Modeling future indeterminacy in possibility semantics

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Abstract

Possibility semantics offers an elegant framework for a semantic analysis of modal logic that does not recruit fully determinate entities such as possible worlds. The present papers considers the application of possibility semantics to the modeling of the indeterminacy of the future. Interesting theoretical problems arise in connection to the addition of object-language determinacy operators. We argue that adding a two-dimensional layer to possibility semantics can help solve these problems. The resulting system assigns to the two-dimensional determinacy operator a well-known logic (coinciding with the logic of universal modalities under global consequence). The paper concludes with some preliminary inroads into the question of how to distinguish two-dimensional possibility semantics from the more established branching framework.

1 Introduction

Possibility semantics offers an elegant framework for a semantic analysis of modal logic that does not recruit fully determinate entities such as possible worlds.[1] This paper investigates conceptual and technical issues emerging from the application of possibility semantics to the modeling of the indeterminacy, or openness, of the

[1]The phrase "possibility semantics" was coined by Humberstone (1981). The tools undergirding the framework have longer histories, including (Fine, 1975, especially §2), Humberstone (1979), as well as deep roots in the algebraic logic tradition. For a contemporary and comprehensive introduction, see Holliday (2022). Possibility semantics is one of a variety of styles of theories that do not rely on worlds, but on coarser objects. In addition to possibility semantics, the general family of "pointless" theories includes various kinds of states-based semantic analyses (Aloni 2018, Willer 2018), truthmaker semantics (Fine, 2017b), as well as several varieties of situation semantics (Barwise and Perry, 1981; Kratzer, 2021). It would be desirable to have a comparative study of these frameworks highlighting the commonalities, as well as the differences, between them.

future, and some related forms of metaphysical indeterminacy. Possibility semantics is plausibly viewed as an alternative to more established branching-time models (Thomason 1970, 1984, 2007, Belnap *et al.* 2001, MacFarlane 2003, 2014) in which indeterminacy is grounded in the overlap of complete possibilities—sometimes referred to as "histories". The key finding is that explicit modeling of indeterminacy within the object language requires the semantics to be two-dimensional.

As understood here, the open-future hypothesis states that some future events and states are objectively, and not merely epistemically, unsettled.[2] The recurring illustrative example will be the proposition that some specific random coin will land heads on its next toss, under the stipulation that the outcome of the coin's toss is not settled by the facts about the past and the present of the tossing apparatus. If in actuality there are no such setups, the case may be entertained as a thought-experiment.

The indeterminacy associated with the future seems unlike other kinds of indeterminacy that have attracted the attention of philosophers. For example, it seems unlike the indeterminacy that some theories associate with vagueness. For one thing, it does not appear to give rise to higher-order indeterminacy. It is generally agreed by those who think that vagueness is grounded in some kind of indeterminacy that it may itself be indeterminate whether Joe is borderline tall. By contrast, it is common to assume that, as far as the unsettledness of the future is concerned, there are no states or events whose determinacy status is itself indeterminate. It might be unsettled whether there will be a sea battle tomorrow, but it cannot be unsettled whether it's unsettled. A second marker of the indeterminacy of the future is that it is not plausibly associated with unusual effects on credence. Many different philosophers have been attracted to the view that there is something non-classical about credence in the contents of vague statements. One form of this is Field's (2000) claim that vague contents require low credence in certain instances of the law of excluded middle; another is Williams's claim that vague contents seem to require imprecise probability (2014).[3] By contrast, statements about the future appear to be paradigmatic examples for the application of classical theories of credence. In prototypical cases, it is perfectly warranted to have a sharp credence that the coin will land heads. The fact that the indeterminacy of the future has these characteristics licenses us to theorize about this specific type of indeterminacy on its own (cf. §2.3 of Torza, forthcoming, on pluralism about indeterminacy).

[2] There is much literature on what constitutes the (alleged) openness of the future. The present discussion leans in various ways on Thomason (1970); Belnap and Green (1994); Belnap *et al.* (2001); MacFarlane (2003, 2014); Barnes and Cameron (2009, 2011); Torre (2011); Cariani and Santorio (2018); Cariani (2021b); Todd (2022).

[3] The matter is complicated, in ways that go beyond the basic demarcation point that is made here. For a sophisticated discussion, see Bacon (2018).

As a last disclaimer, exploring the indeterminacy hypothesis involves no commitment to the claim that the future is open. We only need to assume that the hypothesis is worth taking seriously. As Stalnaker (2019, p.197) puts it, "You don't have to sign on to this metaphysical theory (as I do not) in order to find it intelligible (as I do) and to use it as a kind of precedent for a case where the thesis of metaphysical indeterminacy may be less controversial." Moreover, the discussion is not restricted in scope to the alleged indeterminacy of the future. It pertains to any application of possibility semantics to concepts of indeterminacy that do not give rise to higher-order indeterminacy and are not associated with non-classical effects on credences.[4]

We lead with a general introduction to possibility semantics for a sentential modal language (§2). The next section focuses on the representation of indeterminacy in possibility semantics (§3). The framework itself already incorporates a representation of indeterminacy in the model theory. However, contrary to the inclination of Humberstone (1981), it seems important to capture the notion of indeterminacy in the object language. We observe (§3) that it is not possible to add a determinacy operator with the right profile to the system—not at least without other interventions. After considering some local interventions (§4), we consider an attractive solution to the problem, which lies in the integration of possibility semantics with a two-dimensional framework (§5). The last sections explore the resulting system (§6), extend the approach to incorporate temporal operators (§§7, 8).

The insight behind the approach proposed in §5 is owed to remarks in Fine (1975). The Cliffs notes on Fine's paper focus on the fact that it applies supervaluationist techniques to vague language. However, it is also a central juncture for the logical development of semantics based on partial objects, since Fine builds up to supervaluationism by first analyzing a system in which precisifications of a vague language are viewed as partial. (NB: this account is only considered in passing in Fine 1975, and Fine's theory of vagueness has significantly changed, e.g. in Fine 2017a.) We aim to recast some of those insights about determinacy operators in a different theoretical context, allowing some distinct issues and theoretical choice

[4]Stalnaker (1984) famously suggests that counterfactual selection results in a kind of indeterminacy, and has more recently suggested that this kind of indeterminacy might be viewed as a 'milder' version of the indeterminacy that is associated with the future (Stalnaker, 2019, p.197-ff).

points to come to light.[5]

2 Background on possibility semantics

The basic ideological tenet of possibility semantics is that formulas are not evaluated against worlds, but against "coarser" objects called *possibilities*. This ideology deviates from the standard account of the indeterminacy of the future—which is broadly within the framework of branching time (Thomason, 1970, 1984, 2007; Belnap *et al.*, 2001). According to the branching time picture, indeterminacy is adequately captured by the overlap of multiple complete possibilities with equal claim to fit the settled facts.

Possibility semantics proceeds differentl. Instead of taking a maximally precise representation as its basic modeling object, it deploys primitive objects that are themselves incomplete. That incompleteness is naturally associated with a concept of indeterminacy: possibilities settle the truth values of some sentences of a language, while leaving others unsettled.

The present formulation of possibility semantics originates from Humberstone (1981). The language is a sentential modal language, whose signature features a non-empty countable set of modal operators. (Later, we will add a *determinacy* operator D.) Models for this language are quadruples of the form, $\langle P, \gg, \mathbf{R}, V \rangle$. Here P is a non-empty set of possibilities; $\gg$ is a refinement relation over the possibilities. Structurally, $\gg$ is a *weak partial order* (thus, it is reflexive, transitive and antisymmetric). Intuitively, $Y \gg X$ holds when everything that is settled as either true or false by X is settled in the same way by Y. In short, Y agrees on all the determinate facts that X settles. (Explicit assumptions are needed in order to guarantee that models satisfy this intuition, and they will be provided shortly.) $\mathbf{R}$ is a non-empty set of accessibility relations, and finally V is a partial valuation function: in this setting a valuation function inputs an atomic formula and a possibility, and, if defined, outputs either 0 or 1. When $V(\mathsf{A}, X)$ is undefined, we write $V(\mathsf{A}, X) \uparrow$, otherwise $V(\mathsf{A}, X) \downarrow$. Occasionally, when it is important to disambiguate, and a model $\mathcal{M}$ is salient in context, a subscripted "$\mathcal{M}$" will be used to indicate its coordinates. For example, "$P_{\mathcal{M}}$" refers to the set of possibilities in $\mathcal{M}$.

Models for this language are ordinarily assumed to satisfy two constraints.

