Modeling future indeterminacy in possibility semantics

Fabrizio Cariani

University of Maryland, College Park

draft of May 26, 2023 comments welcome

Abstract

Possibility semantics offers an elegant framework for a semantic analysis of modal logic that does not recruit fully determinate entities such as possible worlds. The present papers considers the application of possibility semantics to the modeling of the indeterminacy of the future. Interesting theoretical problems arise in connection to the addition of object-language determinacy operator. We argue that adding a two-dimensional layer to possibility semantics can help solve these problems. The resulting system assigns to the two-dimensional determinacy operator a well-known logic (coinciding with the logic of universal modalities under global consequence). The paper concludes with some preliminary inroads into the question of how to distinguish two-dimensional possibility semantics from the more established branching framework.

1 Introduction

Possibility semantics offers an elegant framework for a semantic analysis of modal logic that does not recruit fully determinate entities such as possible worlds.[1] This paper

[0]Thanks to David Boylan, Rohan French, Jeff Horty, Arc Kocurek, Stephen Kuhn, John MacFarlane, Matt Mandelkern, Eric Pacuit, Masayuki Tashiro, and Alessandro Torza, as well as two anonymous referees for conversations, exchanges on early drafts of this paper. Also thanks to the audience at the Maryland Work in Progress Workshop, the Logic, Language, and Cognition (LLC) group at the University of Turin, Virlawp working group, and especially to Ginger Schultheis and Malte Willer whose remarks prompted large revisions. Special thanks to Lloyd Humberstone for detailed feedback on a previous version of the paper; to Paolo Santorio for many years of discussions and exchanges of ideas on this topic; and finally to Wes Holliday who came through with multiple suggestions that unlocked significant improvements in the paper.

[1]The phrase "possibility semantics" was coined by Humberstone (1981). The tools undergirding the framework have longer histories, including (Fine, 1975, especially §2), Humberstone (1979), as well as deep roots in the algebraic logic tradition. For a contemporary and comprehensive introduction, see Holliday

develops the application of possibility semantics to the modeling of the indeterminacy, or openness, of the future, and some other related forms of metaphysical indeterminacy. Possibility semantics is plausibly viewed as an alternative to more established branching-time models (Thomason 1970, 1984, 2007, Belnap *et al.* 2001, MacFarlane 2003, 2014) in which indeterminacy is grounded in the overlap of complete possibilities—sometimes referred to as "histories". The key finding is that interesting technical and conceptual problems arise in connection to the explicit modeling of indeterminacy within the object language.

As understood here, the open-future hypothesis is the claim that some future events and states are objectively, and not merely epistemically, unsettled.[2] It is not assumed here that the unsettledness of the future and quantum indeterminacy are one and the same. The recurring illustrative example will be the proposition that some specific random coin will land heads on its next toss, under the stipulation that the outcome of the coin's toss is not settled by the facts about the past and the present of the tossing apparatus. If in actuality there are no such setups, the case may be entertained as a thought-experiment.

The indeterminacy associated with the future seems unlike other kinds of indeterminacy that have attracted the attention of philosophers. For example, it seems unlike the indeterminacy that some theories associate with vagueness. For one thing, it does not appear to give rise to higher-order indeterminacy. It is generally agreed by those who think that vagueness is grounded in some kind of indeterminacy that it may itself be indeterminate whether Joe is borderline tall. By contrast, it is common to assume that, as far as the unsettledness of the future is concerned, there are no states or events whose determinacy status is itself indeterminate. It might be unsettled whether there will be a sea battle tomorrow, but it cannot be unsettled whether it's unsettled. A second marker of the indeterminacy of the future is that it is not plausibly associated with unusual effects on credence. Many different philosophers have been attracted to the view that there is something non-classical about credence in the contents of vague statements. One form of this is Field's (2000) claim that vague contents require low credence in certain

(2022). Possibility semantics is one of a variety of styles of theories that do not rely on worlds, but on coarser objects. In addition to possibility semantics, the general family of "pointless" theories includes various kinds of states-based semantic analyses (Aloni 2018, Willer 2018), truthmaker semantics (Fine, 2017b), as well as several varieties of situation semantics (Barwise and Perry, 1981; Kratzer, 2021). It would be desirable to have a comparative study of these frameworks highlighting the commonalities, as well as the differences, between them.

[2]There is much literature on what constitutes the (alleged) openness of the future. The present discussion leans in various ways on Thomason (1970); Belnap and Green (1994); Belnap *et al.* (2001); MacFarlane (2003, 2014); Barnes and Cameron (2009, 2011); Torre (2011); Cariani and Santorio (2018); Cariani (2021b); Todd (2022).

instances of the law of excluded middle; another is Williams's claim that vague contents seem to require imprecise probability (2014).[3] By contrast, statements about the future appear to be paradigmatic examples for the application of theories of classical credence. In prototypical cases, it seems perfectly warranted to have a sharp credence that the coin will land heads. The fact that the indeterminacy of the future has these characteristics licenses us to theorize about this specific type of indeterminacy on its own (cf. §2.3 of Torza, forthcoming, on pluralism about indeterminacy).[4]

As a last disclaimer, exploring the indeterminacy hypothesis involves no commitment to the claim that the future is open. What we are in fact committed to is the weaker claim that the hypothesis is worth taking seriously. As Stalnaker (2019, p.197) puts it, "You don't have to sign on to this metaphysical theory (as I do not) in order to find it intelligible (as I do) and to use it as a kind of precedent for a case where the thesis of metaphysical indeterminacy may be less controversial."

We lead with a general introduction to possibility semantics for a sentential modal language (§2). The next section focuses on the representation of indeterminacy in possibility semantics (§3). The framework itself already incorporates a representation of indeterminacy in the model theory. However, contrary to the inclination of Humberstone (1981), it seems important to have ways of capturing the notion of indeterminacy in the object language. Unfortunately, it is not possible to add a determinacy operator with the right profile to the system—not at least without other interventions. The main contribution of §3 is an impossibility result to this effect. After considering some theoretical options that would repair the inconsistency by means of local interventions (§4), we consider an attractive solution to the problem, which lies in the integration of possibility semantics with a two-dimensional framework (§5). The last two sections respectively highlight some logical properties of the resulting system (§6) and explore one type of systematic comparison between it and the branching time framework (§7).

Much of the technical substance of the approach proposed in §5 is owed to remarks in Fine (1975). The Cliffs notes on Fine's paper focus on the fact that it is the first application of supervaluationist techniques to vague language. However, it is also a central juncture

[3]The matter is highly complicated, in ways that go beyond the relatively simple demarcation point that is made in this paragraph. For a sophisticated discussion, see Bacon (2018).

[4]It is worth highlighting that that some of the formal discussion to follow is not be restricted in scope to the alleged indeterminacy of the future. It will pertain to any application of possibility semantics to concepts of indeterminacy that do not give rise to higher-order indeterminacy and are not associated with funky effects on credences. As an example, Stalnaker (1984) famously suggests that counterfactual selection results in a kind of indeterminacy, and has more recently suggested that this kind of indeterminacy might be viewed as a 'milder' version of the indeterminacy that is associated with the future (Stalnaker, 2019, p.197-ff).

for the logical development of semantics based on partial objects, since Fine builds up to the supervaluationist machinery by first analyzing a system in which precisifications of a vague language are viewed as partial. (NB: this account is only considered in passing in Fine 1975, and moreover Fine's theory of vagueness has significantly changed, e.g. in Fine 2017a.) The present ambition is to recast some of those insights about determinacy operators in a different theoretical context, allowing some distinct issues and theoretical choice points to come to light.[5]

2 Background on possibility semantics

The basic ideological tenet of possibility semantics is that formulas are not evaluated against worlds, but against "coarser" objects called *possibilities*. This ideology marks a deviation from the standard account of the indeterminacy of the future—which is broadly within the framework of branching time (Thomason, 1970, 1984, 2007; Belnap *et al.*, 2001). According to the branching time picture, indeterminacy is adequately captured by the overlap of multiple complete possibilities with equal claim to fit the settled facts. The exact details of the analysis here depend on deeper metaphysical commitments. For instance, someone with broadly ersatzist leanings might say that the indeterminate reality is represented by multiple, incompatible perfectly determinate representations (Barnes and Cameron, 2009, 2011; Barnes and Williams, 2011).

Possibility semantics proceeds in a different way. Instead of taking a maximally precise representation as its basic modeling object, it deploys primitive objects that are themselves incomplete. That incompleteness is naturally associated with a concept of indeterminacy: possibilities settle the truth values of some sentences of a language, while leaving others unsettled.

The present formulation of possibility semantics originates from Humberstone (1981). The language is a sentential modal language, whose signature features a non-empty countable set of modal operators. (In later sections, we will add a *determinacy* operator D, in addition to these.) Models for this language are quadruples of the form, $\langle P, \gg, \mathbf{R}, V\rangle$. Here P represents a non-empty set of possibilities; $\gg$ is a refinement relation over the possibilities. Structurally, $\gg$ is a *weak partial order* (thus, it is transitive and antisymmetric). From an intuitive standpoint, $Y \gg X$ holds when everything that is settled as either true

[5] The idea of using possibility semantics to model the unsettledness of the future is also explored in a preliminary way in Boylan (forthcoming). However, because Boylan is focused on a different set of problems, he ends up in a theoretical space that is not compatible with the present outlook, especially with regards to the analysis of negation.

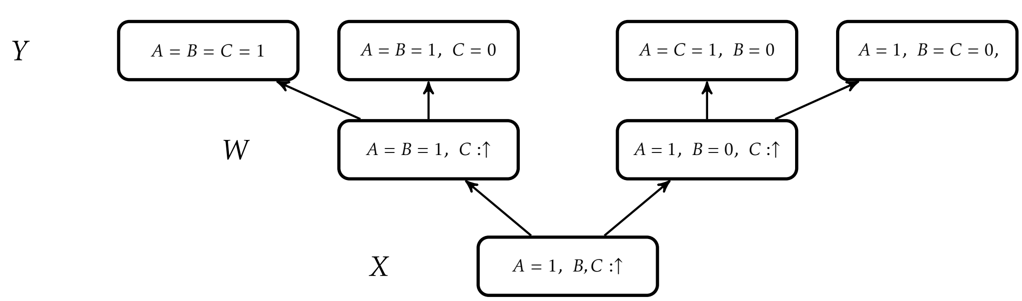
or false by X is settled in the same way by Y. In short, Y agrees on all the determinate facts that X settles. (Explicit structural assumptions are needed in order to guarantee that models satisfy this intuition, and they will be provided in short order.) **R** is a non-empty set of accessibility relations, and finally V is a partial valuation function: in this setting a valuation function inputs an atomic formula and a possibility, and, if defined, outputs either 0 or 1. When $V(\mathsf{A}, X)$ is undefined, we write $V(\mathsf{A}, X)\uparrow$. Occasionally, when it is important to disambuguate, and a model $\mathcal{M}$ is salient in context, a subscripted "$\mathcal{M}$" will be used to indicate its coordinates. For example, "$P_{\mathcal{M}}$" refers to the set of possibilities in $\mathcal{M}$.

