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MODULATED LOGICS AND FLEXIBLE REASONING

Abstract. This paper studies a family of monotonic extensions of first-order logic which we call *modulated logics*, constructed by extending classical logic through generalized quantifiers called *modulated quantifiers*. This approach offers a new regard to what we call *flexible reasoning*. A uniform treatment of modulated logics is given here, obtaining some general results in model theory. Besides reviewing the "Logic of Ultrafilters", which formalizes inductive assertions of the kind "almost all", two new monotonic logical systems are proposed here, the "Logic of Many" and the "Logic of Plausibility", that characterize assertions of the kind "many", and "for a good number of". Although the notion of simple majority ("more than half") can be captured by means of a modulated quantifier semantically interpreted by cardinal measure on evidence sets, it is proven that this system, although sound, cannot be complete if checked against the intended model. This justifies the interest on a purely qualitative approach to this kind of quantification, what is guaranteed by interpreting the modulated quantifiers as notions of *families of principal filters* and *reduced topologies*, respectively. We prove that both systems are conservative extensions of classical logic that preserve important properties, such as soundness and completeness. Some additional perspectives connecting our approach to flexible reasoning through modulated logics to epistemology and social choice theory are also discussed.

Keywords: modulated logics, generalized quantifiers, qualitative reasoning, uncertain reasoning, flexible reasoning.

1. A softened approach to quantification

A long-standing question, that has received a modern formulation, concerns the formalization of reasoning and argumentation based upon assertions or

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sentences which are not absolutely true, but are instead supported by favorable evidence. Such assertions give support to a widespread form of reasoning and argumentation called *uncertain reasoning*, which by its turn helps to support *inductive reasoning* (in the sense of [Car50] and [Pop72]). However, the term "uncertain" may be negatively seen as "questionable" or "inconclusive", while this pejorative connotation makes no justice to its fundamental usage in reasoning; we prefer to refer to "flexible" instead. So, by flexible reasoning we emphasize the positive connotation of any reasoning based on such modifiers as "most", "many", "generally", "plausibly", etc.

This form of inductive reasoning and argumentation is still waiting for a logic foundation. Although Popper (*op.cit*) claimed that induction is never actually used by scientists, and deflated the role of inductive argumentation while distinguishing between science and non-science in favor of falsifiability, it is not said that inductive reasoning and argumentation are useless. On the contrary, uncertain reasoning is commonly used, and in the majority of cases it has to do with assertions and sentences involving a vague quantification of the kind "almost all", "the majority", "most", "many", etc. This kind of statements often occur, not only in ordinary language, but also in new attempts of formalizing abductive reasoning and heuristics.

Some approaches as in [Car50] and [Rei80] concern the question of rational justification of inductive reasoning from the logical viewpoint. However, all proposed solutions to rational justification of inductive reasoning have shown to be insufficient (for different reasons) in the sense that they could not offer a completely satisfactory or definitive answer to the question.

A popular proposal for formalizing certain notions involved in this kind of reasoning is the default logic (cf. [Rei80]). In that paper, the author proposes a non-monotonic logical system which formalizes sentences of the kind "almost always" in terms of the notion of "in the absence of any information to the contrary, assume...". According to [Rei80], it is clear to anyone acquainted with the tools of formal logic that what we have in these situations is a question of quantification, namely how to represent and understand generalized quantifiers[1] of the kind "most", "almost all", etc. Although [Rei80] insists upon the necessity of treating such form of reasoning through generalized quantification, he nonetheless chooses to formalize sentences of the kind "almost always" or "generally" by using extra-logical operators. Several

[1]Mathematically interesting quantifiers which cannot be defined in terms of universal and existential quantifiers([BC81]) were firstly investigated in [Mos57] and have been since then extensively studied; see [Ebb85] and [Flu85]).

impediments and problems of the formalization of sentences through the operator proposed in [Rei80] have already been advanced (e.g. [HM87], [Poo91] and [DW91]). Other criticisms to the non-monotonic approach to the formalization of reasoning under uncertainty have been presented by [SCV99]).

