Assertions and hypotheses: A logical framework for their opposition relations[1]

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Abstract

Following the speech act theory, we take *hypotheses* and *assertions* as *linguistic acts* with different illocutionary forces. We assume that a hypothesis is justified if there is at least a scintilla of evidence for the truth of its propositional content, while an assertion is justified when there is conclusive evidence that its propositional content is true. Here we extend the logical treatment for assertions given by Dalla Pozza and Garola (1995, *Erkenntnis*, 43, 81–109) by outlining *a pragmatic logic for assertions and hypotheses*. On the basis of this extension we analyse the standard logical opposition relations for assertions and hypotheses. We formulate a pragmatic square of oppositions for assertions and a hexagon of oppositions for hypotheses. Finally, we give a mixed hexagon of oppositions to point out the opposition relations for assertions and hypotheses.

Keywords: Assertion, hypothesis, opposition relations, logic for pragmatics.

1 Introduction

Consider the following platitude: the possibility of transforming a hypothesis into an assertion is an essential feature of scientific reasoning. Making an up-to-date example, the existence of the Higgs' boson was hypothesized in 1964 and finally observed in 2012. So, provided with the dynamics of scientific knowledge, we can say that nuclear particle experiments transformed a given hypothesis into a scientific assertion.[2] However, what is the nature of hypotheses and assertions? Following the speech act theory [1, 11], we claim that hypotheses and assertions are linguistic acts with different illocutionary forces. Specifically, we assume that a hypothesis is a linguistic act which is justified when there is at least a *scintilla of evidence* (also in the form of indirect evidence) that its propositional content is true, while an assertion is a linguistic act which is justified when there is conclusive evidence that its propositional content is true.

The above illocutionary perspective on assertions and hypotheses is the starting point for our logical treatment of such notions, carried out by extending Dalla Pozza and Garola's [8] logical system of formal pragmatics. The aim of this article is to enlighten some logical relations between the acts of assertion and hypothesis. In order to do that we first outline a pragmatic logic for assertions and hypotheses, then we propose to handle different forms of assertive and hypothetical reasoning on the basis of this logical framework by means of some diagrams expressing logical opposition relations.

The article is divided into five sections: in Section 2, we present the logical framework for assertions; in Section 3 we analyse the relations of logical opposition for this logical system; then, in Section 4, we extend our pragmatic analysis by introducing a logical framework for hypothesis and, finally, in the last section, we draw some concluding remarks.

[1]This paper is dedicated to Carlo Dalla Pozza (1942–2014).

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[2]Of course, you should observe that empirical assertions always raise an inductive risk of error.

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2 Logic for pragmatics: assertions

In their logical system named *Logic for Pragmatics* (LP), Dalla Pozza and Garola [8] provided a formal treatment of assertion, by introducing some pragmatic connectives, which are required in order to formulate a pragmatic interpretation of intuitionistic propositional logic as a logic of assertions. LP propositions can be either true or false, while the judgements expressed as assertions can be justified (J) or unjustified (U).

Assertions are intended as 'purely logical entities ... without making reference to the speaker's intention or beliefs' [8, p. 83]. LP is composed of two sets of formulas: *radical* and *sentential*. Every sentential formula contains at least a radical formula as a proper sub-formula.

Radical formulas are semantically interpreted by assigning them a (classical) truth value. *Sentential formulas* (briefly, *assertions*), on the other hand, are pragmatically evaluated by assigning them a justification value (J, U), defined in terms of the intuitive notion of proof. The pragmatic language of LP is described below.

Alphabet.

The vocabulary of LP contains the following sets of signs.

Descriptive signs: the propositional letters p, q, r.

Logical signs for radical formulas: $\wedge, \vee, \neg, \rightarrow, \leftrightarrow$.

Logical signs for sentential formulas: the assertion sign $\vdash$ and the pragmatic connectives $\sim$ (negation), $\cap$ (conjunction), $\cup$ (disjunction), $\supset$ (implication), $\equiv$ (equivalence).

Formation rules (FRs).
Radical formulas (*rfs*) are recursively defined by the following FRs.
FR1 (atomic formulas): every propositional letter is a *rf.*
FR2 (molecular formulas):

(i) let γ be a *rf*, then $\neg\gamma$ is a *rf*;
(ii) let γ_1 and γ_2 be *rfs*, then $\gamma_1 \wedge \gamma_2$, $\gamma_1 \vee \gamma_2$, $\gamma_1 \rightarrow \gamma_2$, $\gamma_1 \leftrightarrow \gamma_2$ are *rfs*.

Sentential formulas (*sfs*) are recursively defined by the following FRs.
FR3 (elementary formulas): Let γ be a *rf*, then $\vdash \gamma$ is a *sf.*
FR4 (complex formulas):

(i) let δ be a *sf*, then $\sim \delta$ is a *sf*;
(ii) let δ_1 and δ_2 be *sfs*, then $\delta_1 \cap \delta_2$, $\delta_1 \cup \delta_2$, $\delta_1 \supset \delta_2$, $\delta_1 \equiv \delta_2$ are *sfs*.

Every radical formula of LP has a truth value. Every sentential formula has a justification value, which is defined in terms of the intuitive notion of proof and depends on the truth value of its radical sub-formulas. The semantics of LP is the same as for classical logic, and it provides only the interpretation of the radical formulas, by assigning them a truth value and interpreting propositional connectives as truth functions in a standard way.

To be precise, the semantic rules are the usual classical Tarskian ones and specify the truth conditions (only for radical formulas) through an assignment function σ, thus regulating the semantic interpretation of LP. Let γ_1, γ_2 be radical formulas and 1 = true and 0 = false; then:

(1) $\sigma(\neg \gamma_1) = 1$ iff $\sigma(\gamma_1) = 0$;
(2) $\sigma(\gamma_1 \wedge \gamma_2) = 1$ iff $\sigma(\gamma_1) = 1$ and $\sigma(\gamma_2) = 1$;

(3) $\sigma(\gamma_1 \vee \gamma_2)=1$ iff $\sigma(\gamma_1)=1$ or $\sigma(\gamma_2)=1$;
(4) $\sigma(\gamma_1 \rightarrow \gamma_2)=1$ iff $\sigma(\gamma_1)=0$ or $\sigma(\gamma_2)=1$.