Models for this language are ordinarily assumed to satisfy two constraints.

> **Refinability.** For every atomic formula A and possibility X, if $V(\mathsf{A}, X)\uparrow$, then there are Y, Z such that $Y \gg X$ and $Z \gg X$, s.t. $V(\mathsf{A}, Y) = 1$ and $V(\mathsf{A}, Z) = 0$.
>
> **Persistence.** For all atomic A, if $V(A, X)\downarrow$ and then for every $Y \gg X$, then $V(\mathsf{A}, X) = V(\mathsf{A}, Y)$.

Persistence says that whenever atomic A is settled at X, it stays settled in the same way through X's refinements. Refinability says that whenever an atomic formula A is unsettled at a possibility X, there are Y and Z—both refinements of X—that settle A as true and false respectively. Refinability is related to, but logically distinct from, the assumption that any partial possibility might be refined all the way to a complete one (which Fine 1975 calls "Completability"). In a language with infinitely many atomic sentences, refinability might be satisfied, without completability being satisfied.

Persistence is required to give formal representation to the intuitive conception of refinement. Indeed, under persistence, it is tempting to think of refinement structures as mirroring the structure of the branching models for future contingency,[6] as illustrated by the diagram in Figure 1.



[6] See Thomason (1970); Belnap *et al.* (2001); MacFarlane (2014), Cariani (2021b, ch.2) for discussion of branching models.

Figure 1: The branching structure of refinements ($Y \gg W \gg X$)

However, an important lingering difference — which the formal theory ought to help disentangle — is that standard branching models are built on the idea of maximal *histories*, which at any moment assign a definite truth-value to all the formulas of the language. Indeed, the linear paths through the tree can naturally be viewed as temporally structured possible worlds. No such assumption of completeness is imposed on possibility models.

Humberstone's semantic entries rely on the idea of using a valuation function defined on the atomic formulas of $\mathcal{L}$ to ground a notion of support between possibilities and formulas of the whole language. They are as follows:

- $\mathcal{M}, X \Vdash p$ iff $V_{\mathcal{M}}(p, X) = 1$
- $\mathcal{M}, X \Vdash \mathsf{A} \wedge \mathsf{B}$ iff $\mathcal{M}, X \Vdash A$ and $\mathcal{M}, X \Vdash B$
- $\mathcal{M}, X \Vdash \neg\mathsf{A}$ iff for all $Y \gg X$, $\mathcal{M}, Y \nVdash \mathsf{A}$
- $\mathcal{M}, X \Vdash \Box_i\mathsf{A}$ iff for all $Y \in P$, s.t. $R_i XY$, $\mathcal{M}, Y \Vdash A$

As for other operators, such as $\vee$, $\rightarrow$, $\Diamond$, a common approach recovers entries by fixing some standard equivalences. In the case of disjunction, one might assume it characterized by conjunction, negation and DeMorgan's laws.[7] This results in the following entry:

- $\mathcal{M}, X \Vdash \mathsf{A} \vee \mathsf{B}$ iff for all $Y \gg X$, there is $Z \gg Y$, s.t. $\mathcal{M}, Z \Vdash A$, or there is $Z \gg Y$, s.t. $\mathcal{M}, Z \Vdash B$

Another route to the same goal would be to stipulate some general principles about what it takes for various kinds of possibilities to settle a disjunction as true/false (Holliday, 2022).

- A possibility X settles a disjunction $\mathsf{A} \vee \mathsf{B}$ as false iff it settles A as false and settles B as false.

Assume that a possibility settles A as false iff it settles $\neg\mathsf{A}$ as true. Next, note that the entries for negation and conjunction tell us that:

- A possibility X settles a conjunction $\mathsf{A} \wedge \mathsf{B}$ as true iff it settles both A and B as true.

[7] As an alternative, disjunction could be defined instead by the condition $\mathcal{M}, X \Vdash \mathsf{A} \vee \mathsf{B}$ iff $\mathcal{M}, X \Vdash A$ or $\mathcal{M}, X \Vdash B$. This would have the effect of making the logic of the sentential sub-language non-classical.

- A possibility X settles A as false iff every refinement of X fails to settle A as true.

These assumptions are sufficient to pin down the same entry for disjunction as above. A similar analysis could be carried out for the other operators.[8]

Lastly we follow Humberstone in defining consequence as preservation of support.

Definition 1 *$A_1, ..., A_n \Vdash B$ iff for all models $\mathcal{M}$ and any X in $P_{\mathcal{M}}$, if for all i, $\mathcal{M}, X \Vdash A_i$, then $\mathcal{M}, X \Vdash B$.*

It is a well established fact about this formalism that the logic of the sentential sub-language is classical, both in the sense that the set of logical truths coincides with the set of classical tautologies, and in the sense that the class of valid arguments in this sub-language coincides with the class of tautologically valid arguments (Humberstone, 1981, pp.320-321).

3 Adding object language determinacy operators

It is reasonable to claim that possibility semantics incorporates a model of indeterminacy: an atomic formula A is indeterminate at a possibility X when X leaves A undefined. Imagine a possibility X and an atomic formula, *heads*, which we may take as symbolizing the English sentence *The coin will land heads (on a specific toss that will take place tomorrow at noon toss)*. In a clear sense, the metatheoretic fact that $V_{\mathcal{M}}(heads, X) \uparrow$ represents the relevant indeterminacy from the perspective of the model theory. This warrants the view that indeterminacy is captured in standard possibility semantics at the metatheoretic level.

However, as the system is set up, there is no *object language* device to express the concept of indeterminacy. We do not have, or have not identified, an operator that can properly express things like *it is determinate that the coin landed heads on today's toss*,

[8]While the analysis of necessity simply lifts Kripke semantics to the level of possibilities, Humberstone's account of modality also involves the specification of interplay conditions connecting accessibility and refinement.

(I1) for all X, Y, Z, if $Z \gg X$ and RZY, then RXY

(I2) for all X, Y, Z, if $Z \gg Y$ and RXY, then RXZ

(R) for all X, Y, if RXY then for some $X' \gg X$, for all $Z \gg X'$, RZY

(I1) says that if Z refines X, then X accesses everything Z accesses; (I2) says that if Z refines Y, the any X that accesses Y accesses Z; (I3) says that if X accesses Y, then looking at the (upside-down) tree of refinements of X, there is a branch possibly starting below X itself, where possibility on this branch accesses Y. Holliday (2014, forthcoming) noted that condition (R) is overly strong. In his preferred approach, the model theory for possibility semantics is based on a class of 'functional possibility frames'.

but it is not determinate that it will land heads tomorrow. This is unfortunate because, for various modeling purposes, it's important to have determinacy operators in the object language. For example, determinacy operators may help formulate constraints that involve the interaction of indeterminacy with other concepts. To take just one example drawn from the recent literature, Cariani (2021a) explores interactions between (in)determinacy operators and epistemic operators. In this kind of discussion, certain principles become important that can only be formulated with determinacy operators. An example is: $\neg D\mathsf{A} \rightarrow \neg K\mathsf{A}$—the principle that if A is not determinately true, then it is not known. Such principles, and the constraints they impose on models, are best analyzed from the perspective of a formalized language.[9]

Let us then introduce a determinacy operator D to the formal language—with the interpretation that its argument is determinately *true*. Thus $\neg D\mathsf{A}$ is interpreted as claiming that the proposition expressed by A is not determinately true, while leaving it open that it might be determinately false. To express the claim that A is indeterminate, we add an indeterminacy operator I governed by the condition in Definition 2, which is standardly taken to be definitional of indeterminacy (e.g. in Fine, 1975):

Definition 2 $IA =_{df} \neg DA \wedge \neg D\neg A$

In many respects that are going to be relevant, non-determinacy (which is expressed by '$\neg D$') behaves similarly to indeterminacy. It is important however to keep in mind that in the present terminology 'indeterminacy' denotes a two-sided status, in the sense that it requires that both A and its negation fail to be determinate. By contrast, non-determinacy (the obtaining of $\neg D$) is a one-sided status: a proposition may fail to be determinate, while its negation is determinate.

The addition of determinacy operators to the language of possibility semantics should be guided by some key constraints. To start, object language indeterminacy should, in a precise sense, align with metatheoretic indeterminacy. The simplest statement of this constraint is at the level of atomic formulas:

Constraint 1 (Alignment) *For atomic* A, $\mathcal{M}, X \Vdash IA$ *iff* $V_{\mathcal{M}}(A, X) \uparrow$.

Alignment entails a second constraint: formulas expressing non-determinacy (and indeterminacy) claims must violate (a generalization of) persistence. As initially formulated,

[9] For some additional considerations in favor of introducing object language determinacy operators, see also Barnes and Williams (2011, §5)

persistence applies to the atomic formulas of the language, but there is an entirely natural generalization of it involving the concept of support. A possibility X might support that it's indeterminate whether the coin will land heads, while at the same time it could be refinable into a possibility Y that settles that the coin will land heads. The exact principle that follows from this is:

Constraint 2 (Non-persistence of non-determinacy) *There is a formula* A*, and model* $\mathcal{M}$ *with possibilities* $X, Y \in P_{\mathcal{M}}$ *and* $Y \gg X$ *such that* $\mathcal{M}, X \Vdash \neg D\mathsf{A}$ *but* $\mathcal{M}, Y \nVdash \neg D\mathsf{A}$

Constraint 3 (Non-persistence of indeterminacy) *There is a formula* A*, and model* $\mathcal{M}$ *with possibilities* $X, Y \in P_{\mathcal{M}}$ *and* $Y \gg X$ *such that* $\mathcal{M}, X \Vdash I\mathsf{A}$ *but* $\mathcal{M}, Y \nVdash I\mathsf{A}$

With enough of the possibility framework on board, the route from the alignment constraint to non-persistence is relatively straightforward.