Refinability. For all atomic A and possibilities X, if $V(\mathsf{A}, X) \uparrow$, then there are Y, Z such that $Y \gg X$ and $Z \gg X$, s.t. $V(\mathsf{A}, Y) = 1$ and $V(\mathsf{A}, Z) = 0$.

Persistence. For all atomic A, if $V(A, X) \downarrow$ then for every $Y \gg X$, $V(\mathsf{A}, X) = V(\mathsf{A}, Y)$.

[5] The idea of modeling the unsettledness of the future *via* partial objects is also explored in a preliminary way in Boylan (2021), although Boylan's development is incompatible with the present one.

Persistence says that whenever atomic A is settled at X, it stays settled in the same way through X's refinements. Refinability says that whenever an atomic formula A is unsettled at a possibility X, there are Y and Z—both refinements of X—that settle A as true and false respectively. [6]

Persistence provides formal representation to the intuitive conception of refinement. Indeed, under persistence, it is tempting to think of refinement structures as mirroring the structure of the branching models for future contingency, as illustrated by Figure 1.

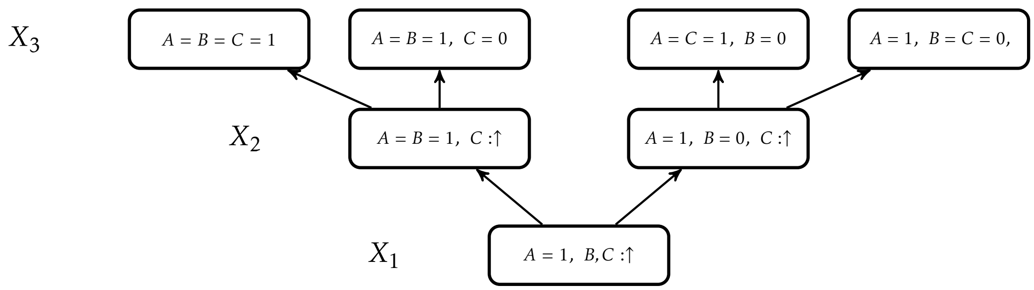

Figure 1: The branching structure of refinements ($X_3 \gg X_2 \gg X_1$)

However, an important lingering difference — which the formal theory ought to help disentangle — is that standard branching models are built on the idea of maximal *histories*, which at any moment assign a definite truth-value to all the formulas of the language. Indeed, the linear paths through the tree can naturally be viewed as temporally structured possible worlds. No such assumption of completeness is imposed on possibility models.

Another important observation is that our assumptions on possibility models do not, by themselves, rule out backwards branching. For example, the model in Figure 2 satisfies Refinability and Persistence and yet, the possibility X_3 has two arrows going into it.

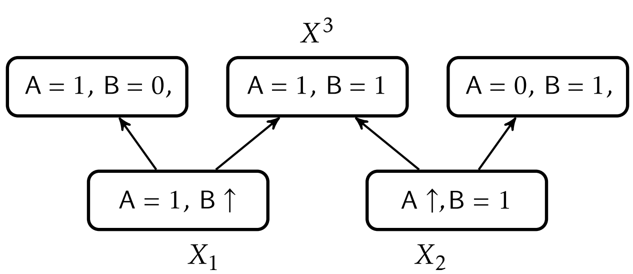

Figure 2: Backwards branching possibility model

[6] Refinability is related to, but logically distinct from, the assumption that any partial possibility might be refined all the way to a complete one (which Fine 1975 calls "Completability"). In a language with infinitely many atomic sentences, refinability might be satisfied, without completability being satisfied.

If we wanted to rule these models out, we would need to impose a "no backwards branching" condition, similar to those that are standardly used in defining branching models. Specifically, we would have to stipulate that whenever $X \gg Y$ and $X \gg Z$, then either $Y \gg Z$ or $Z \gg Y$. We implicitly restrict attention to models that satisfy this condition, but none of our results require imposing it.

Humberstone defines a notion of support between possibilities and formulas of the whole language as follows:

- $\mathcal{M}, X \Vdash p$ iff $V_{\mathcal{M}}(p, X) = 1$
- $\mathcal{M}, X \Vdash \mathsf{A} \wedge \mathsf{B}$ iff $\mathcal{M}, X \Vdash \mathsf{A}$ and $\mathcal{M}, X \Vdash \mathsf{B}$
- $\mathcal{M}, X \Vdash \neg\mathsf{A}$ iff $\forall Y \gg X$, $\mathcal{M}, Y \nVdash \mathsf{A}$
- $\mathcal{M}, X \Vdash \Box_i\mathsf{A}$ iff $\forall Y \in P$, s.t. R_iXY, $\mathcal{M}, Y \Vdash \mathsf{A}$

As for other operators, such as $\vee$, $\rightarrow$, $\Diamond$, a common approach recovers entries by fixing some standard equivalences. In the case of disjunction, one option is to characterize it by conjunction, negation and DeMorgan's laws. This results in the following entry:

- $\mathcal{M}, X \Vdash \mathsf{A} \vee \mathsf{B}$ iff $\forall Y \gg X$, there is $Z \gg Y$, s.t. $\mathcal{M}, Z \Vdash \mathsf{A}$ or $\mathcal{M}, Z \Vdash \mathsf{B}$

Another route to the same goal would be to stipulate some general principles about what it takes for possibilities to settle a disjunction (Holliday, 2022).

- A possibility X settles a disjunction $\mathsf{A} \vee \mathsf{B}$ as false iff it settles A as false and settles B as false.

Assume that a possibility settles A as false iff it settles $\neg$A as true. Next, note that the entries for negation and conjunction tell us that:

- A possibility X settles a conjunction $\mathsf{A} \wedge \mathsf{B}$ as true iff it settles both A and B as true.
- A possibility X settles A as false iff every refinement of X fails to settle A as true.

These assumptions are sufficient to pin down the same entry for disjunction as above. A similar analysis could be carried out for the other operators.[7]

Lastly we follow Humberstone in defining consequence as preservation of support.

[7] While the analysis of necessity lifts Kripke semantics to the level of possibilities, an account of modality also involves the specification of interplay conditions connecting accessibility and refinement. Humberstone proposed:

(uR) $\forall X, Y, Z$, if $Z \gg X$ and RZY, then RXY

Definition 1 *$A_1, ..., A_n \Vdash B$ iff for all models $\mathcal{M}$ and any X in $P_{\mathcal{M}}$, if $\forall i$, $\mathcal{M}, X \Vdash A_i$, then $\mathcal{M}, X \Vdash B$.*

It is a well established fact about this formalism that the logic of the sentential sub-language is classical, both in the sense that the logical truths coincide with the classical tautologies, and in the sense that the valid arguments in this sub-language coincide with the tautologically valid arguments (Humberstone, 1981, pp.320-321).

3 Adding object language determinacy operators

It is reasonable to view possibility semantics as incorporating a model of indeterminacy: an atomic formula A is indeterminate at a possibility X when X leaves A undefined. Imagine a possibility X and an atomic formula, *heads*, symbolizing the English sentence *The coin will land heads (on a specific toss that will take place tomorrow at noon)*. In a clear sense, the metatheoretic fact that $V_{\mathcal{M}}(heads, X) \uparrow$ represents the relevant indeterminacy from the perspective of the model theory. This warrants the view that indeterminacy is captured in standard possibility semantics at the metatheoretic level.

However, as the system is set up, there is no object language device to express the concept of indeterminacy. We have not identified an operator that expresses things like *it is determinate that the coin landed heads on today's toss, but it is not determinate that it will land heads tomorrow*. This is unfortunate because, for various modeling purposes, it's important to have determinacy operators in the object language. For example, determinacy operators help characterize the interaction of indeterminacy with other concepts. To take just one example, Cariani (2021a) explores interactions between (in)determinacy operators and epistemic operators, such as $\neg D\mathsf{A} \rightarrow \neg K\mathsf{A}$—the principle that if A is not determinately true, then it is not known. Such principles, and the constraints they impose on models, are best analyzed from the perspective of a formalized language.[8]

Let us then introduce a determinacy operator D to the formal language—with the interpretation that its argument is determinately *true*. Thus $\neg D\mathsf{A}$ is interpreted as claiming that the proposition expressed by A is not determinately true, while

(Rd) $\forall X, Y, Z$, if $Z \gg Y$ and RXY, then RXZ

(R) $\forall X, Y$, if RXY then $\exists X' \gg X, \forall X'' \gg X', RX''Y$

Holliday (2014, forthcoming) noted that condition (R) is overly strong. One suitable weakening is a condition that Holliday calls R-refinability (see Lemma 5.3.7 of Holliday (2022)).

