Chapter 9
A Multimodal Pragmatic Analysis of the Knowability Paradox

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Abstract The *Knowability Paradox* starts from the assumption that every truth is knowable and leads to the paradoxical conclusion that every truth is also actually known. Knowability has been traditionally associated with both contemporary verificationism and intuitionistic logic. We assume that classical modal logic in which the standard paradoxical argument is presented is not sufficient to provide a proper treatment of the verificationist aspects of knowability. The aim of this paper is both to sketch a language $\mathcal{L}^{P}_{\Box,K}$, where alethic and epistemic classical modalities are combined with the pragmatic language for assertions $\mathcal{L}^{P}$, and to analyse the result of the application of our framework to the paradox.

Keywords Knowability · Logic for pragmatics · Multimodality

9.1 Introduction

A logical argument, known as the *Knowability Paradox*, starts from the assumption that *every truth is knowable* and leads to the paradoxical conclusion that *every truth is actually known*.

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R. Urbaniak and G. Payette (eds.), *Applications of Formal Philosophy*,
Logic, Argumentation & Reasoning 14, DOI 10.1007/978-3-319-58507-9_9

The idea that every truth is knowable is traditionally associated with a verificationist perspective, a perspective that assumes intuitionistic logic as the "correct" one.

The paradox is formulated in a classical modal logic. However, the fact that classical and verificationist/intuitionistic notions are involved into its formulation makes classical modal logic not adequate to handle properly the specific features of the notions of *truth*, *proof* and *knowledge*.

Our idea is to provide a treatment of the paradox in an intuitionistic-like system—a multimodal extension $\mathcal{L}^{P}_{\Box,K}$ of the pragmatic logic for assertions $\mathcal{L}^{P}$—compatible with classical systems; a system where a verificationist interpretation of classic (modal) propositions together with intuitionistic connections (operations) on them is given.[1]

The paper is divided into 7 sections. Section 9.2 is devoted to briefly outlining the structure of the knowability paradox. In Sects. 9.3, 9.4 and 9.5 we present our pragmatic language $\mathcal{L}^{P}_{\Box,K}$. In Sect. 9.6 we analyse the result of the application of our framework to the paradox.

9.2 The Knowability Paradox

The *Knowability Paradox* (KPx) is based on two principles: the *Knowability Principle* (KP) and the *Principle of Non-Omniscience* (Non-Om). (KP) is usually expressed in the following way:

(KP) $\forall p(p \rightarrow \Diamond Kp)$

while (Non-Om) is formulated as:

(Non-Om) $\exists p(p \wedge \neg Kp)$

The expression "Kp" reads "p is, has been, or will be known by somebody". Specifically, the paradox arises because (KP) $\rightarrow \neg$ (Non-Om) is a theorem of classical modal logic, that is:

(KPx) $[\forall p(p \rightarrow \Diamond Kp)] \rightarrow [\forall p(p \rightarrow Kp)]$.

Assume the following two properties of knowledge:

1. the distributive property over conjunction (Dist), i.e., if a conjunction is known, then its conjuncts are also known, and *vice versa*;
2. the factivity (Fact), i.e., if a proposition is known, then it is true.

Moreover, assume the following two standard modal claims, which can be formulated using the usual modal operators $\Diamond$ ("it is possible that") and $\Box$ ("it is necessary that"). The first is the *Rule of Necessitation*:

[1]There is a certain number of multi-modal approaches to the paradox in the literature proposed from a variety of perspectives. See for example: [1, 2, 8, 10, 13, 19, 20].

(Nec) If p is a theorem, then $\Box p$

The second rule establishes the interdefinability of the modal concepts of necessity and possibility:

(ER) $\Box\neg p$ is logically equivalent to $\neg\Diamond p$

From (KP) and (Non-Om) a contradiction follows. The first argument:

(1)	$p \wedge \neg Kp$	from (Non-Om)
(2)	$(p \wedge \neg Kp) \rightarrow \Diamond K(p \wedge \neg Kp)$	substitution of "$p \wedge \neg Kp$" for p in (KP)
(3)	$\Diamond K(p \wedge \neg Kp)$	from (1) and (2) by *Modus Ponens*

The second, independent, argument:

(4)	$K(p \wedge \neg Kp)$	assumption
(5)	$Kp \wedge K\neg Kp$	distributivity of K
(6)	$Kp \wedge \neg Kp$	factivity of K
(7)	$\bot$	contradiction
(8)	$\neg K(p \wedge \neg Kp)$	*reductio*, discharging (4)
(9)	$\Box\neg K(p \wedge \neg Kp)$	(Nec)
(10)	$\neg\Diamond K(p \wedge \neg Kp)$	(ER)

From (3) and (10) a contradiction follows.

Fitch [14] and Church (we follow here [17]) proved that

(*) $\forall p \neg\Diamond\ K\ (p \wedge \neg Kp)$

is a theorem. But if (*) and (Non-Om) hold, then (KP) has to be rejected since the substitution of $p \wedge\neg Kp$ for p in (KP) leads to a contradiction.

Hence, if (KP) is accepted, then (Non-Om) must be denied. However, the negation of (Non-Om) is classically equivalent to $\forall p(p \rightarrow Kp)$.

Therefore, (KP) is sufficient to obtain the paradoxical conclusion of (KPx), a conclusion that is to be particularly problematic for the holders of antirealism who accept knowability.