Whenever only classical metalinguistic procedures of proof are admitted in LP, the pragmatic connectives have a meaning that is explicated by the BHK (Brouwer, Heyting, Kolmogorov) intended interpretation of intuitionistic logical constants. The illocutionary force of an assertion plays an essential role in determining the *pragmatic* component of the meaning of an elementary formula, with the *semantic* component, namely, the meaning of p, interpreted as in a semantic theory.

Justification rules regulate the pragmatic evaluation π, specifying the justification conditions for the sentential formulas in function of the σ-assignments of truth values for their radical sub-formulas. A pragmatic interpretation of LP is an ordered pair $<\{J, U\}, \pi>$, where $\{J, U\}$ is the set of justification values and π is a function of pragmatic evaluation in accordance with the following justification rules.

JR1 – Let γ be a radical formula. $\pi(\vdash\gamma)=J$ iff a proof exists that γ is true, i.e. that σ assigns the value 1 to γ. $\pi(\vdash\gamma)=U$ iff no proof exists that γ is true.
JR2 – Let δ be a sentential formula. Then, $\pi(\sim\delta)=J$ iff a proof exists that δ is unjustified, i.e. that $\pi(\delta)=U$.
JR3 – Let δ_1 and δ_2 be sentential formulas. Then:
1. $\pi(\delta_1 \cap \delta_2)=J$ iff $\pi(\delta_1)=J$ and $\pi(\delta_2)=J$;
2. $\pi(\delta_1 \cup \delta_2)=J$ iff $\pi(\delta_1)=J$ or $\pi(\delta_2)=J$;
3. $\pi(\delta_1 \supset \delta_2)=J$ iff a proof exists that $\pi(\delta_2)=J$ whenever $\pi(\delta_1)=J$.

The soundness criterion (SC) is the following one:

(SC) Let γ be a radical formula, then $\pi(\vdash\gamma)=J$ implies that $\sigma(\gamma)=1$.

SC states that if an assertion is justified, then the content of assertion is true. It is evident from the justification rules that sentential formulas have an intuitionistic-like formal behaviour and can be translated into the modal system S4, where '$\Box\gamma$' means that there is an (intuitive) proof (conclusive evidence) for γ.

The classical fragment of LP, CLP is made up of all the *sfs* that do not contain pragmatic connectives. Axioms for CLP are the following:

A1 $\vdash(\gamma_1 \rightarrow (\gamma_2 \rightarrow \gamma_1))$;
A2 $\vdash((\gamma_1 \rightarrow (\gamma_2 \rightarrow \gamma_3)) \rightarrow ((\gamma_1 \rightarrow \gamma_2) \rightarrow (\gamma_1 \rightarrow \gamma_3)))$;
A3 $\vdash((\neg\gamma_2 \rightarrow \neg\gamma_1) \rightarrow ((\neg\gamma_2 \rightarrow \gamma_1) \rightarrow \gamma_2))$.

Modus ponens rule in CLP is the following:

[MPP] if $\vdash\gamma_1, \vdash(\gamma_1 \rightarrow \gamma_2)$, then $\vdash\gamma_2$.

The intuitionistic fragment of LP, ILP, is made up by all the *sfs* containing only atomic radicals. The axioms of the intuitionistic fragment of ILP, are the following (where $\delta_1, \delta_2, \delta_3$ contain atomic radicals):

A1. $\delta_1 \supset (\delta_2 \supset \delta_1)$;
A2. $(\delta_1 \supset \delta_2) \supset ((\delta_1 \supset (\delta_2 \supset \delta_3)) \supset (\delta_1 \supset \delta_3))$;
A3. $\delta_1 \supset (\delta_2 \supset (\delta_1 \cap \delta_2))$;
A4. $(\delta_1 \cap \delta_2) \supset \delta_1; (\delta_1 \cap \delta_2) \supset \delta_2$;
A5. $\delta_1 \supset (\delta_1 \cup \delta_2); \delta_2 \supset (\delta_1 \cup \delta_2)$;
A6. $(\delta_1 \supset \delta_3) \supset ((\delta_2 \supset \delta_3) \supset ((\delta_1 \cup \delta_2) \supset \delta_3))$;

A7. $(\delta_1 \supset \delta_2) \supset ((\delta_1 \supset (\sim \delta_2)) \supset (\sim \delta_1))$;
A8. $\delta_1 \supset ((\sim \delta_1) \supset \delta_2)$.

Modus ponens rule for ILP is the following:

[MPP'] if $\delta_1, \delta_1 \supset \delta_2$, then δ_2

where, again, δ_1 and δ_2 contain atomic radicals. It is worth noting that the justification rules do not always allow to determine the justification value of a complex sentential formula when all the justification values of its components are known. For instance:

NR1 $\pi(\delta) = J$ implies $\pi(\sim \delta) = U$;
NR2 $\pi(\delta) = U$ does not necessarily imply $\pi(\sim \delta) = J$;
NR3 $\pi(\sim \delta) = J$ implies $\pi(\delta) = U$;
NR4 $\pi(\sim \delta) = U$ does not necessarily imply $\pi(\delta) = J$.

In addition, a formula δ is pragmatically valid or p-valid (respectively, invalid or p-invalid) if, for every π and σ, the formula δ is justified (respectively, δ is unjustified).[3] In any case, no principle analogous to the truth-functionality principle for classical connectives holds for the pragmatic connectives in LP, since pragmatic connectives are partial functions of justification. Moreover, a function $()^*$ mapping the set of *sfs* into an extension of the set of *rfs* obtained by means of the modal operator $\Box$ (*proved*) i.e. a modal translation of pragmatic assertive formulas, is (recursively) induced by the following correspondence:

$(\vdash \gamma)^* = \Box \gamma$;
$(\sim \delta)^* = \Box \neg (\delta)^*$;
$(\delta_1 \cap \delta_2)^* = (\delta_1)^* \wedge (\delta_2)^*$;
$(\delta_1 \cup \delta_2)^* = (\delta_1)^* \vee (\delta_2)^*$;
$(\delta_1 \supset \delta_2)^* = \Box((\delta_1)^* \rightarrow (\delta_2)^*)$.

