The role of epistemological models in Veronese's and Bettazzi's theory of magnitudes

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1 Introduction

The philosophy of mathematics has been accused of paying insufficient attention to mathematical practice: one way to cope with the problem, the one we will follow in this paper on extensive magnitudes, is to combine the 'history of ideas' and the 'philosophy of models' in a logical and epistemological perspective. The history of ideas allows the reconstruction of the theory of extensive magnitudes as a theory of ordered algebraic structures; the philosophy of models allows an investigation into the way epistemology might affect relevant mathematical notions.

The article takes two historical examples as a starting point for the investigation of the role of numerical models in the construction of a system of non-Archimedean magnitudes. A brief exposition of the theories developed by Giuseppe Veronese and by Rodolfo Bettazzi at the end of the 19^{th} century will throw new light on the role played by magnitudes and numbers in the development of the concept of a non-Archimedean order. Different ways of introducing non-Archimedean models will be compared and the influence of epistemological models will be evaluated. Particular attention will be devoted to the comparison between the models that oriented Veronese's and Bettazzi's works and the mathematical theories they developed, but also to the analysis of the way epistemological beliefs affected the concepts of continuity and measurement.

2 Giuseppe Veronese

Giuseppe Veronese is well-known to mathematicians for his studies of projective geometry, but his epistemological contributions to the foundations of geometry have been mostly ignored by contemporary philosophers of science, although they were quite well-known at the beginning of the 20^{th}

century.[1] As I have shown in previous research [9], Veronese's epistemology is neither naive nor inconsistent: it justifies the acceptance of many non-Euclidean geometries, including elliptic, hyperbolic, non-Archimedean geometry and the theory of hyperspaces. Moreover, Veronese's epistemological model, though apparently regressive for its recourse to synthetic tools and its refusal of analytical means, turned out to be fruitful from both a geometrical and an algebraic point of view.[2]

2.1 Veronese's epistemology

Veronese's mathematical theory of continuity and the geometry of hyperspaces contained in his main work – *Fondamenti di geometria* [37] – was strongly influenced by his epistemology and especially by his conceptions of space and intuition, which are exposed in several of his writings, including various articles ([41], [36], [38]) and the prefaces to his geometry textbooks ([39], [40]). His epistemological model is compatible with the development of hyperspaces and non-Archimedean continuity, because it allows the representation of spaces with more than three dimensions and legitimates the intuition of infinitesimals, if not empirically, at least by means of an abstract intuitive capacity that one develops with time, experience and geometrical practice.

Contrary to logic and mathematics, which are formal sciences, and to physics and mechanics, which are experimental sciences, geometry was considered by Veronese as a mixed science [37, p. vii], because its objects are partly abstracted from real objects and partly ideal and because its premises are partly empirical, partly semi-empirical and partly abstract [5]. Empirical premises are evident truths that one grasps by intuition when one observes certain physical objects: for example the property (usually attributed to rectilinear segments) of being determined by a couple of points derives from certain physical features of rigid rectilinear bodies. Semi-empirical premises have an empirical origin but they cannot be verified empirically because they assert something that goes beyond the observable domain: for example the geometrical properties of an unlimited line cannot be verified empirically, because the observable domain is finite, but they derive from

[1]Felix Klein [24] and David Hilbert [22] mentioned Veronese's mathematical results, Paul Natorp [27] and Ernst Cassirer [14] discussed Veronese's non-Archimedean continuum, Bertrand Russell [33] praised Veronese's contribution to the history of the foundations of geometry.

[2]The existence of a system of linear quantities containing infinitely small as well as infinitely great quantities was heavily criticised by Cantor [8], Vivanti [42], and Schoenflies [34], but it was praised by Stolz [35], Bettazzi [4], and Hilbert [22] and proved to be consistent by Levi-Civita ([25], [26]). The fruitfulness of Veronese's approach is clearly visible in the reults of Hans Hahn [21], who built a complete non-Archimedean ordered system of linear quantities [17].

an imaginary extension of the empirical properties of the object. Purely abstract premises concern ideal entities, such as infinitely great and infinitely small quantities, that are not related to any object of the observable domain.