Fact 1 *Given Definition 2 and Refinability, Alignment entails (i) Non-persistence of indeterminacy and (ii) of non-determinacy.*

> *Proof.* Consider a model $\mathcal{M}$ with two possibilities X and Y drawn from its possibility set, such that $Y \gg X$. Suppose in particular that Y settles some atomic formula A that X leaves unsettled. The existence of such a Y is guaranteed by Refinability. Then $V_{\mathcal{M}}(\mathsf{A}, X)\uparrow$ but $V_{\mathcal{M}}(\mathsf{A}, Y) = 1$ or $V_{\mathcal{M}}(\mathsf{A}, Y) = 0$ and so $\mathcal{M}, X \Vdash I\mathsf{A}$ but $\mathcal{M}, Y \nVdash I\mathsf{A}$. For (ii), exploit Refinability to suppose that Y refines X so that $V_{\mathcal{M}}(\mathsf{A}, Y) = 1$. Definition 2 yields $\mathcal{M}, X \Vdash \neg D\mathsf{A}$, but from the way Y refines X it follows that $\mathcal{M}, Y \Vdash D\mathsf{A}$. □

Alignment provides powerful motivation for Non-persistence. It is nonetheless valuable to keep the claims separate, because Non-persistence is weaker and might be motivated in other ways. Another reason to keep these separate is that there are versions of possibility semantics that drop Refinability (see §7.1).

While these constraints seem plausible, important difficulties are lurking under the surface. The just-added ingredients are inconsistent with the framework. In particular, there is tension between the analysis of indeterminacy in Definition 2, the Non-persistence of indeterminacy and the analysis of negation.

Fact 2 *The following are inconsistent (given the framework):*

IN. $IA \equiv_{df} \neg DA \wedge \neg D\neg A$.

NP. *There are* $\mathcal{M}$, A, X, $Y \gg X$ *with* $\mathcal{M}, X \Vdash IA$ *but* $\mathcal{M}, Y \nVdash IA$.

NE. $\mathcal{M}, X \Vdash \neg A$ *iff for all* Y *such that* $Y \gg X$, $\mathcal{M}, Y \nVdash A$

Proof. Consider witnesses, $\mathcal{M}, X, Y, \mathsf{A}$ for NP. By IN, $\mathcal{M}, X \Vdash \neg D\mathsf{A} \wedge \neg D\neg\mathsf{A}$. By the clause for conjunction, $\mathcal{M}, X \Vdash \neg D\mathsf{A}$ and $\mathcal{M}, X \Vdash \neg D\neg\mathsf{A}$. By the clause for negation (NE), $D\mathsf{A}$ and $D\neg\mathsf{A}$ cannot be supported throughout any refinements of X. That is, for all $Z \gg X$, $\mathcal{M}, Z \nVdash D\mathsf{A}$ and $\mathcal{M}, Z \nVdash D\neg\mathsf{A}$. However, since any refinement of Y is a refinement of X, we must also have $\mathcal{M}, Y \Vdash \neg D\mathsf{A}$ and $\mathcal{M}, Y \Vdash \neg D\neg\mathsf{A}$, and hence, by IN, $\mathcal{M}, Y \Vdash I\mathsf{A}$. This contradicts the fact that $\mathcal{M}, \mathsf{A}, X, Y$ were chosen as witnesses for the existential in NP. □

A plausible initial diagnosis is that the problem arises because the negation operator forces persistence. That is to say, the system guarantees that $\neg A$ must always be persistent, whether A is persistent or not. A consequence of this fact is that the indeterminacy operator I cannot be both defined in terms of negation and also such that formulas like $I\mathsf{A}$ are non-persistent.

This inconsistency is related to a less specific unease with object language indeterminacy operators that is already expressed by Humberstone (1981). Humberstone claims that an indeterminacy operator like the one just introduced would go "against the spirit of the present enterprise, since it would give rise to formulas which were not persistent into refinement [...], and thus undermines the idea of refinements as mere resolvers of indeterminacy". Humberstone's exact concern is hard to pin down, and certainly broader than the inconsistency articulated in Fact 2.(He uses this kind of argument to press against other non-persistent operators, including ones that do not give rise to inconsistencies like the one just identified .) But whatever we may think of the broad concern, the inconsistency does show that adding (in)determinacy operators is not entirely innocent.

4 A preliminary journey around the options

Is there a path for integrating possibility semantics with object language determinacy operators? Evidently, any such path requires giving up one of IN, NP, or NE. In other words, it requires either altering the definition of indeterminacy, or giving up non-persistence or modifying the analysis of negation. The option of giving up IN

is a non-starter and may be set aside immediately. The problem is not merely that the definition of indeterminacy captured by Definition 2 is relatively well entrenched, which it is. The real issue is that a version of the inconsistency in Fact 2 arises for $\neg D$A, independently of how IA is defined.

By contrast, it seems more promising to pursue some version of the second option, and so to deny the non-persistence constraint. One might support a plea for persistence by thinking in terms of temporally indexed indeterminacy operators. [10] To illustrate the essence of the approach, start by noting that, in the relevant applications, there is a connection between refinement and temporality. Specifically, advancing through time along a history should result in encountering more and more refined possibilities. Under this temporal interpretation, it might seem attractive to entertain determinacy operators that are relativized to a specific point in time. Under this approach, the object language would feature a collection of operators $\{D_t \mid t \in T\}$, where T is a designated set of times in the model.[11] Simplifying somewhat, imagine that the set of times that are distinguished in a given possibility model is finite. Then consider operators $D_0, D_1, D_2, ...D_n$, each marking what is determinate at a certain time in the development of history, with each D_i anchored to some specific time t_i. To complete the proposal say that the language does not contain any unrelativized determinacy operators, and thus that all determinacy discourse is captured by means of relativized ones.

This model's way out of the inconsistency is to undermine some of the motivation for non-persistence. Suppose again that X_{Mon} represents Monday's possibility, in which the coin has not yet landed heads, and X_{Wed} represents the state of affairs on Wednesday, after the coin has been tossed and has landed heads. In the original approach, with unrelativized determinacy operators, one should approach this by saying that $\neg D(heads)$ is supported at X_{Mon} but unsupported at X_{Wed}. By contrast, the relativized framework opens up a different option: X_{Mon} supports $\neg D_{Mon}(heads)$, while X_{Wed} supports $D_{Wed}(heads)$. Crucially, the formulas $\neg D_{Mon}$ and $\neg D_{Wed}$ can be assumed to be persistent (even when the operator is embedded under negation). The intuitive meaning of D_{Mon}A would be something like "A is/was settled true on Monday". From Wednesday's point of view—i.e., as far as X_{Wed} is concerned—$\neg D_{Mon}$A remains supported. Relatedly, the claim $\neg D_{Monday}(heads) \wedge D_{Wednesday}(heads)$ is perfectly consistent (from any point in time).[12]

[10] I owe this suggestion to [BLINDED] Masayuki Tashiro.

[11] It requires a bit of manipulation to endow standard model of branching time with times. In particular, what is required is a simultaneity relation that connects points on different branches. See chapter 2 of Cariani.2021 for discussion.

[12] A notational variant of this approach maintains that we can have a single concept of indeterminacy that

This approach is valuable, and the solution offered in this paper incorporates some of the insight that motivates it. However, it also seems unsatisfactory in some respects: it is not especially controversial to claim that people possess an unrelativized concept of indeterminacy — plausibly one that can be captured at the level of the theory by an operator that satisfies the alignment constraint. There is no special reason to think that there are barriers to expressing *that* concept in the object language. It is at the very least worth asking whether such a concept is definable.

Before moving to the positive proposal, let us entertain one last option. The initial hunch concerning the incompatibility in Fact 2 was that it is due to the persistence-forcing effect of negation. The obvious alternative would be to introduce a type of negation that does not force persistence.[13] To this end, introduce '$\sim$' as the connective characterized by the clause: $\mathcal{M}, X \Vdash {\sim}\mathsf{A}$ iff $\mathcal{M}, X \nVdash \mathsf{A}$. This alternate negation operator does not have the effect of transforming a non-persistent claim into a persistent one. Indeed, it would make correct predictions for non-determinacy claims in the proof of Fact 2.

An evident problem with this approach is that '$\sim$' cannot be the correct negation operator for the entire language. Outside of determinacy claims, '$\sim$' conflates non-support with rejection. It is undesirable for *it's not the case that the coin will land heads* to be supported by a possibility that merely fails to settle heads. More generally, '$\sim$' is not the correct negation operator for the sentential sub-language of the language. In response, one might consider a language in which the two negation operators, '$\neg$' and '$\sim$', coexist. Footnote 15 of Humberstone (1981) identifies an expressive advantage to having both operators (though Humberstone does not endorse the suggestion currently under consideration): their combination, '$\neg\sim$', is a plausible candidate for a determinacy operator, as it expresses universal quantification over all refinements. (So $\mathcal{M}, X \Vdash \neg \sim \mathsf{A}$ iff all refinements of X support A). However, for the present application, having both operators around is not well-motivated. There is no principled reason for why one negation operator ($\neg$) should apply in the D-free sub-language, while the other operator should apply to formulas involving D. Additionally, any attempt at formulating a generalization concerning which operator is appropriate for a given formula would have to deal with the thorny problem of choosing the correct negation for mixed formulas (like

is *relational*, so that the canonical logical form for determinacy claims is $D(Monday, \mathsf{A})$. From our perspective, this approach is not substantially different from the indexed operator approach, and the critique to be made below applies to both.

[13]Humberstone (1979) considers this alternative negation for a similar application. This is also the negation that Boylan (forthcoming) uses in his application of possibility semantics to the future.

the negation of $\mathsf{A} \wedge D\mathsf{A}$), Ultimately, it is unprincipled to have two negation operators floating around without a systematic account of their distinct roles.

5 Introducing two-dimensional possibility semantics.

This section presents a two-dimensional version of possibility semantics that is capable of addressing the inconsistency.[14] Before presenting it, it will be valuable to collect the desiderata we identified along the way. What is needed is a version of possibility semantics that incorporates a non-persistent, non-relativized determinacy operator that is "aligned" with the metatheoretic concept of indeterminacy that is ordinarily built into the possibility semantics framework. The logic is to be classical within the sentential sub-language, and the D operator must not trivialize. As a specific litmus test, $\mathsf{A} \vee \neg\mathsf{A}$ is to be valid (because the logic is classical) while $D\mathsf{A} \vee D\neg\mathsf{A}$ is not. Finally, the system must avoid conflating failure to support with rejection.