(RR) for all X, Y, if RXY, then $\exists X' \gg X, \forall X'' \gg X', \exists Y' \gg Y, RX''Y'$

In addition to 'RR', the names given here to these conditions are abbreviations of Holliday's names: '(uR)' is for Holliday's 'up-R' for (uR) and '(Rd)' is for Holliday's 'R-down'.

[8] For additional considerations in favor of introducing object language determinacy operators, see also Barnes and Williams (2011, §5)

leaving it open that it might be determinately false. To express the claim that A is indeterminate, add an indeterminacy operator I governed by the condition in Definition 2, which is standardly taken to be definitional of indeterminacy (e.g. in Fine, 1975):

Definition 2 $IA =_{df} \neg DA \wedge \neg D \neg A$

It is important however to keep in mind that in the present terminology 'indeterminacy' denotes a two-sided status, in the sense that it requires that both A and its negation fail to be determinate. By contrast, non-determinacy (the obtaining of $\neg D$) is a one-sided status: a proposition may fail to be determinate, while its negation is determinate.

Determinacy operators should be governed by some key constraints. A natural one within possibility semantics is to suppose that object language indeterminacy is to align with metatheoretic indeterminacy, in the following sense:

Constraint 1 (Alignment) *For atomic* A, $\mathcal{M}, X \Vdash IA$ *iff* $V_{\mathcal{M}}(A, X) \uparrow$.

The trouble is that Alignment is incompatible with the framework we have developed. To see why, note that (given the framework) it entails a second constraint: formulas expressing non-determinacy (and indeterminacy) claims must violate a generalization of Persistence.

Constraint 2 (Non-persistence of non-determinacy) *There is a formula* A, *and model* $\mathcal{M}$ *with possibilities* $X, Y \in P_{\mathcal{M}}$ *and* $Y \gg X$ *such that* $\mathcal{M}, X \Vdash \neg DA$ *but* $\mathcal{M}, Y \nVdash \neg DA$

The route from Alignment to non-persistence is relatively straightforward: possibility X might support that it's indeterminate whether the coin will land heads, while being refinable into a possibility Y that settles that the coin will land heads.

Fact 1 *Given Definition 2 and Refinability, Alignment entails non-persistence of non-determinacy.*

> *Proof.* Consider a model $\mathcal{M}$ with two possibilities X and Y drawn from its possibility set $P_{\mathcal{M}}$, such that $Y \gg X$. Suppose Y settles some atomic formula A that X leaves unsettled. Without loss of generality suppose that Y settles A as true, $V_{\mathcal{M}}(\mathsf{A}, Y) = 1$. The existence of such a Y is guaranteed by Refinability. Then $V_{\mathcal{M}}(\mathsf{A}, X) \uparrow$ but $V_{\mathcal{M}}(\mathsf{A}, Y) = 1$. Definition 2 yields $\mathcal{M}, X \Vdash \neg D\mathsf{A}$ and $\mathcal{M}, Y \Vdash D\mathsf{A}$, so $\mathcal{M}, Y \nVdash \neg D\mathsf{A}$. □

This result does not sit well, or indeed at all, with the analysis of negation.

Fact 2 *The following are inconsistent (given the framework):*

NP. *There are* $\mathcal{M}$, *A*, *X*, $Y \gg X$ *with* $\mathcal{M}, X \Vdash \neg DA$ *but* $\mathcal{M}, Y \nVdash \neg DA$.

NE. $\mathcal{M}, X \Vdash \neg A$ *iff* $\forall Y \gg X$, $\mathcal{M}, Y \nVdash A$

Proof. Consider witnesses, $\mathcal{M}, X, Y, \mathsf{A}$ for NP. So, (i) $\mathcal{M}, X \Vdash \neg D\mathsf{A}$ (ii) $Y \gg X$ and (iii) $\mathcal{M}, Y \nVdash \neg D\mathsf{A}$. By the clause for negation (NE) and (i), $D\mathsf{A}$ cannot be supported throughout any refinements of X. That is, $\forall Z \gg X$, $\mathcal{M}, Z \nVdash D\mathsf{A}$. However, since any refinement of Y is a refinement of X, we must also have $\mathcal{M}, Y \Vdash \neg D\mathsf{A}$. This contradicts (iii). □

This inconsistency is related to a less specific unease with object language indeterminacy operators that is already expressed by Humberstone (1981). Humberstone claims that an indeterminacy operator like the one just introduced would go "against the spirit of the present enterprise, since it would give rise to formulas which were not persistent into refinement [...], and thus undermines the idea of refinements as mere resolvers of indeterminacy".

However, the problem is not with the idea of determinacy operators: a plausible initial diagnosis is that the tension arises because the negation operator forces persistence: $\neg\mathsf{A}$ must always be persistent, whether A is persistent or not. This entails that the indeterminacy operator I cannot be both defined in terms of negation and also such that formulas like $I\mathsf{A}$ and $\neg D\mathsf{A}$ are non-persistent.

4 A preliminary journey around the options

Are there ways of integrating object language determinacy operators within possibility semantics? One option is to introduce persistent determinacy operators. One might support a plea for persistence by thinking in terms of temporally indexed indeterminacy operators. Start by noting that, in the relevant applications, there is a connection between refinement and temporality: when Y strictly refines X ($Y \gg X$ but $X \not\gg Y$), it is both the case that Y resolves some of X's indeterminacy and that Y represents a later moment in time.[9] Perhaps, then we should entertain determinacy operators that are relativized to a specific point in time. Under this approach, the object language would feature a collection of operators $\{D_t \mid t \in T\}$, where T is a designated set of times in the model. Simplifying somewhat, imagine that T is countable. Then consider operators $D_0, D_1, D_2, ..., D_n, ...$, each marking what is determinate at a certain time in the development of history, with each D_i anchored to some specific time t_i. To complete the proposal say that the language does not contain any unrelativized determinacy operators, and thus that all determinacy discourse is captured by means of relativized ones.

[9] This is not essential, and §§7-8 will separate these two traits.

This proposal works by undermining some of the motivation for non-persistence. Suppose again that X_{Mon} represents Monday's possibility, in which the coin has not yet landed heads, and X_{Wed} represents the state of affairs on Wednesday, after the coin has been tossed and has landed heads. In the original approach, with unrelativized determinacy operators, one would say that $\neg D(heads)$ is supported at X_{Mon} but unsupported at X_{Wed}. By contrast, the relativized framework opens up a different option: X_{Mon} supports $\neg D_{Mon}(heads)$, while X_{Wed} supports $D_{Wed}(heads)$. Crucially, the formulas $\neg D_{Mon}$ and $\neg D_{Wed}$ can be assumed to be persistent (even when the operator is embedded under negation). The intuitive meaning of $D_{Mon}\mathsf{A}$ would be something like "A is/was settled true on Monday". From Wednesday's point of view—i.e., as far as X_{Wed} is concerned—$\neg D_{Mon}\mathsf{A}$ remains supported. Relatedly, the claim $\neg D_{Monday}(heads) \wedge D_{Wednesday}(heads)$ is perfectly consistent (from any point in time).[10] Evidently, these operators violate Alignment.

This approach is valuable and instructive, but it also seems flawed: it is not especially controversial to claim that people possess an unrelativized concept of indeterminacy — plausibly one that can be captured at the level of the theory by an operator that satisfies Alignment. There is no special reason to think that there are barriers to expressing *that* concept in the object language. It is at the very least worth asking whether such a concept is definable consistently with the general insights motivating possibility semantics.

Before moving to the positive proposal, let us entertain one more option. The initial hunch was that the tension is due to the persistence-forcing effect of negation. The obvious alternative would be to introduce a negation operator that does not force persistence. To this end, introduce '$\sim$' as the connective characterized by the clause: $\mathcal{M}, X \Vdash {\sim}\mathsf{A}$ iff $\mathcal{M}, X \nVdash \mathsf{A}$. This alternate negation operator does not force persistence, and would make correct predictions for non-determinacy claims in the proof of Fact 2.

An evident problem is that '$\sim$' cannot be the correct negation operator for the entire language. Outside of determinacy claims, '$\sim$' conflates non-support with rejection: *it's not the case that the coin will land heads* should not be supported by a possibility that merely fails to settle the matter. More generally, '$\sim$' is not the correct negation operator for the sentential sub-language. In response, one might consider a language in which the two negation operators, '$\neg$' and '$\sim$', coexist.[11] However, having both operators around is not well-motivated. There is no principled reason

[10] A notational variant of this approach: we can have a single concept of indeterminacy that is *relational*, so that the canonical logical form for determinacy claims is $D(Monday, \mathsf{A})$. The critique to be made below of the indexed operator approach applies to this as well.

