Modeling future indeterminacy in possibility semantics

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Abstract

I consider the application of possibility semantics to the modeling of the indeterminacy of the future. I argue that interesting problems arise in connection to the addition of object-language determinacy operator. I show that adding a two-dimensional layer to possibility semantics can help solve these problems.

1 Introduction

Possibility semantics offers an elegant framework for a semantic analysis of modal logic that does not recruit fully determinate entities such as possible worlds.[1] In this paper, I consider the application of possibility semantics to the modeling of the indeterminacy, or openness, of the future, and some other related forms of metaphysical indeterminacy. I argue that interesting technical and conceptual

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[1]The phrase "possibility semantics" was coined by Humberstone (1981). The conceptual and technical tools undergirding the framework have longer histories, including (Fine, 1975, especially §2), Humberstone (1979), as well as deep roots in the algebraic logic tradition. For a contemporary and comprehensive introduction, see Holliday (forthcoming).

Possibility semantics is one of a variety of styles of theories that do not rely on worlds, but on coarser objects. In addition to possibility semantics, the general family of "pointless" theories includes various kinds of states-based semantic analyses (Aloni 2018, Willer 2018), truthmaker semantics (Fine, 2017b), as well as several varieties of situation semantics (Barwise and Perry, 1981; Kratzer, 2021). It would be desirable to have a comparative study of these frameworks highlighting the commonalities, as well as the differences, between them.

problems arise in connection to the explicit modeling of indeterminacy within the object language. I view possibility semantics as an alternative to more established supervaluationist models (Thomason 1970) in which indeterminacy is grounded in the overlap of complete possibilities.

As I understand it, the open-future hypothesis is the claim that some future events and states are objectively, and not merely epistemically, unsettled.[2] I do not assume that the unsettledness of the future and quantum indeterminacy are one and the same. However, my central illustrative example will be the proposition that some specific radioactive atom will decay in an even number of days. If radioactive decay is a genuinely indeterministic process, then the present truth status of this proposition would appear to be objectively unsettled.

The indeterminacy associated with the future seems unlike other kinds of indeterminacy that have attracted the attention of philosophers. For example, it seems unlike the indeterminacy that some theories associate with vagueness. For one thing, it does not appear to give rise to higher-order indeterminacy. It is generally agreed by those who think that vagueness is grounded in some kind of indeterminacy that it may itself be indeterminate whether Joe is borderline tall. By contrast, it is common to assume that, as far as the unsettledness of the future is concerned, there are no states or events whose determinacy status is itself indeterminate. It might be unsettled whether there will be a sea battle tomorrow, but is cannot be unsettled whether it's unsettled. A second marker of the indeterminacy of the future is that it is not plausibly associated with unusual effects on credence. Many different philosophers have been attracted to the view that there is something non-classical about credence in the contents of vague statements. One form of this is Field's (2000) claim that vague contents require low credence in certain instances of the law of excluded middle; another is Williams's claim that vague contents seem to require imprecise probability (2014).[3] By contrast, statements about the future appear to be paradigmatic examples of classical credence. In prototypical cases, it seems perfectly warranted for me to have a sharp credence that the coin will land heads.

With that said, the formal discussion to follow is not be restricted in scope to the alleged indeterminacy of the future. It will pertain to any application of

[2] There is much literature on what constitutes the (alleged) openness of the future. The present discussion leans in various ways on Thomason (1970); Belnap and Green (1994); Belnap *et al.* (2001); MacFarlane (2003, 2014); Barnes and Cameron (2009, 2011); Torre (2011); Cariani and Santorio (2018); Cariani (2021b); Todd (2022).

[3] The matter is highly complicated, in ways that go beyond the relatively simple demarcation point I am making in this paragraph. For a sophisticated discussion, see Bacon (2018).

possibility semantics to concepts of indeterminacy that do not give rise to higher-order indeterminacy and are not associated with low credence in instances of the law of excluded middle. As an example, Stalnaker (1984) famously suggests that counterfactual selection results in a kind of indeterminacy, and has more recently suggested that this kind of indeterminacy might be viewed as a 'milder' version of the indeterminacy that is associated with the future (Stalnaker, 2019, p.197-ff).

Despite being willing to explore the indeterminacy hypothesis, I am not myself committed to the claim that the future is open. I am inclined to entertain it as a hypothesis I am willing to take seriously, but nothing more than that. As Stalnaker (2019, p.197) puts it, "You don't have to sign on to this metaphysical theory (as I do not) in order to find it intelligible (as I do) and to use it as a kind of precedent for a case where the thesis of metaphysical indeterminacy may be less controversial." My own motivation is to use the open future hypothesis as a testing ground for reflecting on the foundations and potentials of the possibility semantics framework.

I will lead with a general introduction to possibility semantics for a sentential modal language (§2). Next, I focus on the representation of indeterminacy in possibility semantics (§3). The framework itself already incorporates a representation of indeterminacy in the model theory. However, contrary to the inclination of Humberstone (1981), I believe that it is important to have ways of capturing indeterminacy in the object language. Unfortunately, it is not possible to add a determinacy operator with the right profile to the system—not at least without other interventions. The main contribution of §3 is an impossibility result to this effect. After considering some possibilities that would fix the inconsistency by means of local interventions (§4), I suggest that a superior solution to the problem lies in the integration of possibility semantics with a two-dimensional framework. Much of the technical substance of this solution is owed to remarks in Fine (1975). The Cliffs notes on Fine's paper focus on the fact that it is the first application of supervaluationist techniques to vague language.[4] However, it is also a central juncture for the logical development of semantics based on partial objects, since Fine builds up to the supervaluationist machinery by first analyzing a system in which precisifications of a vague language are viewed as partial. My ambition is to recast some of Fine's insights about determinacy operators in a different theoretical context, allowing some distinct issues and theoretical choice points to come to

[4]It is worth highlighting to keep the record straight that Fine no longer endorses this account (Fine, 2017a).

light.[5]

2 Background on possibility semantics

The basic ideological tenet of possibility semantics is that formulas are not evaluated against worlds, but against "coarser" objects called *possibilities*.[6] This ideology differs from that of the standard technology for modeling the indeterminacy of the future, which is broadly supervaluationist (Thomason, 1970, 1984). According to the supervaluationist model, indeterminacy results from there being multiple complete possibilities with equal claim to fit the settled facts. The exact details of the analysis here depend on deeper metaphysical commitments. For instance, someone with broadly ersatzist leanings might say that the indeterminate reality is represented by multiple, incompatible perfectly determinate representations (Barnes and Cameron, 2009, 2011; Barnes and Williams, 2011).