The opening move in crafting such a framework is to distinguish two dimensions of evaluation. In addition to evaluating at a pair consisting of a model and a possibility, consider evaluating at a triple $\mathcal{M}, X, Y$ consisting of the model and *two* possibilities. Doubling the evaluation possibility allows it to play two separate roles: one coordinate of evaluation is operated on by connectives (call this the 'primary possibility'), while the other is read by the determinacy operator D and left untouched by the connectives (call this the 'secondary possibility'). On the basis of the two-dimensional semantics, we can produce a unidimensional entry according to a standard diagonal principle:

Diagonal principle: $\mathcal{M}, X \Vdash \mathsf{A}$ iff $\mathcal{M}, X, X \Vdash \mathsf{A}$

The conceptual motivation for continuing to value unidimensional evaluation is that we continue to focus on a concept of truth, or support, at a possibility as the ultimate target of the theory. Moreover, thanks to the diagonal principle, we the two-dimensional system can inherit the definition of consequence as preservation of support at a model.

Recursive clauses for the connectives and for the determinacy operator are specified at the level of two-dimensional evaluation. Note that the new secondary possibility is largely idle, except for contributing to the interpretation of the determinacy operator.

[14]For some some general surveys on canonical applications of two-dimensional semantics see Humberstone (2004); Kuhn (2013); Schroeter (2021). The suggestion of a two-dimensional treatment of the determinacy operator is first explored in Fine (1975). Fine rightfully questions the ability of such an operator to handle higher-order indeterminacy, but of course this concern is not salient in the present application. The present claim is not that a two-dimensional semantics is anything new, but that it provides an elegant solution to an otherwise extremely thorny puzzle. A slight variation of a two-dimensional determinacy operator is also introduced in Burgess and Humberstone (1987, §6.2).

(i) $\mathcal{M}, X, Z \Vdash p$ iff $V_{\mathcal{M}}(p, X) = 1$

(ii) $\mathcal{M}, X, Z \Vdash \mathsf{A} \wedge \mathsf{B}$ iff $\mathcal{M}, X, Z \Vdash A$ and $\mathcal{M}, X, Z \Vdash B$

(iii) $\mathcal{M}, X, Z \Vdash \neg\mathsf{A}$ iff for all $Y \gg X$, $\mathcal{M}, Y, Z \nVdash \mathsf{A}$

(iv) $\mathcal{M}, X, Z \Vdash \Box_i\mathsf{A}$ iff for all $Y \in P$, s.t. $R_i XY$, $\mathcal{M}, Y, Z \Vdash A$

(v) for $\vee$, $\rightarrow$, $\Diamond$, use standard equivalences to infer clauses.

(vi) $\mathcal{M}, X, Z \Vdash D\mathsf{A}$ iff $\mathcal{M}, Z, Z \Vdash \mathsf{A}$

It is notable that, under this analysis, the determinacy operator resembles an actuality operator in more standard applications of two-dimensional semantics. It evaluates the argument of $D\mathsf{A}$ after setting the primary evaluation possibility so as to match the secondary one.

Logical consequence remains defined as preservation of unidimensional support, as per Definition 1. Furthermore, Refinability and Persistence, understood as constraints on atomic formulas, continue to be in place. While they have generalizations for the full language, the status of those generalizations is not settled by the status of their atomic variants. Thus, saying that the complex formula $I\mathsf{A}$ is non-Persistent is fully compatible with saying that atomic formulas persist through refinements.

6 Victory lap

This section has two objectives: the broad objective is to illustrate that the system fulfills the main desiderata for adding an object language determinacy operator. More narrowly, once those general desiderata are established, it aims to illustrate that the system incorporates a way out of the central incompatibility identified in Fact 2. A key intermediate step in establishing these objective is the characterization of the logic of the system, identified below as Theorem 1.

As noted, the persistence constraint has a natural generalization concerning arbitrary formulas and involving the notion of support.

Definition 3 *(i) An arbitrary formula A is g-persistent in $\mathcal{M}$ iff for all $X, Y \in P_{\mathcal{M}}$ with $Y \gg X$, $\mathcal{M}, X \Vdash A$ but $\mathcal{M}, Y \nVdash A$; (ii) A is g-persistent iff for all $\mathcal{M}$, A is g-persistent in $\mathcal{M}$.*

It is now possible to consider, and establish, the claim that $\neg D\mathsf{A}$ and $I\mathsf{A}$ are not persistent in this generalized sens.

Fact 3 (Non-persistence of non-determinacy and indeterminacy) *Let A be an atomic formula. Then* $\neg DA$ *and* IA *are not g-persistent.*

Proof. Let A be an atomic formula. We want to identify a model $\mathcal{M}$ in which $\neg D$A and IA are not g-persistent. Consider a "minimal fork" model with three possibilities X, Y and Z with $Y, Z \gg X$ and such that $V(\mathsf{A}, X)\uparrow$, $V(\mathsf{A}, Y) = 1$, $V(\mathsf{A}, Z) = 0$. (See Figure 2.) In the model, $\mathcal{M}, X \Vdash \neg D\mathsf{A}$, but $\mathcal{M}, Y \nVdash \neg D\mathsf{A}$ (since $\mathcal{M}, Y \Vdash D\mathsf{A}$).[15]

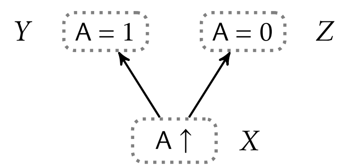


Figure 2: The Minimal Fork Model

The same model also illustrates the non-persistence of IA. □

Violations of g-persistence are limited to the fragment of the language that includes determinacy operators. It is easy to establish by induction that formulas in the D-free fragment are g-persistent.

We can also make quick work of establishing that the present system satisfies the alignment constraint (i.e., Constraint 1).

Fact 4 (Alignment) $\mathcal{M}, X \Vdash IA$ *iff* $\mathcal{M}, X \nVdash A$ *and* $\mathcal{M}, X \nVdash \neg A$

- $\mathcal{M}, X \Vdash I\mathsf{A} \Leftrightarrow \mathcal{M}, X \Vdash \neg D\mathsf{A} \wedge \neg D\neg\mathsf{A} \Leftrightarrow \mathcal{M}, X, X \Vdash \neg D\mathsf{A} \wedge \neg D\neg\mathsf{A} \Leftrightarrow$
- $\mathcal{M}, X, X \Vdash \neg D\mathsf{A}$ and $\mathcal{M}, X, X \Vdash \neg D\neg\mathsf{A} \Leftrightarrow$
- $\mathcal{M}, X, X \nVdash \mathsf{A}$ and $\mathcal{M}, X, X \nVdash \neg\mathsf{A} \Leftrightarrow$
- $\mathcal{M}, X \nVdash \mathsf{A}$ and $\mathcal{M}, X \nVdash \neg\mathsf{A}$ □

There is no way of setting the accessibility relation R to define a modal operator on the primary evaluation coordinate that is equivalent to the determinacy operator. To see this,

[15] The support conditions for $\neg D$A, when A is atomic are as follows.

- $\mathcal{M}, X \Vdash \neg D\mathsf{A} \Leftrightarrow \mathcal{M}, X, X \Vdash \neg D\mathsf{A} \Leftrightarrow \forall W \gg X : M, W, X \nVdash D\mathsf{A} \Leftrightarrow \forall W \gg X : \mathcal{M}, X, X \nVdash \mathsf{A} \Leftrightarrow \mathcal{M}, X, X \nVdash \mathsf{A} \Leftrightarrow V(\mathsf{A}, X) \neq 1$.

It is easy to check that $\mathcal{M}, X \Vdash \neg D\mathsf{A}$ holds but $\mathcal{M}, Y \Vdash \neg D\mathsf{A}$ does not.

note that any modal that operates on the primary evaluation coordinate would collapse the two-dimensional framework into the one-dimensional one. We know from Fact 3 that the two systems do not collapse.

There is, however, an important relationship between the two-dimensional operator, and certain ordinary modals as evaluated in some designated submodels. Given a model $\mathcal{M}$ and possibility X, let $\mathcal{M}_X$ be the submodel of $\mathcal{M}$ that is generated by X. That is, the set of possibilities in $\mathcal{M}_X$ is $\{Y \in P_{\mathcal{M}} \mid Y \gg X\}$, and all the remaining relations are restrictions of the original ones to this set. Let $\Vdash_U$ be the support relation generated by interpreting formulas of our formal language according to the unidimensional rules, while interpreting D as the universal modality in $\mathcal{M}_X$ (i.e. by assuming YRZ for any Y and Z in $P_{\mathcal{M}_X}$.).

Fact 5 *For any refinement Y of X, $\mathcal{M}, Y, X \Vdash \mathsf{B}$ iff $\mathcal{M}_X, Y \Vdash_U \mathsf{B}$.*

Proof: Reason by induction on the complexity of B. If B is atomic, the claim holds because the models have agreeing valuation functions. If B is a negated formula $\neg\mathsf{A}$, and Y is an arbitrary refinement of X, for all $Z \geq Y$, $\mathcal{M}, Z, X \nVdash \mathsf{A}$ iff $\mathcal{M}_X, Z \nVdash_U \mathsf{A}$, but since Z is a refinement of Y, it is also a refinement of X, so this follows from the induction hypothesis. Setting aside the trivial case in which B is a conjunction, the remaining case of interest is where $\mathsf{B} = D\mathsf{A}$ for A satisfying the induction hypothesis. Consider Y refining X: then $\mathcal{M}, Y, X \Vdash D\mathsf{A}$ iff $\mathcal{M}, X, X \Vdash \mathsf{A}$ iff $\mathcal{M}_X, X \Vdash \mathsf{A}$ iff $\mathcal{M}_X, X \Vdash D\mathsf{A}$.