Radical and sentential formulas are related by means of the following 'bridge principles':[4]

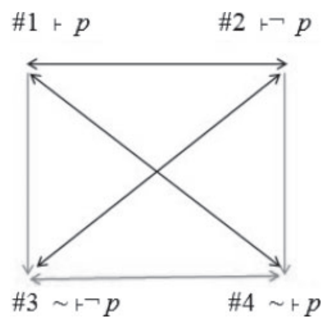
(a) $(\vdash \neg \gamma) \supset (\sim \vdash \gamma)$;
(b) $(\vdash (\gamma_1) \cap \vdash (\gamma_2)) \equiv \vdash (\gamma_1 \wedge \gamma_2)$;
(c) $(\vdash (\gamma_1) \cup \vdash (\gamma_2)) \supset \vdash (\gamma_1 \vee \gamma_2)$;
(d) $(\vdash (\gamma_1 \rightarrow \gamma_2)) \supset (\vdash \gamma_1 \supset \vdash \gamma_2)$.

It is worth observing that **(a)**–**(d)** show the formal relations between classical truth-functional connectives and pragmatic ones. Formula **(a)** states that from the assertion of not-γ the non-assertability of γ can be inferred. **(b)** states that the conjunction of two assertions is equivalent to the assertion of a conjunction; **(c)** states that from the disjunction of two assertions one can infer the assertion of a disjunction. And finally, **(d)** expresses the idea that from the assertion of a classical material implication follows the pragmatic implication between two assertions.

3 A pragmatic square of oppositions for assertions

In LP it is important not to confuse classical negation ($\neg$), pragmatic negation ($\sim$) and justification value (U). The following square of oppositions clarifies their logical relations.

[3] Other pragmatic criteria of validity are presented in [8].
[4] See [8].



Note that #1 and #4 are pragmatic contraries since if #1 is justified then #4 cannot be justified and if #4 is justified then #1 cannot be justified (principle NR1, namely $\pi(\vdash p)=J$ implies $\pi(\sim\vdash p)=U$ and NR3, namely $\pi(\sim\vdash p)=J$ implies $\pi(\vdash p)=U$). Note that if #1 is unjustified then it is not possible to establish the justification value of #4, and vice versa (principle NR2, namely $\pi(\vdash p)=U$ does not necessary imply $\pi(\sim\vdash p)=J$, and NR4, namely $\pi(\sim\vdash p)=U$ does not necessary imply $\pi(\vdash p)=J$). The same holds for #2 and #3.

#1 and #2 are pragmatic contraries since they cannot be both justified but they can be both unjustified (for instance, when p is undecided, namely there is neither a proof of p nor a proof of $\neg p$).

#3 and #4 are pragmatic subcontraries, since they can be both justified (for instance, if p is an undecidable sentence, i.e. when there is a proof that p cannot be proven and there is a proof that $\neg p$ cannot be proven). Indeed no relation exists ‘a priori’ between these justification values because we do not have any pragmatic rule to apply in this case.

#1 and #3 are subalterns since, if the assertion of p is justified, then $\sim\vdash\neg p$ is justified ($\vdash p$ is pragmatically equivalent to $\vdash\neg\neg p$ and, due to the bridge principle **(a)**, it pragmatically implies $\sim\vdash\neg p$).

Finally, #2 and #4 are subalterns since, if the assertion of $\neg p$ is justified, then the assertion that p cannot be proven is also justified, because of the bridge principle **(a)**.

To conclude this section, observe that it is not always the case that every relation of the traditional square of oppositions is completely expressible in the pragmatic square for assertions, e.g. contradictories are not present in the pragmatic square. In the case of assertion, the evidence in play is total, we speak of conclusive evidence or proof. And yet, in the daily epistemic life, such cases are rather rare, as we quite often have to do with a little amount of evidence, i.e. a scintilla of it. Here we do not refer to assertions but to hypotheses. In the next section, we show how to handle hypotheses as primitive pragmatic illocutionary operators.

4 Pragmatic logic for hypotheses

Consider the *hypothesis* as a primitive illocutionary force, indicated by $\mathcal{H}$, which is justified by means of a *scintilla of evidence*. What counts as evidence is contextually specified. The language of pragmatic logic for hypotheses (HLP) is the following:

Alphabet.

The vocabulary of HLP contains the following set of signs.
Descriptive signs: the propositional letters p, q, r.

Logical signs for radical formulas: $\wedge, \vee, \neg, \rightarrow, \leftrightarrow$.
Logical signs for sentential formulas: the sign for hypothesis $\mathcal{H}$ and connectives $\frown$ (negation), $\cap$ (conjunction), $\cup$ (disjunction), $\sqsupset$ (implication), $\equiv$ (equivalence).

Formation rules (FRs).
Radical formulas (*rfs*) are recursively defined by the following FRs.
FR5 (atomic formulas): every propositional letter is a *rf*.
FR6 (molecular formulas):

(i) Let γ be a *rf*, then $\neg\gamma$ is a *rf*.
(ii) Let γ_1 and γ_2 be *rfs*, then $\gamma_1 \wedge \gamma_2$, $\gamma_1 \vee \gamma_2$,$\gamma_1 \rightarrow \gamma_2$, $\gamma_1 \leftrightarrow \gamma_2$ are *rfs*.

Hypothetical formulas (*hpfs*; briefly *hypotheses*) are recursively defined by the following FRs.
FR7 (elementary formulas): Let γ be a *rf*, then $\mathcal{H}\gamma$ is a *hpf*,
FR8 (complex formulas):

(i) let κ be a *hpf*, then $\frown\kappa$ is a *hpf*;
(ii) Let κ_1 and κ_2 be hpf, then $\kappa_1 \cap \kappa_2$, $\kappa_1 \cup \kappa_2$, $\kappa_1 \sqsupset \kappa_2$, $\kappa_1 \equiv \kappa_2$ are *hpfs*.

Every radical formula of HLP has a truth value, which is assigned by classical semantic rules, as in Section 2.

Hypothetical operators for hypothetical formulas formally behave in accordance with the justification rules exposed here below.[5] Observe that ε is a function of evidence from hypothetical formulas to justification values.

4.1 Justification rules

A pragmatic interpretation of HLP is an ordered pair $<\{J, U\}, \varepsilon>$, where $\{J, U\}$ is the set of justification values and ε is a function of pragmatic evaluation for hypothetical formulas such that the following *justification rules* are satisfied:

HJR1 Let γ be a radical formula. $\varepsilon(\mathcal{H}\gamma){=}J$ iff there is a scintilla of evidence that γ is true, while $\varepsilon(\mathcal{H}\gamma){=}U$ iff it does not exist a scintilla of evidence that γ is true.

