F_{\forall^1_x}$ and $B$ meets the conditions for its application. And $\mathsf{L}'$ is closed under $\overline{(x/\omega)}$. So,

$$F_{\forall^1_x}(B) \in \mathsf{L} \text{ and } [B]\overline{(x/\omega)} \in \mathsf{L}'.$$

But $\mathsf{L}'$ is closed under $F^{\omega}_{\forall^1_x}$ and $[B]\overline{(x/\omega)}$ meets the conditions for its application. Thus,

$$F^{\omega}_{\forall^1_x}([B]\overline{(x/\omega)}) \in \mathsf{L}'.$$

Fact 5 then implies that

$$F_{\forall^1_x}(B) \in \mathsf{L}'.$$

Hence,

$$F_{\forall^1_x}(B) \in S$$

**Unrestricted Quantifier Case:**

Suppose that for any $D \in \{B, C\}$, $\omega \not@ D$, $D$ satisfies $(\forall x)$, and that at least one $D \in \{B, C\}$ satisfies $(x)$. For the IH, assume $B, C \in S$. We must show that $F_{\forall^2_x}(B, C) \in S$.

By the IH,

$$B, C \in \mathsf{L} \text{ and } B, C \in \mathsf{L}'.$$

We know that L is closed under $F_{\forall^2_x}$ and $B$ and $C$ meet the conditions for its application. Also, L′ is closed under $\overline{(x/\omega)}$. Therefore,

$$F_{\forall^2_x}(B, C) \in \mathsf{L}, \text{ and } [B]\overline{(x/\omega)}, [C]\overline{(x/\omega)} \in \mathsf{L}'.$$

But L′ is closed under $F^{\omega}_{\forall^2_x}$ and $[B]\overline{(x/\omega)}$ and $[C]\overline{(x/\omega)}$ meet the conditions for its application. Consequently,

$$F^{\omega}_{\forall^2_x}([B]\overline{(x/\omega)}, [C\overline{(x/\omega)}]) \in \mathsf{L}'.$$

This and Fact 5 guarantee that

$$F_{\forall^2_x}(B, C) \in \mathsf{L}'.$$

And so, we obtain:

$$F_{\forall^2_x}(B, C) \in S.$$

By the induction principle, $\mathsf{L} \subseteq \mathsf{L}'$. □

Fact 6, therefore, licences using the induction principle for L′ to prove theorems about all sentence-based sublangages of L, which was the point of its construction in the first place.

**Remark** *Semantic Strangeness:* There is a peculiar feature of the remaining induction proofs that needs addressing. To illustrate, assume that we wanted to prove by induction that for all $A \in \mathsf{L}'$, if $X, s \vDash A$ and $s \sqsubseteq t$, then $X, t \vDash A$. For the restricted quantifier case, we must show that $S$, the set of all $A \in \mathsf{L}'$ with this property, is closed under all $F^{\omega}_{\forall^1_x}$. So, we suppose $[B]\overline{(x/\omega)} \in S$ and show that $F^{\omega}_{\forall^1_x}([B]\overline{(x/\omega)}) \in S$. But there is an issue. The clause for the unrestricted quantifier case requires the existence of *some* $\omega_k \notin X$, but it is not guaranteed that $\omega_k = \omega$. Hence, we cannot be sure that $F^{\omega_k}_{\forall^1_x}$ is applicable to $[B]\overline{(x/\omega)}$. It would seem then that we can only prove that $S$ is closed under $F^{\omega_k}_{\forall^1_x}$. Well, yes and no. Yes: as a consequence of the semantic clause for the unrestricted quantifier, we can only *directly* prove that $S$ is closed under $F^{\omega_k}_{\forall^1_x}$ for the particular $\omega_k$ that is referenced in the clause. Thus, the semantic peculiarity forces us to choose an IH after the fact, so to speak. No: Fact 5 implies that for any $\omega_k, \omega_j \in \Omega$, $F^{\omega_k}_{\forall^1_x}([B](x/\omega_k)) = F^{\omega_j}_{\forall^1_x}([B](x/\omega_j))$. Therefore, if $S$ is closed under at least one formula-building operation for the unrestricted quantifier, then it is closed under all of them vacuously. So too with with the restricted quantifier formula-building operations. We formulate our induction hypotheses for quantifier cases with full generality, but we ask that the reader keep in mind that we are implicitly choosing the relevant IH in our arguments.

*Some Helpful Metalingusitic Conventions:* We continue to suppose $x \in V$ and $P, Q \in Pred^k$, for some contextually specified $k$, and let $A, B, C \in \mathsf{L}'$. We omit the outermost occurrence of (, ), [ and ] when the background deems them unnecessary. We also write $\ulcorner A(a/b)\urcorner$ instead of $\ulcorner [A]\overline{(a/b)}\urcorner$ in similar circumstances.

## 2 Stratified models: the setup

Define a preframe to be a 6-tuple $F = \langle T_F, P_F, \ell_F, \circ_F, \sqsubseteq_F, \star_F \rangle$ where

- $\ell_F \in T_F \supseteq P_F$,
- $\circ_F : (T_F \times T_F) \rightarrow T_F$
- $\sqsubseteq_F \subseteq T_F \times T_F$, and
- $\star_F : P_F \longrightarrow P_F$.

A frame is then a preframe for which all of the following hold:

- $\sqsubseteq_F$ is a partial ordering of $T_F$.
- $\ell_F \circ_F t = t$ for all $t$.
- If $s \sqsubseteq_F t$, then $u \circ_F s \sqsubseteq_F u \circ_F t$ and $s \circ_F u \sqsubseteq_F t \circ_F u$ for all $u \in T_F$.
- If $t \circ_F u \sqsubseteq_F p \in P_F$, then there are $t \sqsubseteq_F q \in P_F$ and $u \sqsubseteq_F r \in P_F$ so that $q \circ_F u \sqsubseteq_F p$ and $t \circ_F r \sqsubseteq_F p$.
- $p^{\star_F \star_F} = p$
- $p \sqsubseteq_F q \Rightarrow q^{\star_F} \sqsubseteq_F p^{\star_F}$.

Given a frame $F$, an $F$-model is a function $M$ mapping each $t \in T_F$ to a set of atomic formulas that satisfies the following two conditions:

- If $t \sqsubseteq_F u$, then $M(t) \subseteq M(u)$;
- If $a \in M(p)$ for all $t \sqsubseteq_F p \in P_F$, then $a \in M(t)$.

Where $M$ is an $F$-model, we define $\vDash$ as follows.

- For atomic $a$, $M, t \vDash a$ iff $a \in M(t)$.
- $M, t \vDash \neg A$ iff $M, p^\star \nvDash A$ for all $t \sqsubseteq p \in P_F$.
- $M, t \vDash A \wedge B$ iff $M, t \vDash A$ and $M, t \vDash B$.
- $M, t \vDash A \vee B$ iff for all $t \sqsubseteq p \in P_F$ either $M, p \vDash A$ or $M, p \vDash B$.
- $M, t \vDash A \rightarrow B$ iff for all $u \in T_F$, if $M, u \vDash A$, then $M, t \circ u \vDash B$.

We say that $A$ is valid when for all frames $F$ and all $F$-models $M$, $M, \ell_F \vDash A$. In Fine (1974), Fine proved that the class of valid formulas is axiomatized as follows:

A1 $A \rightarrow A$
A2 $(A \wedge B) \rightarrow A$; $(A \wedge B) \rightarrow B$
A3 $((A \rightarrow B) \wedge (A \rightarrow C)) \rightarrow (A \rightarrow (B \wedge C))$
A4 $A \rightarrow (A \vee B)$; $B \rightarrow (A \vee B)$
A5 $((A \rightarrow C) \wedge (B \rightarrow C)) \rightarrow ((A \vee B) \rightarrow C)$
A6 $(A \wedge (B \vee C)) \rightarrow ((A \wedge B) \vee (A \wedge C))$
A7 $\neg\neg A \rightarrow A$
R1 $\frac{A \quad A \rightarrow B}{B}$
R2 $\frac{A \quad B}{A \wedge B}$
R3 $\frac{A \rightarrow B \quad C \rightarrow D}{(B \rightarrow C) \rightarrow (A \rightarrow D)}$
R4 $\frac{A \rightarrow \neg B}{B \rightarrow \neg A}$

In addition to validity for formulas, we can define validity for set-fmla sequents.[1] To that end, let $X \cup \{A\}$ be a set of formulas. Then we write $X \vDash B$ when for all frames

[1] This isn't the only option. In Read (1988), an alternative bunch-fmla proof system was proposed; a Read-inspired system aimed at capturing logics in this neighborhood was also explored in Logan (2022).

$F$, all $F$-models $M$, and all $t \in T_F$, if $t \vDash A$ for all $A \in X$, then $t \vDash B$. One can then prove that for finite sets $X$, $X \vDash B$ iff $\bigwedge_{A \in X} A \to B$ is valid.

To reduce notational burden, we will in the remainder tend to omit the '$M$,' in expressions of the form '$M, t \vDash A$', and will also omit most of the subscripts. In addition, we will write things like '$t \in M$, where $M$ is an $F$-model, to mean $t \in T_F$. We rely on common sense and context and the charity of the reader for the interpretation of such expressions.

Intuitively, each model models a collection of theories together with some of the operations and relations they naturally inherit. We won't have space to plumb the details correspondence between the intuitive behavior of a space of theories and the behavior of models so-defined. For that, we refer the reader to the source: Fine (1974) and Fine (1988). That said, we should point out that for each model $M$, the identity point $\ell$ models the theory-building-theory at play in the theories in that model. Given this, the set of valid formulas then picks out one particularly important theory-building-theory: it's the primitive 'most basic' such theory; the one on top of which all the others are built.

This is a second place in the paper where we find ourselves playing a Beall-ian refrain. At least one of the authors was directly inspired to think through relevance logics as universal theory-building-theories explicitly because of Beall's work; especially relevant are Beall (2017), Beall (2018), and Beall (2019). In those papers, Beall presents an argument for FDE as 'the one true logic'. The basic idea is well summarized in the following passage:

> once she has identified her target phenomenon (about which she aims to give the true and as-complete-as-possible theory), the task of the theorist is twofold:
>
> - gather the truths about the target phenomenon
> - construct the right closure relation to 'complete' the true theory—to give as full or complete a true theory as the phenomenon allows.
>
> This twofold task is as basic as it is familiar—and, of course, in no way novel. This is just what we do as truth-seeking theorists, whether we are in mathematics (even if we don't quite know what 'makes true' the theories), physics, theology, biology, philosophy, and more—for every phenomenon that contributes to the overall makeup of reality. Once our target phenomenon has been identified (enough to get on with business, so to speak) we then search and gather whatever truths we can; and after that we aim to give the right closure relation for the theory, all with the aim of giving as full/complete an account of the target phenomenon as possible (Beall 2018, pp. 3–4).