Fact 5 is the key to characterizing the logic of our determinacy operator. Let $\models_{S5}$ denote the S5 consequence relation, and $\models^g_{S5}$ the global consequence relation as characterized on Kripke models (Blackburn *et al.*, 2001, §1.3). The consequence relation on two-dimensional possibility models coincides with the global consequence relation on universal Kripke models.[16]

Theorem 1 *$A_1, ..., A_n \Vdash C$ iff $DA_1, ..., DA_n \models_{\mathrm{S5}} DC$ iff $A_1, ..., A_n \models^g_{\mathrm{S5}} C$*

Proof. We exploit the fact that universal modalities in Kripke frameworks have the same logic (i.e. S5) as universal modalities in the possibility framework. So, let $\Vdash_U$ be the logic of a possibility framework for a language with a single modal D with R_D as the universal relation. So:

[16] See the appendix of Schulz (2010) for a similar result involving the logic of Yalcin's (2007) semantics for epistemic necessity—albeit one that is presented wholly at the level of worlds-based semantics.

$$DA_1, ..., DA_n \Vdash_U DC \text{ iff } DA_1, ..., DA_n \vDash_{S5} DC$$

What is left to prove is:

$$A_1, ..., A_n \Vdash C \text{ iff } DA_1, ..., DA_n \Vdash_U DC$$

This is proven by identifying a chain of equivalences between the claim that an arbirary argument has a countermodel in the two-dimensional framework, and the claim that it has a countermodel in the unidimensional framework with D as universal modality. Let $\Gamma = \{\mathsf{A}_1, ..., \mathsf{A}_n\}$, and $D\Gamma = \{D\mathsf{A}_1, ..., D\mathsf{A}_n\}$.

(i) $\exists \mathcal{M}$ and $X \in P_{\mathcal{M}}$, s.t. $\mathcal{M}, X \Vdash \Gamma$, but $\mathcal{M}, X \nVdash \mathsf{C}$.

(ii) $\exists \mathcal{M}$ and $X \in P_{\mathcal{M}}$, s.t. $\mathcal{M}, X, X \Vdash \Gamma$, but $\mathcal{M}, X, X \nVdash \mathsf{C}$.

(iii) $\exists \mathcal{M}$ and $X \in P_{\mathcal{M}}$, s.t. $\forall Z \gg X$, $\mathcal{M}, Z, X \Vdash \Gamma$, but $\mathcal{M}, X, X \nVdash \mathsf{C}$.

(iv) $\exists \mathcal{M}$ and $X \in P_{\mathcal{M}_X}$, s.t. $\forall Z \gg X$, $\mathcal{M}_X, Z \Vdash_U \Gamma$, but $\mathcal{M}_X, X \nVdash_U \mathsf{C}$.

(v) $\exists \mathcal{M}$ and $X \in P_{\mathcal{M}_X}$, s.t. $\mathcal{M}_X, X \Vdash_U D\Gamma$, but $\mathcal{M}_Y, X \nVdash_U D\mathsf{C}$

The equivalence between (ii) and (iii) is due to the fact that evaluation along the primary coordinate is persistent even in the two-dimensional system. The equivalence between (iii) and (iv) relies on Fact 5 and persistence in the unidimensional framework. For the equivalence between (iv) and (v), note that if $D\mathsf{C}$ fails at X, C must fail at some possibility in the model—i.e. at some refinement of X. But if so, C must also fail at X (or else persistence would force it to hold throughout the entire model). □

Theorem 1 enables us to ascertain the satisfaction of many of our design principles. The remaining facts in this section are all presented without explicity proof, on the understanding that they are elementary corollaries of Theorem 1. First, notice that it entails that the logic in the sentential sub-language remains classical. It is also a simple corollary that the theorems of the two-dimensional theory with D as the sole modal operators are exactly the theorems of S5.

Fact 6 $\Vdash C$ *iff* $\vDash_{S5} DC$ *iff* $\vDash_{S5} C$

Next, we notice that there are consistent statements of indeterminacy, i.e.:

Fact 7 (Non-triviality of indeterminacy)

(i) IA *is consistent (i.e.* $\not\Vdash DA \vee D\neg A$*).*

(ii) $D(A \vee B) \not\Vdash DA \vee DB$

The failure of the entailment in part (ii) of 7 is relevant for comparison with an alternate system involving determinacy operators and two-dimensional semantics (i.e., the one on pp. 220-221 of Burgess and Humberstone, 1987).

Per our design specifications, there is no higher-order indeterminacy in this system.

Fact 8 (No higher-order indeterminacy)

(i) $A \Vdash DA$*, and in particular,* $DA \Vdash DDA$ *and* $\neg DA \Vdash D\neg DA$*.*

(ii) $\not\Vdash A \equiv DA$

(iii) $\Vdash DA \equiv DDA$

(iv) $\Vdash \neg DA \equiv D\neg DA$

Note that establishing the entailments in part (i) is not the same as claiming that truth and settled truth coincide, as the observation in part (ii) highlights. And indeed there is an important difference between higher-order and first-order determinacy claims when it comes to object language collapse facts observed in parts (iii) and (iv)—as contrasted with the non collapse in part (ii).

At the same time, Theorem 1 illustrates that the system has a familiar non-classical profile when it comes to its meta-rules. Though the extension of the consequence relation in the D-free fragment matches that of classical sentential logic, adding expressive capacity to the language in the form of the D-operator results in some non-classical behavior. One example of this behavior is that the consequence relation does not contrapose over the full language: $A \Vdash DA$ holds, as we noted in Fact 8, but $\neg DA \Vdash \neg A$ does not. This phenomenon mirrors the standard behavior of similar systems based on S5 global consequence relations. For example, it is observed in supervaluationist analyses based on the idea of "global" validity (Fine, 1975; Williamson, 1994; Varzi, 2007; Asher *et al.*, 2009; Bacon, 2018) and also in informational analyses of consequence for languages with epistemic modals, as in Yalcin (2007); Bledin (2014); Schulz (2010); Incurvati and Schlöder (2022). More specifically, Theorem 1 entails the following failures:

Fact 9

- *No Conditional proof :* $A \Vdash DA$, *but* $\nVdash A \to DA$
- *No Reductio:* $A \wedge \neg DA \Vdash DA$ *and* $A \wedge \neg DA \Vdash \neg DA$ *but* $\nVdash \neg(A \wedge \neg DA)$
- *No Contraposition:* $A \Vdash DA$ *but* $\neg DA \nVdash \neg A$

Disjunctive syllogism may fail too, depending its exact characterization of disjunctive syllogism.[17]

To conclude this section, it is valuable to reflect on exactly how the system manages to avoid the inconsistency in Fact 2. Recall, that the inconsistency pits the definition of indeterminacy (IN), the claim that indeterminacy is non-persistent (NP) and the analysis of negation (NE) against each other. The technical fact of the matter is that the two-dimensional system avoids the inconsistency by rejecting the negation condition, NE. In particular, in the two-dimensional system, there is no guarantee that if X supports $\neg$A, then X's refinements will fail to support A. This is because unidimensional evaluation is governed by the diagonal principle, and so what's supported at X depends on evaluation triples of the form $\langle M, X, X\rangle$, whereas what's supported at Y depends on evaluation triples of the form $\langle \mathcal{M}, Y, Y\rangle$. These may come apart in ways that undermine the negation clause. Of course, the *effect* of the negation operator is preserved because there is an analogous operator at the level of two-dimensional evaluation. However, that operator only quantifies over refinements along the primary dimension.

This technical gloss is important but it does not illuminate the central mechanics behind the two-dimensional proposal. Instead, the two-dimensional system is better thought of as a more flexible generalization of the idea of indexing determinacy operators. The job of the secondary coordinate of evaluation is to anchor the facts that ground determinacy claims, shielding them from the shifting effects of other operators. The failure of the unidimensional negation clause is a downstream consequence of this intervention.

7 Philosophical Considerations on Framework Choice

The two-dimensional analysis of section 5 threads through the design principles that motivated it. Effectively, in the restricted language with sentential logic and a determinacy

[17] We know immediately that the following form of disjunctive syllogism must fail:

If $A \Vdash C$, $B \Vdash C$, then $A \vee B \Vdash C$

If it didn't, then we would have $A \vee \neg A \Vdash DA \vee D\neg A$ (contradicting Fact 7). Alternatively, it is possible to formulate disjunctive syllogism as follows:

If $\Vdash A \to C$, $\Vdash B \to C$, then $\Vdash (A \vee B) \to C$

It's another consequence of Theorem 1 that this reformulated schema is correct.

operator, it ends up projecting the same logic as a superaluationist analyses based on a branching time framework.[18] The fact that the semantics can go this far raises to salience the question of what distinguishes the possibility approach from these other frameworks, if it's not the logic. This final section advances some considerations about directions that might be pursued to distinguish these approaches. §7.1 is a quick, pessimistic, detour through a more speculative justification, and §7.2 is a more systematic comparison of possibility semantics and branching time semantics that highlights some differences between branching time and possibility frameworks. This comparison is fruitful in identifying some important distinguishing features.

7.1 Deep Metaphysical Indeterminacy?

There is a potentially relevant debate in metaphysics concerning "deep" metaphysical indeterminacy. This is indeterminacy that is not well modeled as indeterminacy concerning which complete world is actual. In particular, Skow (2010) argues that quantum indeterminacy is not well understood in terms of completely precisified possibilities. It is "deep" in the sense that it is not a matter of which absolutely precise world correctly represents the settled facts. (See also Wilson, 2013, for objections along these lines). Here is a suggestive example of this sort of phenomenon, drawn from Torza (2020). The Kochen-Specker theorem in quantum mechanics (Held, 2018) suggests that, under plausible assumptions, there are sets of properties that cannot simultaneously have determinate values. If the electron has a determinate value for one set of properties P_1, then it cannot have determinate values for another set P_2. But if so, there is a problem with representing any given state by means of a world, since it would seem that worlds are fully determinate, and in particular ought to assign determinate values to both the properties in P_1 and the properties in P_2.

It is tempting to think that the possibility semantics framework can offer a model for this sort of unsettledness. However, despite the central role assigned to partial objects, possibility semantics seems also unable to capture the phenomenon of deep indeterminacy. After all, Refinability guarantees that, for any finite combination of properties in the object language, there must be possibilities in any model in which

[18] It probably behooves us to be a bit careful about the distinction between the branching time framework and some specific 'post-semantic' proposals like supervaluationism. MacFarlane (2003, 2014) leans on the branching time framework, but advances a 'relativist' account of future contingents. This model has some elements in common with what is normally termed 'supervaluationism', but also has important elements of distinction. In this paper we are abstracting away from that debate, and zeroing in on the contrasts between evaluation at a (complete) world, or history, and evaluation at a (partial) possibility.

the properties are simultaneously defined. If the moral of the objection from deep metaphysical indeterminacy is that we ought to ban "impossible" states from models, the framework of possibility semantics falls just as short of the requirement as the branching time framework.