In the same vein of Reiter's argument about the need of representing generalized quantifiers in formal language, [BC81] suggest that a semantic theory of natural language has to incorporate generalized quantifiers expressing notions such as "most", "many", "more than half", etc., since quantification in natural languages is not limited to universal and existential quantifiers.

In the present paper we develop some perspectives initiated in [CS94], and present a family of (monotonic) logical systems called *modulated logics* to express rigorously some kinds of inductive assertions involving notions of "most" in the sense of "simple majority", "many" and "a 'good' number of". The starting point of our approach is that one can only agree on what is said about such notions when they are suitably formalized. The systems we investigate here are characterized by adding generalized quantifiers (here denoted by Q) and called *modulated quantifiers* to their syntax; they are semantically interpreted by expanding classical models to include subsets (denoted by q) from the power set of the universe, defined by certain mathematical structures (as filters, topological spaces and their generalizations, and so on). Intuitively, classical models are endowed with sets of sentences which represent positive evidence regarding a knowledge basis. Such expanded models are called *modulated models* or *modulated structures*.

Modulated structures and modulated quantifiers are the mathematical achievement of the intuitive idea of understanding logical constants in a softened way, in the sense of making them more apt to express concepts and reasoning which only seem to be possible in natural language. This is the case of linguistic modifiers like "many" and "few", "much" and "little", "generally" and "rarely", as well as the ability of reasoning by them. If the classical quantifiers are sufficient to model reasoning when the subject matter is well-behaved, as in the universe of numbers, sets, groups and so on – that is, mathematical objects and clear-cut relations on them – usual reasoning is of course much more wide-scoped, dealing with not so tamed objects and not so limpid relations on them. For example, "all numbers that divide $10^{10} + 1$ are odd" is a clear sentence with an uncontroversial interpretation (since odd numbers cannot have even divisors) while, "all numbers greater than $10^{10} + 1$ are big numbers" has a controversial meaning since "big number" is a vague predicate. Also, "most prime multiples of powers of $10^{10} + 1$ are odd" is also controversial, but now for a different reason: although the numerical predicate "odd" is uncontroversial, the modifier "most" complicates everything; it is true that "only" $2 \cdot (10^{10} + 1)^n$ will be even, while $p \cdot (10^{10} + 1)^n$ will be odd for infinitely many odd primes p. On the one hand, one may feel inclined to say that there are more odd than even numbers, but on the other hand, the sets have exactly the same cardinality, although this is not true about initial segments. This plastic, qualitative rather than quantitative, capacity of linguistic modifiers convey reasoning and information, and part of our intention here is to investigate how much of this reasoning and information can be expressed in expanded logical structures, while keeping as much as possible the nice and familiar features of logic. We propose to approach the problem by means of modulated models, which allow talking about modulated logics.

We show how particular modulated models, defined through specific mathematical structures, are able to capture specific kinds of uncertain statements. For example, considering q as the class of subsets of the universe whose cardinal number is greater than their complements, one can formalize assertions of the form "most..." in the precise sense of "simple majority". We call this system the *Logic of Simple Majority*. Another example, which we call the *Logic of Many*, is obtained by identifying q with filters, intended to formalize assertions of the form "many...". A third example, taking q as a weak version of a topological space, can be used to formalize assertions of the form "for a 'good' number of...". This system is called the *Logic of Plausibility*. In particular, we will see that the *Ultrafilter Logic* (extensively studied in [CS94], [CV97], [SCV99], [Vel99a] and [Vel02]) is a special case of modulated logic which formalizes assertions of the form "almost all...".

It is proven that the Logic of Many and the Logic of Plausibility are conservative extensions of first-order classical logic, and that they preserve important properties of classical logic as soundness and completeness with respect to the defined models, among others. The Logic of Simple Majority, however, even if sound, is not complete with respect to the intended models.

We also discuss some fundamental issues about relations among modulated logics, presenting examples of situations where each one is better applied. Concluding, we comment on the particularities of modulated logics, pointing out some possibilities of further work and applications. This paper is a companion to [VC04], in the sense that the essentials of modulated logics are treated here, while a closer investigation on particular logics with applications for qualitative reasoning is carried out there[2].