The idea in short is that theory-building is an activity with two components: a fact-finding component and a closing-up component. Logic is concerned with the second of these and more specifically with what's universal among the closing-up operations. What the account we offer here adds to Beall's conception of logic is a general 'promotion' of the theories we use to construct closure relations. Rather than relegating such theories to a support role, we allow them to be genuine first-class citizens capable of impacting the logic. More to the point, we recognize (in the application operation '$\circ$') that any theory can be applied to any other just as closure relations are applied to facts.

And aside from this (broadly algebraic) modification of Beall's general approach, we otherwise hew quite closely to his ideas, as revealed by the fact that, at the propositional level, the first-degree fragment (that is, the fragment lacking embedded conditionals) of the logic we arrive at (FDE) is exactly the logic that Beall argues for.

To begin to move from the propositional to the first-order level, first define a zero-order frame to be a frame in the above sense together with a set $N$ of names. Given a zero-order frame $F$, an $F$-model is a function $M$ mapping each $t \in T_F$ to a function mapping each $i$-ary predicate $P^i$ to a subset $M_t(P^i)$ of $N^i$; this time we require models to satisfy the following conditions:

- If $t \sqsubseteq_F u$, then $M_t(P^i) \subseteq M_u(P^i)$;
- If $\overline{n} \in M_p(P^i)$ for all $t \sqsubseteq_F p \in P_F$, then $\overline{n} \in M_t(P^i)$.

For zero-order (which is to say, polyadic but unqauntified) formulas, we can then define $\vDash$ just as before, with the exception that in the atomic case we use the following clause:

- $t \vDash Pn_1 \ldots n_i$ iff $\langle n_1, \ldots, n_i \rangle \in M_t(P)$.

Now let us think about universals. A natural expectation is that we should use the following 'Tarskian' clause to interpret them:

- $M, t \vDash \forall x A$ just if $M, t \vDash A(x/n)$ for all $n \in N$.

But, as shown in Fine (1989), this validates too much. The problem, in simplest terms, is this: the fact that a theory $t$ contains $A$ instantiated at each name $n$ is not sufficient license for the conclusion that $t$ contains $\forall x A$. As an easy example: it is a fact about my own theory of the real numbers that (a) all the namable reals are namable, but also that (b) not all the reals are namable. So my theory of the reals contains 'namable($n$)' for all names of reals $n$, but does not (and, indeed, *shouldn't!*) contain '$\forall x$namable($x$)'. Thus, when it comes to *theories*, the Tarskian clause just won't do the job.

What *does* guarantee that $\forall x A$ be in $t$ is the following: take some name $\omega$ not in the language of $t$. Then $\forall x A \in t$ iff $A(x/\omega)$ is in the theory generated by $t$ in a language enriched to include $\omega$. To make sense of this in the semantics, we need more machinery. In particular, we seem to need at least all the following:

- A set $N$ of *ordinary names*.
- A set $\Omega = \{\omega_i\}_{i=1}^{\infty}$ of *additional names*—intuitively, these are names we can extend our language with. And we want them to be *new* names, so we require $N \cap \Omega = \emptyset$.
- A function $M$ mapping each finite $X \subseteq \Omega$ to a zero-order model $M_X$ on the frame $F_X := \langle N \cup X, T_X, P_X, \ell_X, \circ_X, \sqsubseteq_X, \star_X \rangle$.
- A family of functions $\Downarrow$, with one such function $\downarrow_X^Y : T_Y \longrightarrow T_X$ for each finite pair $X \subseteq Y \subseteq \Omega$.
- A family of functions $\Uparrow$, with one such function $\uparrow_X^Y : T_X \longrightarrow T_Y$ for each finite pair $X \subseteq Y \subseteq \Omega$.
- A family of functions $[\![\cdot]\!]_b^a$, with one such function $[-]_b^a : T_X \to T_X$ for each triple $\langle X, a, b \rangle$ with $\{a, b\} \subseteq N \cup X$.

The first three items model, intuitively, the names in our language, the names we could add to our language, and a function that maps each set of names to a space of theories in the language, $\mathsf{L}_X$, that uses names from $N \cup X$. The remaining three items model the following:

- For each $Y \supseteq X$, $\uparrow^Y_X$ models the function that assigns to each $t \in T_X$, a theory $t\uparrow^Y_X$ that behaves like the theory generated from $t$ in the language $\mathsf{L}_Y$.
- each $Y \supseteq X$, $\downarrow^Y_X$ models the function that maps $t \in T_Y$ to $t \cap \mathsf{L}_X$.
- $[-]^a_b$ models the function mapping $t$ to the theory $[t]^a_b$ that extends $t$ so as to make $a$ and $b$ indistinguishable. The idea, roughly, is that nothing in the notion of a theory prevents any name from behaving like any other, and the purpose of the functions $[-]^a_b$ is to ensure that this is the case.

All told, this means that the models we're going to build (stratified models) contain a multitude of zero-order models. These internal models are linked up by the functions $\downarrow$ and $\uparrow$. Finally, to ensure that our names are just names, we require that the various $[-]^n_m$ functions be present.

The tricky part is specifying conditions that ensure everything behaves as it should. And, quoting now directly from the source (see Fine (1988)), "The number of conditions is, I am afraid, rather large." That said, here's what we need:

1. Both $\uparrow$ and $\downarrow$ are covariant: if $t \sqsubseteq u$, then $t\downarrow^Y_X \sqsubseteq u\downarrow^Y_X$ and $t\uparrow^Z_Y \sqsubseteq u\uparrow^Z_Y$.
2. Whenever they make sense, $t\downarrow^Z_Y\downarrow^Y_X = t\downarrow^Z_X$ and $t\uparrow^Y_X\uparrow^Z_Y = t\uparrow^Z_X$
3. For $i$-ary predicates $P$, $M_{t\downarrow^Y_X}(P) = M_t(P) \cap (N \cup X)^i$.
4. $t\uparrow^Y_X\downarrow^Y_X = t$
5. $t\downarrow^Y_X\uparrow^Y_X \sqsubseteq t$
6. $t\downarrow^X_{X\cap Y}\uparrow^Y_{X\cap Y} = t\uparrow^{X\cup Y}_X\downarrow^{X\cup Y}_Y$
7. If $a \in P_Y$, then $a\downarrow^Y_X \in P_X$.
8. If $a \in P_X$, $b \in P_Y$ and $a \sqsubseteq b\downarrow^Y_X$, then for some $c \in P_Y$, $c\downarrow^Y_X = a$ and $c \sqsubseteq b$.
9. If $t\downarrow^Y_X \sqsubseteq a \in P_X$, then for some $c \in P_Y$, $c\downarrow^Y_X = a$ and $t \sqsubseteq c$.
10. If $a \in P_Y$, then $a^\star\downarrow^Y_X = (a\downarrow^Y_X)^\star$.
11. $(t \circ u)\uparrow^Y_X = t\uparrow^Y_X \circ u\uparrow^Y_X$.
12. $(t \circ (u\uparrow^Y_X))\downarrow^Y_X \sqsubseteq t\downarrow^Y_X \circ u$.
13. $\ell_X\uparrow^Y_X = \ell_Y$.
14. $t \sqsubseteq [t]^a_b$
15. If $s \sqsubseteq t$, then $[s]^a_b \sqsubseteq [t]^a_b$.
16. $[t]^a_b = [[t]^a_b]^a_b$
17. If $p \in P_X$, then $[p]^a_b \in P_X$.
18. $[([p]^a_b)^\star]^a_b = ([p]^a_b)^\star$
19. $[t\uparrow^Y_X]^a_b = [t]^a_b\uparrow^Y_X$

    In Fine (1988), Fine gives an intuitive discussion of why we should expect spaces of theories to meet these conditions. For space reasons, we won't reproduce such discussion here, except in the case of the very last condition, which we're not quite to yet—there are a total of three conditions left to give. But for two of them we need a definition:

    For $t \in M_X$ and $a$ and $b$ in $N \cup X$, we say that $t$ is *symmetric* in $a$ and $b$ when $a$ and $b$ are indistinguishable in $t$. That is, when for all $i$-ary predicates $P$, $\langle n_1, \ldots, a, \ldots, n_i\rangle \in M_X(t, P)$ iff $\langle n_1, \ldots, b, \ldots, n_i\rangle \in M_X(t, P)$. With this notion in have, here are two more conditions:

20. $[t]^a_b$ is symmetric in $a$ and $b$.
21. If $t$ is symmetric in $a$ and $b$, and $t \sqsubseteq p \in P$, then there is a $q \in P$ that is symmetric in $a$ and $b$ with $t \sqsubseteq q \sqsubseteq p$.
    The last condition is important, so worth spending a moment on:
22. If $a \in Y - X$ and $b \in N \cup X$ then $[t{\uparrow}^Y_X]^a_b{\downarrow}^Y_X \sqsubseteq t$.

Note what is going on here. $t$ is an $X$ theory. $t{\uparrow}^Y_X$ is the theory generated by $t$ in the $Y$ vocabulary. $a$ is one of the new names we get when we extend to $\mathsf{L}_Y$ from $\mathsf{L}_X$. $b$ is one of the old names. The theory $[t{\uparrow}^Y_X]^a_b$ takes the theory generated by $t$ in the $Y$ vocabulary and extends it so that $a$ and $b$ are indistinguishable. $[t{\uparrow}^Y_X]^a_b{\downarrow}^Y_X$ then restricts the result back to the $X$ vocabulary. What (22) requires is that we don't gain any new information when doing this. Adding a new name for something we've already named is something we *can* do and sometimes there are good reasons to do so. But simply calling something by two different names is not the sort of thing that should get us genuinely new information.

Semantics in hand, we can now state the new semantic clause for the universal:

- If $\forall x A \in \mathsf{L}_X$, then $X, t \vDash \forall x A$ iff $X \cup \{\omega\}, t{\uparrow}^{X\cup\{\omega\}}_X \vDash A(x/\omega)$ for some $\omega \notin X$.

To verify a universal, then, what a theory needs to do is be extendable in certain ways. In particular, it must be the case that every time we extend it with a new name, we get a new instance. This matches precisely what we said above would guarantee that a theory contain a universal.

We deal with a few final matters before moving on to new material. $A \in \mathsf{L}_X$ is valid in a model $M$ just if $M_X, \ell_X \vDash A$. $A$ is valid when it is valid in all models. Finally, where $X \cup \{A\}$ is a set of formulas, we define semantic entailment $X \vDash A$ to mean that for all $M$ and $t \in M$, if $M, t \vDash x$ for all $x \in X$, then $M, t \vDash A$.

