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Volume 490

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Massimiliano Carrara • Enrico Martino

Arbitrary Reference in Logic and Mathematics

Springer

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ISSN 0166-6991 ISSN 2542-8292 (electronic)
Synthese Library
ISBN 978-3-031-66451-9 ISBN 978-3-031-66452-6 (eBook)
https://doi.org/10.1007/978-3-031-66452-6

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Preface

The purpose of this book is to show how arbitrary reference can throw new light to the foundations of logic and mathematics.

Working mathematicians refer very often, in their reasoning, to an arbitrary object of the universe of discourse. They employ expressions such as "Let *a* be an arbitrary set" and reason about *a* as if it were a well-determined set. We have called such a locution a "magic locution", because it puts the puzzling questions:

(Q1) How can one fix the reference of *a*?
(Q2) How can *any* set be capable of being selected among the enormous infinity of sets?

Questions like (Q1) and (Q2) are usually neglected by the working mathematician, because the means of fixing the reference of *a* does not enter in her reasoning. All she needs is to *imagine* that reference has been *somehow* fixed. That's right, but this very consideration suggests the essential role of *imagination* in mathematical reasoning and the opportunity of making it more perspicuous. Following this suggestion, we introduce an ideal agent, free of empirical limitations, who has access to any individual of the domain of quantification. We propose a revision and a development of J.S. Mill's and P. Kitcher's attempts of interpreting the mathematical language as concerning ideal actions instead of abstract mathematical entities. Subsequently we extend the singular agent to an ideal team of infinitely many agents, as many as the individuals of a first order domain of quantification. The job of such agents is that of performing each of them, ad libitum, a simultaneous act of choice of an individual. In this way, singular arbitrary reference is generalized to plural arbitrary reference.

By using this device, we defend G. Boolos' controversial thesis of the ontological innocence of second-order logic. Boolos' argument, based on his reinterpretation of second-order logic in terms of *plural quantification*, has faced a criticism of several authors, moved from the suspicion that plural quantification would surreptitiously exploit a hidden notion of set. We will argue, however, that such a suspicion can be superseded by explaining plural quantification in terms of *plural arbitrary reference* with a suitable semantics for plural choices.

Using our semantics, we analyze and revise Lewis' work on *megethology*, a reconstruction of set theory based on the combination of plural quantification and mereology. In particular, we discuss Lewis' thesis of the ontological innocence of mereology. We argue that the notion of mereological fusion, so as usually understood, is committal to larger and larger infinities, provided a countable infinity of individuals is given. Thereof we maintain that *usual* mereology is far from being ontological innocent. We show, however, how a virtual mereology (**VM**) can be constructed, where the role of fusions is played by suitable plural acts of choice, so that no new entity is generated by the operation of fusion.

We conclude that the ideal team of agents is a powerful source of new insight on the foundations of logic and mathematics.

This book collects and elaborates some ideas found in these papers:

M. Carrara and E. Martino. A Notion of Logical Concept Based on Plural Reference. *Acta Analytica*, 33: 19–33, **2018** (DOI: 10.1007/s12136-017-0327-2).
M. Carrara and E. Martino. The Mereological Foundation of Megethology. *Journal of Philosophical Logic* 45: 227–235, **2016** (DOI: 10.1007/s10992-015-9373-7).
M. Carrara and E Martino. Grounding Megethology on Plural Reference. *Studia Logica* 103: 697–711, **2015** (DOI: 10.1007/s11225-014-9585-9).
M. Carrara and E. Martino. To Be is to Be the Object of a Possible Act of Choice. *Studia Logica* 96: 289–313, **2010** (DOI: 10.1007/s11225-010-9282-2).

Arbitrary reference in logic and mathematics is the result of a rather lengthy collaboration that we began in late 2006. A considerable amount of time has passed since then, although we have never ceased to discuss of arbitrary reference, mereology, and other issues in logic and philosophy of logic, such as the notion of proof and that of negation, which have found only a bit of space in this book.

It narrates a bit of our *research journey* undertaken together over the years; perhaps this is also why it took us so long to finish it! We liked the idea of continuing to discuss these matters, continuing our research journey.

It is for this reason that we chose to conclude the writing of this book with a poem, "Ithaca," that the poet Konstantinos Petrou Kavafis (translated by E. Keely) dedicated to the journey. Kavafis in this poem reminds us not to be in a hurry to reach our destination. These are such beautiful verses that we like to share them with the readers of this book (or more simply the readers of this preface).

Ithaca (by Konstantinos P. Kavafis)[1]

> As you set out for Ithaka
> hope your road is a long one,
> full of adventure, full of discovery.
> Laistrygonians, Cyclops,
> angry Poseidon—don't be afraid of them:
> you'll never find things like that on your way

[1] C. P. Cavafy, *Collected Poems*. Translated by Edmund Keeley and Philip Sherrard. Princeton University Press 1975.

as long as you keep your thoughts raised high,
as long as a rare excitement
stirs your spirit and your body.
Laistrygonians, Cyclops,
wild Poseidon—you won't encounter them
unless you bring them along inside your soul,
unless your soul sets them up in front of you.

Hope your road is a long one.
May there be many summer mornings when,
with what pleasure, what joy,
you enter harbors you're seeing for the first time;
may you stop at Phoenician trading stations
to buy fine things,
mother of pearl and coral, amber and ebony,
sensual perfume of every kind—
as many sensual perfumes as you can;
and may you visit many Egyptian cities
to learn and go on learning from their scholars.

Keep Ithaka always in your mind.
Arriving there is what you're destined for.
But don't hurry the journey at all.
Better if it lasts for years,
so you're old by the time you reach the island,
wealthy with all you've gained on the way,
not expecting Ithaka to make you rich.

Ithaka gave you the marvelous journey.
Without her you wouldn't have set out.
She has nothing left to give you now.

And if you find her poor, Ithaka won't have fooled you.
Wise as you will have become, so full of experience,
you'll have understood by then what these Ithakas mean.

At the end of this preface, there can be no shortage of acknowledgments. We want to give plural thanks to all those who have read parts of this work, in earlier versions, suggesting revisions, cuts, and improvements. We also want to thank anonymous reviewers who read draft versions of the book for their detailed comments, suggestions, and improvements. We hope that this revised version of the book will satisfy their requests. Finally, we want to express our gratitude to Otávio Bueno, Editor-in-Chief of the *Synthese Library series. Studies in Epistemology, Logic, Methodology, and Philosophy of Science*, for supporting with patience (!) the editorial project and for wanting it to be published in the series he directed, and to Christopher Wilby Associate Editor of Springer for the editorial work.

This book is dedicated to our beloved Elena and Livia. And it is also for the beloved son of Massimiliano, Pietro in the hope that he may see the Otaru Harbor dock again.

Padova, Italy
4 April 2024

Massimiliano Carrara
Enrico Martino

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