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[2]This paper is based on the Ph.D. thesis by the second author ([Grá99]) written at the

2. Modulated logics

In this section we introduce a family of monotonic extensions of first order logic dubbed *modulated logics*, which provide a general representation for several kinds of inductive assertions. This family is defined by extending classical logic through the *modulated quantifiers*. Such quantifiers are semantically interpreted by appropriate subsets of the power set of universe; such subsets mean to represent, in intuitive terms, arbitrary sets of assertions supported by evidence related to knowledge basis.

Next section introduces the syntax, semantics and the axiomatics for general modulated logics ($L^{\tau}_{\omega\omega}(Q)$), which constitute thus a general formal approach to inductive or flexible reasoning.

2.1. Syntax, semantics and axiomatics for $L^{\tau}_{\omega\omega}(Q)$

Let $L^{\tau}_{\omega\omega}$ be the usual first-order language of similarity type τ containing symbols for predicates, functions and constants, and closed under the connectives $\wedge, \vee, \rightarrow, \neg$ and under the quantifiers $\exists$ and $\forall$.

By $L^{\tau}_{\omega\omega}(Q)$ we denote the extension of $L^{\tau}_{\omega\omega}$ obtained by including generalized quantifiers Q, called *modulated quantifiers*. The formulas (and sentences) of $L^{\tau}_{\omega\omega}(Q)$ are the ones of $L^{\tau}_{\omega\omega}$ plus those generated by the following clause:

- if φ is a formula in $L^{\tau}_{\omega\omega}(Q)$ then $Qx\varphi$ is also a formula in $L^{\tau}_{\omega\omega}(Q)$.

The notions of free and bound variables in a formula, as well as other syntactical notions, are extended in the usual way for the quantifiers Q.

The result of substituting all free occurrences of a variable x in φ by a term t is denoted by $\varphi(t/x)$. To simplify, when there is no danger of confusion, we write $\varphi(t)$ instead of $\varphi(t/x)$.

The semantical interpretation for the formulas in the modulated logics ($L^{\tau}_{\omega\omega}(Q)$) is defined as follows:

Definition 2.1. Let $\mathcal{A} = \langle A, \{R^{\mathcal{A}_i}\}_{i\in I}, \{f^{\mathcal{A}_j}\}_{j\in J}, \{c^{\mathcal{A}_k}\}_{k\in K}\rangle$ be a classical first-order structure of similarity type $\tau = \langle I, J, K, T_0, T_1\rangle$, and let q be a set of subset of the universe A such that empty set $\varnothing$ does not belong to q, i.e., $q \subseteq \wp(A) - \{\varnothing\}$. The structure $\mathcal{A}^q$ formed by the pair $\langle \mathcal{A}, q\rangle$ is called a *modulated structure* for $L^{\tau}_{\omega\omega}(Q)$.

State University of Campinas and supervised by the first author. A condensed version of this paper was presented at the 6^{th} Kurt Gödel Colloquium held in Barcelona, June 1999.

In [Kei70] this kind of structure is called a *weak structure*. In formal terms:

$$\mathcal{A}^q = \langle \mathcal{A}, q\rangle \text{ where } \mathcal{A} = \langle A, \{R^{\mathcal{A}_i}\}_{i\in I}, \{f^{\mathcal{A}_j}\}_{j\in J}, \{c^{\mathcal{A}_k}\}_{k\in K}, q\rangle$$

is a familiar structure such that A is its universe, R_i is a $T_0(i)$-ary relation defined in A, for $i \in I$, f_j is a function from A^n to A, supposing $T_1(j) = n$, for $j \in J$; c_k is a constant of A, for $k \in K$; and $q \subseteq \wp^n(A)$ is called a *complex*. Usually, $n = 1$ and the complex is just $q \subseteq \wp(A)$; this will be the case in all of our basic examples, except for the suggested applications for formalizing fuzziness in Section 8. Intuitively, as aforementioned, we endow classical first-order logic with a subset of the power set of the universe, which represents arbitrary sets of assertions supported by positive evidence regarding a certain knowledge basis.