However, (KPx) is a *classical* (modal) theorem, and since (KP) has been traditionally associated with both contemporary verificationism and intuitionistic logic, it seems that (KP) is not the correct formalization of knowability. Indeed, according to us, the paradoxical reading of (KPx) can be avoided as soon as we express knowability in an adequate form; to do this we need to take into account the verificationist and intuitionistic features of an antirelist version of knowability.

In order to explain the verificationist and intuitionistic features in a setting *compatible* with classical systems, we introduce a muldimodal pragmatic language for assertions $\mathcal{L}^P_{\Box,K}$. This language is given by an extension of the expressiveness of the pragmatic language for assertion $\mathcal{L}^P$, developed in (Dalla Pozza & Garola, 1995), from propositional contents to modal (propositional) contents, and, in particular, to assertions on (classical) alethic and epistemic contents. Moreover, $\mathcal{L}^P_{\Box,K}$ preserves the main characteristic of $\mathcal{L}^P$; that is, the *integrated* perspective about truth and proof according to the Justification Principle (JP); a principle which captures the intuition that the notion of proof presupposes the classical notion of truth as a regulative concept, since a proof of a proposition amounts to a proof that its truth value is the value "true":

(JP) The assertion of α is justified iff a proof exists that α is true.

In this way, by means of (JP), a verificationist reading of classical propositions is introduced.

9.3 An Outline of $\mathcal{L}^P$

$\mathcal{L}^P$ is a *language for assertions* mainly inspired by Frege and Dummett and by Austin's theory of illocutionary acts. Roughly speaking, the idea is to distinguish propositions from judgments: A proposition is either true or false, while a *judgment*, that can be expressed through the speech act of an *assertion*,[2] is—according to Dalla Pozza and Garola's view—either *justified* or *unjustified*. A justified assertion is defined in terms of the existence of a proof that the asserted content is true. Although the concept of proof is meant to be *intuitive* and *unspecified*, it must always be understood as *correct*: a proof is a proof of a truth. The key ideas behind the language are both the explication of the notion of assertion of a content in terms of justification-values, and the definition of justification-values in terms of the existence of proof for the truth of the asserted content. Therefore, the existence of a proof *is* the ground for a (justified) assertion.

A pragmatic language is the (disjoint) union of two sets of formulas: radicals and sentences. The set of radicals is the descriptive part of the language, while the set of sentences is its pragmatic part.

Radicals represent *propositional contents* of sentences. Sentences express illocutionary acts. A sentence is either *elementary*, i.e., obtained by prefixing a sign of pragmatic mood to a radical, or *complex*, i.e., obtained from other sentences by means of logico-pragmatic connectives ($\sim$, $\cap$, $\cup$, $\supset$, $\equiv$; signs for *pragmatic negation, conjunction, disjunction, implication*, and *equivalence* respectively).

Radicals and sentences have different syntactic forms and are interpreted in different ways. A pragmatic interpretation of a pragmatic language consists of both a semantic interpretation of its descriptive part and a pragmatic interpretation of its pragmatic part. Radicals are interpreted semantically in terms of truth-values, thus every radical is either *true* or *false*. Sentences are interpreted pragmatically in terms of justification-values, so that every sentence is either *justified* or *unjustified*.

[2]Notice that Frege's analysis is extendable to other speech acts such as asking, questioning, etc. So is $\mathcal{L}^P$. Languages where $\mathcal{L}^P$ is expanded so to give rise of other pragmatics acts have been studied. See, for example, [3].

In the pragmatic language for assertions $\mathcal{L}^P$ the descriptive part $\mathcal{L}$ is identified with the language of classical propositional logic (the set of propositional formulas), and the set of sentences is a set of assertions. Elementary sentences are thus built up using only the sign of pragmatic mood of assertion, $\vdash$. So, for example, if α_1 and α_2 are propositional formulas, then $\vdash \alpha_1$ and $\vdash \alpha_2$ are elementary assertions, while $\vdash \alpha_1 \cap \vdash \alpha_2$ or $\vdash \alpha_1 \cup \vdash \alpha_2$ are complex assertions. Intuitionistic language in $\mathcal{L}^P$, is represented as the fragment built up starting from elementary sentences with atomic radicals by means of pragmatic connectives, and, therefore, no classical connective falls under the scope of an assertion [9].

In $\mathcal{L}^P$, there are no assertions whose contents are *modal* propositions. In order to overcome such a limitation, we introduce $\mathcal{L}^P_{\Box,K}$, a pragmatic language for assertions of *modal* propositional contents. In particular, the descriptive part $\mathcal{L}_{\Box,K}$ of $\mathcal{L}^P_{\Box,K}$ is the *fusion* [5] $\mathcal{L}_\Box \oplus \mathcal{L}_K$ of two modal languages, $\mathcal{L}_\Box$ and $\mathcal{L}_K$, endowed with two independent boxes, $\Box$ and K, interpreted as "it is proved that"[3] and "it is known that" respectively.[4] In this way, we have a language that allows us to combine alethic, epistemic and verificationist features within a classically understood framework.

9.4 $\mathcal{L}^P_{\Box,K}$ and Its Interpretations

The set of radical formulas and the set of assertive formulas of $\mathcal{L}^P_{\Box,K}$ are respectively defined recursively through the following formation rules:

$$\alpha := p \mid \top \mid \bot \mid \neg\alpha \mid \alpha_1 \wedge \alpha_2 \mid \alpha_1 \vee \alpha_2 \mid \alpha_1 \rightarrow \alpha_2 \mid \alpha_1 \leftrightarrow \alpha_2 \mid \Box\alpha \mid K\alpha$$
$$\delta := \vdash \alpha \mid \sim \delta \mid \delta_1 \cap \delta_2 \mid \delta_1 \cup \delta_2 \mid \delta_1 \supset \delta_2 \mid \delta_1 \equiv \delta_2$$

In order to get a pragmatic interpretation of $\mathcal{L}^P_{\Box,K}$, we have to semantically interpret $\mathcal{L}^P_{\Box,K}$. This amounts to an interpretation of its descriptive part $\mathcal{L}_{\Box,K}$.