A defender of the framework can respond in one of two ways. The non-concessive reply has it that, although the framework does not ban the unwanted possibilities, it also features primitive representations for all the partial possibilities that represents possible states of the world. This is not so for the rival framework that is grounded in overlap of complete histories. According to this reply, the moral of deep indeterminacy is that we ought to *make room* for partial representations, and it is not that we ought to ban total representations as limit cases of the partial ones. This argument, however, becomes somewhat thinner when the opponent notes that they too have the ability to represent the partial states, for example in terms of sets of possible worlds—or, in the branching framework, sets of histories. At this point it begins to look as if that the only real difference between the frameworks is whether these partial objects are primitive or derived. Nothing much can hang on this kind of choice of primitives in the formal framework. It seems inevitable to conclude that, when it comes to modeling deep indeterminacy, the possibility framework with Refinability does not offer real advantages over a framework based on complete objects.

More concessive replies would start with the recognition that Refinability is indeed inconsistent with the phenomenon of deep metaphysical indeterminacy. At the same time, however one might argue that, if we were to apply possibility semantics to deep indeterminacy, we ought to renounce Refinability. There are existing models of possibility semantics that shun Refinability, and endorse the non-classicality in the logic that follows from it (Holliday, 2022; Holliday and Mandelkern, ms.). It is relevant here that Darby and Pickup (2021) develop a model of deep metaphysical indeterminacy that is based on a version of *situation* semantics in which analogues of Completability and Refinability fail—it is not the case that any situation may be extended into a complete one. This is not the place to develop this direction within the possibility semantics framework, but it strikes me that, if the motivation for possibility semantics was indeed the phenomenon of deep indeterminacy, the most promising route is to interact directly with the literature on metaphysical indeterminacy, and pursue models according to which possibilities are not arbitrarily refinable.[19]

[19]In addition to the references listed above in this discussion, note also Torza (2020, 2021), which contain

7.2 Comparison to branching-time semantics

At a couple of points, we have noted that the possibility frameworks seems to go after the same concepts as the branching time framework. It is worth exploring this comparison in a bit more detail.

Branching time frames are built on *moments*—entities that determine a total distribution of truth-values over the atomic sentences. Moments are arranged in a tree-like branching structure. A relation $<$ is a branching order on a set P of points iff $<$ is a partial order of P (i.e., a transitive, irreflexive, and anti-symmetric relation over P) with the additional property that for any $x, y, z \in P$, if $x \geq z$ and $y \geq z$, then either $x \geq y$ or $y \geq x$. A branching time frame is $\langle \mathcal{S}, \geq \rangle$ is a pair consisting of a set of moments and a branching order.

One key type of entities that emerge from branching time frames are *histories*. The histories within some model $\mathfrak{M}$ are the maximal chains within $\mathfrak{M}$. (Note that we represent branching models and their components with somewhat different notation, so as to make comparison with possibility frames easier to decode.) If m is a moment and h is a history, it is standard to write $m \in h$ to mean that m occurs at some point in the course of the history h, or that h passes through m. The central idea of the framework is that truth is defined recursively at moment-history pairs. Thus a branching time model is a triple $\langle \mathcal{S}, \geq, \mathfrak{k} \rangle$ that expands a branching frame $\langle \mathcal{S}, \geq \rangle$ with a valuation function $\mathfrak{k}$—a mapping from the set of moment/history pairs to $\{T, F\}$.[20] For atomic formulas A, $\mathfrak{M}, m, h \models \mathsf{A}$ iff $\mathfrak{v}(\mathsf{A}, m) = T$. For conjunction, disjunction, and negtion, truth propagates according to the standard Boolean clauses.

Going beyond sentential languages, one might define a *historical necessity* operator:

- $\mathfrak{M}, m, h \Vdash \Box \mathsf{A}$ iff for all h', such that $m \in h'$, $\mathfrak{M}, m, h' \Vdash \mathsf{A}$

This operator is in many respects similar to the determinacy operator we introduced in §5. It is sometimes referred to as the "settledness" or "historical necessity" operator—and often symbolized as *Sett*. Like our determinacy operator, the settledness operator is responsible for the potential failure of the classical meta-rules discussed in Fact 9.

arguments that cast further doubt on the applicability of possibility semantics as a model of quantum indeterminacy.

[20]This has now come to become a standard aspect of the branching framework, canonized for instance in Belnap *et al.* (2001). However, the inputs of the valuation function vary in different presentations of the formalism. Thomason (1984) defines valuations for branching models as constraints on moments (and not on histories). This was, perhaps, an oversight since later presentations of the formalism by Thomason himself (2007) clearly adopt the now standard position that valuations input moment-history pairs.

Similar to our investigation of determinacy in possibility semantics, one can also pin down metalinguistic analogues of this notion of settledness. Say that a formula A is *settled true* (in model $\mathfrak{M}$ at moment m) iff for every history h' going through m, $\mathfrak{M}, m, h' \models$ A; correspondingly, A is settled false iff for every history h' going through m, $\mathfrak{M}, m, h' \not\models$ A.

There are important structural correspondences between branching time and possibility structures. Let us reconsider the diagrammatic representation of a possibility model in figure 1, replicated here as figure 3:

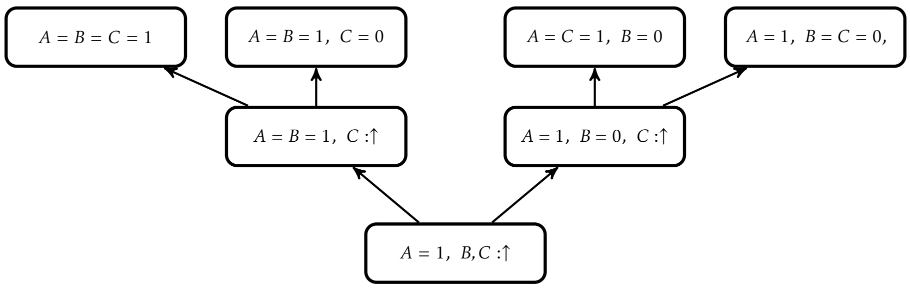


Figure 3: The branching structure of refinements.

This diagram could also be viewed as representing a branching-time model—one with moments corresponding to the seven depicted possibilities.

For the purposes of this discussion, it will be a convenient simplification to assume that the atomic sentences we are modeling have "eternal" truth-values, once they are settled. That is to say, we assume that, while they may go from unsettled to settled, they do not change in truth-value once they are settled. Concretely, we can imagine that in Figure 3, A, B and C are atomic sentences corresponding respectively to the English *On the first day it is sunny*; *On the second day it is sunny*; *On the third day it is sunny*. (Thus the rightmost branch of the tree in Figure 3 represents a world in which it was only sunny on the first day of three.) In other words, we are setting aside the option of having atomic sentences that denote temporally variable statuses such as *it is sunny*—-something that could be true today, but false tomorrow. The reason why this assumption is convenient is that, although the branching framework is neutral between temporalist and eternalist propositions, the temporalist conception is not easily implemented in any possibility semantics subject to the Persistence requirement.

The intuition that there is a correspondence between these two types of structure can be made precise at the level of frames. Moments in branching frames correspond to possibilities and we can define the temporal precedence relation in a branching frame to mirror the refinement relation in the possibility frame.

Definition 4 *A possibility frame $\mathcal{F} = \langle P, \gg \rangle$ induces a branching frame $\mathsf{F} = \langle \mathfrak{S}, \geq \rangle$ as follows. First, identify a set of moments $\mathfrak{S}$ that is in 1-1 correspondence with the set of possibilities of F. Next, choose a 1-1 correspondence that preserves the relations on the points. That is, letting ▸ be the 1-1 correspondence with the possibility frame as domain, and ◂ be its inverse, we must ensure that for any $m, m' \in \mathfrak{S}, m \leq m'$ iff $\blacktriangleleft m \ll \blacktriangleleft m'$.*

Informally, m precedes m' in the branching frame iff the possibility that corresponds to m' refines the possibility that corresponds to m.

Fact 10 *Under ▸ as in Definition 4, every possibility frame matches a branching frame.*

> *Proof.* The proof is a routine matter of checking that the branching property is satisfied, once we generate a branching frame from a possibility frame. □

This correspondence also happens to be unique for finite models and infinite models that have a single origin moment.

There is a clear, and important, structural difference between the two frameworks when it comes to valuations—and thus when it comes to thinking about the relationship between the two types of models. Possibility valuations input possibilities, while branching time valuations input moment/history pairs. In figure 3, where A is an atomic sentence, $v(\mathsf{A}, X) = 1$ would be translated in the branching framework as the claim that A is *settled true* in the point of evaluation "corresponding" to X. After all, as emphasized above, being assigned a truth-value by the valuation function is a way of being determinate in the possibility framework. Specifically, recall that $\blacktriangleright X$ is the moment that corresponds to possibility X in the branching time model $\mathfrak{M}$. Then, we must have $\mathfrak{k}(\mathsf{A}, \blacktriangleright X, h)$ for each h going through $\blacktriangleright X$.[21]

Suppose then that we are given a possibility model: how can we convert it into a branching-time model that is in some sense equivalent? To start we need to be clear conception of what the conversion ought to achieve:

> **Goal:** Suppose $\mathcal{M}$ is a possibility model and $\mathfrak{M}$ the corresponding branching model. Then for any formula A, and any $X \in P_{\mathcal{M}}$,

$$\mathcal{M}, X \Vdash \mathsf{A} \text{ iff the moment } \blacktriangleright X \text{ in } \mathfrak{M} \textit{ settles } \mathsf{A}$$

[21] One way of thinking about what is going on here is that possibility semantics lays out recursive clauses for complex sentences *directly* in terms of possibilities. In the branching time perspective, the points of evaluation in terms of which the recursive analysis is defined represent much finer-grained entities. Because of this, it is not possible to map each point of evaluation for possibility semantics (which is of the form $\langle \mathcal{M}, X \rangle$) to a point for of evaluation in branching time semantics (which is of the form $\langle \mathfrak{M}, m, h \rangle$). The former types of point do not determine the latter type of point of evaluation.

This makes sense in light of Humberstone's idea that being supported by a possibility is itself a kind of determinacy status.