HJR2 Let κ be a hypothetical formula. Then, $\varepsilon(\frown\kappa){=}J$ iff the scintilla of evidence that $\varepsilon(\kappa){=}J$ is smaller than the scintilla of evidence that $\varepsilon(\frown\kappa){=}U$ (i.e. briefly, iff we are more justified in doubting about κ than in believing it).[6]

HJR3 Let κ_1 and κ_2 be hypothetical formulas.

Then:

(i) $\varepsilon(\kappa_1 \cap \kappa_2){=}J$ iff $\varepsilon(\kappa_1){=}J$ and $\varepsilon(\kappa_2){=}J$;
(ii) $\varepsilon(\kappa_1 \cup \kappa_2){=}J$ iff $\varepsilon(\kappa_1){=}J$ or $\varepsilon(\kappa_2){=}J$;
(iii) $\varepsilon(\kappa_1 \sqsupset \kappa_2){=}J$ iff there is a scintilla of evidence that $\varepsilon(\kappa_2){=}J$ whenever $\varepsilon(\kappa_1){=}J$.[7]

HJR1 expresses in particular a soundness criterion for hypotheses:

let be γ a radical formula, then $\varepsilon(\mathcal{H}\gamma){=}J$ implies that there is a scintilla of evidence that γ is true.

[5] Recent developments of LP are pointed out in [2–4, 6].
[6] In other works on pragmatic logic, hypothetical negation has a slightly different meaning.
[7] We consider HJRs intuitive as criteria of justification.

Let us focus now on some notable principles concerning pragmatic hypothetical negation following from **HJR2**:

(HNR1) $\varepsilon(\kappa)=J$ does not imply that $\varepsilon(\frown\kappa)=U$;[8]
(HNR2) $\varepsilon(\kappa)=U$ implies that $\varepsilon(\frown\kappa)=J$;
(HNR3) $\varepsilon(\frown\kappa)=J$ does not imply that $\varepsilon(\kappa)=U$;
(HNR4) $\varepsilon(\frown\kappa)=U$ implies that $\varepsilon(\kappa)=J$.

Rules **HJR1**–**HJR3** can be supplemented with a fuzzy interpretation of the justification rules of hypotheses which seems to be quite natural in order to easily handle our pre-theoretical insights on them. To be precise, let us introduce a new non-classical semantics on the set of radical formulas such that truth values, indicated by $|\,|$, range in degree between 0 and 1 and satisfy the following rules:[9]

$|\neg\gamma|=1-|\gamma|$;
$|\gamma_1\vee\gamma_2|=\mathrm{Max}(|\gamma_1|,|\gamma_2|)$;
$|\gamma_1\wedge\gamma_2|=\mathrm{Min}(|\gamma_1|,|\gamma_2|)$;
$|\gamma_1\rightarrow\gamma_2|=1 \quad \text{if } |\gamma_1|\leq|\gamma_2|$;
$|\gamma_1\rightarrow\gamma_2|=1-(|\gamma_1|-|\gamma_2|)$ otherwise.

Then we establish the following correspondence among justification values of hypothetical formulas and truth values of this fuzzy logic:

(H1)	$\varepsilon(\mathcal{H}\gamma)=J$	$\|\gamma\|\neq 0$;
(H2)	$\varepsilon(\mathcal{H}\gamma)=U$	$\|\gamma\|=0$;
(H3)	$\varepsilon(\frown\mathcal{H}\gamma)=J$	$1-\|\gamma\|>\|\gamma\|$;
(H4)	$\varepsilon(\frown\mathcal{H}\gamma)=U$	$\|\gamma\|\geq 1-\|\gamma\|$;
(H5)	$\varepsilon(\mathcal{H}\gamma_1\sqsupset\mathcal{H}\gamma_2)=J$	$\|\gamma_1\|\leq\|\gamma_2\|$;
(H6)	$\varepsilon(\mathcal{H}\gamma_1\sqsupset\mathcal{H}\gamma_2)=U$	$\|\gamma_1\|>\|\gamma_2\|$;
(H7)	$\varepsilon(\mathcal{H}\gamma_1\cap\mathcal{H}\gamma_2)=J$	$\|\gamma_1\|\neq 0$ and $\|\gamma_2\|\neq 0$;
(H8)	$\varepsilon(\mathcal{H}\gamma_1\cap\mathcal{H}\gamma_2)=U$	$\|\gamma_1\|=0$ or $\|\gamma_2\|=0$;
(H9)	$\varepsilon(\mathcal{H}\gamma_1\cup\mathcal{H}\gamma_2)=J$	$\|\gamma_1\|\neq 0$ or $\|\gamma_2\|\neq 0$;
(H10)	$\varepsilon(\mathcal{H}\gamma_1\cup\mathcal{H}\gamma_2)=U$	$\|\gamma_1\|=0$ and $\|\gamma_2\|=0$.

(H1)–(H10) are consistent with **HJR1**–**HJR3** and integrate them.

One can also obtain a modal translation in S4 of the hypothetical formulas of HLP by introducing an extension of the set of *rfs* constructed by introducing the modal operator $\Diamond$ (here interpreted as *hypothesized*) and a mapping ( )** from *hpfs* to *rfs* recursively induced by the following correspondence:

$(\mathcal{H}\gamma)^{**} = \Diamond\gamma$;
$(\frown\kappa)^{**} = \Diamond\neg(\kappa)^{**}$;
$(\kappa_1\cap\kappa_2)^{**} = (\kappa_1)^{**}\wedge(\kappa_2)^{**}$;
$(\kappa_1\cup\kappa_2)^{**} = (\kappa_1)^{**}\vee(\kappa_2)^{**}$;
$(\kappa_1\sqsupset\kappa_2)^{**} = ((\kappa_1)^{**}\rightarrow(\kappa_2)^{**})$.

[8] **HJR1** implies that it may occur that $\varepsilon(\mathcal{H}\gamma\cap(\frown\mathcal{H}\gamma))=J$. This result seems counterintuitive, but it describes a rather common situation in the scientific practice.

[9] We follow [10].

We introduce now the definition of p-validity in HLP.

A hypothetical formula κ is *pragmatically valid* (or *p-valid*) iff, for every pragmatic evaluation ε, $\varepsilon(\kappa) = J$.