The interpretation of relation, function and constant symbols is the same as in $L^\tau_{\omega\omega}$ with respect to $\mathcal{A}$. The notion of *satisfaction* of a formula of $L^\tau_{\omega\omega}(Q)$ in a structure $\mathcal{A}^q$ is inductively defined in the usual way, by adding the following clause: let φ be a formula in which free variables are among $\{x\} \cup \{y_1, \ldots, y_n\}$ and consider a sequence $\underline{a} = (a_1, \ldots, a_n)$ in A. We define

$$\mathcal{A}^q \vDash Qx\varphi[\underline{a}] \text{ iff } \{b \in A : \mathcal{A}^q \vDash \varphi[b;\underline{a}]\} \in q.$$

Usual semantical notions, such as a model, validity, logical consequence, etc. are appropriately adapted from classical logic.

The axioms of $L^\tau_{\omega\omega}(Q)$ are those of $L^\tau_{\omega\omega}$ including the identity axioms (see, for example, [Men87]), plus the following specific axioms for the quantifier Q:

(Ax1) $\forall x\varphi(x) \to Qx\varphi(x)$;

(Ax2) $Qx(\varphi(x)) \to \exists x(\varphi(x))$;

(Ax3) $\forall x(\varphi(x) \leftrightarrow \psi(x)) \to (Qx(\varphi(x)) \leftrightarrow Qx(\psi(x)))$;

(Ax4) $Qx(\varphi(x)) \leftrightarrow Qy(\varphi(y))$.

Intuitively, the axiom (Ax2) asserts that an inductive statement, formalized in $L^\tau_{\omega\omega}(Q)$, cannot be supported by an empty evidence set. The (Ax1) axiom means that if all individuals of the universe support a statement, then it is an inductive statement in $L^\tau_{\omega\omega}(Q)$. Axiom (Ax3) states that if two statements are equivalent, then they are equivalent inductive statements.

The basic logical rules of the system $L^\tau_{\omega\omega}(Q)$ are the usual rules of classical logic: Modus Ponens (**MP**) and Generalization (**Gen**). Usual syntactical concepts such as a proof, a theorem, non-contradictoriness (or consistency), etc., for $L^{\tau}_{\omega\omega}(Q)$ are also appropriately adapted from classical first-order logic.

As it can be seen below, abstract modulated logics preserve several syntactical properties of classical first-order logic.

THEOREM 2.2. *The system $L^{\tau}_{\omega\omega}(Q)$ is consistent.*

PROOF. Analogous to the proof for classical logic $L^{\tau}_{\omega\omega}$ (see [Men87]), changing the definition of the "forgetting function" by means of the inclusion of the condition $h(Qx\varphi(x)) = h(\varphi(x))$. $\square$

For $\Sigma \cup \{\varphi, \phi\}$ a sentence set and ψ a formula in $L^{\tau}_{\omega\omega}(Q)$, the following properties of theories in $L^{\tau}_{\omega\omega}(Q)$ can be easily proved:

THEOREM 2.3 (Deduction Theorem). *If $\Sigma \cup \{\varphi\} \vdash \psi$, then $\Sigma \vdash \varphi \rightarrow \psi$.*

PROOF. A simple adaptation of the classical arguments. $\square$

THEOREM 2.4. (a) *Σ is consistent iff every finite subset Σ_0 of Σ is consistent.*

(b) *$\Sigma \cup \{\varphi\}$ is inconsistent iff $\Sigma \vdash \neg\varphi$.*

(c) *If Σ is maximal consistent, then:*

  (i) *$\Sigma \vdash \varphi$ iff $\varphi \in \Sigma$;*
  (ii) *$\varphi \notin \Sigma$ iff $\neg\varphi \in \Sigma$;*
  (iii) *$\varphi \wedge \phi \in \Sigma$ iff $\varphi \in \Sigma$ and $\phi \in \Sigma$.*