As a matter of fact, the semantics of the fusion $\mathcal{L}_1 \oplus \mathcal{L}_2$ of two modal languages, $\mathcal{L}_1$ and $\mathcal{L}_2$, endowed with two independent boxes, $\Box_1$ and $\Box_2$, is given within the class of frames of the form $\langle W, R_1, R_2\rangle$ where $\langle W, R_1\rangle$ and $\langle W, R_2\rangle$ are frames for $\Box_1$ and $\Box_2$ respectively. The axiomatic presentation through Hilbert calculus is obtained by merging the axioms and the inference rules of both logics. Moreover, *Bridge Principles* (PBs) can be added, i.e. axioms intended to logically connect the independent boxes, e.g., $\Box_1\alpha \rightarrow \Box_2\alpha$.[5]

We are assuming $\mathcal{L}_{\Box,K}$ to be the *fusion* $\mathcal{L}_\Box \oplus \mathcal{L}_K$ of $\mathcal{L}_\Box$ and $\mathcal{L}_K$ endowed with $\Box$ and K intuitively interpreted as "it is proved that" and "it is known that" respectively.

[3]Where 'proof' has to be understood in its intuitive sense.

[4]The fusion $\mathcal{L}_1 \oplus \mathcal{L}_2$ of two modal languages, $\mathcal{L}_1$ and $\mathcal{L}_2$, endowed with two independent boxes, $\Box_1$ and $\Box_2$, is the *smallest* modal language generated by both boxes. Note also that the fusion of modal languages is commutative.

[5]BPs can be equivalent to conditions on the relations between accessibility relations [5].

Hence, it is intuitive to consider relational structures of the form $\langle W, R_\Box, R_K\rangle \in \mathcal{C}$ where $\mathcal{C}$ is the class of frames with W a set of possible worlds, and $R_\Box, R_K \subseteq W$ x W binary accessibility relations on W such that $R_\Box$ is reflexive and transitive, while R_K is reflexive, symmetric and transitive. In this way, $\Box$ is a **S4**-like alethic modality, and K is a **S5**-like epistemic modality.

In addition, we introduce the following *Bridge Principle* (BP):

(BP) $\Box\alpha \rightarrow \neg K\neg\alpha$

which can be intuitively read as "if it is the case that α is proved to be true, then it is not the case that α is known to be false". (BP) gives a logical connection between $\Box$ and K that turns out to be equivalent to the condition that $R_\Box \subseteq R_K$ [6]. Namely, (BP) is valid (on an appropriate frame) if and only if $R_\Box \subseteq R_K$.

The idea behind (BP) can be made clearer if we consider its equivalent formulation in terms of conjunction:

(BP′) $\neg(\Box\alpha \wedge K\neg\alpha)$

(BP′) identifies the relation expressing a minimal condition holding between proof and knowledge according to our pre-theoretical insights. That is, there must be a logical incompatibility between the proof that α is true and the knowledge that α is false.

The assertable modal contents are given by combining (the *fusion* of) two modal languages, $\mathcal{L}_\Box$ and $\mathcal{L}_K$, endowed with two independent boxes, $\Box$ for the alethic modality and K for the epistemic one. These two boxes are intuitively interpreted as "it is proved that" and "it is known that" and formally realized by means of a **S4**-like and a **S5**-like modality respectively.

Given the intuitive interpretation of the $\Box$, it is possible to read classical alethic contents intuitionistically. We interpret it in this way for two reasons. The first one is a technical reason, and it is related to the fact that any pragmatic language for assertions is a intuitionistic-like system. Indeed, since the pragmatic connectives are interpreted intuitionistically, any pragmatic language is essentially an intuitionistic one. What we are given here is a verificationist interpretation, by means of (JP), of classical modal propositions in terms of assertions together with intuitionistic-like connections of them defined via pragmatic connectives. In this way, a set of Pragmatic Bridge Principles (PBPx) explaining the relations between classical connectives, pragmatic arguments and modal operators is obtained.

The second one is related to the paradox. Indeed, if alethic notions have an intuitionistic-like semantics, then "it is possible that", the dual notion of "it is necessary that", can be interpreted as "there is no proof that not". In such a way, the possibility of something to be true is reduced to the (actual) absence of a proof of its falsity, and $\Diamond K\alpha$ becomes "(at this moment in time), there is no proof that $K\alpha$ is false".

The semantic and the pragmatic interpretation of $\mathcal{L}^P_{\Box,K}$ are given through the following definitions.

(Def.1) [Semantic interpretation of $\mathcal{L}^P_{\Box,K}$]

Let $\mathcal{C}$ be the class of frames $F = \langle W,\ R_\Box,\ R_K\rangle$ such that W is a set of possible worlds, $R_\Box,\ R_K \subseteq W \text{x} W$ are binary accessibility relations on W, $R_\Box$ is reflexive and transitive, R_K is reflexive, symmetric and transitive, and $R_\Box \subseteq R_K$.