The following bridge principles can be proven to be p-valid formulas of HLP by using the fuzzy interpretation provided above:

(a°) $(\frown \mathcal{H}\gamma) \sqsupset (\mathcal{H}\neg\gamma)$;
(b°) $\mathcal{H}(\gamma_1 \wedge \gamma_2) \sqsupset (\mathcal{H}(\gamma_1) \cap \mathcal{H}(\gamma_2))$;
(c°) $\mathcal{H}(\gamma_1 \vee \gamma_2) \sqsupset (\mathcal{H}(\gamma_1) \cup \mathcal{H}(\gamma_2))$;
(d°) $(\mathcal{H}\gamma_1 \sqsupset \mathcal{H}\gamma_2) \sqsupset \mathcal{H}(\gamma_1 \to \gamma_2)$.

Principle **(a°)** shows the relation between hypothetical and classical negation. **(b°)** indicates that the hypothesis of a conjunction entails the conjunction of hypotheses. **(c°)** states that a disjunctive hypothesis entails a disjunction of hypotheses. **(d°)** states that from an implication between hypotheses follows the hypothesis of the implication. The proof of p-validity of these principles can be given as follows.

Principle **(a°)**. If $\frown \mathcal{H}\gamma$ is justified, then $|\gamma| < |\neg\gamma|$ because of H3, hence $|\neg\gamma| \neq 0$, which implies that $\mathcal{H}\neg\gamma$ is justified because of H1.

Principle **(b°)**. If $\mathcal{H}(\gamma_1 \wedge \gamma_2)$ is justified, then $|\gamma_1 \wedge \gamma_2| \neq 0$ because of H1, hence $\mathrm{Min}(|\gamma_1|, |\gamma_2|) \neq 0$, which implies $|\gamma_1| \neq 0 \neq |\gamma_2|$. Therefore, $\mathcal{H}\gamma_1 \cap \mathcal{H}\gamma_2$ is justified because of H7.

Principle **(c°)**. If $\mathcal{H}(\gamma_1 \vee \gamma_2)$ is justified, then $|\gamma_1 \vee \gamma_2| \neq 0$ because of H1, hence $\mathrm{Max}(|\gamma_1|, |\gamma_2|) \neq 0$, which implies $|\gamma_1| \neq 0$ or $|\gamma_2| \neq 0$. Therefore, $\mathcal{H}\gamma_1 \cup \mathcal{H}\gamma_2$ is justified because of H9.

Principle **(d°)**. If $(\mathcal{H}\gamma_1 \sqsupset \mathcal{H}\gamma_2) = J$ is justified then $|\gamma_1| \leq |\gamma_2|$ because of H5, hence $|\gamma_1 \to \gamma_2| = 1$, which implies that $\mathcal{H}(\gamma_1 \to \gamma_2)$ is justified because of H1.

As an instance of application of principle **(d°)** consider the following situation: let $|\gamma_1| = 0.1$ and $|\gamma_2| = 0.2$, then $(\mathcal{H}\gamma_1 \sqsupset \mathcal{H}\gamma_2) = J$ while $|(\gamma_1 \to \gamma_2)| = 1$. However, the converse of **(d°)**, that is $\mathcal{H}(\gamma_1 \to \gamma_2) \sqsupset (\mathcal{H}\gamma_1 \sqsupset \mathcal{H}\gamma_2)$, is not p-valid. For instance, if $|\gamma_1| = 0.8$ and $|\gamma_2| = 0.7$, then $|(\gamma_1 \to \gamma_2)| = 1 - (|0.8| - |0.7|) = 0.9$. Therefore, $\mathcal{H}(\gamma_1 \to \gamma_2)$ is justified. Let us consider now $(\mathcal{H}\gamma_1 \sqsupset \mathcal{H}\gamma_2)$: it is unjustified because of H6, since $|\gamma_1|$ is greater than $|\gamma_2|$. Thus, from a hypothetical implication it is possible to derive the fuzzy one, but not vice versa: the latter is necessary for defining an implication-like connective, usually quite problematic for other dual-intuitionistic systems. However, strictly speaking, HLP shares many aspects of dual-intuitionistic logic even if hypothetical implication is not the dual of pragmatic implication. It is known that the dual of Glivenko's theorem holds for dual-intuitionistic systems and classical logic [13]. On the contrary, the bridge principle **(d°)** does not allow to completely replicate the above argument in our pragmatic framework.

This fact, together with our fuzzy account of the notion of hypothetical inference, allows us to use a (pragmatic hypothetical) implication in the object-language, unlike dual-intuitionistic systems in which it is not definable as any implication-like connective ([9, 13]), but only as $A \div B$, to be read as: *A excludes B* [13]. The intuitive idea is that HLP shows some features of duality with LP, but the hypothetical implication is an implication-like connective rather than a connective expressing exclusion.

Let HLP* be the fragment of HLP containing just complex formulas of atomic radicals. If HLP shows some dualities in comparison with LP, HLP* shows the same aspects in comparison with ILP.

Moreover, let us consider the fuzzy interpretation for the rules governing hypothetical negation in HLP*:

(I) $\varepsilon(\mathcal{H}p) = J$ iff $|p| \neq 0$;
(II) $\varepsilon(\mathcal{H}p) = U$ iff $|p| = 0$;
(III) $\varepsilon(\frown\mathcal{H}p) = J$ iff $|\neg p| > |p|$;
(IV) $\varepsilon(\frown\mathcal{H}p) = U$ iff $|p| \geqslant |\neg p|$.

It is possible to show that HNR1 holds in HLP*, since $\varepsilon(\mathcal{H}p) = J$ means that $|p| \neq 0$, but we cannot conclude that $|p| \geqslant |\neg p|$, i.e. that $\varepsilon(\frown\mathcal{H}p) = U$. The remaining rules for hypothetical negation are in accordance with our interpretation. Specifically, HNR2 holds in HLP* since $\varepsilon(\mathcal{H}p) = U$ means that $|p| = 0$ and thus $|\neg p| = 1$, and therefore $|\neg p| > |p|$, which implies $\varepsilon(\frown\mathcal{H}p) = J$ consistently with HNR2.