Given model $\mathcal{M} = \langle P, \gg, v\rangle$, we can start converting it by first identifying the induced branching frame $\mathfrak{F} = \langle \mathfrak{S}, \geq\rangle$ according to the procedure in Definition 4. Each branching frame $\mathfrak{F}$ determines a class $\mathfrak{H}$ of histories within the frame: in particular, the histories in $\mathfrak{F}$ are the subsets of $\mathfrak{S}$ that are maximal chains in the ordering $\geq$. When the frame is induced by a possibility frame $\mathcal{F} = \langle P, \gg\rangle$, each history also corresponds to a set of possibilities that is maximally linearly ordered by the refinement relation. Call these *possibility chains*. More generally, the 1-1 correspondence between moments and possibilities induces a 1-1 correspondence between sets of possibilities and sets of moments—thus, somewhat less generally, a 1-1 correspondence between histories and possibility chains. If we abuse notation and allow $\blacktriangleright$ to map sets of possibilities to sets of moments, we will also be able to use it to take us from possibility chains to histories.

If we restrict attention to finite models, there is a simple and straightforward way of converting from possibility valuations into to branching valuations in a way that meets our goal.

Definition 5 *Given (i) an atomic sentence* A*, (ii) a possibility model* $\mathcal{M}$ *based on a finite possibility frame* F*, (iii) an induced branching frame* $\mathfrak{F} = \langle \mathfrak{S}, \geq\rangle$*, (iv) a set of histories* $\mathfrak{H}$ *within* $\mathfrak{F}$ *(v) any moment* $m \in \mathfrak{S}_{\mathcal{M}}$*, and (vi) any history* $h \in \mathfrak{H}$ *such that* $m \in h$*, we define*

$$\mathfrak{k}(\mathsf{A}, m, h) = 1 \textit{ iff } \exists Y \in (\blacktriangleleft h) \wedge v(\mathsf{A}, Y) = 1$$

$$\mathfrak{k}(\mathsf{A}, m, h) = 0 \textit{ iff } \exists Y \in (\blacktriangleleft h) \wedge v(\mathsf{A}, Y) = 0$$

Informally, there is a refinement of the possibility corresponding to m that is also on h and settles Y as true.[22] Under this definition, it is possible to establish the following restricted correspondence:

Fact 11 *For every finite possibility model* $\mathcal{M}$*, and any* $X \in P_{\mathcal{M}}$*,* $\mathcal{M}, X \Vdash \mathsf{A}$ *iff* A *is settled true at* $\blacktriangleright X$ *in the induced branching model* $\mathfrak{M}$*.*

> *Proof.* We prove this by induction. Suppose that A is atomic, and consider the chain of equivalences below. Note that each *iff* is numbered and only those that are not obvious are explained.

[22]Note that this approach makes sense if (and only if) we stipulate, as we did, that atomic sentences have "eternal" truth-value. In the branching framework, this means that they have their recursive truth-values directly settled by the history on which they are being evaluated.

- $\mathcal{M}, X \Vdash \mathsf{A}$ iff$_1$ $v(\mathsf{A}, X) = 1$ iff$_2$ for every possibility chain C going through X, $\exists Y \in C, v(\mathsf{A}, Y) = 1$ iff$_3$ for every history h going through $\blacktriangleright X$, $\Bbbk(\mathsf{A}, \blacktriangleright X, h) = 1$ iff$_4$ A is settled true at $\blacktriangleright X$.

For iff$_2$: note that if $v(X, \mathsf{A}) = 1$, X serves as the witness for the existential $\exists Y$. Conversely, if $v(X, \mathsf{A}) \neq 1$, then either $v(X, \mathsf{A}) = 0$ or $v(X, \mathsf{A}) \uparrow$. But in either case there must be a possibility chain that contains a Y that settles A as false (in the case of $v(X, \mathsf{A}) \uparrow$, the existence of this Y requires Refinability).
For iff$_3$: note first that as a result of the frame correspondence, there is a correspondence between the possibility chains going through X and the histories going through $\blacktriangleright X$. Assume the left side of the equivalence, and consider an arbitrary h going through $\blacktriangleright X$. Using the definition of the valuation function $\Bbbk(\mathsf{A}, \blacktriangleright X, h) = 1$ iff $\exists Y \in (\blacktriangleleft h) \wedge v(\mathsf{A}, Y) = 1$. But we know that such a history must exist, since it's the witness to the Y in the assumed left side of *iff*$_3$. With small changes, this reasoning also works in reverse, establishing the converse direction of the equivalence, and thus the base case of the induction.

Suppose next that A is a negation, i.e. $\mathsf{A} = \neg\mathsf{B}$. Assume that the target claim is satisfied for B. Consider:

> $\mathcal{M}, X \Vdash \neg\mathsf{B}$ iff$_1$ $\mathcal{M}, X, X \Vdash \neg\mathsf{B}$ iff$_2$ for all $Y \gg X$, $\mathcal{M}, Y, X \nVdash \mathsf{B}$ iff$_3$ for all $m' \geq m$, B is not settled true at m' in $\mathfrak{M}$ iff$_4$ B is settled false at m in $\mathfrak{M}$ iff$_5$ $\neg$B is settled true at m in $\mathfrak{M}$.

For iff$_3$*:* we appeal the induction hypothesis.
For iff$_4$*:* this follows from the fact that we are restricting to finite models, where every chain must have a terminal point. So, the claim that for all $m' \geq m$, $\mathfrak{M}, m'$ does not settle B true amounts to the claim that each terminal point in the branches that continue on from m settles B false.

The conjunction case is trivial and omitted. As for determinacy, assume that $\mathsf{A} = D\mathsf{B}$ and also that the induction hypothesis is established for B.

> $\mathcal{M}, X \Vdash D\mathsf{B}$ iff$_1$ $\mathcal{M}, X, X \Vdash D\mathsf{B}$ iff$_2$ $\mathcal{M}, X, X \Vdash \mathsf{B}$ iff$_3$ $\mathcal{M}, X \Vdash B$ iff$_4$ B is settled true at $\blacktriangleright X$ in $\mathfrak{M}$

In this chain, the induction hypothesis is applied in *iff*$_4$.

At one critical spot in the proof of Fact 11, we appealed to the restriction to finite models. Indeed, there are infinite models for which the above correspondence breaks down. These turn out to have significant theoretical interest.

Definition 6 *Define the* Beth Comb Frame *as the following possibility frame.*[23] *Suppose our domain is the set H of natural numbers and their halves.* $\{x \in \mathbb{R} \mid \exists y \in \mathbb{N}, x = y + 0.5 \vee x = y\}$. *The refinement structure of the Beth Comb Frame may be defined as follows $Y \gg X$ iff* $(X \leq Y) \wedge (X \in \mathbb{N})$.

This refinement structure might be more easily inspected though a diagram as in Figure 4.

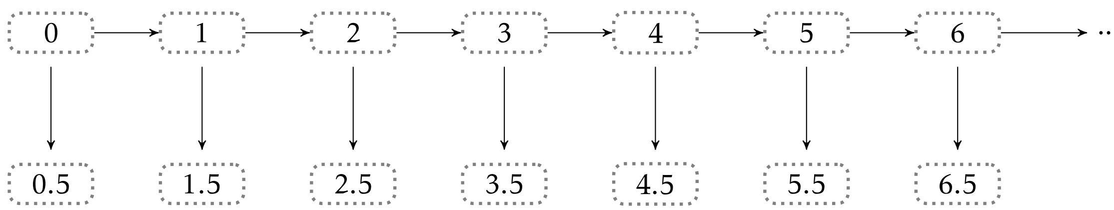


Figure 4: The Beth Comb Frame

The frame contains infinitely many possibilities and infinitely many histories. Among the histories, one is distinguished, namely the one that corresponds to $\mathbb{N}$ itself. Call this the *spine* of the Beth Comb Frame. Continuing with the comb metaphor, the possibilities that are not on the *spine* might be referred as the *teeth* of the frame. In light of our previous work, we can also take note of the fact that, on the branching side, there is a branching frame that corresponds to the Beth Comb frame.

Now consider a possibility model $\mathcal{M}^*$ built on the Beth comb for a language with a single atomic formula p. Set p to undefined at all the possibilities along the spine. Finally, suppose that the truth-value of p alternates at the teeth, in such a way that p is true at the teeth whose integer part is even ($v(p, X) = 1$, for X an "even tooth"), and false at the teeth whose integer part is odd ($v(p, X) = 0$). Call this the Alternating Beth Comb model (Figure 5) Because of the alternation, the refinability condition on the possibility model is met: at any point where p is indeterminate, there is a future refinement where p is evaluated as false, and one where it is evaluated as true. The following observation will be critical in the comparison:

- **observation:** there is no possibility X on the spine at which $v(p, X)$ is defined.

[23]"Beth Comb" is the name of a type of model for intuitionistic logic, as discussed in Beth (1956), Humberstone (2011, p.898), Bezhanishvili and Holliday (2019, example 3.1), Holliday (2022, example 2.18). Here we are going to appropriate the frame of the Beth Comb and put it to a new application.

It follows that $\mathcal{M}^*$ breaks the recipe for generating a branching valuation we provided in Definition 5. The induced branching valuation $\Bbbk^*$ cannot assign either 0 or 1 to $\langle p, m_n^*, h^* \rangle$, but in branching time semantics, every moment/history pair must determine a recursive truth-value for any atomic sentence.

Of course, this could be a flaw in Definition 5. More systematic work that we have no space for here would be required to investigate alternative definition. However, it is plausible that infinite frames like the Beth Comb frame may reveal the difference between the two conceptions underlying the modeling of indeterminacy in possibility semantics and branching semantics. In the branching framework, indeterminacy is represented by the overlap of fully determinate histories. Histories may overlap, but each of them in isolation is a complete object that resolves all indeterminacy. In possibility semantics, indeterminacy may persist even after we fix a history.