In Fine (1988), Fine proved that the set of formulas valid in all models is axiomatized using the above axioms and rules together with the following axioms and rules governing the quantifiers:

A8 $\forall x A \rightarrow A(x/t)$ where $t$ is free for $x$ in $A$.
A9 $\forall x(A \rightarrow B) \rightarrow (A \rightarrow \forall x B)$ where $x$ is not free in $A$.
A10 $\forall x(A \vee B) \rightarrow (A \vee \forall x B)$ where $x$ is not free in $A$.

R5 $\frac{A}{\forall x A}$

The resulting logic is called **BQ**.

We end the section by proving some lemmas.

**Lemma 1** *$A \rightarrow B$ is valid iff $A \vDash B$.*

***Proof*** $A \rightarrow B$ is valid iff for all $t$, if $t \vDash A$, then $\ell \circ t \vDash B$. But $\ell \circ t = t$. So $A \rightarrow B$ is valid iff for all $t$, if $t \vDash A$, then $t \vDash B$ iff $A \vDash B$. □

**Lemma 2** *If $X, t \vDash \forall x A$, then for all $c \in X$, $X, t \vDash A(x/c)$.*

***Proof*** See Corollary 6 in Fine (1988). □

**Lemma 3** $\nvDash (\forall x(A \rightarrow B) \wedge A(x/\omega)) \rightarrow B(x/\omega)$.

***Proof*** Left to the reader, but see the construction in the Appendix for a hint to how you might go about things. □

## 3 Restricted universals

We finally come to the main event: semantics for restricted universals. The philosophical idea is to treat them in much the same way as we treat unrestricted universals. That is, just as (a) we 'check' whether $t$ contains the unrestricted universal $\forall x A$ by checking whether the theory generated by $t$ in a language enriched by a fresh name $\omega$ contains $A(x/\omega)$, so also (b) we 'check' whether $t$ contains the restricted universal $\forall x[A, B]$ by checking whether the theory generated in a language enriched by a fresh name $\omega$ by $t \cup \{A(x/\omega)\}$ contains $B(x/\omega)$.

What this *suggests* is that we should adopt, in addition to the functions $\uparrow$ and $\downarrow$ we've already got, a bunch of further functions ${}^{A}\!\uparrow$ where $t{}^{A}\!\uparrow_X^{X\cup\{\omega\}}$ is, intuitively, the theory generated by $t \cup \{A(x/\omega)\}$. Such a solution *might* work, but we haven't bothered to check because there's a better solution available. The thing to notice is that the theory, $t^+$, generated in $\mathsf{L}_{X\cup\{\omega\}}$ by $t \cup \{A(x/\omega)\}$ will contain $B(x/\omega)$ just if *every* $\mathsf{L}_{X\cup\{\omega\}}$-theory containing both $t$ and $A(x/\omega)$ contains $B(x/\omega)$. And this idea is an idea we already have the machinery to capture; indeed, the following clause does the job:

- If $\forall x[A, B] \in \mathsf{L}_X$, then $X, t \vDash \forall x[A, B]$ iff for some $\omega \notin X$, for all $u \in T_{X\cup\{\omega\}}$, if $u \sqsupseteq t\uparrow_X^{X\cup\{\omega\}}$ and $u \vDash A(x/\omega)$, then $u \vDash B(x/\omega)$.

Semantic clause in hand, we immediately turn to the key lemma we need:

**Lemma 4** *If $C \in \mathsf{L}_Y$, $Y \subseteq X$, and $s \sqsubseteq t$ are $X$-theories and $u$ is a $Y$-theory, then*

*(a) if $X, s \vDash C$, then $X, t \vDash C$,*
*(b) $X, p \vDash C$ for all $t \sqsubseteq p \in P_X$ iff $t \vDash C$,*
*(c) $X, t \vDash C$ iff $Y, t\downarrow_Y^X \vDash C$, and*
*(d) $Y, u \vDash C$ iff $X, u\uparrow_Y^X \vDash C$.*

Before we turn to the proof, a few words. First, usually these are separate lemmas. E.g. in Fine (1988), these occur as Lemmas 2, 8, 4, and 5. But for somewhat subtle and annoying reasons, we'll have to lump them all together and prove them by a simultaneous induction in all four parts at once. That's metatheoretically unfortunate, but isn't, we think, of any serious import.

Our second point is more worrying. It begins from the observation that every treatment of stratified semantics we're aware of proves a lemma stating that every model obeys what we will call the symmetry condition:

> If $t$ is symmetric in $a$ and $b$ and $C'$ is an $a$, $b$-variant of $C$, then if $X, t \vDash C$, then $X, t \vDash C'$.

See, for example, Lemma 3 in Fine (1988) or Lemma 1.3 in Mares (1992) or Lemma 14 in Logan (2019). And this isn't an accident: a symmetric point is a point at which $a$ and $b$ are treated as indistinguishable. So everything a symmetric point has to say about $a$ it says about $b$ as well and vice-versa. Again, in every case we are aware of, symmetry is imposed by directly requiring symmetric points be symmetric on atomic formulas, which one proves via induction is sufficient to ensure the symmetry condition holds for formulas of arbitrary complexity.

But there's a problem: the semantics as given *doesn't* guarantee that symmetric points treat variants of restricted universals the same. In fact, there are easy counterexamples. Consider for example the canonical model for our base logic **BQ**. Note first this isn't—for the simple reason that its points are theories in the wrong language—the *canonical* model for whatever the logic of the extended language is. It's nonetheless a (quite useful) model of the extended language including our new binary restricted universal since we've not modified at all what it takes to be a model, only added a new semantic clause.

With respect to this model, consider e.g. the formula $\forall x[Pxa, Pxa]$. And evaluate this at the $\mathsf{L}_{\emptyset}$-theory $\langle\emptyset\rangle$ generated by the empty set. Clearly $\langle\emptyset\rangle$ is symmetric in any two names. Thus in particular it's symmetric in $a$ and $b$.

Now choose an arbitrary $P\omega_1 a$-containing extension of the $\mathsf{L}_{\{\omega_1\}}$-theory generated by the emptyset. Here are two very obvious things about such extensions:

- Being $P\omega_1 a$-containing, they are all $P\omega_1 a$-containing;
- Some of them are not $P\omega_1 b$-containing.

It follows that $\langle\emptyset\rangle$ verifies $\forall x[Pxa, Pxa]$ but doesn't verify $\forall x[Pxa, Pxb]$. Thus atomically-symmetric theories are not always everywhere-symmetric theories. So the usual trick for enforcing the symmetry condition—impose it on the atoms at let it work its way up—won't do the job here. And the reason for this failure is clear enough to see: $a, b$-symmetry isn't horizontally hereditary. Nor should it be: a point might well be incapable of discriminating $a$ from $b$, but also be extendable, as above, in such a way that such discrimination is possible.

It seems plausible to us that there might be some subtle, recursive way to enforce exactly the sort of extendable symmetry we need. But if there is, it eluded us. More plausible is that symmetry as stated is just the wrong sort of thing to require. Put more bluntly, we (by which we mean just the authors, not the community) think we (by which we mean the community, not just the authors) don't quite grok symmetry and that it is quite likely that it's the wrong thing to demand.

But we don't want to hold progress hostage to an analysis of symmetry, so we'll resort to something as unsubtle as possible to get around the roadblock at hand: we'll simply restrict our attention in the remainder to models that *do* satisfy the symmetry condition. That there are such models is demonstrated by the constructions used in the appendices. That this is a massively noncompositional and hideous way to solve the problem is not lost on us, and we return to discuss the matter at the end of the paper. For now we simply move on with a slightly embarrassed look.

Acts of contrition completed, we return to proving Lemma 4

***Proof*** The proof, as mentioned, is by simultaneous induction on the complexity of $C$ in all four parts.

The base cases for parts (a), (b), and (c) are handled by the fact that each $M_X$ is a zero-order model. For (d) we follow Fine in noting that since $u = u{\uparrow}^X_Y{\downarrow}^X_Y$, (d) is really just a special case of (c). We thus take demonstration of (c) in the remainder to suffice for demonstration of (d). We also note that in all cases, the 'if' part of (b) follows immediately from (a), and is thus omitted. Finally, since they're straightforward and the proof is long enough without them, we leave the $\neg$ cases, the $\wedge$ cases, and the $\vee$ cases to the reader.

**Entailment Case:**

- For (a): Assume that $X, s \models A \to B$ and $s \sqsubseteq t$. Then for all $u \in T_X$, if $X, u \models A$, then $X, s \circ u \models B$. We must show that $X, t \models A \to B$. So, let $u' \in T_X$ and $X, u' \models A$. Thus, $X, s \circ u' \models B$. Since $\circ$ covaries with $\sqsubseteq$, $s \circ u' \sqsubseteq t \circ u'$. Therefore, $t \circ u' \models B$, by the inductive hypothesis (IH). So, $X, t \models A \to B$.
- For (b): 'only if' direction. Suppose that for all $p \in P_X$, if $t \sqsubseteq p$, then $X, p \models A \to B$. We need to establish that $X, t \models A \to B$. So, let $u \in T_X$ and $X, u \models A$. We must show $t \circ u \models B$.
  By part (b) of the IH, it is sufficient to show that for all $p \in P_X$, if $t \circ u \sqsubseteq p$, then $X, p \models B$. Thus, we let $q \in P_X$ and $t \circ u \sqsubseteq q$ to demonstrate that $X, q \models B$. Immediately, we know that there is some $q \in P_X$ such that $t \sqsubseteq q$ and $q \circ u \sqsubseteq q$. Therefore, by assumption, $X, q \models A \to B$. Hence, $q \circ u \models B$. By part (a) of the IH, it follows that $X, q \models B$, which completes the proof.
- For (c):
  *'if' direction.* Assume that $Y, t{\downarrow}^X_Y \models A \to B$. Then for all $u \in T_Y$, if $Y, u \models A$, then $Y, t{\downarrow}^X_Y \circ u \models B$. We must show that $X, t \models A \to B$. So, we let $u' \in T_X$ and suppose $X, u' \models A$ to prove that $X, t \circ u' \models B$. By part (c) of the IH, $Y, u'{\downarrow}^X_Y \models A$. Thus, $Y, t{\downarrow}^X_Y \circ u'{\downarrow}^X_Y \models B$. So, $X, (t{\downarrow}^X_Y \circ u'{\downarrow}^X_Y){\uparrow}^X_Y \models B$, by a special case of part (c) of the IH. And $X, t{\downarrow}^X_Y{\uparrow}^X_Y \circ u'{\downarrow}^X_Y{\uparrow}^X_Y \models B$, given (11). Therefore, $u'{\downarrow}^X_Y{\uparrow}^X_Y \sqsubseteq u'$ and $t{\downarrow}^X_Y{\uparrow}^X_Y \sqsubseteq t$, by (5). It follows that $t{\downarrow}^X_Y{\uparrow}^X_Y \circ u'{\downarrow}^X_Y{\uparrow}^X_Y \sqsubseteq t{\downarrow}^X_Y{\uparrow}^X_Y \circ u' \sqsubseteq t \circ u'$, since $\circ$ covaries with $\sqsubseteq$. But $\sqsubseteq$ is transitive. Hence, $t{\downarrow}^X_Y{\uparrow}^X_Y \circ u'{\downarrow}^X_Y{\uparrow}^X_Y \sqsubseteq t \circ u'$. Consequently, $X, t \circ u' \models B$, by part (a) of the IH. This implies that $X, t \models A \to B$.
  *'only if' direction.* Suppose that $X, t \models A \to B$. Then for all $u \in T_X$, if $X, u \models A$, then $X, t \circ u \models B$. We must show $Y, t{\downarrow}^X_Y \models A \to B$. So, we assume $u' \in T_Y$ and $Y, u' \models A$ to prove $Y, t{\downarrow}^X_Y \circ u' \models B$. By the special case of part (c) of the IH, we know that $X, u'{\uparrow}^X_Y \models A$. So, $X, (t \circ (u'{\uparrow}^X_Y) \models B$. Therefore, $Y, (t \circ (u'{\uparrow}^X_Y){\downarrow}^X_Y \models B$, by part (c) of the IH. Hence, $(t \circ (u'{\uparrow}^X_Y){\downarrow}^X_Y \sqsubseteq t{\downarrow}^X_Y \circ u'$, by (12). Thus, by part (a) of the IH, $t{\downarrow}^X_Y \circ u' \models B$. Therefore, $Y, t{\downarrow}^X_Y \models A \to B$.