Moreover, consider the antecedent of HNR3, namely $\varepsilon(\frown\mathcal{H}p) = J$; this means that $|\neg p| > |p|$ but we cannot conclude that $|p| = 0$, i.e. that $\varepsilon(\mathcal{H}p) = U$. Finally, HNR4 holds in HLP*, hence $\varepsilon(\frown\mathcal{H}p) = U$ means that $|p| \geqslant |\neg p|$, hence $|p| \neq 0$, which implies $\varepsilon(\mathcal{H}p) = J$. So, the intended meaning of hypothetical negation can be elucidated in HLP* by our fuzzy interpretation.

Let $\kappa, \kappa_1, \kappa_2$ be hypothetical formulas of HLP*. We propose the following set of axioms and deduction rules for HLP*:

A1. $\kappa \sqsupset \kappa$;
A2. $(\kappa_1 \cap \kappa_2) \sqsupset \kappa_1$;
A3. $(\kappa_1 \cap \kappa_2) \sqsupset \kappa_2$;
A4. $\kappa_1 \sqsupset (\kappa_1 \cup \kappa_2)$;
A5. $\kappa_2 \sqsupset (\kappa_1 \cup \kappa_2)$;
A6. $(\kappa_1 \cap (\kappa_2 \cup \kappa_3)) \sqsupset ((\kappa_1 \cap \kappa_2) \cup (\kappa_1 \cap \kappa_3))$;
A7. $\kappa_2 \sqsupset (\kappa_1 \cup \frown\kappa_1)$;
A8. $((\kappa_1) \sqsupset (\kappa_2 \sqsupset \kappa_3) \sqsupset ((\kappa_1 \sqsupset \kappa_2) \sqsupset (\kappa_1 \sqsupset \kappa_3))$.

Modus Ponens rule in HLP is the following:

[MPP"] if κ_1, $\kappa_1 \sqsupset \kappa_2$, then κ_2.

By using **HJR1**–**HJR3** and H1–H10 we can now sketch a proof of consistency of A1–A10 and MPP" where $\kappa, \kappa_1, \kappa_2$ are elementary *hpfs* of the form $\mathcal{H}p, \mathcal{H}p_1, \mathcal{H}p_2$, respectively. To be precise, we show that the axioms of HLP* are p-valid hypothetical formulas and that the inference rule of modus pones for hypothetical formulas produces p-valid *hpfs*.

A1. $\mathcal{H}p \sqsupset \mathcal{H}p$. It follows from H5.

A2. $(\mathcal{H}p_1 \cap \mathcal{H}p_2) \sqsupset \mathcal{H}p_1$. The *hpf* $(\mathcal{H}p_1 \cap \mathcal{H}p_2)$ is justified by H7 when $|p_1| \neq 0$ and $|p_2| \neq 0$. But if this is the case, then $\mathcal{H}p_1$ is justified by H1. Therefore, by H5, A2 is justified.

A3. $(\mathcal{H}p_1 \cap \mathcal{H}p_2) \sqsupset \mathcal{H}p_2$. *Similar to* A2.

A4. $\mathcal{H}p_1 \sqsupset (\mathcal{H}p_1 \cup \mathcal{H}p_2)$. The *hpt* $\mathcal{H}p_1$ is justified by H1 when $|p_1| = 0$. But, if this is the case, then $(\mathcal{H}p_1 \cup \mathcal{H}p_2)$ is justified by H9. Therefore, by H5, A4 is justified.

A5. $\mathcal{H}p_2 \sqsupset (\mathcal{H}p_1 \cup \mathcal{H}p_2)$. *Similar to* A4.

A6. $(\mathcal{H}p_1 \cap (\mathcal{H}p_2 \cup \mathcal{H}p_3)) \sqsupset ((\mathcal{H}p_1 \cap \mathcal{H}p_2) \cup (\mathcal{H}p_1 \cap \mathcal{H}p_3))$. The *hpf* $(\mathcal{H}p_1 \cap (\mathcal{H}p_2 \cup \mathcal{H}p_3))$ is justified iff $|p_1| \neq 0$ and ($|p_2| \neq 0$ or $|p_3| \neq 0$) by H1, H7 and H9. But, if this is the case, then $(\mathcal{H}p_1 \cap \mathcal{H}p_2) \cup (\mathcal{H}p_1 \cap \mathcal{H}p_3)$ is justified by H7 and H9. Therefore, by H5, A6 is justified.

A7. $\mathcal{H}p_2 \sqsupset (\mathcal{H}p_1 \cup \frown\mathcal{H}p_1)$. The consequent of A7 is justified by **HJR3** (ii) and H3, when $|p_1| \neq 0$ or $|\neg p_1| > |p_1|$. This condition holds in any case, hence A7 is justified by H5 independently of the justification value of $\mathcal{H}p_2$.

A8. $(\mathcal{H}p_1 \sqsupset (\mathcal{H}p_2 \sqsupset \mathcal{H}p_3)) \sqsupset ((\mathcal{H}p_1 \sqsupset \mathcal{H}p_2) \sqsupset (\mathcal{H}p_1 \sqsupset \mathcal{H}p_3))$. The *hpf* $(\mathcal{H}p_1 \sqsupset (\mathcal{H}p_2 \sqsupset \mathcal{H}p_3))$ is justified by **HJR3** (iii) when $\mathcal{H}p_1$ justified implies that $(\mathcal{H}p_1 \sqsupset \mathcal{H}p_3)$ is justified, i.e. by H1 and H5, when the following condition holds:

(i) $|p_1| \neq 0$ implies $|p_2| \leq |p_3|$.

The *hpf* $(\mathcal{H}p_1 \sqsupset \mathcal{H}p_2) \sqsupset (\mathcal{H}p_1 \sqsupset \mathcal{H}p_3)$ is justified by HJR3, (iii), when $\mathcal{H}p_1 \sqsupset \mathcal{H}p_2$ justified implies that $\mathcal{H}p_1 \sqsupset \mathcal{H}p_2$ is justified, i.e. by H5, when the following condition holds:

(ii) $|p_1| \leq |p_2|$ implies $|p_1| \leq |p_3|$.

Condition (i) implies condition (ii). Therefore, by H5, A8 is justified.

MPP" if $\mathcal{H}p_1$, $\mathcal{H}p_1 \sqsupset \mathcal{H}p_2$, then $\mathcal{H}p_2$. $\mathcal{H}p_1$ justified implies $|p_1| \neq 0$ by H1, $\mathcal{H}p_1 \sqsupset \mathcal{H}p_2$ justified implies $|p_1| \leq |p_2|$ by H5. If both these conditions are satisfied, then $|p_2| \neq 0$, hence $\mathcal{H}p_2$ is justified by H1.