Geometrical premises must satisfy both the requisite of mathematical possibility, i.e. logical soundness, and a specifically geometrical condition, that is, conformity to the intuition of space [37, p. xi-xii]. According to Veronese, a mathematical theory that contradicts the elementary properties of spatial objects that one knows by intuition is not 'geometrical', because geometry is the science of space. For example, Poincaré's theory of a one-sheeted hyperboloid or Hilbert's non-Arguesian geometry are perfectly sound mathematical theories, but they are not geometric theories, for they contain propositions that contradict our spatial intuition. A geometrical theory must have an empirical kernel and be compatible with our intuitions of space: the mathematician can freely determine abstract hypotheses, provided that logical consistency and compatibility with empirical and semi-empirical hypotheses is maintained. The logical study of the independence of axioms is a main tool in order to define abstract hypotheses, for the introduction of new objects is accomplished by a change in the axioms. For example, non-Archimedean geometry arises from the negation of the axiom of Archimedes and the investigation of the properties of continuity that might be independent from it.

Veronese, who was strongly influenced by Moritz Pasch and Felix Klein, aimed at a common foundation of metric and projective geometry and was strongly involved in the project of establishing the theory of extensive magnitudes independently from numbers. His construction of a non-Archimedean geometry cannot be fully understood without considering his epistemological conceptions of space and intuition [9] and his familiarity with different models. Projective geometry led him to the introduction of ideal entities that might extend a system, while preserving its relevant properties. The empirical approach, strictly related to the interest for the origin, the history, and the teaching of geometry, together with the traditional insight that all geometrical properties should be somehow derived from our intuition of space, led him to the the conception of 'the rectilinear continuum' as an ordered system of segments rather than as an ordered system of points. The belief that geometry should be somehow distinguished from pure mathematics and therefore grounded independently from numbers also played a relevant role in Veronese's construction of a new geometrical theory. A further element that strongly influenced Veronese's construction of hyperspaces and infinitesimal quantities derives from Hermann Grassmann's epistemology: the belief that mathematical notions should be genetically connected to specific operations of thought.

Veronese's interest in an abstract and general introduction of the concepts of group, equality, addition and order derives from the belief that primitive mathematical concepts such as unity, plurality, group, order and series reflect relevant characteristics of the way we think. Veronese elaborated a general law that allows the construction of new ideal objects that extend any thinkable domain: "Given a determinate thing A, if we have not established that A is the group of all possible things that we might consider, then we can think of something else that is not included in A (that is outside A) and independent from A". This general law allows human thought to go beyond any given limit, because it makes it possible to assume the existence of a new entity outside the domain that was previously considered as the totality of the existing things.

2.2 Veronese's non-Archimedean continuum

Veronese's exposition of the non-Archimedean continuum is contained in the Introduction to his book *Fondamenti di Geometria*. The continuum is a system of segments endowed with an operation of addition and a relation of order. Unlike Dedekind's definition, Veronese's characterization of continuity is not Archimedean. According to the postulate of Archimedes – so called by Stolz – [35], "given two quantities a and b, if a is bigger than b, there is a number $n \in N$ such that $nb > a$". On the contrary, if a system does not contain any multiple of b being bigger than a, this means that b is infinitely small with respect to a or, vice versa, that a is infinitely great with respect to b. In Veronese's theory the postulate of Archimedes does not hold, for the geometrical continuum contains infinitely small and infinitely great segments, which are introduced by two hypotheses: 1) if a segment AB is taken as a unit, there is a segment AA^1 that is infinite with respect to A, or rather a whole series of segments that are infinite with respect to A, but 2) this series, unlike Cantor's series of transcendent numbers, has no first element. To each segment of the new enlarged system Veronese associates a number, thus obtaining an enlarged numerical system II that preserves the operational properties of real numbers. A generalized version of the Archimedes' postulate still holds if instead of $n \in N$ one considers a number $\eta \in II$: "if $a > b$, there is an η such that $\eta b > a$." Infinitely great and infinitely small numbers are introduced as symbols that can be assigned to infinitely great and infinitely small magnitudes (segments).