Possibility semantics proceeds in a different way. Instead of taking a maximally precise representation as its basic modeling object, it deploys primitive objects that are themselves incomplete. That incompleteness is naturally associated with a concept of indeterminacy: possibilities are understood as settling the truth values of some sentences of a language, while leaving others unsettled.

I adopt the basic template of possibility semantics from Humberstone (1981). The language is a sentential modal language, whose signature features a non-empty countable set of modal operators. Models for this language are quadruples of the form, $\langle P, \sqsupseteq, \mathbf{R}, V \rangle$. Here P represents a non-empty set of possibilities; $\sqsupseteq$ is a refinement relation over the possibilities. Structurally, $\sqsupseteq$ is a *weak partial order* (thus, it is transitive and antisymmetric). From an intuitive standpoint, $Y \sqsupseteq X$ holds when everything that is settled as either true or false by X is settled in the same way by Y. In short, Y agrees on all the determinate facts that X settles. (Explicit structural assumptions are needed in order to guarantee that models satisfy this intuition, and they will be provided in short order.) $\mathbf{R}$ is a non-empty set of accessibility relations,

[5] The idea of using possibility semantics to model the unsettledness of the future is also explored in a preliminary way in Boylan (forthcoming). However, because Boylan is focused on a different set of problems, he ends up in a theoretical space that is not compatible with the present outlook, especially with regards to the analysis of negation.

[6] Once my technical development is complete, I will consider the philosophical question of what kind of philosophical motivation one might have for *adopting* possibility semantics in a semantic model of future discourse. The technical development itself can proceed modularly: what I aim to show from a technical point of view is that the analysis can equally adopt a different starting point than the standard one, and moreover that this alternate starting point is relatively intuitive. Furthermore, there might be reasons of convenience to exploit partial possibilities that would justify the development of their formal theory, much as there might be reasons of convenience for developing temporal semantics using intervals instead of instants (as explored by Humberstone, 1979).

and finally V is a partial valuation function: in this setting a valuation function inputs an atomic formula and a possibility, and if defined outputs either 0 or 1.

Models for this language are ordinarily assumed to satisfy two constraints.

> **Refinability.** For every atomic formula A and possibility X, if $V(A, X)$ is undefined—which we'll write as $V(A, X)\uparrow$—then there are Y, Z such that $Y \sqsupseteq X$ and $Z \sqsupseteq X$, s.t. $V(\mathsf{A}, Y) = 1$ and $V(A, Z) = 0$.
>
> **Persistence.** For all atomic A, if $V(A, X)\downarrow$ and then for every $Y \sqsupseteq X$, then $V(\mathsf{A}, X) = V(\mathsf{A}, Y)$.

Persistence says that whenever atomic A is settled at X, it stays settled in the same way through refinements. Refinability says that whenever an atomic formula A is unsettled at a possibility X, there are Y and Z both refinements of X that settle A as true and false respectively. Refinability is related to, but logically distinct from, the assumption that any partial possibility might be refined all the way to a complete one (which Fine 1975 calls "Completability").

Persistence is required to give formal representation to the intuitive conception of refinement. Indeed, under persistence, it is tempting to think of refinement structures as mirroring the structure of the branching models for future contingency,[7] as illustrated by the diagram in Figure 2.

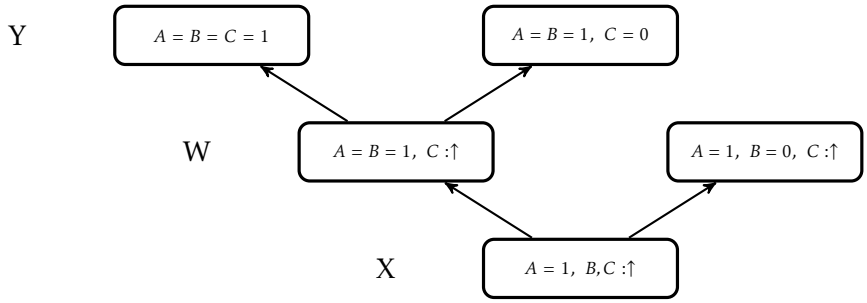

Figure 1: The branching structure of refinements ($Y \sqsupseteq W \sqsupseteq X$)

However, an important lingering difference — which the formal theory ought to help disentangle — is that standard branching models are built on the idea of maximal *histories*, which at any moment assign a definite truth-value to all the formulas of the language. Indeed, the linear paths through the tree can naturally be viewed as temporally structured possible worlds. No such assumption of completeness is imposed on possibility models.

[7] See Thomason (1970); Belnap *et al.* (2001); MacFarlane (2014), Cariani (2021b, ch.2) for discussion of branching models.

Humberstone's semantic entries rely on the idea of using a valuation function defined on the atomic formulas of $\mathcal{L}$ to ground a notion of support between possibilities and formulas of the whole language. They are as follows:

- $M, X \Vdash p$ iff $V_M(p, X) = 1$
- $M, X \Vdash \mathsf{A} \wedge \mathsf{B}$ iff $M, X \Vdash A$ and $M, X \Vdash B$
- $M, X \Vdash \neg\mathsf{A}$ iff for all $Y \sqsupseteq X$, $M, Y \nVdash \mathsf{A}$
- $M, X \Vdash \Box_i\mathsf{A}$ iff for all $Y \in P$, s.t. $R_i XY$, $M, Y \Vdash A$

As for other operators, such as $\vee$, $\rightarrow$, $\Diamond$, a common approach recovers entries by fixing some standard equivalences. In the case of disjunction, one might assume it characterized by conjunction, negation and DeMorgan's laws.[8] This results in the following entry:

- $M, X \Vdash \mathsf{A} \vee \mathsf{B}$ iff for all $Y \sqsupseteq X$, there is $Z \sqsupseteq Y$, s.t. $M, Z \Vdash A$, or there is $Z \sqsupseteq Y$, s.t. $M, Z \Vdash B$

Another route to the same goal would be to stipulate some general principles about what it takes for various kinds of possibilities to settle a disjunction as true/false (Holliday, forthcoming).