The logic of hypothesis HLP resembles a dual-intuitionistic logic with an implication-like connective, while the fragments ILP plus HLP* shows some features of an (illocutionary) bi-intuitionistic logic.[10] A meta-theoretical duality between assertions and hypotheses can be guaranteed once we deal with assertions that have a positive content and hypotheses with a negative content, as proposed by Bellin *et al*. [3]. A variant of this logic can be a logic where the content of hypotheses is positive, while the content of assertions is negative.

A possible general principle connecting assertions and hypotheses is the following:

(GP1) $\pi(\vdash \neg p) = J$ iff $\varepsilon(\mathcal{H}p) = U$

Let us consider separately the two implications of (GP1):

(GP1a) if $\pi(\vdash \neg p) = J$ then $\varepsilon(\mathcal{H}p) = U$;
(GP1b) if $\varepsilon(\mathcal{H}p) = U$ then $\pi(\vdash \neg p) = J$.

GP1a is intuitively plausible for any interpretation of the notion of hypothesis, while **GP1b** is plausible if we assume that $\mathcal{H}p$ may be justified (in a minimal sense) by the existence of a mere cognitive possibility of a situation (no matter how unlikely it might be) where p is true.[11] This epistemic interpretation is validated by the $S4$ modal translation, while stronger interpretations of hypotheses may not allow to infer that $\pi(\vdash \neg p) = J$ from $\varepsilon(\mathcal{H}p) = U$, since the ground associated to $\varepsilon(\mathcal{H}p) = U$ may not be strong enough to justify $\vdash \neg p$. Thus, following the epistemic interpretation for hypotheses, **GP1** holds and states that what grounds the justification of an assertion $\vdash p$ is also necessary and sufficient to consider the hypothesis $\mathcal{H}(\neg p)$ as unjustified. Once again, the fuzzy interpretation can make this principle clear.

Intuitively, consider a justified assertion $\vdash p$. The truth value of p is 1 because we have conclusive evidence (a proof) for p in this case. If, instead, $\vdash p$ is unjustified, then the truth value of p either is 0 (when we have a proof that p does not hold) or it is not determined (when there is contingently no proof of p). The interpretation of **GP1a** is trivial, since it simply states that from $(\vdash \neg p) = J$, hence $\sigma(\neg p)$=1, it is possible to infer $\varepsilon(\mathcal{H}p) = U$, that is $|p| = 0$ in the fuzzy semantics for *rfs* of HLP. Let us consider **GP1b**. When $\varepsilon(\mathcal{H}p) = U$, $|p| = 0$ and therefore $|\neg p| = 1$. It follows that, in the classical semantics for *rfs*, $\sigma(\neg p)$=1, hence $\pi(\vdash \neg p) = J$ which proves **GP1b**.

[10]On the distinction between bi-intuitionistic and dual-intuitionistic logic, see [14].
[11]See on this [2].

The second general principle connecting assertions and hypotheses is the following:

(GP2) The justification of $\vdash p$ implies the justification of $\mathcal{H}p$.

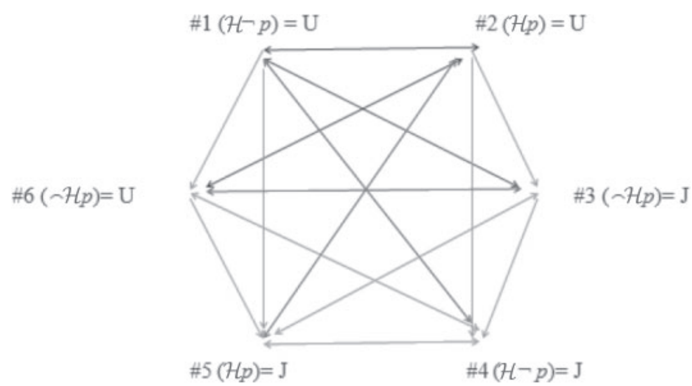
Namely, what we need to justify an assertion that p is sufficient to justify the hypothesis that p.

It is worth noting that the possibility to develop logical systems with some dual-intuitionistic features with richer languages (possibly containing implication-like connectives as hypothetical implications) was one of the open issues indicated by Shramko [12]. Moreover, as for dual-intuitionistic systems, there is no correspondence between all classical contradictions and pragmatic hypothetical contradictions.

For instance, $\mathcal{H}p \cap \frown \mathcal{H}p$ has a fuzzy interpretation according to which the first conjunct is justified when $|p| \neq 0$ and the second conjunct is justified when $|\neg p| > |p|$: the two conjuncts do not contradict each other. Thus, HLP* shows some paraconsistent formal features as other dual-intuitionistic systems (e.g. [12]). As for assertions, the opposition relations among hypotheses can be geometrically represented. In fact, it is possible to construct a complete Sherwood–Czeżowski hexagon of opposition[12] for hypothetical formulas of Czezowski [7].

Notice that unlike the pragmatic square for assertions, the opposition relations in the pragmatic hexagon for hypotheses hold among formulas for which the justification value is explicitly stated.

For sake of brevity, we omit the symbols π and ε in the following whenever no ambiguity occurs.



#1–#6
They are subalterns. Indeed, $(\mathcal{H}\neg p) = U$, by H2, iff $|\neg p| = 0$, that is $|p| = 1$, while $(\frown \mathcal{H}p) = U$, by H4, iff $|p| \geqslant |\neg p|$. Therefore, #1 implies #6, but not vice versa.

#6–#5
They are subalterns. Indeed, $(\frown \mathcal{H}p) = U$, by H4, iff $|p| \geqslant |\neg p|$, while $(\mathcal{H}p) = J$, by H1, iff $|p| \neq 0$. Therefore #6 implies #5 but not vice versa.

#5–#4
They are sub-contraries. Indeed, $(\mathcal{H}p) = J$, by H1, iff $|p| \neq 0$, while $\mathcal{H}(\neg p) = J$, by H1, iff $|\neg p| \neq 0$. Therefore, $\mathcal{H}p$ and $\mathcal{H}(\neg p)$ can be both justified, for instance, when we have some contingent propositions. At the same time, $\mathcal{H}p$ and $\mathcal{H}(\neg p)$ cannot be both unjustified. Indeed, $(\mathcal{H}p) = U$ by H2, iff $|p| = 0$, that is $|\neg p| = 1$, while $(\mathcal{H}\neg p) = U$, by H2, iff $|\neg p| = 0$.