Veronese's continuity is a generalization of Dedekind's principle: if one does not assume the Archimedes' postulate, there might not exist a segment being the limit of each partition of the straight line in two parts A and B so that each segment of A is to the left of each segment of B. This holds only if a further condition also holds: "There is a segment x in the part A and a segment y in the part B so that the difference between y and x becomes infinitely small". If a is infinitely small with respect to b, the difference between a and b does not become infinitely small and the continuum contains a gap, but if we restrict ourselves to finite segments, then Dedekind's condition holds and there are no gaps.

2.3 How the epimestological model affects continuity

The difference between the approaches of Dedekind and Veronese is relevant not only from a theoretical but also from an epistemological point of view. Veronese believed that the geometrical continuum should not be defined as a system of points but as a system of segments that should not and could not be reduced to a system of numbers. Refusing the idea of defining the continuity of space by means of the continuity of real numbers, Veronese did not assume the Archimedes principle as a necessary element for the continuity of a geometrical system of magnitudes. If Veronese had assumed the real number system as the privileged model for the description of geometrical magnitudes, this would have hindered the discovery of an alternative description of spatial continuity.

Veronese's results, stemming from a combination of an empirical model for continuity, a thought model for order and equality, and a projective model for the foundation of the theory of extensive magnitudes, affected the meaning of the concepts of order, continuity, group, magnitude and number. Numbers were considered as essentially ordinal (cardinal numbers being, as in Cantor's perspective, the result of an operation of abstraction from ordered sets) and were introduced in two independent ways. Natural numbers were introduced as the result of an act of thought - the counting of the elements of an ordered set. Continuous numbers – real and non-Archimedean numbers – were introduced by association to a given system of geometrical magnitudes. The properties of numbers derive from the properties of magnitudes and not vice versa. According to Veronese, the continuity of numbers should be modelled on (since it is derived from) the continuity of the geometrical rectilinear line. Which numerical system should be associated to a given system of magnitudes depends on the properties of the magnitudes, that is to say, on the properties of the spatial continuum that one is not able to perceive but that one can represent to oneself by means of an abstract intuition.

3 Rodolfo Bettazzi

Rodolfo Bettazzi's mathematical works did not receive much attention from contemporaries and have been largely ignored both by historians of mathematics and by philosophers of science. Apart from some studies on Bettazzi's criticism of the axiom of choice [13] and from recent historical research on Peano's school, there is scarcely any literature on Bettazzi's writings.[3] Bettazzi's main work is a monograph on magnitudes that was awarded a prize by the Accademia Nazionale dei Lincei in 1888. Betti, Beltrami, Cremona, and Battaglini – members of the Prize Commitee – remarked that Bettazzi's *Teoria delle grandezze* was an original study in line with Grassmann's, Hankel's, Stolz's, and Cantor's writings [20].

The volume [2] appeared just before Veronese's *Fondamenti di geometria* [37], but the two authors reached their results independently.[4] A comparison between the two works shows remarkable similarities in the results and in the epistemological background, but also a marked difference in the mathematical approach to the enlargement of the numerical domain and in the general aim [11].

3.1 The epistemological model

According to Bettazzi, the objects of a science are ideal and under-determined with respect to the properties of the objects of the real world, for only certain properties are defined and considered as relevant. Such objects are mere concepts and their properties might have similarities with the properties of real objects (for example in geometry) but might also be introduced independently according to certain specific aims. If the existence of an object is accompanied by the determination of the properties of the object, one has a definition of the object itself: so, if one says that there exists an object with certain characteristics, that is a definition of the object [2, p. 3 ff.]. Before Frege or Peano commented on the topic, Bettazzi distinguished between a direct definition, that is to say a definition that aims at defining what an entity is in itself, from a relational definition that defines entities by their reciprocal relations. Every definition is an existential definition asserting the possibility of the attribution of certain properties to a given concept: some properties are attributed to the introduced entity in itself or to its relations to other previously introduced entities; some properties express relations between entities that belong to the same category one wants to define.