- A possibility X settles a disjunction $\mathsf{A} \vee \mathsf{B}$ as false iff it settles A as false and settles B as false.

Assume that a possibility settles A as false iff it settles $\neg\mathsf{A}$ as true. Next, note that the entries for negation and conjunction tell us that:

- A possibility X settles a conjunction $\mathsf{A} \wedge \mathsf{B}$ as true iff it settles both A and B as true.
- A possibility X settles A as false iff every refinement of X fails to support A.

These assumptions are sufficient to pin down the same entry for disjunction as above. A similar analysis could be carried out for the other operators.[9]

The last component of Humberstone's system to require comment is the consequence relation, which is straightforwardly defined as preservation of support.

[8] As an alternative, disjunction could be defined instead by the condition $M, X \Vdash \mathsf{A} \vee \mathsf{B}$ iff $M, X \Vdash A$ or $M, X \Vdash B$. This would have the effect of making the sentential fragment of the logic non-classical.

[9] It is important to note that, while the analysis of the necessity operator simply lifts Kripke semantics to the level of possibilities, Humberstone's account of modality also requires the specification of interplay conditions connecting accessibility and refinement.

(I1) for all X, Y, Z, if $Z \sqsupseteq X$ and RZY, then RXY

Definition 1 *$A_1, ..., A_n \vdash B$ iff for all models M and any X in P_M, if for all i, $M, X \Vdash A_i$, then $M, X \Vdash B$.*

It is a well established fact about this formalism that the logic of the sentential fragment is classical, both in the sense that the set of logical truths coincides with the set of classical tautologies, and in the sense that the class of valid arguments in this fragment coincides with the class of tautologically valid arguments (Humberstone, 1981, pp.320-321).

3 Adding object language determinacy operators

A proponent of possibility semantics might reasonably claim that the framework incorporates a model of indeterminacy: an atomic formula A is indeterminate at a possibility X when X leaves A undefined. Imagine a possibility X and an atomic formula, *even*, for "Bob will decay in an even number of days". In a clear sense, the metatheoretic fact $V_M(even, X) \uparrow$ represents the relevant indeterminacy from the perspective of the model theory. In light of this, it would be warranted to say that possibility semantics incorporates a metatheoretic representation of indeterminacy.

However, as the system is set up, there is no *object language* device that allows us to talk about indeterminacy. That is: there is no operator for expressing things like *it is determinate that the coin landed heads today, but it is not determinate that it will land heads tomorrow*. This is unfortunate because, for various modeling purposes, it's important to have determinacy operators in the object language. Determinacy operators are crucial devices in the application to vagueness for characterizing higher-order vagueness — for example, to distinguish between the claim that a proposition **A** is indeterminate from the claim that it is indeterminate whether **A** is indeterminate (Fine, 1975). They also have important roles to play outside of that context, because determinacy interacts non-trivially with other concepts. To take just one example drawn from the recent literature, Cariani (2021a) explores interactions between (in)determinacy operators and epistemic operators. In this kind of discussion, certain lifting principles become important, such as $\neg D\mathsf{A} \rightarrow \neg K\mathsf{A}$—the principle that if A is not determinate, then it is not known. Such principles, and the kinds of constraints they impose on models, are best analyzed from the perspective

(I2) for all X, Y, Z, if $Z \sqsupseteq Y$ and RXY, then RXZ

(R) for all X, Y, if RXY then for some $X' \sqsupseteq X$, for all $Z \sqsupseteq X'$, RZY

(I1) says that if Z refines X, then X accesses everything Z accesses; (I2) says that if Z refines Y, the any X that accesses Y accesses Z; (I3) says that if X accesses Y, then looking at the (upside-down) tree of refinements of X, there is a branch possibly starting below X itself, where possibility on this branch accesses Y.

of the object language, and of course carrying out this analysis requires the language to have this kind of expressive resource.[10]

Let us then introduce a determinacy operator D to the formal language, as well as an indeterminacy operator I governed by the condition in Definition 2, which is standardly taken to be definitional of indeterminacy (e.g. in Fine, 1975):

Definition 2 $IA =_{df} \neg DA \wedge \neg D\neg A$

In many respects that are going to be relevant, non-determinacy (which is expressed by '$\neg D$') behaves similarly to indeterminacy. It is important however to keep in mind that in the present terminology 'indeterminacy' denotes a two-sided status, in the sense that it requires that both A and its negation fail to be determinate. By contrast, non-determinacy is a one-sided status: a proposition may fail to be determinate, while its negation is determinate.

The addition of determinacy operators to the language of possibility semantics should be guided by some key constraints. The first is that object language indeterminacy should, in a precise sense, align with metatheoretic indeterminacy. The simplest statement of this constraint is at the level of atomic formulas:

Constraint 1 (Alignment) *For atomic* A, $M, X \Vdash IA$ *iff* $V_M(A, X)\uparrow$.

Alignment entails a second constraint: formulas expressing non-determinacy (and indeterminacy) claims must violate (a generalization of) persistence. Our initial formulation of persistence applied to the atomic formulas of the language, but there is an entirely natural generalization of it involving the concept of support. A possibility X might support that it's indeterminate whether the atom will decay in an even number of days, while at the same time it could be refinable into a possibility Y that settles that the atom will decay in an even number of days. The exact principle that follows from this is:

Constraint 2 (Non-persistence of non-determinacy) *There is a formula* A, *and model* M *with possibilities* $X, Y \in P_M$ *and* $Y \sqsupseteq X$ *such that* $M, X \Vdash \neg DA$ *but* $M, X \nVdash \neg DA$

Constraint 3 (Non-persistence of indeterminacy) *There is a formula* A, *and model* M *with possibilities* $X, Y \in P_M$ *and* $Y \sqsupseteq X$ *such that* $M, X \Vdash IA$ *but* $M, X \nVdash IA$

[10] For some additional considerations in favor of introducing object language determinacy operators, see also (Barnes and Williams, 2011, §5)

With enough of the possibility framework on board, the route from Alignment to non-persistence is relatively straightforward.