[11] Footnote 15 of Humberstone (1981) identifies a minor expressive advantage to having both operators (though Humberstone does not endorse the suggestion currently under consideration): their combination, '$\neg\sim$', is a plausible candidate for a determinacy operator, as it expresses universal quantification over all refinements. (So $\mathcal{M}, X \Vdash \neg \sim \mathsf{A}$ iff all refinements of X support A).

why one negation operator ($\neg$) should apply in the D-free sub-language, while the other should apply to formulas involving D. Additionally, any attempt at formulating a generalization concerning which operator is appropriate for a given formula would have to deal with the thorny problem of choosing the correct negation for mixed formulas (like $\mathsf{A} \wedge D\mathsf{A}$).

5 Introducing two-dimensional possibility semantics.

This section presents a two-dimensional version of possibility semantics that is capable of retaining the fundamental motivation for the framework while incorporating a non-persistent, non-relativized determinacy operator that is "aligned" with the metatheoretic concept of indeterminacy.[12]

The opening move in crafting such a framework is to distinguish two dimensions of evaluation. The support relation is relativized to a triple $\langle \mathcal{M}, X, Y \rangle$ consisting of the model and *two* possibilities. The first, 'primary', possibility is operated on by connectives, while the other, 'secondary', coordinate is read by the determinacy operator D and unaffected by the connectives. The unidimensional notion of support is still part of the theory because we continue to focus on a concept of truth, or support, at a possibility as the target of the theory. A standard diagonal principle helps us pin it down:

Diagonal principle: $\mathcal{M}, X \Vdash \mathsf{A}$ iff $\mathcal{M}, X, X \Vdash \mathsf{A}$

As an additional benefit, the diagonal principle allows the two-dimensional system to inherit the earlier definition of logical consequence.

Recursive clauses for the connectives and for the determinacy operator are specified at the level of two-dimensional evaluation. Note that the secondary possibility only affects the entry for the determinacy operator.

(i) $\mathcal{M}, X, Z \Vdash p$ iff $V_{\mathcal{M}}(p, X) = 1$

(ii) $\mathcal{M}, X, Z \Vdash \mathsf{A} \wedge \mathsf{B}$ iff $\mathcal{M}, X, Z \Vdash \mathsf{A}$ and $\mathcal{M}, X, Z \Vdash \mathsf{B}$

(iii) $\mathcal{M}, X, Z \Vdash \neg\mathsf{A}$ iff $\forall Y \gg X$, $\mathcal{M}, Y, Z \nVdash \mathsf{A}$

(iv) $\mathcal{M}, X, Z \Vdash \Box_i\mathsf{A}$ iff $\forall Y \in P$, s.t. $R_i XY$, $\mathcal{M}, Y, Z \Vdash \mathsf{A}$

(v) for $\vee$, $\rightarrow$, $\Diamond$, use standard equivalences to infer clauses.

[12] For general surveys two-dimensional semantics see Humberstone (2004); Kuhn (2013); Schroeter (2021). A two-dimensional treatment of the determinacy operator is explored in Fine (1975). Fine rightfully questions the ability of such an operator to handle higher-order indeterminacy, but of course this is not salient in the present application. An alternative two-dimensional determinacy operator is also introduced in Burgess and Humberstone (1987, §6.2).

(vi) $\mathcal{M}, X, Z \Vdash D\mathsf{A}$ iff $\mathcal{M}, Z, Z \Vdash \mathsf{A}$

Under this analysis, the determinacy operator resembles an actuality operator in more standard applications of two-dimensional semantics. It evaluates the argument of $D\mathsf{A}$ after setting the primary evaluation possibility so as to match the secondary one.

Logical consequence remains defined as preservation of unidimensional support, as per Definition 1. Furthermore, Refinability and Persistence, understood as constraints on atomic formulas, continue to be in place. While they have generalizations for the full language, the status of those generalizations is not settled by the status of their atomic variants. Thus, saying that the complex formula $I\mathsf{A}$ is non-Persistent is fully compatible with saying that atomic formulas persist through refinements.

6 Temporary victory lap

This section illustrates how the system fulfills the main desiderata for adding an object language determinacy operator, and offers a way out of the central incompatibility from Fact 2. A key intermediate step in establishing these objectives is the characterization of the logic of the system, identified below as Theorem 1.

Recall the suggestion that non-determinacy and indeterminacy should violate a generalization of Persistence.

Fact 3 (Non-persistence of non-determinacy and indeterminacy) *Let A be an atomic formula. Then there is a model $\mathcal{M}$ and a possibility $X \in P_X$, such that $\mathcal{M}, X \Vdash \neg DA$ but $\exists Y \gg X$, $\mathcal{M}, Y \nVdash \neg DA$. (Same for IA.)*

Proof. Consider a model with three possibilities X_1, X_2 and X_3 with $X_2, X_3 \gg X_1$ and such that $V(\mathsf{A}, X_1)\uparrow$, $V(\mathsf{A}, X_2) = 1$, $V(\mathsf{A}, X_3) = 0$. Then, $\mathcal{M}, X_1 \Vdash \neg D\mathsf{A}$, but $\mathcal{M}, X_2 \nVdash \neg D\mathsf{A}$ (since $\mathcal{M}, X_2 \Vdash D\mathsf{A}$).[13]

The same model also illustrates the non-persistence of $I\mathsf{A}$. □

An easy induction establishes that this non-persistent behavior does not apply to formulas that lack determinacy operators.

It is also easy to establish that the Alignment constraint is met.

Fact 4 (Alignment) *$\mathcal{M}, X \Vdash IA$ iff $\mathcal{M}, X \nVdash A$ and $\mathcal{M}, X \nVdash \neg A$*

[13] The support conditions for $\neg D\mathsf{A}$, when A is atomic are as follows.

- $\mathcal{M}, X \Vdash \neg D\mathsf{A} \Leftrightarrow \mathcal{M}, X, X \Vdash \neg D\mathsf{A} \Leftrightarrow \forall W \gg X : M, W, X \nVdash D\mathsf{A} \Leftrightarrow \forall W \gg X : \mathcal{M}, X, X \nVdash \mathsf{A} \Leftrightarrow \mathcal{M}, X, X \nVdash \mathsf{A} \Leftrightarrow V(\mathsf{A}, X) \neq 1$.

It is easy to check that $\mathcal{M}, X_1 \neg D\mathsf{A}$ holds but $\mathcal{M}, X_2 \Vdash \neg D\mathsf{A}$ does not.

$$\mathcal{M}, X \Vdash I\mathsf{A} \Leftrightarrow \mathcal{M}, X \Vdash \neg D\mathsf{A} \wedge \neg D\neg\mathsf{A} \Leftrightarrow \mathcal{M}, X, X \Vdash \neg D\mathsf{A} \wedge \neg D\neg\mathsf{A} \Leftrightarrow$$

$$\mathcal{M}, X, X \Vdash \neg D\mathsf{A} \text{ and } \mathcal{M}, X, X \Vdash \neg D\neg\mathsf{A} \Leftrightarrow$$

$$\mathcal{M}, X, X \nVdash \mathsf{A} \text{ and } \mathcal{M}, X, X \nVdash \neg\mathsf{A} \Leftrightarrow \mathcal{M}, X \nVdash \mathsf{A} \text{ and } \mathcal{M}, X \nVdash \neg\mathsf{A} \qquad \square$$

This shows that there is no way of setting the accessibility relation R to define a modal operator on the primary evaluation coordinate that is equivalent to the determinacy operator. Any modal that operates on the primary evaluation coordinate would collapse the two-dimensional framework into the one-dimensional one. However, Facts 3 and 4 assure us that the systems do not collapse.

There is, however, an important relationship between the two-dimensional determinacy operator in a language with no other modals, and certain ordinary modals as evaluated in some designated submodels. Given a model $\mathcal{M}$ and $X \in P_X$, let $\mathcal{M}_X$ be the submodel of $\mathcal{M}$ that is generated by X. This is the model $\langle P_X, \gg', \mathbf{R}', V' \rangle$ where P_X is the closure of $\{X\}$ under $\gg$ and any accessibility relation in $\mathcal{M}$, and all the other elements of the model are restrictions of the remaining relations and functions in $\mathcal{M}$ to this set. Let $\Vdash_U$ be the support relation generated by interpreting formulas of our formal language according to the unidimensional rules, while interpreting D as the universal modality in $\mathcal{M}_X$ (i.e. by assuming YRZ for any Y and Z in $P_{\mathcal{M}_X}$). Note that in our specific case there are no modals other than D, and so $\mathbf{R}$ is empty, and so is $\mathbf{R}'$.