Before introducing a precise definition of the concept of magnitude, Bettazzi makes some remarks on mathematical entities. All scientific entities

[3] The oblivion of Bettazzi's works might be partly due to the fact that he never attained an academic position nor published in international reviews. A recent historical study on non-Archimedean mathematics that dedicates a whole paragraph to Bettazzi is Ehrlich 2006 [18] and analyses the originality of his contribution to the topic. Since it is not focused on Bettazzi, it does not discuss Bettazzi's epistemology nor the details of Bettazzi's numerical systems.

[4] Veronese remarked in a footnote that the work of Bettazzi came to his notice when his own book was already getting into print.

need to be well defined, at least with respect to their relevant properties. Scientific entities are ideal because only some of their properties are taken as relevant. These properties might or might not be similar to the properties of certain objects of the real world, for certain entities derive from the observation of the external world while other entities are introduced according to special purposes. Like Veronese, Bettazzi makes a distinction between entities that are somehow connected to our experience and entities that are independent from it.

Scientific entities are pure concepts whose properties are expressed by contemporaneity of certain concepts with others: Bettazzi's terminology here is similar to that of Grassmann. Non-contradiction means 'possible contemporaneity' of the concepts; Grassmann used the expression 'Vereinstimmung' to express the coherence of different acts of thought. The properties of the entities are called postulates and the existence of the entities is itself a postulate. Bettazzi is a conceptualist, because he considers scientific entities as ideal and believes that their properties might be arbitrarily chosen, provided that no contradiction arises. On the other hand Bettazzi, like Veronese, is very much concerned with experience and seems to believe that most mathematical concepts are derived from the observation of an external reality. Space and time cannot be a priori concepts but are rather relational concepts that have to be introduced by defining what it means that two spaces or two times are equal. Time cannot be defined in itself. Analogously all concepts should be introduced by defining relations of equality or inequality.

Refusing the idea of deriving the properties of magnitudes from the properties of the real numbers that are used to measure them, Bettazzi intends to build a rigourous system of magnitudes without presupposing the notion of number. He aims at deducing the properties of real numbers from the properties of magnitudes. In an article on the concept of number [1, p. 98 ff.] Bettazzi gives some reasons for introducing magnitudes independently from numbers. He recalls the distinction between two ways of introducing real numbers: a synthetic and an analytic way. According to the synthetic way, a number represents the ratio of a magnitude to a magnitude of the same species, the unity. According to this point of view, the number indicates the way a magnitude can be obtained from the unity of its category. Examples of magnitudes are aggregates of equal, separated objects, aggregates of their parts, segments, angles, surfaces, solids, times, weights, and so on. The notion and the properties of numbers (such as commutativity or transitivity) must derive from the correspondent properties of magnitudes and have to be demonstrated as theorems rather than be assumed as definitions.

While in the synthetic approach numbers have a concrete meaning that derives from their being introduced as ratios of magnitudes, according to the analytic point of view, numbers are devoid of any concrete meaning.The properties of numbers depend on the formal properties of certain abstract operations, because numbers are first introduced as the elements of the given operations and can be generalized only if the properties of those operations are preserved and certain impossibilities eliminated. For example, natural numbers are generalized into integers so as to make subtraction possible, integer numbers are generalized into rational numbers so as to make division possible, rational numbers are extended by the introduction of certain real numbers so as to allow the operation of extracting the root of any positive number, and so on. A main difficulty of this approach consists, according to Bettazzi, in the fact that one does not know exactly where one should stop in this procedure of generalization or when one would have enough numbers to measure magnitudes.

Advantages and disadvantages of the synthetic and analytic approach are discussed in an article entitled *Sui sistemi di numerazione per i numeri reali* [3], where Bettazzi argues that the definition of real numbers as an extension of rational numbers is not convincing for two reasons: 1) it introduces a dishomogeneity, for it is not based on the closure of certain operations that should be made possible, but rather on a completely different concept: the limit; 2) it presupposes a property of extensive magnitudes, i.e. their undenumerable continuity. As a result, Bettazzi argues that those who intend to define the real numbers as successive enlargements of the natural numbers can never obtain a unitarian notion of number, but rather only give many different and separate constructions of rational, irrational, and negative numbers, so that including them all into a single concept of real numbers would be quite arbitrary. This criticism sheds doubts on the legitimacy of the arithmetization of analysis.