Fact 1 *Given Definition 2 and Refinability, Alignment entails Non-persistence of indeterminacy and of non-determinacy.*

> *Proof.* Consider a model M with two possibilities X and Y drawn from its possibility set, such that $Y \sqsupseteq X$. Suppose in particular that Y settles some atomic formula A which X leaves unsettled. The existence of such a Y is guaranteed by Refinability. Then $V_M(\mathsf{A}, X)\uparrow$ but $V_M(\mathsf{A}, Y) = 1$ or $V_M(\mathsf{A}, Y) = 0$ and so $M, X \Vdash I\mathsf{A}$ but $M, Y \nVdash I\mathsf{A}$. For (ii), exploit Refinability to suppose that Y refines X so that $V_M(\mathsf{A}, Y) = 1$. Definition 2 yields $M, X \Vdash \neg D\mathsf{A}$, but from the way Y refines X it follows that $M, Y \Vdash D\mathsf{A}$. □

Alignment provides powerful motivation for Non-persistence. It is nonetheless valuable to keep the claims separate, because Non-persistence is weaker and might motivated in other ways. Another reason to keep these separate is that there will be some reason to consider versions of possibility semantics that drop Refinability (§7).

While these constraints seem plausible, important difficulties are lurking under the surface. Some of the ingredients that constitute the framework as presented up to this point are in fact jointly inconsistent. In particular, there is a tension between the analysis of indeterminacy in Definition 2, the Non-persistence of indeterminacy and the analysis of negation.

Fact 2 *The following are inconsistent (given the framework):*

IN. $IA \equiv_{df} \neg DA \wedge \neg D\neg A$.

NP. *There are* M, A, X, $Y \sqsupseteq X$ *with* $M, X \Vdash IA$ *but* $M, Y \nVdash IA$.

NE. $M, X \Vdash \neg A$ *iff for all* Y, $Y \sqsupseteq X$, $M, Y \nVdash A$

> *Proof.* Consider witnesses, M, X, Y, A for NP. By IN, $M, X \Vdash \neg D\mathsf{A} \wedge \neg D\neg\mathsf{A}$. By the clause for conjunction, $M, X \Vdash \neg D\mathsf{A}$ and $M, X \Vdash \neg D\neg\mathsf{A}$. By the clause for negation (NE), $D\mathsf{A}$ and $D\neg\mathsf{A}$ cannot be supported throughout any refinements of X. That is, for all $Z \sqsupseteq X$, $M, Z \nVdash D\mathsf{A}$ and $M, Z \nVdash D\neg\mathsf{A}$. However, since any refinement of X is a refinement of Y, we must also have $M, Y \Vdash \neg D\mathsf{A}$ and $M, Y \Vdash \neg D\neg\mathsf{A}$, and hence, by IN, $M, Y \Vdash I\mathsf{A}$. This contradicts the fact that M, A, X, Y were chosen as witnesses for the existential in NP. □

A plausible initial diagnosis here is that the problem arises because $\neg$ forces persistence—that is to say $\neg A$ must be persistent, whether A is persistent or not. A consequence of this fact is that the indeterminacy operator I cannot be both defined in terms of negation and also such that formulas of like IA are non-persistent.

This inconsistency is related to a less specific unease with object language indeterminacy operators that is already expressed by Humberstone (1981). Humberstone claims that an indeterminacy operator like the one I just introduced would go "against the spirit of the present enterprise, since it would give rise to formulas which were not persistent into refinement [...], and thus undermines the idea of refinements as mere resolvers of indeterminacy". Humberstone's exact concern is hard to pin down, and certainly broader than the inconsistency articulated in Fact 2.[11] But whatever we may think of the broad concern, the inconsistency does show that adding (in)determinacy operators is not entirely innocent.

4 A preliminary journey around the options

I propose to push through these difficulties and identify a path for integrating possibility semantics with object language determinacy operators. Evidently, this path requires giving up one of IN, NP, or NE. In other words, it requires either altering the definition of indeterminacy, or giving up non-persistence or modifying the analysis of negation. The option of giving up IN is a non-starter and may be set aside immediately. The problem is not merely that the definition of indeterminacy captured by Definition 2 is relatively well entrenched, which it is. The real issue is that a version of the inconsistency in Fact 2 arises for $\neg D$A, independently of how IA is defined.

By contrast, some versions of the second option—denying the non-persistence constraint—seem more promising. One might motivate persistence by thinking in terms of temporally indexed indeterminacy operators.[12] To illustrate the essence of the approach, start by noting that in the relevant applications there is a connection between refinement and temporality: advancing through time along a history should be result in encountering more and more refined possibilities. Under this temporal interpretation, it might seem attractive to entertain determinacy operators that are relativized to a specific point in time. Under this approach, the object language would feature a family of operators $\{D_t \mid t \in T\}$, where T is a designated

[11] This is easily noted he uses this kind of argument to press against other non-persistent operators, including ones that do not give rise to inconsistencies like the one I identified above.

[12] I owe this suggestion to [BLINDED]

set of times in the model.[13] Simplifying a bit, imagine that the set of times that are distinguished in a given possibility model is finite. Then imagine a a family of operators $D_0, D_1, D_2, ...D_n$ each marking what is determinate at a certain point in the development of history, with each D_i anchored to some specific time t_i. To complete the proposal say that the language does not contain any unrelativized determinacy operators, and thus that all determinacy discourse is expressed by means of relativized ones.

This sort of model would undermine some of the motivation for non-persistence assumptions. Suppose again that X_{Mon} represents Monday's possibility, in which Bob has not yet decayed, and X_{Wed} represents the state of affairs on Wednesday. If all one had was unrelativized determinacy operators, one should approach this by saying that $\neg D(even)$ is supported at X_{Mon} but unsupported at X_{Wed}. The relativized framework opens up a different option: X_{Mon} supports $\neg D_{Mon}$, while X_{Wed} supports D_{Wed}. Crucially, each of D_{Mon} and D_{Wed} can be assumed to be persistent even under negation. The intuitive meaning of D_{Mon}A would be something like "A is/was settled true on Monday". From Wednesday's point of view—i.e., as far as X_{Wed} is concerned—$\neg D_{Mon}$A remains supported. Relatedly, the claim $\neg D_{Monday}(even) \wedge D_{Wednesday}(even)$ is perfectly consistent (from any point in time).