Fact 5 *For any* $Y \gg X$, $\mathcal{M}, Y, X \Vdash \mathsf{B}$ *iff* $\mathcal{M}_X, Y \Vdash_U \mathsf{B}$.

> *Proof:* Reason by induction on the complexity of B. If B is a negated formula ¬A, and Y is an arbitrary element of P_X, $\forall Z \geq Y$, $\mathcal{M}, Z, X \nVdash \mathsf{A}$ iff $\mathcal{M}_X, Z \nVdash_U \mathsf{A}$, but since Z is a refinement of Y, it is also a refinement of X, so this follows from the induction hypothesis. Setting aside the trivial case in which B is a conjunction, the remaining case of interest is where $\mathsf{B} = D\mathsf{A}$ for A satisfying the induction hypothesis. Consider Y refining X: then $\mathcal{M}, Y, X \Vdash D\mathsf{A}$ iff $\mathcal{M}, X, X \Vdash \mathsf{A}$ iff $\mathcal{M}_X, X \Vdash_U \mathsf{A}$ iff $\mathcal{M}_X, Y \Vdash_U D\mathsf{A}$. □

Fact 5 is key to characterizing the logic of D, at least in the special case in which the language does not contain other modal operators. Let $\models_{S5}$ denote the S5 consequence relation, and $\models^g_{S5}$ the global consequence relation as characterized on Kripke models (Blackburn *et al.*, 2001, §1.3). The consequence relation on two-dimensional possibility models coincides with the global consequence relation on universal Kripke models.[14]

[14] See the appendix of Schulz (2010) for a similar result involving the logic of Yalcin's (2007) semantics for epistemic necessity—albeit one that is presented wholly at the level of worlds-based semantics.

Theorem 1 *If the language does not contain modals other than D, $A_1, ..., A_n \Vdash C$ iff $DA_1, ..., DA_n \vDash_{S5} DC$ iff $A_1, ..., A_n \vDash^g_{S5} C$*

Proof. Universal modalities in Kripke frameworks have the same logic (i.e. S5) as universal modalities in the possibility framework. So, let $\Vdash_U$ be the logic of a possibility framework for a language with a single modal D with R_D as the universal relation. Then:

$$DA_1, ..., DA_n \Vdash_U DC \text{ iff } DA_1, ..., DA_n \vDash_{S5} DC$$

What is left to prove is:

$$A_1, ..., A_n \Vdash C \text{ iff } DA_1, ..., DA_n \Vdash_U DC$$

This is proven by a chain of equivalences. Let $\Gamma = \{A_1, ..., A_n\}$, and $D\Gamma = \{DA_1, ..., DA_n\}$.

(i) $\exists \mathcal{M}$ and $X \in P_{\mathcal{M}}$, $\mathcal{M}, X \Vdash \Gamma$, but $\mathcal{M}, X \nVdash C$.

(ii) $\exists \mathcal{M}$ and $X \in P_{\mathcal{M}}$, $\mathcal{M}, X, X \Vdash \Gamma$, but $\mathcal{M}, X, X \nVdash C$.

(iii) $\exists \mathcal{M}$ and $X \in P_{\mathcal{M}}$, $\forall Z \gg X$, $\mathcal{M}, Z, X \Vdash \Gamma$, but $\mathcal{M}, X, X \nVdash C$.

(iv) $\exists \mathcal{M}$ and $X \in P_{\mathcal{M}_X}$, $\forall Z \gg X$, $\mathcal{M}_X, Z \Vdash_U \Gamma$, but $\mathcal{M}_X, X \nVdash_U C$.

(v) $\exists \mathcal{M}$ and $X \in P_{\mathcal{M}_X}$, $\mathcal{M}_X, X \Vdash_U D\Gamma$, but $\mathcal{M}_Y, X \nVdash_U DC$

Ad (ii)⇔(iii): evaluation along the primary coordinate is persistent even in the two-dimensional system. *Ad (iii)⇔(iv)*: this follows from Fact 5 and persistence in the unidimensional framework. *Ad (iv)⇔(v)*: if DC fails at X, C must fail at some possibility in the model—i.e. at some refinement of X. But if so, C must also fail at X (or else persistence would force it to hold throughout the entire model). □

Theorem 1 enables us to ascertain the satisfaction of many of our design principles. The remaining facts in this section are all presented without explicit proof, on the understanding that they are elementary corollaries of Theorem 1. First, notice that it entails that the logic in the sentential sub-language remains classical. It is also a simple corollary that the theorems of the two-dimensional theory with D as the sole modal operators are exactly the theorems of S5.

Fact 6 $\Vdash C$ *iff* $\vDash_{S5} DC$ *iff* $\vDash^g_{S5} C$

Another important consequence of the theorem is that there are consistent statements of indeterminacy, in the object language. Thus $I\mathsf{A}$ is consistent (i.e. $\nVdash D\mathsf{A} \vee D\neg\mathsf{A}$) and $D(\mathsf{A} \vee \mathsf{B}) \nVdash D\mathsf{A} \vee D\mathsf{B}$.[15]

Per our design specifications, there is no higher-order indeterminacy in this system. We have $\mathsf{A} \Vdash D\mathsf{A}$, and in particular, $D\mathsf{A} \Vdash DD\mathsf{A}$ and $\neg D\mathsf{A} \Vdash D\neg D\mathsf{A}$. This is not, of course, the same as the claim that truth and settled truth coincide. Indeed, we also have: $\nVdash \mathsf{A} \leftrightarrow D\mathsf{A}$; $\Vdash D\mathsf{A} \leftrightarrow DD\mathsf{A}$; $\Vdash \neg D\mathsf{A} \leftrightarrow D\neg D\mathsf{A}$.

Theorem 1 illustrates that the system has a familiar non-classical profile when it comes to its metarules. Though the consequence relation in the D-free fragment matches that of classical sentential logic, adding expressive capacity to the language in the form of the determinacy operator results in some non-classical behavior. For example, the consequence relation does not contrapose over the full language: as noted, $\mathsf{A} \Vdash D\mathsf{A}$ holds, but $\neg D\mathsf{A} \Vdash \neg\mathsf{A}$ does not. This mirrors the standard behavior of systems based on S5 global consequence relations. For example, it is observed in supervaluationist analyses based on the idea of "global" validity (Fine, 1975; Williamson, 1994; Varzi, 2007; Asher *et al.*, 2009; Bacon, 2018) and also in informational analyses of consequence for languages with epistemic modals, as in (Yalcin, 2007; Bledin, 2014; Schulz, 2010; Incurvati and Schlöder, 2022). More specifically, Theorem 1 entails the following failures:

Fact 7

- *No Conditional proof :* $A \Vdash DA$, *but* $\nVdash A \rightarrow DA$
- *No Reductio:* $A \wedge \neg DA \Vdash DA$ *and* $A \wedge \neg DA \Vdash \neg DA$ *but* $\nVdash \neg(A \wedge \neg DA)$
- *No Contraposition:* $A \Vdash DA$ *but* $\neg DA \nVdash \neg A$

Disjunctive syllogism may fail too, depending on its exact characterization.[16]

It is valuable to reflect on exactly how the system avoids the inconsistency in Fact 2. Recall, that the inconsistency pits the claim that indeterminacy is non-persistent (NP) against the analysis of negation (NE). The two-dimensional system avoids the inconsistency by violating the negation condition, NE. There is no guarantee that if X supports $\neg\mathsf{A}$, then all of X's refinements will fail to support A.

[15]The failure of this last entailment is relevant for comparison with an alternate system involving determinacy operators and two-dimensional semantics (Burgess and Humberstone, 1987, pp. 220-221).

[16]The following form of disjunctive syllogism fails:

If $\mathsf{A} \Vdash \mathsf{C}$, $\mathsf{B} \Vdash \mathsf{C}$, then $\mathsf{A} \vee \mathsf{B} \Vdash \mathsf{C}$

If it didn't, then we would have $\mathsf{A} \vee \neg\mathsf{A} \Vdash D\mathsf{A} \vee D\neg\mathsf{A}$ (contradicting the finding that determinacy is non-trivial). Alternatively, it is possible to formulate disjunctive syllogism as follows:

If $\Vdash \mathsf{A} \rightarrow \mathsf{C}$, $\Vdash \mathsf{B} \rightarrow \mathsf{C}$, then $\Vdash (\mathsf{A} \vee \mathsf{B}) \rightarrow \mathsf{C}$

Theorem 1 entails that this reformulated schema is correct.