Let $\mathcal{V}_F$ be the class of valuations

$$v : \begin{cases} PROP \to \wp(W) \\ p \mapsto v(p) \subseteq W \end{cases}$$

on a frame $F \in \mathcal{C}$ where $PROP$ is the set of atomic propositional radicals.

Let $\mathcal{M} = \{M = \langle F, v\rangle \,|\, F \in \mathcal{C}\ \&\ v \in \mathcal{V}_F\}$ be the class of models on a frame F.

Let $M = \langle\langle W,\ R_\Box,\ R_K\rangle, v\rangle \in \mathcal{M}$.

Then, a *semantic interpretation* σ_v of $\mathcal{L}^P_{\Box,K}$ on M is any function

$$\sigma_v : \begin{cases} (\mathcal{L}_\Box \oplus \mathcal{L}_K)\text{x}W \to \{T, F\} \\ (\alpha, w) \mapsto \sigma_v(\alpha, w) \in \{T, F\} \end{cases}$$

which satisfies the following truth-rules:

(TR1) Let $p \in PROP$ and $w \in W$. Then:
(i) $\sigma_v(\top, w) = T$
(ii) $\sigma_v(\bot, w) = F$
(iii) $\sigma_v(p, w) = T \Leftrightarrow p \in v(p)$

(TR2) Let $\alpha, \alpha_1, \alpha_2 \in \mathcal{L}_{\Box,K}$ and $w \in W$. Then:
(i) $\sigma_v(\neg\alpha, w) = T \Leftrightarrow \sigma_v(\alpha, w) = F$
(ii) $\sigma_v(\alpha_1 \wedge \alpha_2, w) = T \Leftrightarrow \sigma_v(\alpha_1, w) = T$ and $\sigma_v(\alpha_2, w) = T$
(iii) $\sigma_v(\alpha_1 \vee \alpha_2, w) = T \Leftrightarrow \sigma_v(\alpha_1, w) = T$ or $\sigma_v(\alpha_2, w) = T$
(iv) $\sigma_v(\alpha_1 \to \alpha_2, w) = T \Leftrightarrow \sigma_v(\alpha_2, w) = T$ whenever $\sigma_v(\alpha_1, w) = T$
(v) $\sigma_v(\alpha_1 \leftrightarrow \alpha_2, w) = T \Leftrightarrow \sigma_v(\alpha_1 \to \alpha_2, w) = T$ and $\sigma_v(\alpha_2 \to \alpha_1, w) = T$

(TR3) Let $\alpha \in \mathcal{L}_{\Box,K}$ and $w \in W$. Then:
(i) $\sigma_v(\Box\alpha, w) = T \Leftrightarrow$ for all v belonging to W, $\sigma_v(\alpha, v) = T$ whenever $wR_\Box v$
(ii) $\sigma_v(K\alpha, w) = T \Leftrightarrow$ for all v belonging to W, $\sigma_v(\alpha, v) = T$ whenever $wR_K v$

(*Def. 2*) [Pragmatic Interpretation of $\mathcal{L}^P_{\Box,K}$]

Let σ_v be a *semantic interpretation* of $\mathcal{L}^P_{\Box,K}$ on a model M. Then a *pragmatic interpretation* π_{σ_v} of $\mathcal{L}^P_{\Box,K}$ on M is any (partial) function

$$\pi_{\sigma_v} : \begin{cases} \mathcal{L}^P_{\Box,K}\text{x}W \to \{J,\ U\} \\ (\delta, w) \mapsto \pi_{\sigma_v}(\delta, w) \in \{J,\ U\} \end{cases}$$

which satisfies the following Justification Rules (JRs) and the Correctness Criterion (CC):

(JR1) Let $\alpha \in \mathcal{L}_{\Box,K}$ and $w \in W$. Then:
$\pi_{\sigma_v}(\vdash \alpha, w) = J \Leftrightarrow$ a proof exists that $\sigma_v(\alpha, w) = T$

Hence, $\pi_{\sigma_v}(\vdash \alpha, w) = U \Leftrightarrow$ no proof exists that $\sigma_v(\alpha, w) = T$

(JR2) Let $\delta, \delta_1, \delta_2 \in \mathcal{L}^P_{\Box,K}$ and $w \in W$. Then:
(i) $\pi_{\sigma_v}(\sim \delta, w) = J \Leftrightarrow$ a proof exists that $\pi_{\sigma_v}(\delta, w) = U$
(ii) $\pi_{\sigma_v}(\delta_1 \cap \delta_2, w) = J \Leftrightarrow \pi_{\sigma_v}(\delta_1, w) = J$ and $\pi_{\sigma_v}(\delta_2, w) = J$
(iii) $\pi_{\sigma_v}(\delta_1 \cup \delta_2, w) = J \Leftrightarrow \pi_{\sigma_v}(\delta_1, w) = J$ or $\pi_{\sigma_v}(\delta_2, w) = J$
(iv) $\pi_{\sigma_v}(\delta_1 \supset \delta_2, w) = J \Leftrightarrow$ a proof exists that $\pi_{\sigma_v}(\delta_2, w) = J$ whenever $\pi_{\sigma_v}(\delta_1, w) = J$
(v) $\pi_{\sigma_v}(\delta_1 \equiv \delta_2, w) = J \Leftrightarrow \pi_{\sigma_v}(\delta_1 \supset \delta_2, w) = J$ and $\pi_{\sigma_v}(\delta_2 \supset \delta_1, w) = J$

(CC) Let $\alpha \in \mathcal{L}_{\Box,K}$ and $w \in W$. Then $\pi_{\sigma_v}(\vdash \alpha, w) = J \Rightarrow \sigma_v(\alpha, w) = T$

9.5 Semantic and Pragmatic Validity for $\mathcal{L}^P_{\Box,K}$

We consider now the semantic and pragmatic notions of validity for $\mathcal{L}^P_{\Box,K}$. Intuitively, a radical is semantically valid (s-valid) for $\mathcal{L}^P_{\Box,K}$ if it is a *classical logical law* for its descriptive part.