PROOF. Analogous to the classical case. $\square$

THEOREM 2.5. *Any consistent theory in $L^{\tau}_{\omega\omega}(Q)$ can be extended to a maximal consistent theory.*

PROOF. Similar to the proof of the well-known Lindembaum's theorem. $\square$

2.2. Rudimentary model theory for $L^{\tau}_{\omega\omega}(Q)$

Considering that [Kei70] proved soundness and completeness for an axiomatic system analogous to $L^{\tau}_{\omega\omega}(Q)$ with respect to weak models, in this section we are interested in establishing some other prime results of model theory for modulated logics. We discuss here, thus, an analogous result of Łoś ultraproduct theorem for modulated models, and a counter-example to the problem of interpolation in $L^{\tau}_{\omega\omega}(Q)$. A similar result for topological models was given in [Sgr77].

A *modulated ultraproduct* for $L^{\tau}_{\omega\omega}(Q)$ is defined by extending the definition of ultraproduct in classical first-order logic (see [Men87]) by means of the following procedure.

Let J be a nonempty set, and for each $j \in J$, let $\mathfrak{M}_j = \langle \mathcal{A}_j, q_j \rangle$ be a modulated model for $L^{\tau}_{\omega\omega}(Q)$. Let F be an ultrafilter on J. For each $j \in J$, let A_j denote the universe of the model $\mathfrak{M}_j$. The *cartesian product* $\Pi_j A_j$ is defined as the set of all functions f with domain J such that $f(j) \in A_j$, for any $j \in J$. In the cartesian product $\Pi_j A_j$, the following equivalence relation[3] is defined:

$$f \sim_F g \text{ iff } \{j : f(j) = g(j)\} \in F.$$

On the basis of this equivalence relation, $\Pi_j A_j$ can be split into equivalence classes: for any $f \in \Pi_j A_j$, its equivalence *class* f_F is defined as:

$$\{g : f \sim_F g\}.$$

Denote the set of all equivalence classes f_F by $\Pi_F A_j$, and define a model $\mathfrak{M} = \langle \mathcal{A}, q \rangle$ of $L^{\tau}_{\omega\omega}(Q)$, with universe $\Pi_F A_j$, in the usual way, including the clause:

- Let ε_j be any element of q_j, for any $j \in J$. Then $\varepsilon_j \subseteq A_j$, thus we have that if $a \in \varepsilon_j$, then $a = f(j)$ for some $f \in \Pi_j A_j$. The subset q (where $q \subseteq \Pi_F A_j$) is generated by:

  $$\{[\Pi_F \varepsilon_j]_F : \{j : \varepsilon_j \in q_j\} \in F\}$$

  where $[\Pi_F \varepsilon_j]_F = \{f_F \in \Pi_F A_j : \{\beta : f(\beta) \in \varepsilon_\beta\} \in F\}$

We call the model $\mathfrak{M}$ just defined a *modulated ultraproduct* for $L^{\tau}_{\omega\omega}(Q)$ and denote it by $\Pi_F \mathfrak{M}_j$. The definition allows us to express the following theorem.

Theorem 2.6. *Let F be an ultrafilter on a set J and $\{\langle \mathcal{A}_j, q_j \rangle\}$ be a family of modulated models for $L^{\tau}_{\omega\omega}(Q)$. Let $\mathfrak{M} = \Pi_F \mathfrak{M}_j$ be a modulated ultraproduct. Then for any formula φ, whose free variables are among $v_1, \ldots, v_n$, and for any sequence $(g_1)_F, \ldots, (g_n)_F$ of elements of $\Pi_F A_j, \mathfrak{M} \models \varphi[(g_1)_F, \ldots, (g_n)_F]$ if and only if $\{j \in J : \mathfrak{M}_j \models \varphi[g_1(j), \ldots, g_n(j)]\} \in F$.*

[3] It is easy to see that $f \sim_F g$ is indeed an equivalence relation ([Men87]).