This approach is valuable, and my own solution incorporates some of its insight. However, it also seems unsatisfactory: it is not controversial that people possess an unrelativized concept of indeterminacy — plausibly, the very same concept that's captured at the level of the metatheory by the idea of valuations leaving some formulas unsettled. There is no special reason to think that there are barriers to expressing it in the object language.

Let us then consider a third option. The initial hunch concerning the incompatibility in Fact 2 was that it is due to the persistence-forcing effect of negation. The obvious alternative would be to introduce a type of negation that does not force persistence.[14] To this end, introduce '$\sim$' as the connective characterized by the clause: $M, X \Vdash \sim$A iff $M, X \nVdash$ A. This alternate negation operator does not have the effect of transforming a non-persistent claim into a persistent one. Indeed, it would make correct predictions for non-determinacy claims. The problematic inconsistency arises because $\neg D$A is forced to be persistent. This alternate operator

[13] It requires a bit of manipulation to endow standard model of branching time with times. In particular, what is required is a simultaneity relation that connects points on different branches. See [BLINDED] for discussion.

[14] Humberstone (1979) considers this alternative negation for a similar application. This is also the negation that Boylan (forthcoming) uses in his application of possibility semantics to the future.

avoids the problem: if A is indeterminate at X but becomes determinate at Y, then $\sim D$A is non-persistent.

An evident problem with this approach is that '$\sim$' cannot be the correct negation operator for the entire language. Outside of determinacy claims, '$\sim$' conflates non-support with rejection. It is evidently undesirable for *it's not the case that the coin will land heads* to be supported by a possibility that merely fails to settle heads. More generally, '$\sim$' is not the correct negation operator for the sentential fragment of the language.

In response, one might consider a language in which the two negation operators, '$\neg$' and '$\sim$', coexist. Footnote 15 of Humberstone (1981) identifies an expressive advantage to having both operators (though he would not, and does not endorse the suggestion that is currently being explored): their combination, '$\neg\sim$', is a plausible candidate for a determinacy operator, as it expresses universal quantification over all refinements. (So $M, X \Vdash \neg\sim$ A iff all refinements of X support A). However, for the present application, having both operators around is not well-motivated. There is no principled reason for why one negation operator ($\neg$) should apply in the D-free language, while the other operator should apply to formulas involving D. Additionally, any attempt at formulating a generalization concerning which operator is appropriate for a given formula would have to deal with the thorny problem of choosing the correct negation for mixed formulas (like the negation of A $\wedge$ DA), Ultimately, it is unprincipled to have two negation operators floating around without a systematic justification for their distinct roles.

5 Introducing two-dimensional possibility semantics.

In this section, I argue that a two-dimensional version of possibility semantics provides the way out of the problem.[15] Before presenting it, let me collect the desiderata we picked up along the way. What is needed is a version of possibility semantics that incorporates a non-persistent, non-relativized determinacy operator that is "aligned" with the metatheoretic concept of indeterminacy that is ordinarily built into the possibility semantics framework. The logic is to be classical in the

[15]Thanks to [BLINDED] for pointing me in this direction. For some some general surveys on canonical applications of two-dimensional semantics see Humberstone (2004); Kuhn (2013); Schroeter (2021). The suggestion of a two-dimensional treatment of the determinacy operator is first explored in Fine (1975). Fine rightfully questions the ability of such an operator to handle higher-order indeterminacy, but of course this concern is not salient in the present application. The present claim is not that a two-dimensional semantics is anything new, but that it provides an elegant solution to an otherwise extremely thorny puzzle. A slight variation of a two-dimensional determinacy operator is also introduced in (Burgess and Humberstone, 1987, §6.2).

sentential fragment, and the D operator must not trivialize. As a specific litmus test, $\mathsf{A} \vee \neg\mathsf{A}$ is to be valid (because the logic is classical) while $D\mathsf{A} \vee D\neg\mathsf{A}$ is not. Finally, the system must avoid conflating failure to support with rejection.

The opening move in crafting such a framework is to distinguish two dimensions of evaluation. In addition to evaluating at a pair consisting of a model and a possibility, consider evaluating at a triple M, X, Y consisting of the model and *two* possibilities. Doubling the evaluation possibility allows it to play two separate roles: one coordinate of evaluation is operated on by connectives (call this the 'primary possibility'), while the other is read by the determinacy operator D and left untouched by the connectives (call this the 'secondary possibility'). As noted, two-dimensional evaluation is "in addition to" the standard unidimensional evaluation. The relation between two-dimensional and unidimensional evaluation is governed by a standard diagonal principle:

Diagonal principle: $M, X \Vdash \mathsf{A}$ iff $M, X, X \Vdash \mathsf{A}$

The conceptual motivation for continuing to value unidimensional evaluation is that the ultimate explanans remains a concept of truth, or support, at a possibility—possibly representing a temporal snapshot of an unsettled world. Indeed, two-dimensional possibility semantics inherits the definition of consequence as preservation of support at a model thanks to the diagonal principle.

Recursive clauses for the connectives and for the determinacy operator are specified at the level of two-dimensional evaluation. Note that the new dimension of evaluation is largely idle, except for contributing to the interpretation of the determinacy operator.

(i) $M, X, Z \Vdash p$ iff $V_M(p, X) = 1$

(ii) $M, X, Z \Vdash \mathsf{A} \wedge \mathsf{B}$ iff $M, X, Z \Vdash A$ and $M, X, Z \Vdash B$

(iii) $M, X, Z \Vdash \neg\mathsf{A}$ iff for all $Y \sqsupseteq X$, $M, Y, Z \nVdash \mathsf{A}$

(iv) $M, X, Z \Vdash \Box_i\mathsf{A}$ iff for all $Y \in P$, s.t. $R_i XY$, $M, Y, Z \Vdash A$

(v) for $\vee$, $\rightarrow$, $\Diamond$, use standard equivalences to infer clauses.

(vi) $M, X, Z \Vdash D\mathsf{A}$ iff $M, Z, Z \Vdash \mathsf{A}$

It is notable that, under this analysis, the determinacy operator resembles an actuality operator in more standard applications of two-dimensional semantics. It

evaluates the argument of DA after setting the primary evaluation possibility so as to match the secondary one.