It worth noting that by taking into account soundness and completeness results about modal logics [4] and their fusions [12, 15]—given the fact that the axiomatic presentation through Hilbert calculus of the fusion of modal logics is obtained by merging the axioms and the inference rules of the fused logics, and considering our (BP)—it is possible to introduce definitions and to obtain results about s-validity into the following proposition.

(Def. 3) [Semantic validity for $\mathcal{L}^P_{\Box,K}$]
Let $\Lambda \subseteq \mathcal{L}_{\Box,K} := \mathcal{L}_{\Box} \oplus \mathcal{L}_K$ be the normal modal logic given by the following axioms and closed under the following rules:

(TAUT) all the schemas of propositional tautologies belong to Λ;
(AX) the following axiom schemas belong to Λ:
(1) $\Box(\alpha_1 \to \alpha_2) \to (\Box\alpha_1 \to \Box\alpha_2)$ (P-Distributivity)
(2) $\Box\alpha \to \alpha$ (Correctness)
(3) $\Box\alpha \to \Box\Box\alpha$
(4) $K(\alpha_1 \to \alpha_2) \to (K\alpha_1 \to K\alpha_2)$ (K-Distributivity)
(5) $K\alpha \to \alpha$ (Factivity)
(6) $K\alpha \to KK\alpha$ (Positive Introspection)
(7) $\neg K\alpha \to K\neg K\alpha$ (Negative Introspection);
(BP) $\Box\alpha \to \neg K\neg\alpha$ (Proof-Knowledge Compatibility);
(MP) If $\alpha_1 \in \Lambda$ and $\alpha_1 \to \alpha_2 \in \Lambda$, then $\alpha_2 \in \Lambda$ (Modus Ponens);
(N) If $\alpha \in \Lambda$, then $\Box\alpha,\ K\alpha \in \Lambda$ (Generalization);
(UF) Uniform substitution.

Let $\alpha \in \mathcal{L}_{\Box,K}$. Then

(**SV**) $\alpha \in \mathcal{L}_{\Box,K}$ is s-valid for $\mathcal{L}^P_{\Box,K} \Leftrightarrow \alpha \in \Lambda$

Similarly for the s-validity for radicals; since pragmatic sentences are interpreted in terms of justification-values, it makes sense to define a sentence as pragmatically valid (p-valid) if it is justified in any case. Therefore:

(Def. 4) An assertion $\delta \in \mathcal{L}^P_{\Box,K}$ is p-valid iff $\pi_{\sigma_v}(\delta, w) = J$ for every π_{σ_v}.

Moreover, it turns out that the two notions of validity are closely related to each other by the *Justification Lemma*. The lemma is based on a *modal translation* of the pragmatic part of $\mathcal{L}^P_{\Box,K}$ into its descriptive part $\mathcal{L}_{\Box,K}$, *viz.* a syntactic translation of any assertion into a *modal* radical formula. There are two intuitive motivations behind the translation. The first one is that an (elementary) assertion is justified just in case there is a proof, intuitively left unspecified, that what is asserted is true. The second reason is that the way proofs, and so justification-values, are combined into the (JRs) is captured by the BHK interpretation of intuitionistic logic [18]. Therefore, the justification of assertions can be formalized by means of the Gödel-McKinsey-Tarski *modal* translation of intuitionistic logic in terms of a **S4**-like modality [16]. Statements such as "a proof exists that α is true" can be translated with the formula $\Box\alpha$, and any complex assertion can be translated into a modal formula according to the modal translation of its BHK reading. This clarifies also the way the pragmatic connectives are introduced. That is, *via* the intended reading of Gödel-McKinsey-Tarski translation which captures the BHK proof interpretation of intuitionistic logic.

The following definition makes the translation precise.

(Def. 5) [Modal Translation of $\mathcal{L}^P_{\Box,K}$ into $\mathcal{L}_{\Box,K}$]
Let $\mathcal{L}^P_{\Box,K}$ and let $(-)^*$ be the function:

$$(-)^* : \begin{cases} \mathcal{L}^P_{\Box,K} \rightarrow \mathcal{L}_{\Box,K} \\ \delta \mapsto (\delta)^* \end{cases}$$

such that:

(MT1) Let $\alpha \in \mathcal{L}_{\Box,K}$. Then:
$(\vdash \alpha)^* = \Box\alpha$

(MT2) Let $\delta, \delta_1, \delta_2 \in \mathcal{L}^P_{\Box,K}$. Then:
(i) $(\sim \delta)^* = \Box\neg(\delta)^*$
(ii) $(\delta_1 \cap \delta_2)^* = (\delta_1)^* \wedge (\delta_2)^*$
(iii) $(\delta_1 \cup \delta_2)^* = (\delta_1)^* \vee (\delta_2)^*$
(iv) $(\delta_1 \supset \delta_2)^* = \Box((\delta_1)^* \rightarrow (\delta_2)^*)$
(v) $(\delta_1 \equiv \delta_2)^* = \Box((\delta_1)^* \leftrightarrow (\delta_2)^*)$

(Lemma 1) [Justification Lemma for $\mathcal{L}^P_{\Box,K}$]
Let $\delta \in \mathcal{L}^P_{\Box,K}$ and $(\delta)^*$be its modal translation. Then, for every *pragmatic interpretation* π_{σ_v} of $\mathcal{L}^P_{\Box,K}$ we have that:

(JL) $\pi_{\sigma_v}(\delta, w) = J \Leftrightarrow \sigma_v((\delta)^*, w) = T$

On the basis of the justification lemma, we get a criterion for p-validity. The idea is the following: if justification-values can be reduced to truth-values, then p-validity can be reduced to s-validity as well.