Since unidimensional evaluation follows the diagonal principle, what's supported at X depends on evaluation triples of the form $\langle M, X, X\rangle$, whereas what's supported at Y depends on evaluation triples of the form $\langle \mathcal{M}, Y, Y\rangle$. These may come apart in ways that undermine (NE). Of course, much of the semantic effect of the negation operator is preserved because there is an analogous operator at the level of two-dimensional evaluation. However, that operator only quantifies over refinements along the primary dimension.

Relatedly, although the two-dimensional framework allows violations of persistence understood as a constraint on the unidimensional support relation, it continues to satisfy a local version at the level of two dimensional evaluation.

Local Persistence. If $\mathcal{M}, X, Y \Vdash \mathsf{A}$ then $\forall Z \gg X$, $\mathcal{M}, Z, Y \Vdash \mathsf{A}$.

7 Adding tense operators.

The framework of section 5 is founded on a distinctive structural assumption concerning the relationship between refinement and temporality: the refinement relation corresponds to the temporal precedence. For example, we have supposed that a possibility in which the coin toss is indeterminate splits, after a step of refinement, into two possibilities that *immediately* follow the initial one.

This assumption has both simplifying and heuristic value, but it may, with reason, be viewed with some suspicion. For example, it obfuscates how one might sensibly add temporal operators to the language. Relatedly, the assumption seems to force an eternalist understanding of propositions—the idea that propositions do not vary in their truth-value from time to time. Eternalism is not obviously mistaken, and indeed it is probably the dominant theory of propositions, but it also is not obviously mandated by any arguments we have made. This concluding section explores the prospects for lifting this assumption, and sketches a theory of tenses that integrates with two-dimensional possibility semantics.[17] This will serve as proof of concept that the integration is possible but we will stop short of developing the theory in full.

Following (Holliday, 2022, §5.3), we sever the connection between refinement and temporality, by giving temporal operators their own accessibility relations. Specifically, add temporal operators $\langle \mathsf{F}\rangle$, for *sometime in the future*, and $\langle \mathsf{P}\rangle$ for *sometime in the past*, respectively governed by accessibility relations R^F and R^P. (So, for example $R^F XY$ means that Y is in the future of X, and $R^P XY$ means that Y is in the past of X) We assume R^F and R^P to be at least irreflexive, transitive,

[17] Fans of eternalist propositions can also convert the present discussion into their framework, by introducing quantificational operators with the appropriate logical properties.

and asymmetric. For reasons that will become clear momentarily, we refrain from assuming that R^P is always the converse of R^F. Suppose 'p' is an atomic formula of our object language, to be interpreted as meaning that it's raining. Then the model

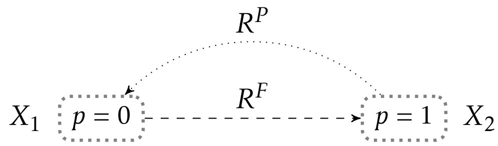

Figure 3: Simple temporal model

in Figure 3 diagrams a situation in which X_1 precedes X_2, and such that from X_1's perspective it will be raining in the future and from X_2's perspective it was not raining in the past.

To improve readability, introduce the notation $R^F(X)$ to denote $\{Y \mid R^F XY\}$ — the set of possibilities in the future of X. With this we can informally read $Y \in R^F(X)$ as "Y is in the future of X". It is worth keeping in mind that, under the open future view, multiple possibilities might be in the future of a given base possibility (see Figure 4 below).

One immediate idea is to apply the standard analysis of modal operators to the semantics of ⟨F⟩ and ⟨P⟩. We begin by applying the three interplay conditions entertained in §5.3 of Holliday (2022) to R^F and R^P. Letting R be some arbitrary accessibility relation, these are:

(uR) $\forall X, Y, Z$, if $Y \gg X$ and $Z \in R(Y)$, then $Z \in R(X)$

(Rd) $\forall X, Y, Z$, if $Y \gg X$ and $X \in R(Z)$, then $Y \in R(Z)$

(RR) $\forall X, Y$, if $Y \in R(X)$, then $\exists X' \gg X, \forall X'' \gg X', \exists Y' \gg Y, Y' \in R(X'')$

Applied to R^F, (uR) says that any possibility Z that is in the future of a refinement of X is also in the future of X; (Rd) says that any possibility Y that refines a possibility that is in the future of Z is also in the future of Z; (RR) states that for any Y in the future of X, there is a refinement X' of X every refinement of which has some refinement of Y in its future.

Treat ⟨F⟩ and ⟨P⟩ as duals of universal modals [F] and [P] with the standard semantics from §5.

- $\mathcal{M}, X, Z \Vdash [\mathsf{F}]\mathsf{A}$ iff $\forall Y \in R^F(X)$, $\mathcal{M}, Y, Z \Vdash \mathsf{A}$
- $\mathcal{M}, X, Z \Vdash [\mathsf{P}]\mathsf{A}$ iff $\forall Y \in R^P(X)$, $\mathcal{M}, Y, Z \Vdash \mathsf{A}$
- $\langle\mathsf{F}\rangle\mathsf{A} =_{def} \neg[\mathsf{F}]\neg\mathsf{A}$

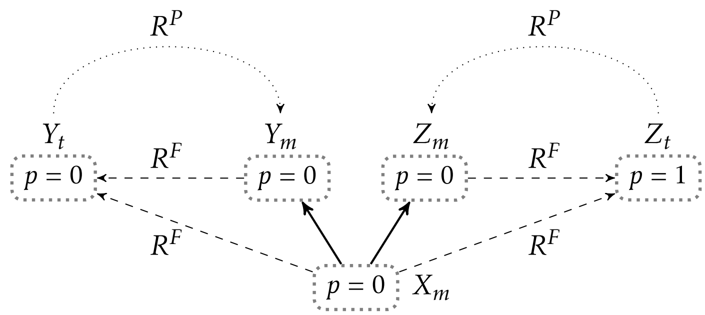

Figure 4: Holliday's model of the Sea Battle puzzle

- $\langle \mathsf{P} \rangle \mathsf{A} =_{def} \neg [\mathsf{P}] \neg \mathsf{A}$

Under condition (Rd), the support conditions for $\langle \mathsf{F} \rangle$ and $\langle \mathsf{P} \rangle$ simplify to:[18]

(F1) $\mathcal{M}, X, Z \Vdash \langle \mathsf{F} \rangle \mathsf{A}$ iff $\forall Y \gg X, \exists K \in R^F(Y), \mathcal{M}, K, Z \Vdash \mathsf{A}$

(P1) $\mathcal{M}, X, Z \Vdash \langle \mathsf{P} \rangle \mathsf{A}$ iff $\forall Y \gg X, \exists K \in R^P(Y), \mathcal{M}, K, Z \Vdash \mathsf{A}$

Call (F1) and (P1) "unidimensional tenses". A notable — and ultimately problematic — feature of the unidimensional tenses is that their support conditions are insensitive to the secondary coordinate of evaluation.

For now, we note the clean separation between temporality and refinement. Refinement relations track the resolution of metaphysical indeterminacy, without shifting the locus of temporal evaluation. This idea may be best illustrated with Holliday's own model of the sea battle puzzle. Suppose that there are two salient times, Monday and Tuesday and that p represents the proposition that there is a sea battle on Tuesday. Define a model $\mathcal{SB}$ (for "sea battle") as follows, and as diagrammed in Figure 4. Let X_m be Monday's state of affairs both with regards to categorical facts and also with regards to which facts are determinate. Now, X_m can be refined into two possibilities Y_m, and Z_m without changing the temporal perspective from which we evaluate. That is, both Y_m and Z_m represent the world as it is on Monday. Where they differ is that they resolve (some of) Monday's indeterminacy in different ways: in Z_m's future there is a sea battle (i.e., at Z_t); in Y_m's future (Y_t) there isn't one. In light of condition (uR), both Y_t and Z_t must also be futures of X_m: after all, Z_m (viz. Y_m) refines X_m and Z_t (viz. Y_t) are in the future of Z_m (viz. Z_m).