**Unrestricted Universal Case:**

- For (a): Suppose $X, s \models \forall x A$ and $s \sqsubseteq t$. Then for some $\omega \notin X$, $X \cup \{\omega\}, s{\uparrow}^{X\cup\{\omega\}}_X \models A(x/\omega)$. By the covariance of $\sqsubseteq$, $s{\uparrow}^{x\cup\{\omega\}}_X \sqsubseteq t{\uparrow}^{x\cup\{\omega\}}_X$. Thus, $X \cup \{\omega\}, t{\uparrow}^{X\cup\{\omega\}}_X \models A(x/\omega)$, by part (a) of the IH. Consequently, $X, t \models \forall x A$, which is our goal.
- For (b): 'only if' direction. We prove the contrapositive. So, suppose $X, t \not\models \forall x A$. Then, letting $\omega \notin X$, $X \cup \{\omega\}, t{\uparrow}^{X\cup\{\omega\}}_X \not\models A(x/\omega)$. Thus by the IH, for some $t{\uparrow}^{X\cup\{\omega\}}_X \sqsubseteq p \in P_{X\cup\{\omega\}}$, $X \cup \{\omega\}, p \not\models A(x/\omega)$. But now observe that since $p{\downarrow}^{X\cup\{\omega\}}_X{\uparrow}^{X\cup\{\omega\}}_X \sqsubseteq p$ and $X \cup \{\omega\}, p \not\models A(x/\omega)$, we get that $X \cup \{\omega\}, p{\downarrow}^{X\cup\{\omega\}}_X{\uparrow}^{X\cup\{\omega\}}_X \not\models A(x/\omega)$. So $X, p{\downarrow}^{X\cup\{\omega\}}_X \not\models \forall x A$. And since $t{\uparrow}^{X\cup\{\omega\}}_X \sqsubseteq p \in P_{X\cup\{\omega\}}$, $t \sqsubseteq p{\downarrow}^{X\cup\{\omega\}}_X \in P_X$. So there is a $t \sqsubseteq q \in P_X$ so that $X, q \not\models \forall x A$.
- For (c):
  *'if' direction*. Suppose $Y, t{\downarrow}^X_Y \models \forall x A$. Then $Y \cup \{\omega\}, t{\downarrow}^X_Y{\uparrow}^{Y\cup\{\omega\}}_Y \models A(x/\omega)$ for some $\omega \notin Y$. We claim that it follows from that that for any $\omega' \notin X$, we also have that $Y \cup \{\omega'\}, t{\downarrow}^X_Y{\uparrow}^{Y\cup\{\omega'\}}_Y \models A(x/\omega')$.

To see this, choose $\omega' \notin X$ and notice that since we have that

$$t{\downarrow}_Y^X{\uparrow}_Y^{Y\cup\{\omega\}}{\uparrow}_{Y\cup\{\omega\}}^{Y\cup\{\omega,\omega'\}}{\downarrow}_{Y\cup\{\omega\}}^{Y\cup\{\omega,\omega'\}} = t{\downarrow}_Y^X{\uparrow}_Y^{Y\cup\{\omega\}},$$

the IH gives that $t{\downarrow}_Y^X{\uparrow}_Y^{Y\cup\{\omega\}}{\uparrow}_{Y\cup\{\omega\}}^{Y\cup\{\omega,\omega'\}} \vDash A(x/\omega)$. Thus by the symmetry condition, $\left[t{\downarrow}_Y^X{\uparrow}_Y^{Y\cup\{\omega\}}{\uparrow}_{Y\cup\{\omega\}}^{Y\cup\{\omega,\omega'\}}\right]_\omega^{\omega'} \vDash A(x/\omega')$. So (again by the IH), $\left[t{\downarrow}_Y^X{\uparrow}_Y^{Y\cup\{\omega\}}{\uparrow}_{Y\cup\{\omega\}}^{Y\cup\{\omega,\omega'\}}\right]_\omega^{\omega'}{\downarrow}_{Y\cup\{\omega'\}}^{Y\cup\{\omega,\omega'\}} \vDash A(x/\omega')$.
Now observe that

$$\begin{aligned}\left[t{\downarrow}_Y^X{\uparrow}_Y^{Y\cup\{\omega\}}{\uparrow}_{Y\cup\{\omega\}}^{Y\cup\{\omega,\omega'\}}\right]_\omega^{\omega'}{\downarrow}_{Y\cup\{\omega'\}}^{Y\cup\{\omega,\omega'\}} &= \left[t{\downarrow}_Y^X{\uparrow}_Y^{Y\cup\{\omega,\omega'\}}\right]_\omega^{\omega'}{\downarrow}_{Y\cup\{\omega'\}}^{Y\cup\{\omega,\omega'\}} \\ &= \left[t{\downarrow}_Y^X{\uparrow}_Y^{Y\cup\{\omega'\}}{\uparrow}_{Y\cup\{\omega'\}}^{Y\cup\{\omega,\omega'\}}\right]_\omega^{\omega'}{\downarrow}_{Y\cup\{\omega'\}}^{Y\cup\{\omega,\omega'\}} \\ &\sqsubseteq t{\downarrow}_Y^X{\uparrow}_Y^{Y\cup\{\omega'\}}\end{aligned}$$

Thus, by (a) of the IH, $Y \cup \{\omega'\}, t{\downarrow}_Y^X{\uparrow}_Y^{Y\cup\{\omega'\}} \vDash A(x/\omega')$.
But also $t{\downarrow}_Y^X{\uparrow}_Y^{Y\cup\{\omega'\}} = t{\uparrow}_X^{X\cup\{\omega'\}}{\downarrow}_{Y\cup\{\omega'\}}^{X\cup\{\omega'\}}$. So

$$Y \cup \{\omega'\}, t{\uparrow}_X^{X\cup\{\omega'\}}{\downarrow}_{Y\cup\{\omega'\}}^{X\cup\{\omega'\}} \vDash A(x/\omega').$$

So by the IH, $X \cup \{\omega'\}, t{\uparrow}_X^{X\cup\{\omega'\}} \vDash A(x/\omega')$. And since $\omega' \notin X$, it follows from this that $X, t \vDash \forall x A$.
*'only if' direction:* Let $X, t \vDash \forall x A$. Then $X \cup \{\omega\}, t{\uparrow}_X^{X\cup\{\omega\}} \vDash A(x/\omega)$. So by the IH, $Y \cup \{\omega\}, t{\uparrow}_X^{X\cup\{\omega\}}{\downarrow}_{Y\cup\{\omega\}}^{X\cup\{\omega\}} \vDash A(x/\omega)$. But $t{\uparrow}_X^{X\cup\{\omega\}}{\downarrow}_{Y\cup\{\omega\}}^{X\cup\{\omega\}} = t{\downarrow}_Y^X{\uparrow}_Y^{Y\cup\{\omega\}}$. Thus $Y \cup \{\omega\}, t{\downarrow}_Y^X{\uparrow}_Y^{Y\cup\{\omega\}} \vDash A(x/\omega)$. Finally, note that since $\omega \notin X$, $\omega \notin Y$. Thus $Y, t{\downarrow}_Y^X \vDash \forall x A$.

**Restricted Universal Case:**

- For (a): Suppose that $X, s \vDash \forall x[A, B]$ and $s \sqsubseteq t$. Then for some $\omega \notin X$ and any $u \in T_{X\cup\{\omega\}}$, if $s{\uparrow}_X^{X\cup\{\omega\}} \sqsubseteq u$ and $X \cup \{\omega\}, u \vDash A(x/\omega)$, then $X \cup \{\omega\}, u \vDash B(x/\omega)$. We must show that $X, t \vDash \forall x[A, B]$. So, let $u' \in T_X^{X\cup\{\omega\}}$, and suppose $t{\uparrow}_X^{X\cup\{\omega\}} \sqsubseteq u'$ and $X \cup \{\omega\}, u' \vDash A(x/\omega)$. By the covariance of $\sqsubseteq$, it follows that $s{\uparrow}_X^{X\cup\{\omega\}} \sqsubseteq t{\uparrow}_X^{X\cup\{\omega\}}$. Hence, $s{\uparrow}_X^{x\cup\{\omega\}} \sqsubseteq u'$, since $\sqsubseteq$ is transitive. Therefore, $X \cup \{\omega\}, u' \vDash B(x/\omega)$. This suffices to show that $X, t \vDash \forall x[A, B]$.
- For (b): 'only if' direction. Let $X, p \vDash \forall x[A, B]$ for all $t \sqsubseteq p \in P_X$. Let $t{\uparrow}_X^{X\cup\{\omega\}} \sqsubseteq u$ and $u \vDash A(x/\omega)$. We want to show that $u \vDash B(x/\omega)$. We will do so by instead showing that $q \vDash B(x/\omega)$ for all $u \sqsubseteq q \in P_{X\cup\{\omega\}}$. By the IH this will suffice.
So choose $u \sqsubseteq q \in P_{X\cup\{\omega\}}$. Then by (a) of the IH, $q \vDash A(x/\omega)$. But we also have that $t \sqsubseteq q{\downarrow}_X^{X\cup\{\omega\}} \in P_X$. Thus by assumption, $q{\downarrow}_X^{X\cup\{\omega\}} \vDash \forall x[A, B]$. But then since $q{\downarrow}_X^{X\cup\{\omega\}}{\uparrow}_X^{X\cup\{\omega\}} \sqsubseteq q$ and $q \vDash A(x/\omega)$, $q \vDash B(x/\omega)$.