Proof. Using induction on the number of connectives and quantifiers in φ, it is simple to carry out the proof similarly to the one for classical logic $L^{\tau}_{\omega\omega}$. The only case that remains to be proven is for formulas φ of the form $Qy\psi$. The other cases are analogous to those of classical first-order logic (see [Men87], p. 105).

Let $\varphi(\underline{x})$ be a formula of the form $Qy\psi(\underline{x})$ whose free variables belong to $\underline{x} = \{x_1, \ldots, x_n\}$. Firstly, suppose $\{j \in J : \mathfrak{M}_j \vDash Qy\psi[g_1(j), ..., g_n(j)]\} \in F$. Then, by the definition, $\{j \in J : \{a \in A_j : \mathfrak{M}_j \vDash \psi[a; g_1(j), ..., g_n(j)]\}\} \in F$ Denoting the set $\{a \in A_j : \mathfrak{M}_j \vDash \psi[a; g_1(j), \ldots, g_n(j)]\}$ by ε^{ψ}_j, we have that $\{\{j \in J : \varepsilon^{\psi}_j\} \in q_j\} \in F$, hence

$$[\Pi_F \varepsilon^{\psi}_j]_F = \{f_F \in \Pi_F A_j : \{\beta : f(\beta) \in \varepsilon^{\psi}_j\} \in F\} \text{ and } [\Pi_F \varepsilon^{\psi}_j]_F \in q.$$

Therefore $\mathfrak{M} \vDash Qy\psi[(g_1)_F, \ldots, (g_n)_F]$.

Conversely, suppose that $\mathfrak{M} \vDash Qy\psi[(g_1)_F, \ldots, (g_n)_F]$. Then, by the definition of satisfaction, $\{f_F \in \Pi_F A_j : \mathfrak{M} \vDash \psi[f_F; (g_1)_F, ..., (g_n)_F]\} \in q$ hence, by the inductive hypothesis,

$$\varepsilon^{\psi} = \{f_F \in \Pi_F A_j : \{j : \mathfrak{M}_j \vDash \psi[f(j); g_1(j), \ldots, g_n(j)]\} \in F\} \in q.$$

But, by definition of q, then $\forall j \in \{j : \mathfrak{M}_j \vDash \psi[f(j), g_1(j), \ldots, g_n(j)]\}$ and $\{f \in \varepsilon^{\psi} : \mathfrak{M}_j \vDash \psi[f(j); g_1(j), \ldots, g_n(j)]\} \in q_j$. So,

$$\{j \in J : \{f \in \varepsilon^{\psi} : \mathfrak{M}_j \vDash \psi[f(j); g_1(j), \ldots, g_n(j)]\} \in q_j\} \in F$$

Therefore, by the definition, $\{j \in J : \mathfrak{M}_j \vDash Qy\psi[g_1(j), \ldots, g_n(j)]\} \in F$. □

We conclude this section presenting a counter-example to the validity of an interpolation theorem in $L^{\tau}_{\omega\omega}(Q)$. The problem of interpolation in $L^{\tau}_{\omega\omega}(Q)$ can be formulated in the following way. Let φ and ψ be sentences of $L^{\tau}_{\omega\omega}(Q)$ such that $\varphi \vDash \psi$. Then there exists a sentence θ of $L^{\tau}_{\omega\omega}(Q)$ such that $\varphi \vDash \theta, \theta \vDash \psi$ and, except for the equality, all extra-logical symbols (relations, functions and constants) occurring in θ occur both in φ and ψ. We call the sentence θ an *interpolate* of φ and ψ.

Let φ and ψ be the sentences of $L^{\tau}_{\omega\omega}(Q)$. Suppose $L^{\tau}_{\omega\omega}(Q)$ is the language containing just symbols occurring in φ or ψ. Let:

- L_1 = sublanguage of $L^{\tau}_{\omega\omega}(Q)$ containing just extralogical symbols in φ;
- L_2 = sublanguage of $L^{\tau}_{\omega\omega}(Q)$ containing just extralogical symbols in ψ;
- $L_0 = L_1 \cap L_2$.

