A Notion of *Logical Concept* based on Plural Reference

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Abstract

In *To be is to be the object of a possible act of choice* (6) the authors defended Boolos' thesis that plural quantification is part of logic. To this purpose, plural quantification was explained in terms of plural reference, and a semantics of plural acts of choice, performed by an ideal team of agents, was introduced. In this paper, following that approach, we develop a theory of *concepts* that – in a sense to be explained – can be labelled as a theory of *logical concepts.* Within this theory we propose a new logicist approach to natural numbers. Then, we compare *our logicism* with Frege's traditional logicism.

1 Introduction

As it is well known, Boolos ((2), (3)) proposed a reinterpretation of second-order monadic logic in terms of *plural quantification* and argued that such interpretation shows – against Quine's criticim – that second order monadic logic is a genuine logic.[1]

Boolos' view, although very attractive, is highly controversial. It has faced the criticisms of several philosophers of mathematics (see, for example, Parsons (14), Resnik (15) and, more recently, Linnebo in (11)). Quine's old claim that second-order logic is set theory in disguise (metaphorically a wolf in sheep clothing) does not seem to have lost all its advocates.

[1] A general introduction to plural quantification is in (12).

We think that, independently of the use of plural quantification in the natural language, the role of plural quantification in logic and mathematics can be better understood within the frame of a highly idealized notion of reference, which can be seen behind the mathematical use of arbitrary reference (For a detailed discussion of this notion see (4) and (7)). Such a framework has been developed in *To be is to be the object of a possible act of choice* (6). In section (2) we will resume its main features. Then, following that approach, we develop a theory of *logical concepts* (sec. 3–7) that leads to a new *logicist* theory of natural numbers. The logicist flavor of the theory is discussed in section 8.

2 A team of agents for a new semantics of second-order logic

In *To be is to be the object of a possible act of choice* (6) the authors argued that the role of plural quantification in logic (and mathematics) can be better understood within the frame of a highly idealized notion of *plural reference.*

This approach to second order logic started from the observation that the possibility, in principle, of referring to any individual of the universe of discourse of a mathematical theory is essentially, although implicitly, presupposed in mathematical reasoning. In order to make such presupposition explicit and to clarify the sense of the locution "in principle," a team of *ideal agents*, which are supposed to have direct access to any individual, has been introduced. Plural reference to certain individuals is realized through an act of *plural choice*, i.e. an act of choosing an individual performed simultaneously by each agent.[2]

Let us suppose that the team of agents is composed by as many agents as the individuals of the universe of discourse. Our agents are supposed to be able to perform the following actions:

1. Singular selecting choice (*s.s.c.*): one of the agents chooses an individual *ad libitum*;

2. Plural selecting choice (*p.s.c.*): it is performed by all agents *simultaneously*: each agent chooses an individual *ad libitum* (independently one

[2]In (6) it has shown that such an approach is suitable for defending the claim that second order logic is free of any ontological commitment to second order entities, so that, in so far as first order logic is *genuine logic*, so is second order logic too.

of the others) or refrains from choosing;

3. Plural relating choice (*p.r.c.*) of degree $n \geq 2$: it is performed by all agents *simultaneously*: each agent chooses n (not necessarily distinct) individuals in a certain order or refrains from choosing. (Abstention from choosing serves the purpose of introducing empty pluralities and relations).

We imagine to be the *ideal leader* of the team, so that we can order at will the execution of one of the foregoing actions. By means of such actions, we can refer to a single individual or to a plurality of individuals or to a *plural relation* among individuals, without submitting to abstract entities the job of collecting and correlating individuals.

A locution as "Let X be an arbitrary plurality of individuals" is to be rephrased as "Suppose that we have ordered a *p.s.c* and call X the chosen individuals." That is, the locution in question is to be understood as an act of reference to the individuals chosen by a certain plural choice. Similarly, a universal quantification "for every plurality X" is to be read as "however a *p.s.c.* of certain individuals Xs is performed ... "; an existential quantification "there is a plurality such that ... " is to be read as "it is possible that such a *p.s.c.* of certain Xs be performed that ...."