(Lemma 2) [Pragmatic validity for $\mathcal{L}^P_{\Box,K}$]
Let $\delta \in \mathcal{L}^P_{\Box,K}$ and $(\delta)^*$be its modal translation. Then:

(**PV**) δ is p-valid for $\mathcal{L}^P_{\Box,K} \Leftrightarrow (\delta)^*$ is s-valid for $\mathcal{L}^P_{\Box,K}$

[Remark 1] Let us show an application of (PV). Consider (BP). It is not difficult to see that from $\Box\alpha \to \neg K\neg\alpha$, it is possible to derive $\Box(\Box\alpha \to \Box\neg K\neg\alpha)$, and that $\Box(\Box\alpha \to \Box\neg K\neg\alpha) = (\vdash \alpha \supset \vdash \neg K\neg\alpha)^*$. Namely, $\Box(\Box\alpha \to \Box\neg K\neg\alpha)$ is the modal translation of $\vdash \alpha \supset \vdash \neg K\neg\alpha$. It follows that $\vdash \alpha \supset\vdash \neg K\neg\alpha$ is p-valid, and it could be read as the pragmatic version of (BP).

[Remark 2] In $\mathcal{L}^P_{\Box,K}$ two versions of *Modus Ponens* can be formulated in the following ways:

(**PMP1**) If $\vdash (\alpha_1 \to \alpha_2)$ and $\vdash \alpha_1$, then $\vdash \alpha_2$
(**PMP2**) If $\delta_1 \supset \delta_2$ and δ_1, then δ_2

[Remark 3] Here it is a list of p-valid formulas and *Pragmatic Bridge Principles* that explain the relations between the semantic logical operators and the pragmatic ones.

(**PBP1**) $(\vdash \neg\alpha) \supset (\sim\vdash \alpha)$
(**PBP2**) $\vdash (\alpha_1 \wedge \alpha_2) \equiv (\vdash \alpha_1 \cap \vdash \alpha_2)$
(**PBP3**) $(\vdash \alpha_1 \cup \vdash \alpha_2) \supset \vdash (\alpha_1 \vee \alpha_2)$
(**PBP4**) $\vdash (\alpha_1 \to \alpha_2) \supset (\vdash \alpha_1 \supset \vdash \alpha_2)$
(**PBP5**) $\vdash (\alpha_1 \leftrightarrow \alpha_2) \supset (\vdash \alpha_1 \equiv \vdash \alpha_2)$
(**PBP6**) $\vdash \Box\alpha \equiv \vdash \alpha$
(**PBP7**) $\vdash K\alpha \supset \vdash \alpha$
(**PBP8**) $\vdash K\alpha \supset \vdash \Box\alpha$
(**PBP9**) $\vdash K\alpha \equiv \vdash KK\alpha$
(**PBP10**) $\vdash \alpha \supset \vdash \neg K\neg\alpha$
(**PBP11**) $\vdash \neg K\alpha \supset \vdash K\neg K\alpha$

Notice that $\vdash \alpha$, $\vdash \Box\alpha$, and $\vdash \Box\Box\alpha$ are p-equivalent assertions, that $\vdash K\alpha$ and $\vdash KK\alpha$ are p-equivalent as well, but neither $\vdash K\alpha$ nor $\vdash KK\alpha$ is p-equivalent to $\vdash \alpha$. Indeed, $\Box(\Box K\alpha \leftrightarrow \Box\alpha)$ is not s-valid since $\Box(\Box K\alpha \to \Box\alpha)$ is s-valid, but $\Box(\Box\alpha \to \Box K\alpha)$ is not. Therefore, by (PV), (PBP7) $\vdash K\alpha \supset \vdash \alpha$ is p-valid, but $\vdash \alpha \supset \vdash K\alpha$ is *not* p-valid.

In other words, because factivity holds, a proof that α is known to be true is transformed into a proof that α is true, but not vice versa.

[Remark 4] Here it is a list of some p-valid formulas that could be of some interest.

(**P1**) $\delta_1 \supset (\delta_2 \supset \delta_1)$
(**P2**) $(\delta_1 \supset (\delta_2 \supset \delta_3)) \supset ((\delta_1 \supset \delta_2) \supset (\delta_1 \supset \delta_3))$

(P3) $(\delta_1 \cap \delta_2) \supset \delta_{1/2}$
(P4) $\delta_1 \supset (\delta_2 \supset (\delta_1 \cap \delta_2))$
(P5) $\delta_{1/2} \supset (\delta_1 \cup \delta_2)$
(P6) $(\delta_1 \supset \delta_3) \supset ((\delta_2 \supset \delta_3) \supset (\delta_1 \cup \delta_2 \supset \delta_3))$
(P7) $(\delta_1 \supset \delta_2) \supset ((\delta_1 \supset\sim \delta_2) \supset\sim \delta_1)$
(P8) $(\delta_1 \equiv \delta_2) \supset ((\delta_1 \supset \delta_2) \cap (\delta_2 \supset \delta_1))$
(P9) $((\delta_1 \supset \delta_2) \supset ((\delta_2 \supset \delta_1) \supset (\delta_1 \equiv \delta_2))$

It is worth noting that it is now possible to formalize the verificationist features of knowledge and proof in our multimodal pragmatic language $\mathcal{L}^P_{\Box,K}$. Furthermore, these verificationist features are compatible with the classical interpretation of intuitionism, as indicated by the modal translation in *(Def. 4)*.