- Before moving to (c) we need a supplemental result:

  $t \vDash \forall x[A, B]$ iff for all $t\uparrow_Y^{Y\cup\{\omega\}} \sqsubseteq p \in P_{Y\cup\{\omega\}}$, if $p \vDash A(x/\omega)$ then $p \vDash B(x/\omega)$.
  *proof* The 'only if' direction is immediate. For the 'if' direction, suppose that for all $t\uparrow_Y^{Y\cup\{\omega\}} \sqsubseteq p \in P_{Y\cup\{\omega\}}$, if $p \vDash A(x/\omega)$ then $p \vDash B(x/\omega)$. To see that $t \vDash \forall x[A, B]$, let $t\uparrow_Y^{Y\cup\{\omega\}} \sqsubseteq u$ and $u \vDash A(x/\omega)$. Now let $u \sqsubseteq p \in P_{Y\cup\{\omega\}}$. Then $t\uparrow_Y^{Y\cup\{\omega\}} \sqsubseteq p$ and $p \vDash A(x/\omega)$. Thus $p \vDash B(x/\omega)$. So by (b) of the IH, $u \vDash B(x/\omega)$ as required.

- For (c): 'if' direction. Let $t\downarrow_X^Y \vDash \forall x[A, B]$. Choose $t\uparrow_Y^{Y\cup\{\omega\}} \sqsubseteq p \in P_{X\cup\{\omega\}}$ and suppose $p \vDash A(x/\omega)$. Then $p\downarrow_{X\cup\{\omega\}}^{Y\cup\{\omega\}} \in P_{X\cup\{\omega\}}$ and by the IH, $p\downarrow_{X\cup\{\omega\}}^{Y\cup\{\omega\}} \vDash A(x/\omega)$. But note by (6) that $t\downarrow_X^Y\uparrow_X^{X\cup\{\omega\}} = t\uparrow_Y^{Y\cup\{\omega\}}\downarrow_{X\cup\{\omega\}}^{Y\cup\{\omega\}}$. So since $t\uparrow_Y^{Y\cup\{\omega\}} \sqsubseteq p$, we also have that
  $$t\downarrow_X^Y\uparrow_X^{X\cup\{\omega\}} = t\uparrow_Y^{Y\cup\{\omega\}}\downarrow_{X\cup\{\omega\}}^{Y\cup\{\omega\}} \sqsubseteq p\downarrow_{X\cup\{\omega\}}^{Y\cup\{\omega\}}.$$
  But then since $t\downarrow_X^Y \vDash \forall x[A, B]$ and $p\downarrow_{X\cup\{\omega\}}^{Y\cup\{\omega\}} \vDash A(x/\omega)$ we also have that $p\downarrow_{X\cup\{\omega\}}^{Y\cup\{\omega\}} \vDash B(x/\omega)$. Thus by the supplemental result, $t \vDash \forall x[A, B]$.
- For (c): 'only if' direction. suppose $t \vDash \forall x[A, B]$. By the supplemental result, to show $t\downarrow_X^Y \vDash \forall x[A, B]$ it suffices to show for all $t\downarrow_X^Y\uparrow_X^{X\cup\{\omega\}} \sqsubseteq p \in P_{X\cup\{\omega\}}$ that if $p \vDash A(x/\omega)$, then $p \vDash B(x/\omega)$.
  So we let $t\downarrow_X^Y\uparrow_X^{X\cup\{\omega\}} \sqsubseteq p \in P_{X\cup\{\omega\}}$ and $p \vDash A(x/\omega)$. Note by (6) that $t\downarrow_X^Y\uparrow_X^{X\cup\{\omega\}} = t\uparrow_Y^{Y\cup\{\omega\}}\downarrow_{X\cup\{\omega\}}^{Y\cup\{\omega\}}$. Thus $t\uparrow_Y^{Y\cup\{\omega\}}\downarrow_{X\cup\{\omega\}}^{Y\cup\{\omega\}} \sqsubseteq p$. So by (9) there is $q \in P_{Y\cup\{\omega\}}$ so that $q\downarrow_{X\cup\{\omega\}}^{Y\cup\{\omega\}} = p$ and $t\uparrow_Y^{Y\cup\{\omega\}} \sqsubseteq q$. But then since $p \vDash A(x/\omega)$, $q\downarrow_{X\cup\{\omega\}}^{Y\cup\{\omega\}} \vDash A(x/\omega)$. So by the IH, $q \vDash A(x/\omega)$. Thus since $t \vDash \forall x[A, B]$, $q \vDash B(x/\omega)$. So again by the IH, $q\downarrow_{X\cup\{\omega\}}^{Y\cup\{\omega\}} = p \vDash B(x/\omega)$.

□

We can now show that the account we've provided is, as promised, a **hope**ful account of restricted quantification.

**Lemma 5** *(**hope**fulness) If $t \vDash \forall x[A, B]$ and $t \vDash A(x/c)$, then $t \vDash B(x/c)$.*

***Proof*** Let $c \in N$. Suppose $X, t \vDash \forall x[A, B]$ and $X, t \vDash A(x/c)$. We must show that $X, t \vDash B(x/c)$. By assumption, there is some $\omega \notin X$ such that for all $u \in T_{X\cup\{\omega\}}$, if $t\uparrow_X^{X\cup\{\omega\}} \sqsubseteq u$ and $X \cup \{\omega\}, u \vDash A(x/\omega)$, then $X \cup \{\omega\}, u \vDash B(x/\omega)$, and with Lemma 4, part (d): $X \cup \{\omega\}, t\uparrow_X^{X\cup\{\omega\}} \vDash A(x/c)$. Now, (14) guarantees that $t\uparrow_X^{X\cup\{\omega\}} \sqsubseteq [t\uparrow_X^{X\cup\{\omega\}}]_c^\omega$. Therefore, $[t\uparrow_X^{X\cup\{\omega\}}]_c^\omega \vDash A(x/c)$, by Lemma 4, part (a). But the symmetry condition and (20) yield $[t\uparrow_X^{X\cup\{\omega\}}]_c^\omega \vDash A(x/c)(c/\omega)$. Fact 4 then implies $[t\uparrow_X^{X\cup\{\omega\}}]_c^\omega \vDash A(x/\omega)$. So, $[t\uparrow_X^{X\cup\{\omega\}}]_c^\omega \vDash B(x/\omega)$. And $[t\uparrow_X^{X\cup\{\omega\}}]_c^\omega \vDash B(x/\omega)(\omega/c)$, by the symmetry condition and (20), again. This means that $[t\uparrow_X^{X\cup\{\omega\}}]_c^\omega \vDash B(x/c)$, given Fact 4. Hence, with Lemma 4, part (c), we have $X, [t\uparrow_X^{X\cup\{\omega\}}]_c^\omega\downarrow_X^{X\cup\{\omega\}} \vDash B(x/c)$. But $[t\uparrow_X^{X\cup\{\omega\}}]_c^\omega\downarrow_X^{X\cup\{\omega\}} \sqsubseteq t$, by (22). Thus, $X, t \vDash B(x/c)$, given Lemma 4, part (a).□

## 4 Evaluating the account

As the discussion at the top of §3 makes clear, the account we've provided here is quite natural from the stratified perspective. And, modulo the pesky little matter of symmetry, the formal results proved immediately above demonstrate that the account is semantically well-behaved. These are both nice features. But they aren't decisive—there's also the question of how the account fares *as an account of restricted quantification*, which we now turn to discussing.

Our answer has two parts, each with two subparts. For the first part, we first examine to what extent the account we've provided can meet the desiderata found in the BBHPR paper. The results are mixed: of the five purely universal conditions they give, we fully satisfy three, partially satisfy one, and entirely fail on one of them. But (now to the second subpart of the first part) we argue that the BBHPR desiderata that we either don't or don't fully satisfy were erroneous in the first place. So we claim a win here—all the desiderata we had any actual reason to worry about we satisfy.

For the second part we turn to thinking about relevance. After all, both the BBHPR account and Beall's simplified account were meant as accounts of *relevant* restricted quantification. So there's a natural question to ask about the extent to which what we provide is relevant. Here the results are entirely negative: our account is, wholeheartedly and unrestrictedly, irrelevant. After explaining why this is the case, we spend a bit of time thinking about just how bad a problem it actually is.

### 4.1 The BBHPR Desiderata

The BBHPR team examines in their paper a total of 16 different desiderata on a theory of relevant restricted quantification. But given that we've only proposed an account of relevant restricted universals, we'll focus only on the solely-universal desiderata, of which there are five:

A1 $A(x/c), \forall x[A, B] \vdash B(x/c)$
A2 $\forall x B \vdash \forall x[A, B]$
A3 $\forall x(A \rightarrow B) \vdash \forall x[A, B]$
B1 $\forall x[A, B] \nvdash \forall x[\neg B, \neg A]$
B2 $\forall x[A, \forall x[A, B]] \nvdash \forall x[A, B]$

Before going further, we need to remind the reader of the distinction between *global* and *local* validity for rules. The distinction—due mostly to Humberstone (see Humberstone (1996)) who, in turn appeals to work by Garson (see Garson (1990))—is well-summarized in Da Ré et al. (2021) as follows:

> Local validity means that any counterexample to the conclusion is a counterexample to one of the premises as well. Global validity means that if there is a counterexample to the conclusion, there must also be a counterexample to one of the premises (where the two counterexamples need not be the same).

It's less than totally obvious whether the authors of the BBHPR paper intended their turnstiles to pick out local validity or global validity. Most of what they say seems to follow only on the assumption that they pick out claims about global validity. That said—and, as we'll see below—some of their claims seem to presuppose that they're picking out local validity. Rather than turning aside into exegesis, we'll sidestep the problem by simply considering both interpretations. Thus, we'll bifurcate each of the five desiderata given above into two distinct options: a global option for which we will continue to use the single turnstile and a local option for which we will use '$\vDash$' since, as the reader can easily verify, our definition of this relation from above precisely matches the definition of local validity from the offset quote.