Formally, we consider a full second-order language $\mathcal{L}$ with identity, with first-order variables x, y, z and second-order variables X^n, Y^n, Z^n (of any degree $n \geq 1$). We omit the superscripts for variables of degree 1.

Let us recall (following (6)) the *semantics of acts of choice.*

An *assignment* to a formula A is realized by ordering, for every free variable v (of any sort) in A, an appropriate act of choice, i.e., a *s.s.c.* for every first-order variable, a *p.s.c.* for every second-order variable of degree 1, a *p.r.c.* of degree n for every variable of degree $n \geq 2$. With reference to an assignment to a formula A, if v is a free variable of A of any sort, we indicate by v^* the relative act of choice.

We define inductively the truth of a formula relative to an assignment:

(i) $x = y$ is true if x^* and y^* choose the same individual;

(ii) Xy is true if the individual chosen by y^* is one of the individuals chosen by X^*;

(iii) $X^n y_1 ... y_n$ is true if the individuals chosen respectively by $y_1^*, ..., y_n^*$ are chosen in the order by X^{n^*};

(iv) usual clauses for the propositional connectives;

(v) $\forall vB$ is true if, however the assignment may be extended to B by an appropriate act of choice v^* for v, B turns out to be true;

(vi) $\exists vB$ is true if it is performable an act of choice v^* for v such that B turns out to be true.

Now, it is clear that a *p.s.c.* does not create any entity that collects the chosen individuals. Speaking of pluralities as if they were genuine entities is a mere *façon de parler*, paraphrasable in terms of plural choices. Thus, plural reference does not involve the notion of set. On the contrary, singular reference to a set presupposes plural reference to its members. In this sense, Boolos' claim that second-order logic, interpreted in terms of plural quantification, does not involve second-order entities is vindicated.

Acts of choice, unlike agents, are to be understood in a mere *potential* way.[3] In (6) it has been argued that the notion of possibility at issue is perfectly compatible with the use of classical logic and justifies the *comprehension principle of second-order logic* (PCP):

(PCP) (*Plural comphrehension principle*)
$$\exists X^n \forall y_1 ... y_n (X^n y_1 ... y_n \leftrightarrow A) \ (X \text{ not free in } A)$$

In particular, for $n = 1$:

$$\text{(PCP*)} \ \exists X \forall y (Xy \leftrightarrow A).$$

PCP* says the trivial truth that it is possible to perform a *p.s.c.* such that the chosen individuals are just those satisfying the formula A. For, in virtue of the arbitrarity of choices allowed by the choice rule, nothing can prevent the possibility of a *p.s.c.* such that the chosen individuals are just the ones satisfying A (remember that our approach endorses also empty pluralities). Possible occurrences in A of second-order quantifications cannot produce any circularity, since acts of choice, unlike properties, are all independent one of the other. Anagously, for the justification of PCP.

[3]For a detailed analysis and discussion of the modality involved see (6)).

3 A theory of logical concepts

Using our theory of plural reference, we now develop a predicative theory of *logical concepts* (LC). The leading idea is that logical concepts are definible in a logical language and have as range of application the domain of their linguistic expressions. To this purpose we will use a device introduced by Nino Cocchiarella in ((8), (9)).

He adopted a language of second order logic with nominalization for revising Frege's ill-famed *Law V*, according to which every concept is correlated with an object. In such a language second order variables range both over concepts and objects. A second order variable can occur both in predicate and in subject position: when occurring in a formula both in subject and predicate position, it represents, in subject position, the object correlated with the concept that it represents in predicate position. Similarly, our concept variables will range over our closed abstracts both in their roles of concepts and of mere strings of signs. Their positions, in a formula, will distinguish the two roles.