Similar remarks can be found also in Cesare Burali-Forti and Sebastiano Catania's works, which were, like Bettazzi's, influenced by Grassmann's writings. In his book on numbers and magnitudes [15, p. vi-vii], Catania wanted to "deduce the whole class of absolute real numbers from magnitudes and the partial classes of integers and rational numbers therefrom. It is an inversion of the usual procedure, which first defines different entities in different ways and then identifies them afterwards to preserve the ordinary properties". Burali-Forti wrote similar remarks in his note on magnitudes [6] and in his book *Logica matematica* (especially in the 1919 edition) [7, p. 323-4]. He argued that since defining real numbers from natural numbers is quite complicated and inconvenient, the simplest and clearest way to introduce numbers is to define them as corresponding to magnitudes.

3.2 Bettazzi's theory of magnitudes

In his book *Teoria delle grandezze* [2] Bettazzi defines magnitudes as the entities of a category that can be compared with respect to a relation of equality or inequality. A class of magnitudes is defined as a structure composed by a set and an additive operation that is associative, commutative, monotonic and univocal. In modern parlance, a class of magnitudes is an abelian additive semigroup. The introduction of an order relation allows a distinction between one-dimensional (linear ordered abelian monoids), multi-dimensional (complex), and non-dimensional classes.

Bettazzi considers several properties of classes, such as that of being one-directional, limited, proper, isolated. One-directional classes correspond to what is now called positive or negative cone of a linearly ordered group. A class is limited if it has an inferior limit which is different from the neutral element. It is proper if the difference of two magnitudes belongs to the class whenever the minuend is greater than the subtrahend. A limited proper class can be ordered by a repetitive application of the additive operation to the limiting magnitude (it is right-solvable). A limited proper class is discrete: it contains the neutral element, a least element (the unit) and its multiples. A class is isolated if it contains no magnitudes that are smaller than any magnitude of the class except the neutral element and if it contains no magnitudes that are greater than any magnitude but the absorbing element (i.e. a class is isolated if the neutral element is the only least element and the absorbing element is the only greatest element). Should an isolated class be embedded into another class, any least element will be considered as equal to the neutral element 0 and any greatest element will be considered as equal to the absorbing element Ω:

(*) for any a in G, if $a^* < a$ then $a^* = 0$ and for any b in G if $b^* > b$ then $b^* = \Omega$.

The procedure of isolating a class is very interesting, for it explains how the same class might be considered as containing or not containing infinitesimal magnitudes. For example, Veronese's non-Archimedean system would be Dedekind-continuous if the class containing the unit were considered as isolated. Bettazzi remarks that a new definition of equality is at stake when one considers a class as isolated: two magnitudes of a class H are equal to 0 when they diverge by a magnitude that is smaller than any magnitude of a subclass G and are equal to Ω if they contain a magnitude that is greater than any magnitude of the subclass G. If one does not want to modify the definition of equality, then one must assume the postulate (*) in order to consider a class as isolated. Bettazzi acutely observes that the postulate is implicitly assumed whenever one applies a specific name to the magnitudes of a certain category, because the exclusive name means that other things should not be considered as comparable to the given magnitudes. For example, if one defines segments as sets of consecutive points, one is thereby using an exclusive name that 'isolates' the class of magnitudes that are called 'segments': infinitely small or infinitely great entities are thus considered as not comparable to segments, that is, as equal to 0 or to Ω respectively.

Other features of dimensional classes are related to how an ordered class can be divided into subclasses. Connected classes contain only links or sections. Closed classes do not contain sections but contain the limit of every section of their subclasses. Continuous classes, being connected and closed, contain only links. Archimedean classes contain no gaps and are called classes of the 1^{st} species, while non-Archimedean classes are called classes of the 2^{nd} species.