Finally, note that the local and global versions of the claims made above are connected in fairly obvious ways that are worth stating aloud. In particular, we note that if $X \vDash A$, then $X \vdash A$ and thus, if $X \nvdash A$, then $X \nvDash A$.

With all that said, here's what we have: in A1 and A2 both hold in local form and thus also in global form. A3 holds in global but not local form. The nonentailment in B1 holds in global form, and thus also in local form. Finally, the nonentailment in B2 fails in both local and global form. Before discussing the significance of the successes and failures thus adumbrated, we'll pause to prove these claims.

#### 4.1.1 Proofs of results

**Theorem 6** $A(x/c), \forall x[A, B] \vDash B(x/c)$.

***Proof*** This follows immediately from Lemma 5. □

**Corollary 7** $A(x/c), \forall x[A, B] \vdash B(x/c)$.

**Theorem 8** $\forall x B \vDash \forall x[A, B]$

***Proof*** Suppose that $X, t \vDash \forall x B$ and let $A \in \mathsf{L}_X$. Thus, there is some $\omega \notin X$ such that $X \cup \{\omega\}, t{\uparrow}_X^{X\cup\{\omega\}} \vDash B(x/\omega)$.

Since we must show that $x, t \vDash \forall x[A, B]$, let $u \in T_{X\cup\{\omega\}}$, and let's assume that $t{\uparrow}_X^{X\cup\{\omega\}} \sqsubseteq u$, and $X \cup \{\omega\}, u \vDash A(x/\omega)$. Therefore, $X \cup \{\omega\}, u \vDash B(x/\omega)$, by Lemma 4, part (a). So, $X, t \vDash \forall x[A, B]$. □

**Corollary 9** $\forall x B \vdash \forall x[A, B]$

**Theorem 10** $\forall x(A \to B) \vdash \forall x[A, B]$

***Proof*** Let $X, \ell_X \vDash \forall x(A \to B)$. To see that $\ell_X \vDash \forall x[A, B]$, let $\omega \notin X$, $\ell_{X\cup\{\omega\}} \sqsubseteq t$ and $t \vDash A(x/\omega)$. By part (d) of Lemma 4, $\ell_{X\cup\{\omega\}} \vDash \forall x(A \to B)$. So by Lemma 2, $\ell_{X\cup\{\omega\}} \vDash A(x/\omega) \to B(x/\omega)$. Thus $\ell_{X\cup\{\omega\}} \cdot t = t \vDash B(x/\omega)$. So $\ell_X \vDash \forall x[A, B]$. □

**Theorem 11** $\forall x(A \to B) \nvDash \forall x[A, B]$.

***Proof*** See Appendix. □

**Theorem 12** $\forall x[A, B] \nvdash \forall x[\neg B, \neg A]$

***Proof*** *Proof Sketch.* Here we allow ourselves to merely sketch the proof since the full result is somewhat tedious. First, one can verify by e.g. using the metavaluational machinery in Slaney (1987) that in the propositional fragment, **B**, of the logic **BQ**, given distinct atoms $P$ and $Q$, $\mathbf{B} \cup \{\neg(P \to P)\} \nvdash \neg Q$. Thus, in the canonical model for **B**, there is an extension of the logic that contains $\neg(P \to P)$ that does not contain $Q$. We can then construct from the canonical model for propositional **B** a model for $\mathsf{L}^+$ that verifies the symmetry condition using the same tricks we use in Appendix. Supposing now that $A$ and $B$ are appropriate $\mathsf{L}^+$-atoms that correspond to $P$ and $Q$, we see that this model is a counterexample to $\forall x[\neg(A \to A), \neg B]$, even though $\forall x[B, A \to A]$ is valid by Corollary 9.[2] □

At a referee's suggestion, we think it worth pointing out the following interesting fact about this proof: it works even in logics that admit axiomatic contraposition. That is, the fact that the restricted universal fails to contrapose seems to be a genuine feature *of the restricted universal* rather than a feature that the restricted universal merely inherits from the logic. All that is required, in fact, for the above argument to go through is that the logic be (with apologies for the neologism) 'pseudo-non-explosive'. That is that there be formulas $B$ and theorems of the logic $A$ so that the union of the logic and the negation of $A$ fails to entail the negation of $B$. Given Corollary 9 and the construction in the Appendix, any such will demonstrate the failure of contraposition for the restricted universal.

**Corollary 13** $\forall x[A, B] \nvDash \forall x[\neg B, \neg A]$.

**Theorem 14** $\forall x[A, \forall x[A, B]] \vDash \forall x[A, B]$.

***Proof*** Let $X, t \vDash \forall x[A, \forall x[A, B]]$. In order to see that $t \vDash \forall x[A, B]$, choose $\omega \notin X$ and let $t{\uparrow}_X^{X\cup\{\omega\}} \sqsubseteq u$ and $u \vDash A(x/\omega)$. Then since $t \vDash \forall x[A, \forall x[A, B]]$, $u \vDash \forall x[A, B]$. So $u \vDash A(x/\omega)$ and $u \vDash \forall x[A, B]$. It follows by Lemma 5 that $u \vDash B(x/\omega)$. Thus $t \vDash \forall x[A, B]$. □

**Corollary 15** $\forall x[A, \forall x[A, B]] \vdash \forall x[A, B]$.

**Theorem 16** $\forall x[A, B] \vDash \forall x[A, \forall x[A, B]]$

***Proof*** Let $X, t \vDash \forall x[A, B]$, $\omega \notin X$, $t{\uparrow}_X^{X\cup\{\omega\}} \sqsubseteq u$ and $X \cup \{\omega\}, u \vDash A(x/\omega)$. Note that since $X, t \vDash \forall x[A, B]$, (d) of Lemma 4 gives $X \cup \{\omega\}, t{\uparrow}_X^{X\cup\{\omega\}} \vDash \forall x[A, B]$. So by (a) of Lemma 4, $X \cup \{\omega\}, u \vDash \forall x[A, B]$. Thus $X, t \vDash \forall x[A, \forall x[A, B]]$. □

The reader may have the sense that this proof was a bit too easy. If so, know that (a) you are correct, and (b) we will address the issue below.

**Corollary 17** *If* $\forall x[A, B]$ *is valid, then* $\forall x[A, \forall x[A, B]]$ *is valid.*

### 4.1.2 Discussion

From the bifurcated BBHPR perspective we mentioned above, the account we have provided thus has three 'failures' to address: the local failure of A3 and the double failure of B2. We will deal with these in turn.

[2] Thanks are due to Andrew Tedder for suggestions that significantly simplified this proof.

On the first failure, we should immediately note that, given our interpretation of BBHPR is correct, they will not classify the local failure of A3 as a failure at all. That said, it still seems to us worth saying a few words about it.

The phenomenon in need of explanation is this: there are theories $t$ for which $t \vDash \forall x(A \rightarrow B)$ and yet $t \nvDash \forall x[A, B]$. Taking a high-level perspective, the reason this can happen is because (a) $\forall x(A \rightarrow B)$ is an essentially *intentional* formula—it tells us how theories that contain it will behave when applied to other theories, and tells us relatively little about how the contents of such theories are themselves organized, while (b) $\forall x[A, B]$ is an essentially *extensional* formula—it tells us a great deal about how the contents of a theories that contain it are organized, and tells us relatively little about how such theories will behave when applied to other theories. Both of these claims have to be interpreted cautiously—claims of the form $\forall x[A, B \rightarrow C]$ do, after all have intensional content that they inherit from the intensional content of their consequent formulas and even $\forall x(Px \rightarrow Qx)$ has *some* extensional content, as borne out by Lemma 2. But the general point holds: the two formulas are, *for a given theory*, largely orthogonal to each other.

We now move on to the double failure of B2, about which readers might be a bit more concerned. After all, in the BBHPR paper, the stated reason for requiring the nonentailment in B2 concerned a rather troubling bit of Curry-esque reasoning. For the sake of our discussion, it is useful to include the BBHPR discussion in its entirety:

> The second inference that must fail concerns contraction. Define $\alpha \rightsquigarrow \beta$ as $\forall \nu[\alpha, \beta]$ (again, add vacuous variables if desired). Then we cannot have $\alpha \rightsquigarrow (\alpha \rightsquigarrow \beta) \vdash \alpha \rightsquigarrow \beta$ else we would be able to reason as follows. Let $\gamma$ be of the form $T\langle\gamma\rangle \rightsquigarrow \bot$, where $\bot$ is a constant—often written as $F$ in the relevant-logic literature—such that $\gamma \vdash \alpha$ for all $\alpha$.
>
> $$\cfrac{\cfrac{\cfrac{T\langle\gamma\rangle \leftrightarrow (T\langle\gamma\rangle \rightsquigarrow \bot)}{T\langle\gamma\rangle \rightsquigarrow (T\langle\gamma\rangle \rightsquigarrow \bot)}}{T\langle\gamma\rangle \rightsquigarrow \bot}}{\cfrac{T\langle\gamma\rangle}{\bot}}$$
>
> Hence, Contraction for $\rightsquigarrow$ must fail:
>
> - B2 $\forall \nu[\alpha, \forall \nu[\alpha, \beta]] \nvdash \forall \nu[\alpha, \beta]$

We find the argument as given entirely unconvincing. To begin, suppose we read B2 in its global form. Then the derivation in question is only problematic if read in global form. But now there is no clear reason to accept its initial premise. Indeed, the argument as given seems to give us good reason not to accept the initial premise. That is, what the derivation seems to demonstrate is that we ought not—on pain of triviality!—adopt *as theorems of the logic* all instances of the T-schema. This is compatible, of course, with all of the instances of the T-schema being part of the correct theory of truth. It is just that there is a difference between being a part of the correct theory of truth and being a theorem of the logic, with membership the latter category being, however you slice the cake, not the sort of thing that follows from membership in the former.

On the other hand, if we read B2 in its local form, then the argument is only troubling if it too is read in local form. But then we can simply reject that the second line follows from the first. After all, as already discussed, A3 is not (and oughtn't be) valid in local form.

Altogether, we do not think that BBHPR have given us particularly good justifications for B2. Thus, our double failure to accommodate it is, we think, no failure at all.

### 4.2 Relevance

Now we turn to the more troubling bit: relevance. To see the trouble, let us follow the vocabulary that, in Fuhrmann (1990) is attributed to Humberstone, and call a formula *ubiquitous* just when it is verified at every point in a model. Ubiquitous formulas are problematic from the relevant point of view: given that $A$ is ubiquitous, it follows from the semantic clause for the conditional that for all $B$, $B \to A$ is also ubiquitous. Thus, so also is $C \to (B \to A)$, etc. And since each of these is ubiquitous, each is, in particular, verified at the identity point of each model. So the logic captured by a class of models in which $A$ is ubiquitous will turn out to be *disastrously* irrelevant, as it will contain as theorems all formulas of the form $B \to A$, all formulas of the form $C \to (B \to A)$, etc. This is a well-known kind of problem for relevance logicians. Extremely similar problems arise in other contexts; see Standefer (2022) and Standefer (2023) for worked out details of a few such cases.