3.1 The formal language

We introduce the following language with three sorts of variables:

Variables

x, y, z... singular individual variables;

X, Y, Z... plural individual variables;

F, G, H... singular concept variables.

Logical constants

1. ε: a relation of form "$b \varepsilon X$" to be read as: "b is one of the Xs".

We prefer to use in the sequel the notation "$b \varepsilon X$" instead of "Xb", in order to reserve juxtaposition for predication. Indeed, when intepreted in terms of plural quantification, *second order logic* could be more properly called *plural first order logic.*

2. The usual connectives $(\neg, \vee, \wedge, \rightarrow, \leftrightarrow)$ and quantifiers $(\exists, \forall)$ for all variables.

Terms and formulas

We define *individual terms*, *concept terms* and *formulas* by a simultaneous induction:

(a) Each term is an individual term (concept terms included);

(b) each singular individual variable is a term;

(c) each concept variable is a concept (and individual) term;

(d) if A is a concept term and b is an arbitrary term, Ab is a formula;

(e) if X is a plural (individual) variable and b is an arbitrary term, $b\varepsilon X$ is a formula;

(f) if A, B are formulas, $\bullet$ is a binary connective, and v is an arbitrary variable, $\neg A$, $A \bullet B$, $\exists v A$, $\forall v A$ are formulas;

(g) if A is a formula without quantified concept variables and free plural variables, and x is a singular individual variable, then $[x\colon A]$ is a concept term. Such a term is said an *abstract*: x and A are, respectively, the *defining variable* and the *defining formula* of the abstract. The abstract is *closed* if x is the only free variable of A.

So a concept term is a concept variable or an abstract: it can occur in a formula in subject position or in predicate position. A singular individual variable is an individual term but not a concept term: it cannot occur in predicate position.

A *bound occurrence* of a variable in a term or in a formula is either (i) a quantified occurrence or (ii) a defining occurrence of an abstract. (An occurrence of) a variable is *free* if it is not bound. A term or a formula *is closed iff* all its variables are bound. The restrictions imposed to the defining formulas of the abstracts will be motivated in the description of the intended model in the next section.

Identity

Identity is defined as follows:

$$\text{if } a,\ b \text{ are terms, then } a = b =_{df.} \forall X(a\varepsilon X \leftrightarrow b\varepsilon X).$$

3.2 The intended model

The domain D of individuals is constituted by all *closed abstracts.*

Two *abstracts* are regarded as the same entity when they are typographically identical up to alphabetic variants (i.e. when one is obtained from the other by a change of *bound variables*). The abstracts play both the role of individuals, mere linguistic expressions, and of the concepts they express. A *closed* abstract $[x\colon A]$ is regarded as the concept of all individuals satisfying the formula A. The *satisfaction* relation will be determined in the next section by an inductive *truth* definition on the complexity of a sentences.

According to these lines, the linguistic use of individual (singular and plural) variables is different from that of the concept variables. A singular individual variable cannot occur in predicate position, because its intended range is the domain of the closed abstracts, understood as mere syntactical entities, which do not play any predicative role. Similarly, no predicative role is played by plural variables: a formula as $t\varepsilon X$ means that t is one of the Xs chosen by a certain plural act of choice, performed by a team of countably many agents (as many as the closed abstracts).

The restrictions imposed to the *abstracts* are motivated by a definitional predicative conception of logical concepts. The definition of a *concept* is understood as *constitutive* of the concept, so no quantification over all concepts (on pain of vicious circularity) is allowed in the *defining formula* of an abstract. In contrast, in such a formula, concept variables and singular individual variables can occur free: an abstract with such variables is a schema that yields a concept for every substitution of them with closed abstracts.