Logical consequence remains defined as preservation of unidimensional support, as per Definition 1. Furthermore, Refinability and Persistence continue to be in place. It is important to emphasize again that, in their original formulation, these principles are explicitly restricted to atomic formulas. While they have generalizations for the full language, the status of those generalizations is not settled by the status of their atomic variants. Thus, saying that the complex formula IA is non-Persistent is fully compatible with saying that atomic formulas persist through refinements.

6 Victory lap

This section has two objectives: the broad objective is to illustrate that the system matches the main desiderata for adding an object language determinacy operator. More narrowly, once those general features are established, it aims to illustrate that the system incorporates a way out of the central incompatibility identified in Fact 2.

First, notice that the logic in the sentential fragment remains classical.

Fact 3 *The consequence relation in the sentential fragment matches the extension of classical sentential logic.*

> *Proof.* In the D-free fragment of the language, the second evaluation coordinate is idle, and thus all of Humberstone's results in unidimensional possibility semantics carry over, including the classicality of the logic in the sentential fragment.

As noted, the persistence constraint has a natural generalization concerning arbitrary formulas and involving the notion of support.

Definition 3 *(i) An arbitrary formula A is g-persistent in M iff for all* $X, Y \in P_M$ *with* $Y \sqsupseteq X$, $M, X \Vdash A$ *but* $M, Y \nVdash A$*; (ii) A is g-persistent iff for all M, A is g-persistent in M.*

It is now possible to consider, and establish, the claim that $\neg D$A and IA are not persistent.

Fact 4 (Non-persistence of non-determinacy and indeterminacy) *Let A be an atomic formula. Then* $\neg DA$ *and* IA *are not g-persistent.*

Proof. I illustrate this with the atomic formula E, but nothing depends on the choice of formula. We want to identify a model M in which $\neg D\mathsf{E}$ and $I\mathsf{E}$ are not g-persistent. Consider a model with three possibilities X, Y and Z with $Y, Z \sqsupseteq X$ and such that $V(\mathsf{E}, X)\uparrow$, $V(\mathsf{E}, Y) = 1$, $V(\mathsf{E}, Z) = 0$.

- $M, X \Vdash \neg D\mathsf{E} \Leftrightarrow M, X, X \Vdash \neg D\mathsf{E} \Leftrightarrow \forall W \sqsupseteq X : M, W, X \nVdash D\mathsf{E} \Leftrightarrow \forall W \sqsupseteq X : M, X, X \nVdash \mathsf{E} \Leftrightarrow M, X, X \nVdash \mathsf{E} \Leftrightarrow V(\mathsf{E}, X) \neq 1$.
- $M, Y \Vdash \neg D\mathsf{E} \Leftrightarrow M, Y, Y \Vdash \neg D\mathsf{E} \Leftrightarrow \forall W \sqsupseteq Y : M, W, Y \nVdash D\mathsf{E} \Leftrightarrow \forall W \sqsupseteq X : M, Y, Y \nVdash \mathsf{E} \Leftrightarrow M, Y, Y \nVdash \mathsf{E} \Leftrightarrow V(\mathsf{E}, Y) \neq 1$.

These chains of equivalence, together with the model we defined, show that $M, X \Vdash \neg D\mathsf{E}$ holds but $M, Y \Vdash \neg D\mathsf{E}$ does not. The model also serves to illustrate the non-persistence of $I\mathsf{E}$. □

Violations of *g*-persistence are limited to the fragment of the language that includes determinacy operators. It is easy to establish by induction that formulas in the D-free fragment are *g*-persistent.

We can also make quick work of establishing that the present system satisfies the alignment constraint (i.e., Constraint 1).

Fact 5 (Alignment) *$M, X \Vdash IA$ iff $M, X \nVdash A$ and $M, X \nVdash \neg A$*

- $M, X \Vdash I\mathsf{A} \Leftrightarrow M, X \Vdash \neg D\mathsf{A} \wedge \neg D\neg\mathsf{A} \Leftrightarrow M, X, X \Vdash \neg D\mathsf{A} \wedge \neg D\neg\mathsf{A} \Leftrightarrow$
- $M, X, X \Vdash \neg D\mathsf{A}$ and $M, X, X \Vdash \neg D\neg\mathsf{A} \Leftrightarrow$
- $M, X, X \nVdash \mathsf{A}$ and $M, X, X \nVdash \neg\mathsf{A} \Leftrightarrow$
- $M, X \nVdash \mathsf{A}$ and $M, X \nVdash \neg\mathsf{A}$

The diagonal determinacy operator is unlike ordinary relational modal operators. Of course, any modal that operates on the primary evaluation coordinate would collapse the two-dimensional framework into the one-dimensional one. As for necessity operators that are defined on the secondary evaluation coordinate D coincides with stipulation that the accessibility relation relates each possibility to itself and to no other possibility, i.e. $RXY \equiv X = Y$. As a consequence, there are no operators meeting the secondary analogues of the conditions in footnote 9 that are equivalent to D.

Let us then establish some facts concerning the logic of this determinacy operator. As a preliminary point, note that there are consistent ascriptions of indeterminacy. In other words:

Fact 6 (Non-triviality) *$DA \vee D\neg A$ is not valid.*

Proof. Let us first work out the semantic derivation for $D\mathsf{A} \vee D\neg\mathsf{A}$.

- $M, X \Vdash D\mathsf{A} \vee D\neg\mathsf{A} \Leftrightarrow M, X, X \Vdash D\mathsf{A} \vee D\neg\mathsf{A} \Leftrightarrow$
- for all $Y \sqsupseteq X$, either $\exists Z \sqsupseteq Y$, $M, Z, X \Vdash D\mathsf{A}$ or $\exists Z \sqsupseteq Y$, $M, Z, X \Vdash D\neg\mathsf{A} \Leftrightarrow$
- for all $Y \sqsupseteq X$, either $\exists Z \sqsupseteq Y$, $M, X, X \Vdash \mathsf{A}$ or $\exists Z \sqsupseteq Y$, $M, X, X \Vdash \neg\mathsf{A} \Leftrightarrow$
- either $M, X, X \Vdash \mathsf{A}$ or $M, X, X \Vdash \neg\mathsf{A}$

But if X leaves A unsettled (e.g. if A is atomic, and $V(\mathsf{A}, X)\uparrow$), then this condition can fail. □

A notable consequence of Fact 6 is that the system does not validate the problematic inference from $D(\mathsf{A} \vee \mathsf{B})$ to $D\mathsf{A} \vee D\mathsf{B}$ (see pp. 220-221 of Burgess and Humberstone, 1987, for discussion of this putative entailment in their alternate two-dimensional system).