Ubiquitousness in hand, the problem is this: given that $\forall x(A \to B)$ is a theorem, $\forall x[A, B]$ is ubiquitous. This is not hard to see: given that $\forall x(A \to B)$ is a theorem, every theory is closed under all its instances. Thus, given any theory $t$, all of its $A(x/n)$-extensions—where $n$ can be any name at all—are, since closed under the logic, $B(x/n)$-extensions. So $t$ validates $\forall x[A, B]$, and thus $\forall x[A, B]$ is ubiquitous.

Before saying why we are happy to live with this problem, it is worth first noting that the BBHPR account has it is own problems with relevance. To explain, first recall that on the BBHPR account 'all $A$s are $B$s' is interpreted as '$\forall x(Ax \mapsto Bx)$' where $A \mapsto B$ is true at a point $p$ just if for all points $q$ at which $A$ is true, $B$ is true at all points that extend both $p$ and the application of $p$ to $q$.

Now observe that if $B$ is true at $x$ and $x \sqsubseteq z$, then $B$ is true at $z$. Thus if $B$ is true at $x$, so is $A \mapsto B$ for any $A$. It follows that $B \to (A \mapsto B)$ is valid for all $A$ and $B$. Call this Observation 1. Next observe that if $A \to B$ is true at $p$, $q$ is a point where $A$ is true, and $r$ extends both $p$ and the application of $p$ to $q$, then (solely by virtue of the fact that $r$ extends the application of $p$ (where $A \to B$ is true) to $q$ (where $A$ is true)) $B$ will be true at $r$. It follows from this observation that $(A \to B) \to (A \mapsto B)$ is will be valid for all $A$ and $B$.

Now recall that (a) relevance is often interpreted as at least requiring variable sharing and (b) in the literature there are four different types of variable sharing that have been identified:

- Ordinary variable sharing,
- Strong variable sharing, (see Anderson and Belnap (1975)

- Ordinary depth relevance, (see Brady (1984) and
- Strong depth relevance (see Logan (2021)).

The first of the four results says that if $A \rightarrow B$ is a theorem of any sublogic of the very strong relevant logic **R**, then there is a variable $p$ that occurs in both $A$ and $B$. In both cases, the 'strong' version of the variable sharing feature in question strengthens the given feature by requiring preservation of sign. The 'depth' versions, on the other hand, require preservation not just of variables, but of variables occurring within the scope of a given number of conditionals. For details, see the above papers; it suffices for our purposes to note that $p$ occurs positively in $p$ and that in $p \rightarrow q$, $p$ occurs negatively and $q$ occurs positively.

Unsurprisingly, there is nothing said in the BBHPR paper about whether $\mapsto$ should increase depth and nothing's said about whether it changes the signs of its subformulas. But while we would naturally expect '$\mapsto$' to be a depth-changing connective, we do not actually need to know whether it *is* in order to make our point. To begin, suppose '$\mapsto$' *doesn't* change depth. That is, suppose that if $C$ occurs in $A$ (or in $B$) with a given depth $n$, the corresponding occurrence of $C$ in $A \mapsto B$ is also a depth $n$ occurrence of $C$. Then $(p \rightarrow p) \rightarrow (p \mapsto p)$ is a conditional formula where no variable occurs at the same depth in both antecedent and consequent. Thus if '$\mapsto$' doesn't change depths, then the logics of systems that contain it will lack either of the depth variable sharing features.

On the other hand, if '$\mapsto$' *does* change depth, then we can instead look at $B \rightarrow (A \mapsto B)$ where we again see that logics of systems that contain it will lack either of the depth variable sharing features. This is troubling on its own in light of the fact that the BBHPR account is meant to give a version of relevant restricted quantification that works in logics that typically *do* exhibit depth relevance, yet whichever way things go, depth relevance and strong depth relevance are out. That said, we should acknowledge that the BBHRP account does manage to capture more variable sharing than we do with our (totally and completely irrelevant) account.

We promised above that we would offer a discussion of why we are okay with our account's irrelevance, and we will turn to that at last. We have two points to make. First: we think there is good reason to wonder whether what we are witnessing is simply a consequence of the fact that we have entered the sort of terrain where relevance peters out. More to the point, it is well known that the semantics of relevance logics can be extended with relevance-destroying connectives—see e.g. Meyer and Routley (1973) for an enlightening discussion of the matter. The above considerations make one wonder whether perhaps capturing restricted quantification, like capturing boolean negation, is something that relevance logicians can only do at the cost of the relevant bits of their souls.

Putting that aside, the actual reason we are happy to accept our account's irrelevance is more simple and can be summed up as follows: relevance was always a (happy) accident. The motivation we gave for our semantics has nothing to do with relevance. It has to do with theory building and universal theory-buidling theories. As it turned out, the universal theory-building theory was relevant and, while we restricted our attention to the sorts of vocabulary where this remained the case, this was a nice thing to have. But it we didn't build the logic by aiming for it and the fact that we now have to leave it aside is, while a bit sad, nonetheless tolerable. And it is preferable, in any case, to either not doing logic or to doing it in an unnatural way, which seem to be the only alternatives.

## Declarations

**Conflict of interest** The authors declare no competing interests.

## Appendix: Proof of the fourth Lemma from Section 4

The idea is to build a first-order model $U$ from a model $M$ for the propositional logic B.[3] To that end, choose a model $M$ based on a frame $F$ containing a point $t$ at which $(p \to q) \wedge p$ is verified but $q$ isn't. That there are such points follows from the fact that $((p \to q) \wedge p) \to q$ isn't valid which can be seen by relying on, e.g. Depth Relevance—see Brady (1984) for details. We then construct the model $U$ by letting each $U_X$ be a copy of $\langle (N \cup X), F\rangle$—that $N_X$ is just the tuple whose first element is the appropriate set of names and whose remaining elements are a copy of the frame on which $M$ is based. We take all the functions $\downarrow$, $\uparrow$, and $[-]^a_b$ to be identity functions. Finally, we choose a bijection $f$ between predicates of the first-order language and atoms of the propositional language. Using this, we define $U^X_t(P)$ for $j$-ary $P$ to be $(N \cup X)^j$ if $f(P) \in M(t)$, and $U^X_t(Q) = \emptyset$ otherwise—put otherwise, if $f(P)$ is true at $t$, then every instance of $P$ is true in $U$, and if $f(P)$ is false at $t$ then so is every instance of $P$. Thus every point in every model is symmetric in every variable. With a bit of elbow grease, one can verify that this is in fact a (rather uninteresting and hackneyed) stratified model. Of more interest is the following fact:

**Lemma 18** *$U$ satisfies the symmetry condition.*

***Proof*** The symmetry condition holds of $U$ just in case whenever a theory $t$ of $U$ is symmetric in $a, b \in (N \cup X)$ and $A \in \mathsf{L}_X$, then $t \vDash A$ only if $t \vDash A'$, where $A'$ is the $a, b$-variant of $A$. Now, any $t \in U_X$ is symmetric in $a, b \in (N \cup X)$. Also, given the definition of $a, b$-variant, we know that $A' = A(a/b)$. So, it is sufficient to show that for any $t \in U_X$, $t \vDash A$ only if $t \vDash A(a/b)$ by induction on the construction of $A \in \mathsf{L}_X$. Let $a, b \in (N \cup X)$ and let $t \in U_X$. In order to do this, we prove a stronger claim that holds true of $U$: $t \vDash A$ if and only if $t \vDash A(a/b)$.

**Base Case:** Let $Pn_1 \ldots n_j \in Atomic_X$ and $n_1 \ldots n_j \notin V$.

*'if' direction.* Suppose $t \vDash Pn_1 \ldots n_j(a/b)$. So, $t \vDash Pn_1(a/b) \ldots n_j(a/b)$. And $\langle n_1(a/b) \ldots n_j(a/b)\rangle \in U^X_t(P) \neq \emptyset$. Hence, $U^X_t(P) = (N \cup X)^j$. Since $Pn_1 \ldots n_j \in Atomic_X$ and $n_1 \ldots n_j \notin V$, we know that $\langle n_1 \ldots n_j\rangle \in (N \cup X)^j$, which is to say that $\langle n_1 \ldots n_j\rangle \in U^X_t(P)$. Thus, we obtain what we wanted to show: $t \vDash Pn_1 \ldots n_j$.

*'only if' direction.* Suppose that $t \vDash Pn_1 \ldots n_j$. Then $\langle n_1 \ldots n_j\rangle \in U^X_t(P)$ which is thus nonempty. And $U^X_t(P) = (N \cup X)^j$. Since $n_i \in (N \cup X)^j$, $n_i(a/b) \in (N \cup X)^j$: if

[3] For more on the model theory of propositional B, the reader is referred to Read (1988) or Routley et al. (1982).

$n_i = a$, then $n_i(a/b) = a(a/b) = b \in (N \cup X)$; otherwise, $n_i(a/b) = n_i \in (N \cup X)$. So, $\langle n_1 \ldots n_j \rangle \in (N \cup X)^j = U_t^X(P)$. Therefore, $t \vDash Pn_1(a/b) \ldots n_j(a/b)$ -i.e., $t \vDash Pn_1 \ldots n_j(a/b)$, which is what we wanted to establish.

**Induction Step:** Let $\{B, C\} \subseteq \mathsf{L}_X$. For our IH, we assume that for all $D \in \{B, C\}$, all $s \in T_X$, and all $c, d \in (N \cup X)$, (i) $s \vDash D$ if and only if $s \vDash D(c/d)$, and (ii) for all $\omega \notin X$, all $Y \subseteq \Omega$ such that $X \subseteq Y$ and $\omega \in Y$, all $x \in V$, and all $u \in T_Y$, $u \vDash D(x/\omega)$ if and only if $u \vDash D(x/\omega)(c/d)$.

**Conditional Case:**

$$
\begin{aligned}
t \vDash (B \to C) &\text{ iff if } u \vDash B, \text{ then } t \circ u \vDash C;\\
&\text{ iff if } u \vDash B(a/b), \text{ then } t \circ u \vDash C(a/b), \text{ by (i)};\\
&\text{ iff } t \vDash (B(a/b) \to C(a/b));\\
&\text{ iff } t \vDash (B \to C)(a/b).
\end{aligned}
$$

In order to prove the final cases, we need an auxiliary claim, which is easily proved by induction: for any $D \in \mathsf{L}_X$, $c, d \in (N \cup X)$, all $x \in V$, and all $\omega \in \Omega - X$, if $c \neq \omega \neq d$, then $D(x/\omega)(c/d) = D(c/d)(x/\omega)$. We invite the reader to verify it on their own. But for the remainder, note that $a \neq \omega \neq b$, for all $\omega \notin X$.