On the other hand, it is possible to use singular and plural quantification on individuals because, as purely syntactical objects, they are completely determined by the syntactical formation rules. We allow that concept variables occur also in subject position in order to express in the formal language the fact that any individual defines a concept and any concept is defined by an individual. We adopt, therefore, the following principles:

$$\forall x \exists F(F = x).$$

and:

$$\forall F \exists x(F = x).$$

In this way one can express the fact that the concept defined by x applies to y: $\exists F(F = x \wedge Fy)$. Observe, however, that such a formula, because of the quantification over all concepts, cannot be the *defining formula* of an abstract: it has a mere *descriptive* (and not *constitutive*) role. This explains the reason of the restrictions for concept variables in the defining formula of an abstract.

As to the restrictions to plural variables, these cannot occur free in an abstract, because we do not suppose that any plurality of individuals can be selected by a concept. In our perspective a concept must be linguistically determined, so that it cannot depend on the outcomes of an arbitrary plural choice. Observe that, if the formula $x \varepsilon X$ were a defining formula of an abstract, then every plurality would be the extension of a concept. In fact, in this case, the abstract $[x\colon x \varepsilon X]$ would yield a concept for every *p.s.c.* and one would easily reproduce Russell's paradox.

In contrast, the occurrence in an abstract of a free singular variable is allowed because such abstract is a schema of the concept, linguistically definable, obtained by replacing the variable with a closed abstract. Besides, a plural variable is allowed to occur quantified in the defining formula of an abstract. In fact, while the language cannot be able to describe the individuals selected by a specific plural choice, nothing prevents the linguistic construction of a concept from using locutions of kind "however certain individuals are chosen..." or "it is possible to perform a choice such that... ."

3.3 Formal semantics

From here on we particularize the above semantics by taking as individuals the closed abstracts of the language so that every individual is definable in the language. An *assignment* a for a formula A assigns to each singular (individual or concept) free variable of A a closed abstract, to each plural free variable of A none, one or more (even infinitely many) closed abstracts. If a does not assign any abstract to X, this variable represents the *empty plurality*. So it is clear that an assignment can be realized by performing a singular or a plural choice for each free variable of A.

Let us define the *truth-conditions* for a formula A relative to an assignment a:

a) suppose, for the moment, that A has no concept variables and proceed by induction on the complexity of A:

   1. $A := t \varepsilon X$. A is true if the term t^*, obtained from t by substituting

its free variables with the terms assigned to them by a, is, up to alphabetic variants, one of the individuals assigned to X by a.

2. $A := [x : B]t$. A is true if $B[t/x]$ is true.
3. The truth conditions for the propositional connectives ($\neg, \vee, \wedge, \rightarrow, \leftrightarrow$) are as usual.
4. $A := \exists vB$ (where v is a singular or plural individual variable). A is true if a can be extended to an assignment a^* for B such that B is true.

b) Let A be an arbitrary formula with concept variables. We proceed by induction on the number of concept variables:

1. A is a formula without quantified concept variables. A is true if the formula A^*, obtained by substituting to the free concept variables the closed terms assigned by a, is true. Notice that A^* has no concept variables.
2. Usual clauses for propositional connectives ($\neg, \vee, \wedge, \rightarrow, \leftrightarrow$).
3. $A := \exists FB$. A is true if there is a closed term t such that $B[t/F]$ is true.
4. $A := \forall FB$. A is true if, for all closed term t, $B[t/F]$ is true.

3.4 Logical Principles

Let us introduce the crucial logical principles of our theory. The general comprehension principle (PCP*)

$$\text{(PCP*)}\ \exists X \forall y (y \varepsilon X \leftrightarrow A)\ (X \text{ not free in } A).$$

holds for pluralities of individuals, as already observed in section 4.

As to concepts, the following restricted comphrehension principle holds:

(SCP) (*Singular comphrehension principle*) $\exists F(F = t)$, where t is a term.