Lastly, I will note here that, per our design specification, there is no higher-order indeterminacy in this system.

Fact 7 (No higher-order indeterminacy) *(a) $DA \vdash DDA$ (b) $\neg DA \vdash D\neg DA$*

Proof. For part (a), suppose $M, X \Vdash D\mathsf{A}$, by the diagonal principle $M, X, X \Vdash D\mathsf{A}$; the target is $M, X, X \Vdash DD\mathsf{A}$, but by entry (vi) in the recursive analysis this is just $M, X, X \Vdash D\mathsf{A}$. For part (b), run the same reasoning with $\neg D\mathsf{A}$ in place of $D\mathsf{A}$. □

In fact, something stronger is true. The very same reasoning in part (a) of Fact 7 establishes:

Fact 8 *$A \vdash DA$*

Combining this with Fact 6 immediately entails that the logic in the full language has a familiar non-classical profile. Though the extension of the consequence relation in the D-free fragment matches that of classical sentential logic, adding expressive capacity to the language results in some non-classical behavior. One example of this behavior is that the consequence relation does not contrapose over the

full language: $\mathsf{A} \Vdash D\mathsf{A}$ holds but $\neg D\mathsf{A} \Vdash \neg\mathsf{A}$ does not.[16] Thus, at least one between the deduction theorem and disjunctive syllogism must fail, depending on whether we are going to import the analysis of the conditional from possibility semantics and also on the exact characterization of disjunctive syllogism.[17]

To conclude this section, it is valuable to reflect on exactly how the system manages to avoid the inconsistency in Fact 2. Recall, that the inconsistency pits the definition of indeterminacy (IN), the claim that indeterminacy is non-persistent (NP) and the analysis of negation (NE) against each other. The technical fact of the matter is that the two-dimensional system avoids the inconsistency by rejecting the negation condition, NE. In particular, in the two-dimensional system, there is no guarantee that if X supports $\neg\mathsf{A}$, then X's refinements will fail to support A. This is because unidimensional evaluation is governed by the diagonal principle, and so what's supported at X depends on evaluation triples of the form $\langle M, X, X\rangle$, whereas what's supported at Y depends on evaluation triples of the form $\langle M, Y, Y\rangle$. These may come apart in ways that undermine the negation clause. Of course, the *effect* of the negation operator is preserved because there is an analogous operator at the level of two-dimensional evaluation. However, that operator only quantifies over refinements along the primary dimension.

This technical gloss is important but, in my view, it does not illuminate the central mechanics behind the two-dimensional proposal. Instead, the two-dimensional system is better thought of as a more flexible generalization of the idea of indexing determinacy operators. The job of the secondary coordinate of evaluation is to anchor the facts that ground determinacy claims, shielding them from the shifting effects of other operators. The failure of the unidimensional negation clause is a

[16] This is parallel to what happens in supervaluationist analyses based on the idea of "global" validity (Fine, 1975; Williamson, 1994; Asher *et al.*, 2009; Bacon, 2018) and also in informational analyses of consequence for languages with epistemic modals, as in Yalcin (2007); Bledin (2014).

[17] We know immediately that the following form of disjunctive syllogism must fail:

If $\mathsf{A} \vdash \mathsf{C}$, $\mathsf{B} \vdash \mathsf{C}$, then $\mathsf{A} \vee \mathsf{B} \vdash \mathsf{C}$

If it didn't, then we would have $\mathsf{A} \vee \neg\mathsf{A} \vdash D\mathsf{A} \vee D\neg\mathsf{A}$ (contradicting Fact 6). Alternatively, it is possible to formulate disjunctive syllogism as follows:

If $\vdash \mathsf{A} \supset \mathsf{C}$, $\vdash \mathsf{B} \supset \mathsf{C}$, then $\mathsf{A} \vee \mathsf{B} \vdash \mathsf{C}$

The tenability of this constraint would depend on the semantics of the conditional. The conditional in the possibility semantics in Humberstone (1981) is:

$M, X \Vdash \mathsf{A} \rightarrow \mathsf{B}$ iff for all $Y \sqsupseteq X$, if $Y \Vdash \mathsf{A}$ there is $Z \sqsupseteq Y$, s.t. $M, Z \Vdash B$

Its two-dimensional generalization would appear to be:

$M, X, Z \Vdash \mathsf{A} \rightarrow \mathsf{B}$ iff for all $Y \sqsupseteq X$, if $M, Y, Z \Vdash \mathsf{A}$ there is $Y' \sqsupseteq Y$, s.t. $M, Y', Z \Vdash B$

Given this analysis, $\mathsf{A} \rightarrow D\mathsf{A}$ is not a logical truth: start with an X that does not settle A, then a Y that refines X and supports A. It is still not the case that Y has refinements that support $D\mathsf{A}$, since $D\mathsf{A}$ continues to depend on the original value of X.

downstream consequence of this intervention.

7 Philosophical Considerations on Framework Choice

The two-dimensional analysis of section 5 threads through the design principles that motivated it. In this section, I focus on the philosophical upshot of these technical developments. The supervaluationist treatment and the possibility semantics recruit very different primitives, but they have much in common. What sorts of consideration may distinguish them, and thus function potentially as reasons for preferring one over the other?

There is a relevant debate in metaphysics concerning "deep" metaphysical indeterminacy. This is indeterminacy that is not well modeled as indeterminacy concerning which complete world is actual. In particular, Skow (2010) argues that quantum indeterminacy is not well understood in terms of completely precisified possibilities. It is "deep" in the sense that it is not a matter of which absolutely precise world correctly represents the settled facts. (See also Wilson, 2013, for objections along these lines). In particular, the Kochen-Specker theorem in quantum mechanics (Held, 2018) suggests that, under plausible assumptions, there are sets of properties that cannot simultaneously have determinate values. If the electron has a determinate value for one set of properties P_1, then it cannot have determinate values for another set P_2. But if so, there is a problem with representing any given state by means of a world, since it would seem that worlds are fully determinate, and in particular ought to assign determinate values to both the properties in P_1 and the properties in P_2.