Having introduced all these properties of classes of magnitudes independently of numbers, Bettazzi turns to the introduction of *numbers*: given a one-dimensional class of magnitudes, Bettazzi associates a number to each magnitude, then introduces a relation of equality and an operation of addition, and shows that these numbers form a class with the same properties of the correspondent class of magnitudes. Since numbers are associated to magnitudes, a relation of equality might be defined between numbers on the basis of the equality of the corresponding magnitudes. Bettazzi analogously defines other properties of numbers. In modern parlance, one could say that he introduces numbers by means of a homomorphism μ between a class of magnitudes G (which is an ordered abelian monoid) and a class of numbers K. The system of numbers associated to a class of magnitudes is thus itself an ordered abelian monoid.

Bettazzi remarks that numbers are mathematically relevant because the same numerical system can be associated to different classes of magnitudes that have something in common – or, as Bettazzi expresses it, belong to the same category. Bettazzi defines two classes as belonging to the same category if they can be shown to have a correspondence that preserves the relation of order, the additive operation, the module magnitude and the infinite magnitude. In modern parlance, two ordered monoids belong to the same category, if they are homomorphic. The homomorphic function f that establishes the correspondence between the two classes of magnitudes is called a *metrical correspondence*; it allows a partition of classes of magnitudes in different categories: discrete, rational, continuous, ... The concept of metrical correspondence is an abstract algebraic notion that does not presuppose the notion of number: examples of metrical correspondence are both the mapping of a discrete (rational, continuous) class into any other discrete (rational, continuous) class and the mapping of a class of magnitudes into a numerical system.

Numerical systems are introduced as the systems that correspond to a class of a given category (and thus to all classes of the same category, for each class is homomorphic to each other) and can thus be used to represent distinct categories. The class of integer numbers is the class of numbers associated to discrete classes: it contains 0 (module magnitude), 1 (the unity), all multiples of the unity and ω (infinite magnitude): $I = 0, 1, (1 + 1), \ldots, (1 + 1 + 1 + +1), \ldots, \omega$. The class F of fractional numbers is the class of numbers associated to rational classes and it contains 0, the number associated to a rational magnitude, its multiples and ω: $F = 0, a, (a + a), \ldots, (a + a + a + +a), \ldots, \omega$. The class of fractional numbers contains the class of integer numbers as a subclass (for $a = 1$). A continuous class of numbers (real numbers) is associated to continuous classes of magnitudes.

Bettazzi finally introduces a representation theorem, which asserts that any continous class can be put into a metrical correspondence with the class of real numbers. Metrical correspondence is clearly distinguished, in Bettazzi's terminology, from measurement. The distinction is quite subtle but denotes a profound algebraic insight: a metrical correspondence is an homomorphism of a class of magnitudes (a specific set with a certain structure) into another class of magnitudes (a specific set with a certain structure), whereas measurement is the mapping of any class of magnitudes of a certain kind (a generic structure) into a class of numbers (a numerical structure) [12]. Measurement can thus be defined only after both metrical correspondence and numerical systems have been introduced: the representation theorem asserts that all continuous classes can be put into a metrical correspondence with the system of real numbers. In the last paragraphs of *Teoria delle grandezze* Bettazzi associates numbers to one-dimensional classes of 2^{nd} species and generalizes the representation theorem to non-Archimedean classes of magnitudes.[5]

3.3 How the epistemological model affects measurement and magnitude

An implication of the epistemological choice to introduce the properties of numbers synthetically is that they can be derived from the properties of magnitudes, which are assumed by definition. Bettazzi considers the synthetic method as more simple, intuitive, and comprehensible, but he acknowledges the risk of limiting the possible extensions of the notion of number, if the last is rooted to certain concrete classes of magnitudes. The risk might be avoided if one includes the study of classes of magnitudes that

[5]For a more detailed analysis of Bettazzi's numerical systems, see [12].

cannot be concretely imagined: this is exactly what Bettazzi does when he considers classes of more dimensions or one-dimensional classes of 2^{nd} species. Bettazzi's synthetic approach is an abstract approach to the study of algebraic ordered structures. Although the epistemological background is similar to that of Veronese, Bettazzi's aim is quite different: a general investigation of magnitudes rather than a geometrical description of the intuitive continuum. Bettazzi is more influenced by the conceptualism of Grassmann than by the empiricism of Pasch.