It is satisfied in the intended model because, for any assignment to the free variables of t, this becomes a closed abstract that defines a concept. It is also evident the principle

$$\exists x(x = t).$$

The *conversion principle*

$$\text{(Conv) } [x\colon A]t \leftrightarrow A[t/x]$$

holds in the intended model in virtue of clause a2) of the truth definition. One can reformulate (SCP) in the form:

$$\text{(SCP')} \ \exists F \forall y (Fy \leftrightarrow A) \ (F \text{ not free in } A)$$

where A is without quantified concept variables and free plural variables. For, these restrictions on A are the same required for the defining formula of an abstract. Since closed abstracts play the double role of concepts and individuals, it is evident the following Frege's principle:

$$\text{(FP) } \forall F \exists x (x = F).$$

In the present theory FP is the *intensional* counterpart of the famous *Law V*. As is well-known, it is debatable whether the original *Law V* counts as a logical principle; furthermore, there is a large consensus that *Law V* is responsible for Russell's paradox. In fact, it is not clear at all the nature of the Fregean correlation between concepts and objects. Frege himself, once he knew the paradox, expressed some doubts on *Law V*.

In contrast, in our present system, FP is, as already observed, a self-evident principle: the relation between concepts and individuals is the mere identity. Unlike Frege's *Law V*, FP is intensional: it does not require that the correlates of coextensive concepts are identical individuals. In fact, in the intended model, two distinct abstracts may play the role of co-extensive concepts. For instance, consider the abstracts $[x\colon x = x]$ and $[x\colon x = x \wedge x = x]$: since they are not typographical identical (up to alphabetic variants) they are distinct individuals, while as concepts they are coextensive.

Assume the usual principles of identity (*reflexivity* and *substitutivity*). These trivially hold for the typographical identity (up to alphabetic variants). Define a singleton $[x]$ and an ordered pair $[x, y]$ by putting

$$[x] = [z\colon z = x]$$

$$[x, y] =_{df.} [z\colon z = x \vee z = y]$$

Assume the axiom

$$[x, y] = [x', y'] \leftrightarrow x = x' \wedge y = y'.$$

Again, this trivially holds for the intended typographical identity.

4 The consistency of the theory

One can easily show that our intended model is interpretable in the standard model $N$of classical second order Peano arithmetic PA. Let us introduce a numerical codification of our language. We choose the set K of codes of the closed abstracts in such a way that two abstracts have the same code *iff* they differ at most for alphabetic variants.

Then the inductive truth definition for the sentences is expressible in PA. For the purpose, let the variables X, Y,... range over the subsets of K and the variables x, y,... F, G,... range over the members of K. Then take as an assignment any second order binary relation between the set $V \subset N$ of (the codes of) variables and K that assigns a unique member of K to every singular variable and without any restriction on plural variables. The true sentences are the members of the least subset T of the set of (the codes of) the sentences satisfying all clauses a) and b) of section 5.3.

Now, observe that the logical principles in section 5.4 are satisfied by the above truth definition, so that they are consistent with PA. The *plural comprehension principle* PCP* becomes the classical second order comprehension principle endorsed in PA. SCP and FP are trivial because both variables x and F range over the same set K. The conversion principle Conv holds by clause a2.)

$$[x\colon B]t \text{ is true } \textit{iff } B[t/x] \text{ is true.}$$

SCP' holds because the restrictions on A are the same required for the defining formula of an abstract.

The principles of identity and the axiom of ordered pairs hold because of the above conditions on the codes of the closed abstracts.

5 Relations and Sets

In our system LC we can express two kinds of relations: *plural relations*, generated by plural relating choices *p.r.c.* and *singular relations,* i.e. concepts of ordered pairs. An important singular relation is the relation η defined as follows:

$x\eta y =_{df.} \exists F(F = y \wedge Fx)$ (to be read: "x falls under y" or "y applies to x").

As intensional entities, concepts do not satisfy the extensionality principle: coextensive concepts may not be identical. However, we will define in the next section an extensional identity between concepts, with respect to which they can play the role of sets.

The extensional identity for concepts ($\cong$) is inductively definable by means of the following clauses:

(a) If x, y are intensionally identical ($x = y$), then they are extensionally identical ($x \cong y$);

(b) if x, y satisfy the condition that every individual falling under one of them is extensionally identical to an individual falling under the other, then $x \cong y$.