[12] A different hexagon of oppositions is in [5].

#3–#4

They are subalterns. Indeed, $(\frown\mathcal{H}p)=J$, by H3, iff $|\neg p| > |p|$, while $\mathcal{H}(\neg p)=J$, by H1, iff $|\neg p| \neq 0$. Therefore, #3 implies #4 (and this is what (**a°**)) states, but not vice versa.

#2–#3

They are subalterns. Indeed, $(\mathcal{H}p)=U$, by H2, iff $|p|=0$, that is $|\neg p|=1$, while $(\frown\mathcal{H}p)=J$, by H3, iff $|\neg p| > |p|$. Therefore, #2 implies #3 but not vice versa.

#2–#1

They are contraries. Indeed, $(\mathcal{H}\neg p)=U$, by H2, iff $|\neg p|=0$, while $\mathcal{H}p=U$, by H2, iff $|p|=0$. Thus there is a contradiction. But, $(\mathcal{H}\neg p)=J$, by H1, iff $|\neg p|\neq 0$, while $(\mathcal{H}p)=J$, by H1, iff $|p|\neq 0$. Then, no contradiction follows in this case.

#1–#4

They are contradictories. Indeed $(\mathcal{H}\neg p)=U$, by H2, iff $|\neg p|=0$, while $(\mathcal{H}\neg p)=J$, by H1, iff $|\neg p|\neq 0$. Thus, there is a contradiction.

#2–#5

They are contradictories. Indeed, $(\mathcal{H}p)=U$, by H2, iff $|p|=0$, while $(\mathcal{H}p)=J$, by H1, iff $|p|\neq 0$. Thus, there is a contradiction.

#3–#6

They are contradictories. Indeed $(\frown\mathcal{H}p)=U$, by H4, iff $|p|\geqslant|\neg p|$, while $(\frown\mathcal{H}p)=J$, by H3, iff $|\neg p| > |p|$. Thus, there is a contradiction.

#2–#6

They are contraries. Indeed $(\mathcal{H}p)=U$, by H2, iff $|p|=0$, while $(\frown\mathcal{H}p)=U$, by H4, iff $|p|\geqslant|\neg p|$. Thus, there is a contradiction. But, $(\mathcal{H}p)=J$, by H1, iff $|p|\neq 0$, while $(\frown\mathcal{H}p)=J$, by H3, iff $|\neg p| > |p|$. Then, no contradiction follows in this case.

#1–#3

They are contraries. Indeed, $(\mathcal{H}\neg p)=U$, by H2, iff $|\neg p|=0$, while $(\frown\mathcal{H}p)=J$, by H3, iff $|\neg p| > |p|$. Thus, there is a contradiction. But, $(\mathcal{H}\neg p)=J$, by H1, iff $|\neg p|\neq 0$, while $(\frown\mathcal{H}p)=U$, by H4, iff $|p|\geqslant|\neg p|$. Then non contradiction follows in this case.

#6–#4

They are sub-contraries. Indeed, $(\frown\mathcal{H}p)=U$, by H4, iff $|p|\geqslant|\neg p|$, while $(\mathcal{H}\neg p)=J$ by H1, iff $|\neg p|\neq 0$. Thus, there is no contradiction. But $(\frown\mathcal{H}p)=J$, by H3, iff $|\neg p| > |p|$, while $(\mathcal{H}\neg p)=U$, by H2, iff $|\neg p|=0$, that is $|p|=1$. Thus, there is a contradiction.

#3–#5

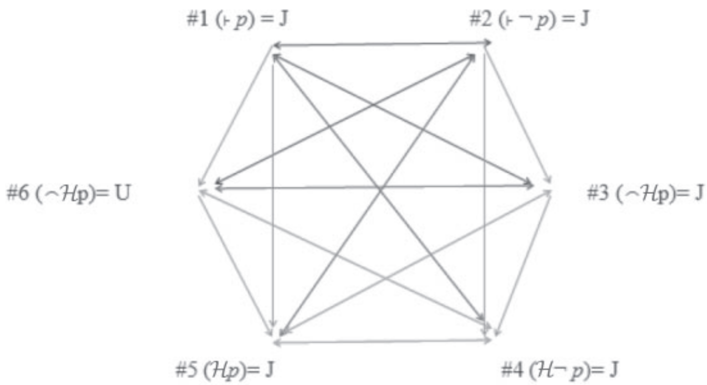
They are sub-contraries. Indeed, $(\frown\mathcal{H}p)=J$, by H3, iff $|\neg p| > |p|$, while $\mathcal{H}p=J$, by H1, iff $|p|\neq 0$. Thus, there is no contradiction. But, $(\frown\mathcal{H}p)=U$, by H4, iff $|p|\geqslant|\neg p|$, while $(\mathcal{H}p)=U$, by H2, iff $|p|=0$. Thus, there is a contradiction.

#1–#5

They are subalterns. Indeed, $(\mathcal{H}\neg p)=U$ by H2, iff $|\neg p|=0$, while $(\mathcal{H}p)=J$, by H1, iff $|p|\neq 0$. Therefore, #1 implies #5, but not vice versa.

#2–#4
They are subalterns. Indeed, $(\mathcal{H}p) = U$, by H2, iff $|p| = 0$, that is $|\neg p| = 1$, while $(\mathcal{H}\neg p) = J$, by H1, iff $|\neg p| \neq 0$. Therefore, #2 implies #4, but not vice versa.

Notice that a hypothetical square of oppositions is contained within the hypothetical hexagon. It is also possible to construct a mixed hexagon with both assertions and hypotheses if we replace the unjustified hypotheses with the corresponding justified assertions, as stated by **GP1**. In this way, it is possible to get the following mixed hexagon:



5 Conclusion

In the present article, we have pointed out the main features of a pragmatic logic of assertions, LP, and we have constructed a pragmatic square of oppositions for assertions. Moreover, we have provided a pragmatic logic for hypotheses with hypothetical pragmatic operators, in order to cope with the linguistic act of hypothesizing. Then, we have provided a new interpretation of the pragmatic logic for hypotheses and a hexagon of oppositions for hypotheses. Finally, we have constructed a mixed hexagon of oppositions in order to point out some formal relations and the possible combinations of assertions and hypotheses.

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