Remarkably, in this model R^P is not the converse of R^F. The intuition behind this is that, "looking backwards" from Z_t's perspective, the past has a new veneer of

[18]See Lemma 5.3.8 in Holliday 2022. In the current setup: $\mathcal{M}, X, Z \Vdash \neg \Box_i \neg \mathsf{A}$ iff $\forall Y \gg X$, $\mathcal{M}, Y, Z \not\Vdash \Box_i \neg \mathsf{A}$ iff $\forall Y \gg X, \exists K \in R_i(Y), \mathcal{M}, K, Z \not\Vdash \neg \mathsf{A}$ iff $\forall Y \gg X, \exists K \in R_i(Y), \exists K' \gg K, \mathcal{M}, K', Z \Vdash \mathsf{A}$. Now, fix $Y \gg X$ and $K \in R_i(Y)$. Suppose $K' \gg K$. Then, by (Rd), $K' \in R_i(Y)$. Thus K' would be available as witness to the existential in (F1) and (P1).

determinacy. Here is Holliday's gloss, with adaptations to our notation in square brackets:

> Thus, the future is presently [i.e. at X_m] open. Yet if there is a sea battle, so [Z_t] is realized, then the past will turn out to be [Z_m], in which there would be a future sea battle, whereas if there is no sea battle, so [Y_t] is realized, then the past will turn out to be [Y_m], in which there would be no future sea battle. Come tomorrow, we might say, "the past is not what it used to be." (Holliday, 2022, §5.3)

While we carry along this interesting assumption, it does bear further scrutiny, so as to determine the theoretical tradeoffs it involves.

The model verifies what we might call **minimal openness**:

(MO) $\mathcal{SB}, X_m \Vdash \langle \mathsf{F} \rangle p \vee \neg\langle \mathsf{F} \rangle p$ while $\mathcal{SB}, X_m \nVdash \langle \mathsf{F} \rangle p$ and $\mathcal{SB}, X_m \nVdash \neg\langle \mathsf{F} \rangle p$.

With our determinacy operator, we further verify $\mathcal{SB}, X_m \Vdash \neg D\langle \mathsf{F} \rangle p \wedge \neg D \neg\langle \mathsf{F} \rangle p$.[19] This is indeed a minimal standard for capturing a concept of openness of the future. However, the addition of two-dimensional determinacy operators to the temporalist version of the theory presents some difficulties. Here is one: in the combined theory determinacy statuses are permanent.

The two-dimensional system augmented with unidimensional tenses incorrectly predicts that $D\mathsf{A} \Vdash [\mathsf{P}]D\mathsf{A}$, and so that $D\mathsf{A}, \langle \mathsf{P} \rangle \top \Vdash \langle \mathsf{P} \rangle D\mathsf{A}$. More generally:

Fact 8 (Determinacy is forever) *For any model $\mathcal{M}$, $X, Z \in P_{\mathcal{M}}$,*

(i) $\mathcal{M}, X, Z \Vdash DA$ *iff* $\mathcal{M}, X, Z \Vdash [P]DA$

(ii) $\mathcal{M}, X, Z \Vdash DA$ *iff* $\mathcal{M}, X, Z \Vdash [F]DA$

> *Proof of (i):* the left side reduces to $M, Z, Z \Vdash \mathsf{A}$ given the support conditions for D. Unpacking on the right side: $\mathcal{M}, X, Z \Vdash [\mathsf{P}]D\mathsf{A}$ iff $\forall Y \in R^P(Y)$, $\mathcal{M}, Y, Z \Vdash D\mathsf{A}$, which also reduces to $M, Z, Z \Vdash \mathsf{A}$. For (ii), replace ' $[\mathsf{P}]$ ' with ' $[\mathsf{F}]$ ' in this argument. □

Since the D operator overwrites the primary coordinate of evaluation with the value of the secondary coordinate, temporal operators scoping over D are irrelevant.

Within the present framework, there is an obvious alternative: treat the tenses as two-dimensional operators in their own right.

[19] While Holliday (2022) discusses this model within a unidimensional possibility semantics (with operators $\langle \mathsf{F} \rangle$ and $\langle \mathsf{P} \rangle$ also being given standard unidimensional entries), his discussion can easily be exported to the two-dimensional setting with no essential alteration. The observation concerning the determinacy operators is not in Holliday (2022), but it is also noted in slides for Holliday's NASSLLI course on possibility semantics. To verify the first of these claims: $\mathcal{SB}, X_m, X_m \Vdash \neg D\langle \mathsf{F} \rangle p$ iff $\forall X' \gg X_m$, $\mathcal{SB}, X', X_m \nVdash D\langle \mathsf{F} \rangle p$ iff $\mathcal{SB}, X_m, X_m \nVdash \langle \mathsf{F} \rangle p$, which is indeed the case since it's not the case that every refinement of X_m has a future in which p holds (e.g. Y_m does not).

(F2) $\mathcal{M}, X, Z \Vdash [\mathsf{F}]\mathsf{A}$ iff $\forall K \in R^F(X)$, $\mathcal{M}, K, K \Vdash \mathsf{A}$

$\mathcal{M}, X, Z \Vdash \langle\mathsf{F}\rangle\mathsf{A}$ iff $\forall Y \gg X$, $\exists K \in R^F(Y)$, $\mathcal{M}, K, K \Vdash \mathsf{A}$

(P2) $\mathcal{M}, X, Z \Vdash [\mathsf{P}]\mathsf{A}$ iff $\forall K \in R^P(X)$, $\mathcal{M}, K, K \Vdash \mathsf{A}$

$\mathcal{M}, X, Z \Vdash \langle\mathsf{P}\rangle\mathsf{A}$ iff $\forall Y \gg X$, $\exists K \in R^P(Y)$, $\mathcal{M}, K, K \Vdash \mathsf{A}$

It can be quickly checked that this modified system retains the duality relationship between $\langle\mathsf{F}\rangle$ and $[\mathsf{F}]$.[20]

The modified system also fixes the bug with the previous proposal by blocking the analogue of Fact 8. The model in Figure 4 illustrates that determinacy is not forever. Specifically: $\mathcal{SB}, Z_t, Z_t \Vdash Dp$, but $\mathcal{SB}, Z_m, Z_m \Vdash \neg Dp$, and yet $\{Z_m\} = R^P(Z_t)$, and so $\mathcal{SB}, Z_t, Z_t, \Vdash \langle\mathsf{P}\rangle\neg Dp$. Furthermore, after this modification, the semantics continues to meet the minimal openness benchmark.

8 Preliminary analysis of the tensed system

This is progress towards the goal of keeping a theoretical distinction between refinement and temporal precedence. Whether this integration is successful requires substantial additional work. Here we touch on two questions: are the operators in (F2) and (P2) properly called temporal operators? And how much of the open future lore can be retrieved within the modified theory?

Both questions may be addressed by exploring the logic of the system. Having added temporal operators, Fact 5 fails, and so does the path we followed to characterize the logic of D. Even without a characterization, however, we can answer our questions in a more piecemeal fashion. To start, D continues to mirror the logic of universal necessity modals — e.g. by validating axioms T ($D\mathsf{A} \to \mathsf{A}$), 4 ($D\mathsf{A} \to DD\mathsf{A}$), and 5 ($\neg D\mathsf{A} \to D\neg D\mathsf{A}$). These can be checked directly by applying the evaluation rule for material conditionals, which is derived from $(A \to B) =_{def} \neg(A \wedge \neg B)$, and simplifies to:[21]

(MC) $\mathcal{M}, X, Z \Vdash \mathsf{A} \to \mathsf{B}$ iff $\forall X' \gg X, \mathcal{M}, X', Z \Vdash \mathsf{A}, \mathcal{M}, X', Z \Vdash \mathsf{B}$.

With regards to the temporal logic, we can study whether the system satisfies at least the "minimal temporal logic" (Goranko and Rumberg, 2024), consisting of the axioms:

[20] $\mathcal{M}, X, Z \Vdash \neg[\mathsf{P}]\neg\mathsf{A}$ unpacks to $\forall Y \gg X, \exists K \in R^P(Y), \exists Q \gg K, \mathcal{M}, Q, K \Vdash \mathsf{A}$ Under (Rd) this is equivalent to $\forall Y \gg X$, $\exists K \in R^P(Y), \mathcal{M}, K, K \Vdash \mathsf{A}$, since $Q \gg K$, $K \in R^P(Y)$, and (Rd) entail $Q \in R^P(Y)$.