Such inductive definition is formalizable in terms of plural relations:

$$x \cong y =_{df.} \forall X(\forall u \forall v(u = v \vee (\forall z(z\eta u \rightarrow \exists w([z, w]\varepsilon X \wedge w\eta v) \wedge \forall z(z\eta v \rightarrow \exists w([z, w]\varepsilon X \wedge w\eta u)) \rightarrow [u, v]\varepsilon X) \rightarrow [x, y]\varepsilon X)$$

(in words: the extensional identity is the least equivalence relation correlating two individual x and y provided that it correlates every individuals falling under one of them with an individual falling under the other). Then, we define the membership relation as follows:

$$x \in y =_{df.} \exists z(z \cong x \wedge z\eta y).$$

With respect to the defined extensional identity and membership our individuals form a set-theoretical structure: in this role we will call them *logical sets*. They satisfy the extensional principle:

(EXT) *Two sets are identical iff their members are identical* (as sets).

A formula will be said to be set-theoretic if it is expressible in terms of the membership relation. In this way, we obtain the *logicist fragment* of pure set theory. In particular, this fragment includes the *hereditarily finite sets* (*h.f.s.*), inductively defined by the following clauses:

(i) $\emptyset = [x\colon\ x \neq x]$ is a *h.f.s.*;

(ii) if $[x\colon\ A]$ and $[x\colon\ B]$ are *h.f.s.*, so are $[x\colon\ A \vee B]$ and $[x\colon\ A \wedge B]$;

(iii) if u is a *h.f.s.*, so is $[u]$.

6 Natural numbers

As is well known, arithmetic is interpretable in the theory of *h.f.s.* For example, one can define the set $\mathbb{N}$ of natural numbers à la Zermelo, identifying the successor operation with the singleton operation:

$0 =_{df} \emptyset,$

$Sx =_{df} [x],$

$\mathbb{N} =_{df.} [x : \forall X(0 \in X \wedge \forall y(y \in X \rightarrow Sy \in X) \rightarrow x \in X)].$

By using plural relations, one can easily reformulate the inductive definitions of the usual arithmetical operations and prove second order Peano axioms.

So we obtain a theory of natural numbers that may be considered a form of structural logicism, insofar as it is grounded on the theory of logical concepts. We will make later some further comments on this form of logicism.

7 Russell's paradox

Since LC is consistent no paradox can arise. It may be instructive, however, to stress how Russell paradox is avoided. Russell set-theoretical paradox does not follow because the formula $\neg x \in x$ cannot occur in an abstract, since membership $\in$ is defined by means of quantification over concept variables.

As to the intensional version of Russell's paradox, concerning the alleged concept of all concepts that are not falling under themselves, observe that the condition of not falling under itself $\neg(x\eta x)$ is expressed by the formula: $\exists F(x = F \wedge \neg Fx)$, which, again, because of the existential quantification over concept variables, cannot be the defining formula of an abstract. Observe that, in virtue of (PCP)*

$$\exists X \forall x(x\eta X) \text{ } \textit{iff} \text{ } \exists F(x = F \wedge \neg Fx)$$

So, if every plurality were the extension of a logical concept one could reproduce Russell's paradox.

However, in our theory the alleged concept of all individuals satisfying the formula $\exists F(x = F \wedge \neg Fx)$ does not count as a logical concept in accordance with our predicative understanding of logical concepts.

8 The logicist flavor of the theory of logical concepts

Our theory of logical concepts can be regarded as a form of logicism. We want to emphasize, however, that our form of logicism is not to be understood according to the Fregean view, which aims to supply a logical definition of natural numbers in terms of the *general* notions of *concept* and *object*.