9.6 The Paradox of Knowability in $\mathcal{L}^P_{\Box,K}$

We use $\mathcal{L}^P_{\Box,K}$ to provide a fine-grained analysis of the Knowability Paradox (KPx). Notice that only justified assertions can be expressed in $\mathcal{L}^P_{\Box,K}$ while in $\mathcal{L}_{\Box,K}$ it is also possible to express unjustified assertions by means of their modal translation. Observe also that the validity of a deduction in $\mathcal{L}^P_{\Box,K}$ is guaranteed by making reference to the corresponding multimodal logical steps expressible in $\mathcal{L}_{\Box,K}$ in virtue of the *Justification Lemma*.

The Paradox is based on the Knowability Principle (KP) and the Principle of Non-Omniscience (Non-Om), and it is of particular interest from an antirealist perspective. As we have seen, the view that all truths are knowable is logically incompatible with the reasonable idea that we are non-omniscient. Specifically, the paradox arises because (KP) $\rightarrow \neg$ (Non-Om) is a theorem of a classical modal logic:

(KPx) $[\forall p(p \rightarrow \Diamond Kp)] \rightarrow [\forall p(p \rightarrow Kp)]$

Hence, leaving quantification aside, it turns out that $(p \rightarrow \Diamond Kp) \rightarrow (p \rightarrow Kp)$ is a logical law of the descriptive part of the language $\mathcal{L}_{\Box,K}$, i.e., it is s-valid. And, by Justification Lemma, $\vdash [(p \rightarrow \Diamond Kp) \rightarrow (p \rightarrow Kp)]$ is p-valid.

Consider the following derivation:

$$
\begin{array}{ll}
(1) \vdash [(p \rightarrow \Diamond Kp) \rightarrow (p \rightarrow Kp)] \supset & \\
\quad (\vdash (p \rightarrow \Diamond Kp) \supset \vdash (p \rightarrow Kp)) & \text{(PBP4)} \\
(2) \vdash [(p \rightarrow \Diamond Kp) \rightarrow (p \rightarrow Kp)] & p - \text{valid} \\
(3) \vdash (p \rightarrow \Diamond Kp) \supset \vdash (p \rightarrow Kp) & \text{(PMP2) 1,2} \\
(4) \vdash (p \rightarrow Kp) \supset (\vdash p \supset \vdash Kp) & \text{(PBP4)} \\
(5) \vdash (p \rightarrow \Diamond Kp) \supset (\vdash p \supset \vdash Kp) & \text{transitivity of} \supset 3, 4
\end{array}
$$

Thus, 5 $\vdash (p \rightarrow \Diamond Kp) \supset (\vdash p \supset \vdash Kp)$ is the pragmatic reading of the paradox. It is clear that, in $\mathcal{L}^P_{\Box,K}$, in order to obtain $\vdash Kp$, that is, *a proof that p is actually* *known*, both a justification of the asserted version of (KP), (KPPC) $\vdash (p \rightarrow \Diamond Kp)$, *and* of $\vdash p$ are required. In this way, in (5), the paradoxical conclusion that *every truth is actually known* seems to disappear because the conclusion is $\vdash p \supset \vdash Kp$, i.e., *every asserted truth is an assertion of an actually known truth*.

However, even if $\vdash p \supset \vdash Kp$ would be considered a paradoxical conclusion, there would be a problem with the justification of $\vdash (p \rightarrow \Diamond Kp)$. The only justification an antirelist could give for it would be the one depending on the fact that an antirealist has, somehow, to accept it. And, according to us, this does not count as a proof.

Moreover, given that the antirealist conception of truth is paired with intuitionistic logic, (KPPC) does not seem to be an adequate formulation of the antirealist version of Knowability. Indeed, as shown in [9], intuitionistic language is represented as the fragment built up starting from elementary sentences with only atomic radicals by means of pragmatic connectives. Therefore, no classical connective has to fall under the scope of an assertion. Hence, if a justification of (KPPC) would be available, then it would be a justification of something that does seem compatible with the antirealist form of knowability.

A version of (KP) compatible with antirealism is needed. Consider the following formulation:

(KPI) for all p, if p is intuitionistically true, then $\Diamond Kp$ is intuitionistically true.

It is not entirely clear how a modality should be interpreted from the intuitionistic point of view. However, the intuitionistic double negation ($\sim\sim$) seems to be a form of possibility. Indeed, there is a bridge between the intuitionistic double negation and the verificationist reading the (classical) possibility of a proposition p, $\Diamond p$, obtained by (JP). That is, $\sim\sim\vdash p \supset \vdash \Diamond p$ is p-valid. It must also be noticed that the verificationist reading of the (classical) possibility of a proposition p, $\Diamond p$, introduced by (JP) - i.e., the assertion of the truth of $\Diamond p$ - cannot be reduced to intuitionistic logic because $\vdash \Diamond p \equiv \sim\vdash \neg p$ is p-valid. And, the classical negation that falls under the scope of an assertion cannot be eliminated.