**Unrestricted Universal Case:**

$$
\begin{aligned}
t \vDash \forall x B &\text{ iff for some } \omega \notin X \text{ such that } t{\uparrow}_X^{X\cup\{\omega\}} \vDash B(x/\omega);\\
&\text{ iff for some } \omega \notin X \text{ such that } t{\uparrow}_X^{X\cup\{\omega\}} \vDash B(x/\omega)(a/b), \text{ by (ii)};\\
&\text{ iff for some } \omega \notin X \text{ such that } t{\uparrow}_X^{X\cup\{\omega\}} \vDash B(a/b)(x/\omega);\\
&\text{ iff } t \vDash \forall x B(a/b).
\end{aligned}
$$

**Restricted Universal Case:**
$t \vDash \forall x[B, C]$ iff

$$
\begin{aligned}
&\text{for some } \omega \notin X, t{\uparrow}_X^{X\cup\{\omega\}} \sqsubseteq u \text{ and } u \vDash B(x/\omega) \text{ only if } u \vDash C(x/\omega);\\
&\text{iff for some } \omega \notin X,\ t{\uparrow}_X^{X\cup\{\omega\}} \sqsubseteq u \text{ and } u \vDash B(x/\omega)(a/b) \text{ only if } u \vDash C(x/\omega)(a/b);\\
&\text{iff for some } \omega \notin X,\ t{\uparrow}_X^{X\cup\{\omega\}} \sqsubseteq u \text{ and } \vDash B(a/b)(x/\omega) \text{ only if } u \vDash C(a/b)(x/\omega);\\
&\text{iff } t \vDash \forall x[B(a/b), C(a/b)];\\
&\text{iff } t \vDash \forall x[B, C](a/b).
\end{aligned}
$$

We leave the other cases to the reader. □

Thus, while the example is (we think objectively) hackneyed, it's nonetheless helpful insofar as it shows that our restriction to models that satisfy the symmetry condition doesn't amount to a restriction to the empty set of models.

We can also extend $f$ to $\overline{f}$, which is a function from quantifier-free formulas to propositional formulas defined in the expected way, by leaving connectives alone and replacing all instances of $Pn_1 \ldots n_j$, where $n_1 \ldots n_j \notin V$, with $f(P)$; stipulated recursively:

$$\overline{f}(Pn_1 \ldots n_j) = f(P);$$
$$\overline{f}(\neg A) = \neg \overline{f}(A);$$
$$\overline{f}(A \Box B) = (\overline{f}(A) \Box \overline{f}(B)).$$

**Lemma 19** *If $A \in \mathsf{L}_X$ is a quantifier-free formula, then $U, X, t \vDash A$ iff $M, t \vDash \overline{f}(A)$.*

***Proof*** By induction on the construction of $A \in \mathsf{L}_X$. As usual we restrict our attention to only the interesting cases and leave the rest to the reader.

**Base Case.** Let $Pn_1 \ldots n_j \in Atomic_X$ and $n_1 \ldots n_j \notin V$.

*'if' direction.* Suppose that $M, t \vDash \overline{f}(Pn_1 \ldots n_j)$. Then $M, t \vDash f(P)$ - i.e., $f(P) \in M(t)$. It follows that $U_t^X(P) = (N \cup X)^j$. Since $\langle n_1 \ldots n_j \rangle \in (N \cup X)^j$, $\langle n_1 \ldots n_j \rangle \in U_t^X(P)$. Therefore, $U, X, t \vDash Pn_1 \ldots n_j$, which is our desired result.

*'only if' direction.* Suppose that $U, X, t \vDash Pn_1 \ldots n_j$. This implies that $\langle n_1 \ldots n_j \rangle \in U_t^X(P)$. So, $U_t^X(P) \neq \emptyset$, which means that $f(P) \in M(t)$. Hence, $M, t \vDash f(P)$ and $M, t \vDash \overline{f}(Pn_1 \ldots n_j)$, our goal.

**Induction Step:** Let $\{B, C\} \subseteq \mathsf{L}_X$. For our IH, we assume that for all $D \in \{B, C\}$, if $D$ is quantifier-free, then for all $s \in T_X$ $U, X, s \vDash D$ iff $M, s \vDash \overline{f}(D)$. *Nota bene*: $T_M = T_X$ and $P_M = P_X$, given the construction of $U_X$.

**Conditonal Case.** Suppose $(B \to C)$ is quantifier-free. Thus, B and C are too.

$$\begin{aligned} U, X, t \vDash (B \to C) &\text{ iff if } U, X, u \vDash B, \text{ then } U, X, t \circ u \vDash C; \\ &\text{ iff if } M, u \vDash \overline{f}(B), \text{ then } M, t \circ u \vDash \overline{f}(C); \\ &\text{ iff } M, t \vDash (\overline{f}(B) \to \overline{f}(C)); \\ &\text{ iff } M, t \vDash \overline{f}(B \to C). \end{aligned}$$

□

**Lemma 20** *Suppose for simplicity that $f^{-1}(p) = P$ and $f^{-1}(q) = Q$ are unary. Then for some $t \in T_{\{y\}}$ $U, \{y\}, t \vDash \forall x(Px \to Qx) \wedge Py$ but $U, \{y\}, t \nvDash Qy$.*

***Proof*** We suppose the antecedent. Recall that $U_{\{y\}}$ is constructed from a propositional model $M$ in such a way that we are guaranteed the existence of a theory $t \in T_{\{y\}}$ such that $M, t \vDash (p \to q) \wedge p$ but $M, t \nvDash q$. This fact is necessary for establishing that $t \vDash \forall x(Px \to Qx) \wedge Py$ but $t \nvDash Qy$.

First, we demonstrate that $t \vDash \forall x(Px \to Qx) \wedge Py$. It is sufficient to show that (i) $t \vDash \forall x(Px \to Qx)$ and that (ii) $t \vDash Py$.

For (i): We know that $M, t \vDash (p \to q) \wedge p$. So, $M, t \vDash (p \to q)$. This implies that for any $u \in T_M$, if $M, u \vDash p$, then $M, t \circ u \vDash q$. Now, pick any $\omega \in \Omega - \{y\}$. Assume that $u' \in T_{\{y,\omega\}}$ and $u' \vDash P\omega$ in order to prove $t{\uparrow}_{\{y\}}^{\{y,\omega\}} \circ u' \vDash Q\omega$.

Hence, $\omega \in U_{u'}^{\{y,\omega\}}(P) \neq \emptyset$. So, $U_{u'}^{\{y,\omega\}}(P) = (N \cup \{y, \omega\})$. That implies: $y \in U_{u'}^{\{y,\omega\}}(P)$ and $u' \vDash Py$. By Lemma 4, part (c) then: $u'{\downarrow}_{\{y\}}^{\{y,\omega\}} \vDash Py$. But $\downarrow$ is the identity function. Thus, $u' \vDash Py$. By Lemma 19 then: $M, u' \vDash \overline{f}(Py)$. That is, $M, u' \vDash f(P)$ -

and $M, u' \vDash p$, by the antecedent hypothesis. But since $M, t \vDash (p \to q)$, $M, t \circ u' \vDash q$. Therefore, $M, t \circ u' \vDash f(Q)$ - and $M, t \circ u' \vDash \overline{f}(Qy)$. Again, by Lemma 19, it follows that $t \circ u' \vDash Qy$ and Lemma 4, part (d) gives us: $(t \circ u')\uparrow^{\{y,\omega\}}_{\{y\}} \vDash Qy$. But $\uparrow$ is also the identity function. So, $t \circ u' \vDash Qy$. Consequently, $y \in U^{\{y,\omega\}}_{t\circ u'}(Q) \neq \emptyset$, which means: $U^{\{y,\omega\}}_{t\circ u'}(Q) = (N \cup \{y, \omega\})$. Hence, $\omega \in U^{\{y,\omega\}}_{t\circ u'}(Q)$. From this, we obtain: $t \circ u' \vDash Q\omega$. Thus, $t\uparrow^{\{y,\omega\}}_{\{y\}} \circ u' \vDash Q\omega$, given that $\uparrow$ is the identity function.

It follows that:

$$\begin{aligned} U, \{y, \omega\}, t\uparrow^{\{y,\omega\}}_{\{y\}} &\vDash (P\omega \to Q\omega); \\ &\vDash (Px(x/\omega) \to Qx(x/\omega)); \\ &\vDash ([Px](x/\omega)) \to [Qx](x/\omega)); \\ &\vDash (Px \to Qx)(x/\omega). \end{aligned}$$

Therefore, (i) $U, \{y\}, t \vDash \forall x(Px \to Qx)$.

For (ii): Luckily, the reasoning involved here is much simpler.

Observe:

$$\begin{aligned} M, t &\vDash p, \text{ given our selection of } t; \\ &\vDash f(P), \text{ by the antecedent hypothesis;} \\ &\vDash \overline{f}(Py); \\ U, \{y\}, t &\vDash Py, \text{ by } \textbf{Lemma 19}. \end{aligned}$$

We have obtained (i) and (ii). All that remains to prove is that $U, \{y\}, t \nvDash Qy$, which is similarly easy to do.

Note that:

$$\begin{aligned} M, t &\nvDash q, \text{ given our selection of } t; \\ &\nvDash f(Q), \text{ by the antecedent hypothesis;} \\ &\nvDash \overline{f}(Qy); \\ U, \{y\}, t &\nvDash Qy, \text{ by } \textbf{Lemma 19}. \end{aligned}$$

□

Note now that the proof of Theorem 11 is in hand. By the above Lemma, there is a $t \in T_{\{y\}}$ for which $U, \{y\}, t \vDash \forall x(Px \to Qx)$ and $U, \{y\}, t \vDash Py$, but $U, \{y\}, t \nvDash Qy$. It follows that $t$ is itself a $Py$-containing extension of $t$. But it's not a $Qy$-containing extension of $t$. So not all $Py$-containing extensions of $t$ are $Qy$-containing extensions of $t$. Thus $U, \emptyset, t\downarrow^{\{y\}}_{\emptyset} \nvDash \forall x[Px, Qx]$. On the other hand, since $U, \{y\}, t \vDash \forall x(Px \to Qx)$, it follows by part (c) of Lemma 4 that $U, \emptyset, t\downarrow^{\{y\}}_{\emptyset} \vDash \forall x(Px \to Qx)$. Thus not all theories that verify $\forall x(Px \to Qx)$ verify $\forall x(Px, Qx)$, as promised.

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