It is tempting to think that the possibility semantics framework can offer a model for this sort of unsettledness. However, despite the central role assigned to partial objects, possibility semantics seems also unable to capture the phenomenon of deep indeterminacy. After all, Refinability guarantees that, for any combination of properties in the object language, there must be possibilities in any model in which the properties are simultaneously defined. If the moral of the objection from deep metaphysical indeterminacy is that we ought to ban "impossible" states from models, the framework of possibility semantics falls just as short of the requirement as the supervaluationist.

A defender of the framework can respond in one of two ways. The non-concessive one would be to maintain that, although the framework does not ban the unwanted possibilities, it also features primitive representations for all the partial possibilities that represents possible states of the world. This is not so for the rival

framework that is grounded in complete worlds. According to this reply, the moral of deep indeterminacy is that we ought to *make room* for partial representations, and it is not that we ought to ban total representations as limit cases of the partial ones. This argument, however, becomes pretty thin when the opponent notes that they too have the ability to represent the partial states, for example in terms of sets of possible world. The only real difference between the frameworks is whether these partial objects are primitive or derived. I am tempted to agree witht he opponent that nothing much can hang on this kind of choice of primitives in the formal framework.

More concessive replies would start with the recognition that Refinability is indeed inconsistent with the phenomenon of deep metaphysical indeterminacy. At the same time, however one might argue that, if we were to apply possibility semantics to deep indeterminacy, we ought to renounce Refinability. There are existing models of possibility semantics that shun Refinability, and endorse the non-classicality in the logic that follows from it (Holliday, forthcoming; Holliday and Mandelkern, ms.). It is relevant here that Darby and Pickup (2021) develop a model of deep metaphysical indeterminacy that is based on a version of *situation* semantics in which analogues of Completability and Refinability fail—it is not the case that any situation may be extended into a complete one. This is not the place to develop this direction within the possibility semantics framework, but it strikes me that, if the motivation for possibility semantics was indeed the phenomenon of deep indeterminacy, the most promising route is to interact directly with the literature on metaphysical indeterminacy, and pursue models according to which possibilities are not arbitrarily refinable.[18]

Aside from this, it is valuable to ask what other considerations might distinguish possibility semantics (understood for the purposes of this discussion as including the Refinability constraint) from supervaluationism. There are two structural differences between the two frameworks. The first is that, as Humberstone notes, "the leading idea behind any supervaluations treatment is that of relating truth on a bivalent valuation (the primary concept inductively defined for formulae) to truth *simpliciter* by means of universal quantification". Possibility semantics does not require this two-level structure, and allows for direct evaluation for truth at a possibility.

The other structural difference is that there is a gap between Refinability and Completability. Refinability requires the existence of certain refinements, which

[18] In addition to the references listed above in this discussion, I also note: Torza (2020, 2021).

may themselves continue to be partial. By contrast Completability requires the existence of complete refinements. In this context, one might attempt to argue for possibility semantics from a couple different perspectives. From the perspective of the overall logic, Holliday (forthcoming) provides examples of modal logics that can be characterized semantically in a possibility framework but not in a Kripke-semantics based on possible worlds. The first example of such a logic uses propositional quantifiers (and is discussed in §5.1), the second does not, but involves a possibility semantics specified in terms of neighborhood functions (Holliday, forthcoming, §5.2).

From a more local perspective, one might prefer the possibility framework for reasons having to do with theory design, regardless of expressive capacity. Consider the underlined phrases in these examples:

(1) Naomi *won the match*.

(2) Corinne *passed the test*.

The correct application of these predicates has preconditions. Roughly speaking, Naomi cannot win the match unless and until the match is over, and Corinne cannot pass the test, unless and until the test is completed.

There are two options for thinking about the inference from a predicate to its precondition. Roughly speaking, either it is an entailment, or it is a presupposition. If *Naomi won the match* entails *The match is over*, we should expect the former to be false when the latter is false. If on the other hand, it presupposes it we should expect it to be semantically defective—similar to *Naomi stopped smoking* in a context in which Naomi never smoked.[19] On a semantic understanding of presuppoistion this would mean that the semantic value of a precondition predicate is undefined

[19]In practice, these preconditions seem a bit different from standard presuppositions in their projection behavior. Here we contrast *pass the test* with *stop smoking*. All of the sentences in (i) entail that Geraldo used to smoke.

(i) a. Geraldo has not stopped smoking.
b. Did Geraldo stop smoking?
c. It is possible that Geraldo has stopped smoking.
d. Leia believes Geraldo stopped smoking.

By contrast, the sentences in (ii) do not uniformly suggest that Corinne has completed the test.

(ii) a. Corinne has not passed the test.
b. Did Corinne pass the test?
c. It is possible that Corinne has passed the test.
d. Han believes that Corinne has passed the test.

Each of these seems compatible with Corinne not having taken the test at all. With that said,

until the precondition is satisfied. The defender of complete possibilities would, in my view, have no trouble modeling these predicates if their relationship to their preconditions is entailment.

However, I argue that if the presuppositional view was correct, it would introduce an instability in the complete possibility view. To see this, focus on future directed versions of (1) and (2):

(3) Naomi *will win the match*.

(4) Corinne *will pass the test*.

If the presuppositional analysis of preconditions is correct, the semantics must build in a kind of partiality for these predicates. This partiality is not unlike what is provided by possibility semantics: some future conditions would need to hold to secure a determinate truth-value for the relevant sentences. Furthermore, this is not *just* the partiality of standard presuppositions: presupposition failure can project to render *Geraldo will stop smoking or he won't* defective. By contrast, sentences like *Naomi will win the match or she won't* or *Corinne will pass the test or she will fail* have the ring of tautology. The upshot is this: if precondition predicates are partial, both supervaluationism and possibility semantics must already incorporate the kind of partiality that the possibility theorist invoke. From that point of view, the possibility semantics appears to be the more principled choice.

8 Conclusion

My main conclusions are as follows: there is a clear path for the application of possibility semantics to the metaphysical hypothesis of the open future. That path must include the characterization of object language determinacy operators. Finally, and crucially, this path requires, on pain of inconsistency, a two-dimensional analysis of determinacy operators.

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