Frege's foundation of arithmetic can be labeled as "logicist" because, *inter alia*, it exploits only very general aspects of such notions and generality is usually considered as a peculiar feature of logic. However, his theory quantifies over *all* concepts and *all* objects, so that its ontological commitment, according to Quine's criterion, is not restricted to any sort of logical entities (in any reasonable sense of the word "logical"), but includes an enormous infinity of entities of any nature. In this respect, Frege's numbers are not defined in terms of *logical* entities.

On the contrary, our interpretation of second order logic—using *acts of choice* instead of *second order entities*—allows a strong reduction of the ontology involved in a logicist foundation of natural numbers. Our *abstracts*, the individuals we quantify over, are countably many types of finite strings of signs. Such entities, called by C. Parsons *quasi-concrete objects* (as types of concrete entities), are capable of being chosen by *ostension*, as required by our theory of plural choices. Furthermore, they are very basic objects and the human ability to grasp them is essential to the development of logic, as was clearly emphasized by Hilbert in the following passage:

> [A]s a condition for the use of logical inferences and performance of logical operations, something must already be given to our faculties of representation, certain extralogical concrete objects that are intuitively present as immediate experience prior to all thought. If logical inference is to be reliable, it must be possible to survey these objects completely in all their parts, and the fact that they occurr, that they differ from one another, and that they follow each other, or are concatenated, is immediately given intuitively, togheter with the objects, as something that neither can be reduced to anything else nor requires reduction. This is the basic philosophical position that I consider requisite for mathematics and, in general, for all scientific thinking, understanding, and communication. And in mathematics, in particular, what we

> consider is the concrete signs themselves, whose shape, according to the conception we have adopted, is immediately clear and recognizable. (10, p. 376)

In this sense our abstracts, though *extra-logical* entities, can be labelled as *pre-logical* entities. Besides, as we saw, they can play the role of certain particular concepts with respect to a suitable relation of application. We want to enlighten a sense according to which such concepts can be labeled as *logical concepts*.

In virtue of the restrictions imposed to the abstracts, the application relation among abstracts can be inductively defined, by means of logical connectives and quantifiers, in terms of the syntactical identity among abstracts, considered as mere strings of signs. Individual identity, though formally definable in terms of plural quantification, is essentially a *primitive relation*. Indeed, it is implicitly presupposed in the very same notion of individual, and hence in *singular* and *plural quantification* over individuals. For this reason identity is usually regarded as a *logical relation*. So the concepts represented by our abstracts are in turn of logical nature, insofar as they are logically defined in terms of a logical relation through logical constants. In this sense, the Hilbert's extra-logical entities (finite strings of signs) can be structured as *logical concepts*. Similarly for their role as logical sets (i.e. extensions of logical concepts). And our approach to number theory can be regarded as a form of *logicism*, insofar as all arithmetical notions are definable within our theory of logical concepts.

It may be worthwhile to compare our logicism with *neologicism* (for an introduction to neologicism see (13)). We just sketch here the most relevant for our purposes. Second-order arithmetic is derivable in second-order logic from what is usually referred to as Hume's Principle:

$$\text{(HP)}\ \#F = \#G\ \textit{iff}\ F \text{ and } G \text{ are equinumerous.}$$

In the standard interpretation F and G range over second-order entities (as concepts or classes), while # is a function mapping second-order entities onto individuals. Neologicism attempts to revise Frege's logicism by replacing Frege's ill-famed *Basic Law V* with *HP*. In fact, Frege exploits Law V only to derive HP. The latter is of the same kind as Law V but, unlike Law V, it turns out to be consistent with classical second-order arithmetic. Crispin Wright and other neologicists hold the highly controversial thesis that HP counts as a logical definition of numbers (see for example (16), (17). See also (13)).

The main argument against this thesis is that HP has ontological consequences: the existence of infinitely many numbers. But, following Boolos, it is common to argue that pure logic should have no ontological consequences. If so, there is no hope to derive the existence of numbers from logical principles: arithmetic requires infinitely many individuals, whose existence cannot be guaranteed by logic. Besides, HP has been charged to be in "bad company", because of its similarity with Frege's inconsistent *Basic Law V.*

In contrast, our logicism is compatible with the thesis that the existence of no entity can be granted by mere logical means. In fact, we do not claim that the existence of the entities we are committed to is guaranteed by logic. What we claim is that such entities are of a peculiar logical significance insofar as they are functional to the development of logic. In fact, our abstracts as syntactical entities are pre-logical in the explained sense.