Hence, if Kp is considered as an atomic radical proposition, then we have an intuitionistically valid representation of (KP):

(KPPI) for all p $(\vdash p \supset \sim\sim\vdash Kp)$.[6]

Moreover, since $(\vdash p \supset \sim\sim\vdash Kp)$ implies $(\vdash p \supset \vdash \Diamond Kp)$ and $\vdash \Diamond Kp \equiv\sim\vdash \neg Kp$, also

(KPPI') for all p, $(\vdash p \supset \vdash \Diamond Kp)$

and

(KPPI") for all p, $(\vdash p \supset \sim\vdash \neg Kp)$,

---

[6]On (KPPI) see [11].

would be more adequate versions of an antirealist form of Knowability than (KP).[7] However, neither of them, (KPPI), (KPPI') and (KPPI"), would be sufficient to obtain the paradoxical conclusion.

To summarize. In classical (modal) logic, (KP) is *sufficient* to obtain the paradoxical conclusion of (KPx), i.e., ¬(Non-Om). Within the framework of $\mathcal{L}^P_{\Box,K}$, the paradoxical conclusion *every truth is actually known* seems to disappear because in 5 the conclusion is $\vdash p \supset \vdash Kp$, where the *proofs* of both the truth of p and of Kp are involved, not only the mere being truth of these formulas.

Nonetheless, assuming that $\vdash p \supset \vdash Kp$ is a paradoxical conclusion since, according to us, there exist no poof of (KP), the conclusion could not be given either. And, even if a proof would be available, then it would turn out to be a justification for (KPPC). But (KPPC) is not an adequate representation of an antirelist version of Knowability. Hence, according to 5, the paradoxical conclusion would follow from an assumption that it is not an antirealist version of knowability. A version of (KP) compatible with intuitionistic logic should be (KPPI). But then it would not be sufficient to get the (supposed) paradoxical conclusion either. And the same holds for the other versions of knowability compatible with antirealism, that is (KPPI') and (KPPI").

9.7 Conclusions

In the paper we have outlined a multimodal pragmatic language $\mathcal{L}^P_{\Box,K}$ for the analysis of the Knowability Paradox. The language is based on the identification, by the Justification Principle (JP), of the verificationist notion of truth with the notion of justified assertion - the proof of a classical (modal) truth. This verificationist interpretation of classical (modal) propositions in terms of assertions is integrated with intuitionistic-like connections of them defined via pragmatic connectives. In this way, a set of Pragmatic Bridge Principles (PBPx), explaining the relations between classical connectives, pragmatic ones and (classical) modal operators, is obtained. We have assumed a knowledge-proof compatibilist Bridge Principle (BP). Justifications do not always warrant for knowledge, but, according to (BPB7), justified knowledge is a sufficient condition for the lack of any proof of the contrary. That is to say that justified knowledge is a sufficient condition for the assertion of the falsity being unjustified: there is not any proof of the contrary of what it is known.

Because of its analytic expressiveness, $\mathcal{L}^P_{\Box,K}$ provides a fine-grained analysis of the notions of truth, proof, knowledge, and their relations. Our approach takes into account the fact that the Knowability Principle is mainly associated with a verificationist perspective in epistemology, and that an important connection between verificationism and intuitionistic logic is usually recognized. Indeed, in our pragmatic language it is possible to develop a compatibilist perspective on classical and intuitionistic systems: a notable aspect concerning the communicability among dif-

[7] On (KPPI') and (KPPI") see [7].

ferent logics that comes to be an advantage for the interpretation of the paradox from the antirelistic perspective.

Within $\mathcal{L}^{P}_{\Box,K}$, the Knowability Paradox seems to disappear because the pragmatic version of the paradoxical conclusion that *every truth is actually known* turns out to be *every asserted truth is an assertion of an actually known truth*, i.e., $\vdash p \supset \vdash Kp$, where the proofs of the truth of p and of the truth of Kp are required, not only their being true. Even assuming that $\vdash p \supset \vdash Kp$ is a paradoxical conclusion since, according to us, there exist no proof of (KP), the conclusion could not be given either. And, if a proof was available, then it would be a justification for (KPPC), which is not an adequate representation of the antirelist version of Knowability. Hence, the paradoxical conclusion would follow from the assumption that it is not an antirealist version of knowability. A version of (KP) compatible with intuitionistic logic should be (KPPI). But then it would not be sufficient to get the (supposed) paradoxical conclusion either. And the same holds for the other versions of knowability compatible with antirealism, that is (KPPI′) and (KPPI″).

Thus, we conclude that the pragmatic reading of the paradoxical conclusion is not really paradoxical because the notion of justification is involved. Nonetheless, if it is assumed that it is paradoxical, then one has to face the problem of the correct formulation of knowability. Within our pragmatic multimodal language, we have listed four pragmatic representations of it: (KPPC), (KPPI), (KPPI′) and (KPPI″). On the one hand, despite the fact that (KPPC) would be sufficient to obtain the supposed paradoxical conclusion, it is not an antirealistically adequate representation of knowability, so it cannot be taken into account. On the other hand, even if (KPPI), (KPPI′) and (KPPI″) are antirealistically adequate representations of knowability, none of them is sufficient to obtain the supposed paradoxical conclusion, and, therefore, the paradox does not arise.

Acknowledgments * We would like to thank the referees of the volume for their helpful comments and suggestions. The research of Daniele Chiffi is supported by the Estonian Research Council, PUT1305 2016-2018, PI: Pietarinen. Massimiliano Carrara's research was conducted while he was in his sabbatical year.

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