Bettazzi extends the notion of measurement to non-Archimedean classes but assumes a continuous class of magnitudes to be Archimedean. The notion of measurement does not entail Dedekind's continuity nor monotonicity, but it cannot be defined in classes with n dimensions, because they lack an ordering. The definition of measurement presupposes the definition of a class of magnitudes as an abelian ordered monoid. That is a reason why Bettazzi's abstract approach marks a significant step towards the axiomatization of the theory of magnitudes, which is usually attributed to Otto Hölder [23].

4 The role of epistemological models

4.1 A comparison

Both Bettazzi and Veronese adopt old epistemological models in an original and fruitful way. Bettazzi follows Grassmann's algebraic approach and develops a general theory of magnitudes independently of numbers, associating numerical systems to categories of magnitudes. Veronese follows Grassmann in the effort of developing geometry without numbers and associates a given system of numbers to a particular system of geometrical magnitudes. Both consider systems of numbers as something that has to be associated to previously defined classes of magnitudes.

Both Bettazzi and Veronese are concerned with the notion of ordinal number rather than with the notion of cardinal number. Veronese does not intend to derive cardinal numbers from ordinals: he explicitly introduces natural numbers as concepts deriving from the act of counting. Bettazzi tries to define real numbers without any reference to natural numbers, but he ends by presupposing their existence in several passages of his text, as Peano critically remarked [28].

In the writings of Veronese a new epistemological model begins to emerge: instead of the result of a successive enlargement of the domain of natural numbers, real numbers are considered as entities that can be defined in terms of richer systems of numbers: they are a subclass of the non-Archimedean numbers. This is due not only to the fact that attention is drawn to order but also to the fact that real numbers are considered as the arrival point of the enterprise rather than as its point of departure.

In the writings of Bettazzi the properties of continuous classes of magnitudes are similarly derived from abstract properties of general categories of classes of magnitudes. Nonetheless real numbers play a relevant role in Bettazzi's system, because Bettazzi, unlike Veronese, conceives of non-Archimedean numbers as hypercomplex numbers. Real numbers play a similar role in some works of Grassmann, especially in the second edition of the *Ausdehnungslehre*, where the philosophical approach is abandoned in favour of a widespread analytical notation [10].

Even if, from a strictly foundational perspective, neither Bettazzi nor Veronese develop a theory of magnitudes without numbers, what is radically new in their effort is the conception of numbers as a special case of an algebraic structure and the conception of the properties of real numbers as a special case of more general properties of ordered structures.

4.2 Different ways to enlarge the domain of numbers

The abstract approach promoted by the synthetic models of Bettazzi and Veronese did not only contribute to a better understanding of the notion of magnitude but also induced an inversion of the defining techniques. The construction of the real numbers is obtained by a one-to-one correspondence with a previously given domain of magnitudes. The introduction of abstract categories of magnitudes allows the construction of new numerical systems that do not necessarily result from the analytical need to make certain operations possible, as in the usual procedures for enlarging the numerical domain.

The approach is top-down rather than bottom-up. Instead of enlarging smaller systems, one starts from larger systems and isolates subsystems by the introduction of new conditions. Following this approach real numbers can be identified as the larger Archimedean sub-field of an ordered field. This approach is radically different from the construction of hyperreal numbers by the enlargement of the system of real numbers: instead of assuming real numbers as a starting point and trying to insert new entities in the given domain, one starts from general properties of classes of magnitudes (ordered fields) and then isolates real numbers by means of the Archimedean property.

This approach has the advantage of avoiding ontological questions. Moreover, it is intrinsically devoted to the comparison of a plurality of models rather than to the search for 'the' model of a categorical theory. Studies concerning the definition of real and hyperreal numbers as real closed fields are fruitful results of such an approach, which is interested not only in isomorphism but also in the study of common properties of non-isomorphic models (such as R and R*).

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