Concerning the ontological commitment we think, unlike Quine, that a theory is committed not only to the entities it quantifies over, but to all entities assumed in the object language and in the meta-language. Accordingly, our ontological commitment consists both of the syntactical entities, including the close abstracts (our individuals) and of the agents (the entities we appeal to in the meta-language).

Following Hilbert we have argued for the logical relevance of finite strings of signs. Besides our abstracts are structured, by means of a plural logic, in such a way to play, in turn, the roles of concepts, sets, and numbers.

As to our agents they are, of course, fictional characters, so in this respect, our philosophical perspective is a form of *fictionalism.*[4] By the way, we think that any idealization is fictional. We think of as fictional the very same infinitely many syntactical entities of a formal system.

One may wonder: if one wants to go fictionalist, why should one not prefer a fictionalism about sets rather than about *agents* and *abstracts*? Is the commitment to agents not so heavy as a commitment to sets? Our answer is that it is not.

First of all, our team of agents is countably infinite. Assuming only countably many entities we supply a categorical theory of natural numbers, while classical second order arithmetic uses a higher infinity of sets and classical first order arithmetic is non categorical.

[4]On *fictionalism* see (1), (5) and (18).

Secondly, logical reasoning about sets presupposes the ideal possibility of referring to any of them, and, since a set is determined by its members, this possibility presupposes, in turn, the possibility of referring to all its members simultaneously. The team of agents serves just the purpose of providing a suitable way for understanding an ideal act of referring to them. On the other hand, once the team of agents has been introduced, in order to develop second order logic, the further assumption that for any plural choice the chosen objects form a set becomes superfluous. Plural choices make the job of sets without the need of any extension of the ontology. Thus the commitment to agents is much more fundamental than the commitment to arbitrary sets of individuals.

We conclude that our ontological commitment is restricted to entities of a crucial relevance for explainning a way of understanding the idealization underlying logical reasoning.

Observe that the fictional flavor of our theory consists in the fact that individuals (i.e. our agents and abstracts) are treated as if they actually existed. That is essential in order to make acts of plural *simultaneous* choices performable. In contrast, acts of choice, unlike agents, are to be understood in a mere *potential* way: as said there is no realm of possible acts; an act exists only insofar as it is performable.

Again, as said, our notion of combinatorial possibility justifies immediately, without any circularity, the *comprehension principle of our language* (where A is any formula of our language):

$$(PCP) \quad \exists X^n \forall y_1 ... y_n (X^n y_1 ... y_n \leftrightarrow A)$$

Possible occurrences in A of plural quantifications cannot produce any circularity, since acts of choice, unlike properties, are all independent one of the other. So, the so called *impredicative comprehension principle* of second order logic escapes, in our interpretation, the well known criticism to impredicativity.

9 Conclusions

Following (6) we have revised Boolos' interpretation of second order logic in terms of plural quantification by introducing a notion of *plural reference*, characterized through *acts of plural choice*, performable by a suitable team of *ideal agents*. On the basis of this approach to second order logic, we have proposed a new form of *logicism* characterized by the following claims:

1. Finite strings of signs are entities essential for the development of logic (as emphasized by Hilbert).

2. Our theory of plural choices supports the claim that plural logic is pure logic.

3. Plural logic is adequate to structuring our closed abstracts (particular finite strings of signs) in such a way that they play the role of logical concepts of our theory LC.

4. The concepts at issue count as logical concepts, insofar as they are defined with logical means on the base of the logical relation of identity among individuals.

5. Natural numbers are logical entities, insofar as they are defined in terms of logical concepts.

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