[21] See Holliday (2022), §4.2.2. Holliday's proof is for the unidimensional system and requires appeals to Persistence and Refinability across the language. The analogue of this proof in the two-dimensional system, goes through by appeal to the local versions of Persistence and Refinability as identified at the end of §6.

(KF) $[\mathsf{F}](\mathsf{A} \to \mathsf{B}) \to ([\mathsf{F}]\mathsf{A} \to [\mathsf{F}]\mathsf{B})$

(KP) $[\mathsf{P}](\mathsf{A} \to \mathsf{B}) \to ([\mathsf{P}]\mathsf{A} \to [\mathsf{P}]\mathsf{B})$

(PF) $\mathsf{A} \to [\mathsf{P}]\langle\mathsf{F}\rangle\mathsf{A}$

(FP) $\mathsf{A} \to [\mathsf{F}]\langle\mathsf{P}\rangle\mathsf{A}$

The first two follow immediately from the support conditions for the tenses and (MC). The case of the conversion principles is more complex. Instances of (PF) where A does not involve determinacy operators (or tenses) — call them *basic instances* — can be proven by adopting the standard frame condition that if $X \in R^P(Y)$, then $Y \in R^F(X)$. (Its non-basic instances can be proven given a principle I am about to introduce modeling the Fixity of the Past, but I won't provide the proof here.) As for (FP), its basic instances can be proven on the basis of a weaker frame condition:[22]

> **Step conversion.** If $Y \in R^F(X)$, then either $X \in R^P(Y)$ or $\exists X' \gg X$ s.t. $X' \in R^P(Y)$.

However, in full generality, these may fail. Consider this instance of (FP):

$$\neg D\langle\mathsf{F}\rangle\mathsf{A} \to [\mathsf{F}]\langle\mathsf{P}\rangle\neg D\langle\mathsf{F}\rangle\mathsf{A}$$

Informally: if it's not determinate that the future verifies A, it will always be the case that in the past it's not determinate that the future verifies A. This may fail for reasons related to Holliday's quote from earlier: it may now be indeterminate that the future contains a sea battle, but as we advance forward into the future, we may land into a possible future where the past 'has changed'. The model $\mathcal{SB}$ gives formal content to this counterexample.[23]

With regards to the open future lore, a theorist in that vein might aim to validate $D\langle\mathsf{P}\rangle\mathsf{A} \leftrightarrow \langle\mathsf{P}\rangle D\mathsf{A}$ (because the past is settled) without validating $D\langle\mathsf{F}\rangle\mathsf{A} \leftrightarrow \langle\mathsf{F}\rangle D\mathsf{A}$. After all, if the future is open, one might agree that at some time it will be determinate that there is a sea battle, while denying that it is presently determinate that there will be one.

These are exactly the predictions of our system provided we adopt a frame condition that represents the fact that the past is fixed.

[22] Proof: Fix $X \gg Y$, $\mathcal{M}, X, Y \Vdash \mathsf{A}$ then:

(i) $\mathcal{M}, X, Y \Vdash [\mathsf{F}]\langle\mathsf{P}\rangle\mathsf{A} \Leftrightarrow \forall K \in R^F(X), \forall K' \gg K, \exists Z \in R^P(K'), \mathcal{M}, Z, Z \Vdash \mathsf{A}$

Fix $K \in R^F(X)$ and $K' \gg K$. By (Rd), $K' \in R^F(X)$. By step conversion, either $X \in R^P(K')$ or $\exists X' \gg X$, $X' \in R^P(K)$. Either way $\exists X' \in R^P(K'), \mathcal{M}, X', Y \Vdash \mathsf{A}$. Provided that A is from the Boolean sublanguage $\exists X' \in R^P(K'), \mathcal{M}, X', X' \Vdash \mathsf{A}$, which combined with (i) entails $\mathcal{M}, X, Y \Vdash [\mathsf{F}]\langle\mathsf{P}\rangle\mathsf{A}$.

[23] Check that $\mathcal{SB}, X_m, X_m \Vdash \neg D\langle\mathsf{F}\rangle\mathsf{A}$, but $\mathcal{SB}, X_m, X_m \nVdash [\mathsf{F}]\langle\mathsf{P}\rangle\neg D\langle\mathsf{F}\rangle\mathsf{A}$. Towards the latter note: $Z_t \in R^F(X_m)$, and Z_t's only past is Z_m. At Z_m, it *is* determinate that in the future there will be a sea battle.

Fixity of the past. If $X \gg Y$, $Z \in R^P(Y) \to Z \in R^P(X)$.

Given (uR), this can be strengthened to a biconditional. But the part that is *not* entailed by (uR) induces a key asymmetry between past and future. The principle states that if X refines Y, and Z is in the past of Y, Z must also be in the past of X.

Fact 9 *Given Fixity of the past:*

(i) $\not\Vdash \langle F\rangle DA \to D\langle F\rangle A$

(ii) $\Vdash D\langle F\rangle A \to \langle F\rangle DA$ *and* $\Vdash D\langle P\rangle A \to \langle P\rangle DA$

(iii) $\Vdash \langle P\rangle DA \to D\langle P\rangle A$

Part (i): the model $\mathcal{SB}$ is a counterexample with X_m as base possibility and p as instance. Among the refinements of X_m that support $\langle \mathsf{F}\rangle Dp$ pick Z_m. So, $\mathcal{M}, Z_m, X_m \Vdash \langle \mathsf{F}\rangle Dp$. However, $\mathcal{M}, Z_m, X_m \not\Vdash D\langle \mathsf{F}\rangle p$. To see this recall that $\mathcal{M}, Z_m, X_m \Vdash D\langle \mathsf{F}\rangle p$ iff $\mathcal{M}, X_m, X_m \Vdash \langle \mathsf{F}\rangle p$, and note that $\mathcal{M}, X_m, X_m \not\Vdash \langle \mathsf{F}\rangle p$, since Y_m refines X_m, but Y_m does not have a future at which p is true.

Part (ii): It is easy to check the validity of $\langle \mathsf{F}\rangle \mathsf{A} \to \langle \mathsf{F}\rangle D\mathsf{A}$, which together with an instance of the T axiom, $D\langle \mathsf{F}\rangle \mathsf{A} \to \langle \mathsf{F}\rangle \mathsf{A}$, yields (ii).

Part (iii): Suppose (a) $\mathcal{M}, X, Y \Vdash \langle \mathsf{P}\rangle D\mathsf{A}$ for $X \gg Y$. We want to check (b) $\mathcal{M}, X, Y \Vdash \langle \mathsf{P}\rangle D\mathsf{A}$ for $X \gg Y$. Respectively these are equivalent to:

(a') $\forall Z \gg X, \exists K \in R^P(Z), \mathcal{M}, K, K \Vdash \mathsf{A}$.

(b') $\forall Z' \gg Y, \exists K' \in R^P(Z'), \mathcal{M}, K', K' \Vdash \mathsf{A}$

Fix $Z' \gg Y$. Because $X \gg X$, $\exists K \in R^P(X)$, as in (a'). By $X \gg Y$ and (uR), $K \in R^P(Y)$. By Fixity $K \in R^P(Z')$. □

Let us take stock. Two-dimensional tense operators make for a better behaved system. Although the analysis fell short of characterizing the logic of this full system, it establishes that the two-dimensional tenses behave in important respects like *bona fide* tense operators. Specifically, they satisfy the core of tense logic, albeit within limits. Moreover, we noticed that under a frame condition capturing the fixity of the past, we were able to capture core principles of open future lore. Much more would need to be said to provide a full vindication of the two-dimensional possibility semantics as a model of the open future, especially in comparison to branching time approaches. But the subject is beyond the scope of the present work.

9 Conclusion

The main conclusions are as follows: there is a clear path for the application of possibility semantics to the metaphysical hypothesis of the open future. That path must include the characterization of object language determinacy operators. Introducing such operators under something like the alignment constraint requires, on pain of inconsistency, some modifications to the original framework. A two-dimensional variant of possibility semantics is one path to relieve this theoretical pressure. In its natural interpretation, the logic of determinacy under the two-dimensional analysis is the global version of S5.

The most immediate development of this idea is feasible under a broadly eternalist conception of propositions, and under the hypothesis that the refinement relation and the (reflexive closure of the) temporal precedence relation collapse. It appears important to explore the prospects for the two-dimensional analysis in a context that does not involve these structural assumptions. Holliday (2022) has already provided key insights for how to think about facets of the open future without collapsing refinement and temporal precedence. Integrating these insights within the two-dimensional framework highlighted the promise of thinking of tenses two-dimensional operators in their own right.

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