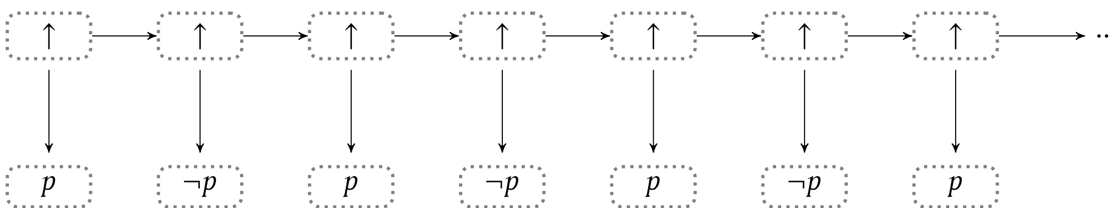


Figure 5: The Alternating Beth Comb Model

To bolster this diagnosis, it is helpful to make the point in a less abstract way. There might be conceivable scenarios that can be represented by models like $\mathcal{M}^*$ that have no corresponding branching models. Suppose that Bea will flip a coin every day, possibly for eternity. If the coin lands heads on an even day, the game is over and Bea wins. If the coin lands heads on an odd day, the game is over and Bea loses. If the coin lands tails, she continues flipping the next day. Let us symbolize the proposition that *Bea win the game (at some unspecified point)* as p, and the proposition that *Bea loses (at some unspecified point)* as $\neg p$. It is evident that this scenario is captured by the Alternating Beth Comb model. It is also evident that this scenario cannot be represented in the branching model, since the branching model needs to make p either true or false on at the spine. However, neither of these choices is intuitively right. Intuitively, every point on the spine represents a moment at which the outcome of Bea's game is unsettled. Notice finally, that the problem is not that we have incorrectly used $\neg p$ to represent the proposition that *Bea loses*. For suppose that that proposition was associated with another atomic formula, say q. Then we would have to decide how to treat p and q on the spine. Under that setup, and it won't do to claim that they are both false on the spine. That stipulation would contradict the otherwise plausible assumption that is is determinate that either Bea will win the game or she will lose. The two-dimensional possibility framework has no problem making this prediction.

This brief discussion is not presented here as an argument against branching frameworks. It is a virtual certainty that proponents of the branching framework can propose alternate symbolizations of the scenario in which they can deliver the correct verdict. The claim being advanced is more modestly that the Alternating Beth Comb model is diagnostic of the fact that the possibility framework incorporate a different model of indeterminacy than the branching framework, and is not a mere notational variant of it.

8 Conclusion

The main conclusions are as follows: there is a clear path for the application of possibility semantics to the metaphysical hypothesis of the open future. That path must include the characterization of object language determinacy operators. Introducing such operators under something like the alignment constraint requires, on pain of inconsistency, some modifications to the original framework. A two-dimensional variant of possibility semantics is one path to relieve this theoretical pressure. In its natural interpretation, the logic of determinacy under the two-dimensional analysis is the global version of S5.

From a philosophical point of view, a pressing question remains concerning what sorts of considerations might distinguish branching-time and possibility-based treatments of the indeterminacy of the future. We argued that a key to the difference might lie in the different treatments of indeterminacy in the new framework.

Bibliography

Aloni, Maria (2018), 'Fc disjunction in states-based semantics', University of Amsterdam, currently available at https://www.marialoni.org/resources/Aloni2018.pdf.

Asher, Nicholas, Dever, Josh and Pappas, Chris (2009), 'Supervaluations debugged', *Mind*, 118(472), 901–933.

Bacon, Andrew (2018), *Vagueness and Thought*, Oxford University Press.

Barnes, Elizabeth and Cameron, Ross (2009), 'The open future: bivalence, determinism and ontology', *Philosophical Studies*, 146, 291–309.

Barnes, Elizabeth and Cameron, Ross (2011), 'Back to the open future', *Philosophical Perspectives*, 25(1), 1–26.

Barnes, Elizabeth and Williams, J.R.G. (2011), 'A theory of metaphysical indeterminacy', *Oxford studies in metaphysics*, 6, 103–148.

Barwise, Jon and Perry, John (1981), 'Situations and attitudes', *The Journal of Philosophy*, 78(11), 668–691.

Belnap, Nuel and Green, Mitchell (1994), 'Indeterminism and the thin red line', *Philosophical perspectives*, 3, 365–388.

Belnap, Nuel, Perloff, Michael and Xu, Ming (2001), *Facing the Future*, Oxford University Press.

Beth, E.W. (1956), 'Semantic construction of intuitionistic logic', *Mededelingen der Koninklijke Nederlandse Akademie van Wetenschappen*, 19(11), 357–388.

Bezhanishvili, Guram and Holliday, Wesley H (2019), 'A semantic hierarchy for intuitionistic logic', *Indagationes Mathematicae*, 30(3), 403–469.

Blackburn, Patrick, De Rijke, Maarten and Venema, Yde (2001), *Modal logic*, Cambridge University Press.

Bledin, Justin (2014), 'Logic informed', *Mind*, 123(490), 277–316.

Boylan, David (forthcoming), 'Does success entail ability?', *Noûs*.

Burgess, J. A. and Humberstone, I. L. (1987), 'Natural deduction rules for a logic of vagueness', *Erkenntnis*, 27(2), 197–229.

CARIANI, FABRIZIO (2021a), 'Human foreknowledge', *Philosophical Perspectives*.

CARIANI, FABRIZIO (2021b), *The Modal Future*, Cambridge University Press.

CARIANI, FABRIZIO AND SANTORIO, PAOLO (2018), '*Will* done better: Selection semantics, future credence, and indeterminacy', *Mind*, 127(505), 129–165.

DARBY, GEORGE AND PICKUP, MARTIN (2021), 'Modelling deep indeterminacy', *Synthese*, 198(2), 1685–1710.

FIELD, HARTRY (2000), 'Indeterminacy, degree of belief, and excluded middle', *Noûs*, 34(1), 1–30.

FINE, KIT (1975), 'Vagueness, truth, and logic', *Synthese*, 30, 265–300.

FINE, KIT (2017a), 'The possibility of vagueness', *Synthese*, (10), 3699–3725.

FINE, KIT (2017b), 'Truthmaker semantics', *A Companion to the Philosophy of Language*, 2, 556–577.

HELD, CARSTEN (2018), 'The Kochen-Specker Theorem', in Edward N. Zalta (ed.), 'The Stanford Encyclopedia of Philosophy', Metaphysics Research Lab, Stanford University, Spring 2018 edition.

HOLLIDAY, WESLEY H. (2014), 'Partiality and adjointness in modal logic', *Advances in Modal Logic*, 10.

HOLLIDAY, WESLEY H. (2022), 'Possibility semantics', in Melvin Fitting (ed.), 'Selected Topics from Contemporary Logic (Landscapes in Logic Vol. 2)', 363–476.

HOLLIDAY, WESLEY H. (forthcoming), 'Possibility frames and forcing for modal logic', *Australasian Journal of Logic*.

HOLLIDAY, WESLEY H. AND MANDELKERN, MATTHEW (ms.), 'The orthologic of epistemic modals', University of California, Berkeley and NYU.

HUMBERSTONE, LLOYD (1979), 'Interval semantics for tense logic: Some remarks', *Journal of philosophical logic*, 8(1), 171–196.

HUMBERSTONE, LLOYD (1981), 'From worlds to possibilities', *Journal of Philosophical Logic*, 10(3), 313–339.

Humberstone, Lloyd (2004), 'Two-dimensional adventures', *Philosophical Studies*, 118(1), 17–65.

Humberstone, Lloyd (2011), *The Connectives*, MIT Press.

Incurvati, Luca and Schlöder, Julian J. (2022), 'Meta-inferences and supervaluationism', *Journal of Philosophical Logic*, 1549–1582.

Kratzer, Angelika (2021), 'Situations in Natural Language Semantics', in Edward N. Zalta (ed.), 'The Stanford Encyclopedia of Philosophy', Metaphysics Research Lab, Stanford University, Winter 2021 edition.

Kuhn, Steven T (2013), 'Two-dimensional logic and Two-dimensionalism in philosophy', in Delia Graff Fara and Gillian Russell (eds.), 'Routledge Companion to Philosophy of Language', 640–651, Routledge.

MacFarlane, John (2003), 'Future contingents and relative truth', *Philosophical Quarterly*, 53, 321–336.

MacFarlane, John (2014), *Assessment Sensitivity*, Oxford University Press.

Schroeter, Laura (2021), 'Two-Dimensional Semantics', in Edward N. Zalta (ed.), 'The Stanford Encyclopedia of Philosophy', Metaphysics Research Lab, Stanford University, Fall 2021 edition.

Schulz, Moritz (2010), 'Epistemic modals and informational consequence', *Synthese*, 174, 385–395.

Skow, Bradford (2010), 'Deep metaphysical indeterminacy', *The Philosophical Quarterly*, 60(241), 851–858.

Stalnaker, Robert (1984), *Inquiry*, MIT Press (Bradford Books).

Stalnaker, Robert C. (2019), *Knowledge and conditionals: essays on the structure of inquiry*, Oxford University Press.

Thomason, Richmond H. (1970), 'Indeterminist time and truth-value gaps', *Theoria*, 18(3), 264–281.

Thomason, Richmond H. (1984), 'Combinations of tense and modality', in D. Gabbay and F. Guenthner (eds.), 'Handbook of philosophical logic (Volume II: Extensions of Classical Logic)', 135–165, Springer.

Thomason, Richmond H. (2007), 'Truth after the fact in indeterminist tense logic', URL https://web.eecs.umich.edu/~rthomaso/documents/indeterminism/indettns.pdf, unpublished Manuscript: University of Michigan.

Todd, Patrick (2022), *The Open Future*, Oxford University Press.

Torre, Stephan (2011), 'The open future', *Philosophy Compass*, 6(5), 360–373.

Torza, Alessandro (2020), 'Quantum metaphysical indeterminacy and worldly incompleteness', *Synthese*, 197, 4251–4264.

Torza, Alessandro (2021), 'Quantum metametaphysics', *Synthese*, 199(3), 9809–9833.

Torza, Alessandro (forthcoming), *Indeterminacy in the world*, Cambridge University Press.

Varzi, Achille C (2007), 'Supervaluationism and its logics', *Mind*, 116(463), 633–676.

Willer, Malte (2018), 'Simplifying with free choice', *Topoi*, 37(3), 379–392.

Williams, J Robert G (2014), 'Decision-making under indeterminacy', *Philosophers' Imprint*, 14, 1–34.

Williamson, Timothy (1994), *Vagueness*, Routledge.

Wilson, Jessica M. (2013), 'A determinable-based account of metaphysical indeterminacy', *Inquiry*, 56(4), 359–385.

Yalcin, Seth (2007), 'Epistemic modals', *Mind*, 116(4), 983–1027.