We will construct sentences φ and ψ and modulated models $\langle \mathcal{A}, q_1\rangle$ and $\langle \mathcal{A}, q_2\rangle$ in such a way that $\varphi \vDash \psi$, $\langle \mathcal{A}, q_1\rangle \vDash \varphi$, $\langle \mathcal{A}, q_2\rangle \vDash \neg\psi$ and $\langle \Re, q_1\rangle \equiv \langle \Re, q_2\rangle$, where $\Re$ is the reduct of $\mathcal{A}$ to L_0. This construction violates the interpolation property since any interpolate is in $L_1 \cap L_2$ and the models above are elementarily equivalent in $L_1 \cap L_2$. So if $\langle \mathcal{A}, q_2\rangle \vDash \neg\psi$, then it is true that $\langle \mathcal{A}, q_2\rangle \vDash \neg\theta$; but $\langle \mathcal{A}, q_1\rangle \vDash \theta$ too, which generates a contradiction (since $\langle \Re, q_1\rangle \equiv \langle \Re, q_2\rangle$) ([Sgr77]).

Let $L_1 = \{B(x), C(x), P(x)\}$ and $L_2 = \{B(x), C(x), R(x)\}$. Let φ be the sentence $QxP(x) \wedge \forall x(P(x) \leftrightarrow (B(x) \vee C(x))) \wedge \neg QxC(x)$ and ψ be the sentence $Qx(B(x) \vee C(x)) \vee \neg\forall x(C(x) \to R(x))$.

We can verify, on the basis of axiom (Ax1), that $\vDash \varphi \to \psi$ (or, equivalently, by the deduction theorem, $\varphi \vDash \psi$).

Let $\mathbb{N}$ be the set of natural numbers,

$$
\begin{aligned}
P^{\mathcal{A}} &= \{2n: \; n \in \mathbb{N}\},\\
C^{\mathcal{A}} &= \{4n: \; n \in \mathbb{N}\},\\
B^{\mathcal{A}} &= \{n: \; n \in P^{\mathcal{A}} - C^{\mathcal{A}} \text{ or } n = 8k, \text{ for some } k \in \mathbb{N}\}\\
R^{\mathcal{A}} &= \{n: \; n \in C^{\mathcal{A}} \text{ or } n = 2k+1, \text{ for some } k \in \mathbb{N}\}.
\end{aligned}
$$

By defining

$$
\begin{aligned}
\langle \mathcal{A}, q_1\rangle &= \langle \mathbb{N}, B^A, C^A, P^A, R^A, \{P^A\}\rangle,\\
\langle \mathcal{A}, q_2\rangle &= \langle \mathbb{N}, B^A, C^A, P^A, R^A, \{P^A\}\rangle
\end{aligned}
$$

and then

$$
\begin{aligned}
\langle \Re, q_1\rangle &= \langle \mathbb{N}, B^A, C^A, \{P^A\}\rangle\\
\langle \Re, q_2\rangle &= \langle \mathbb{N}, B^A, C^A, \{R^A\}\rangle.
\end{aligned}
$$

we can easily verify that $\langle \mathcal{A}, q_1\rangle \vDash \varphi$ and $\langle \mathcal{A}, q_2\rangle \vDash \neg\psi$. The above construction represents a counter-example to the interpolation property in $L^{\tau}_{\omega\omega}(Q)$, since $\langle \Re, q_1\rangle \equiv \langle \Re, q_2\rangle$[4].

In the following section we present some specific forms of flexible reasoning formalized by means of the notion of modulated logics.

3. The Logic of Simple Majority

This section is devoted to particularizing $L^{\tau}_{\omega\omega}(Q)$ for a system which formalizes inductive statements of the type "most..." in the sense of "simple

[4]Result given in [Sgr77], p.188.

majority", that is, "more than half". Under the approach to be formalized in this section, we say that *the simple majority* of individuals satisfy a sentence φ when the set formed by the individuals satisfying it is greater than the one formed by the individuals not satisfying it. So this non-qualitative approach intends to explain the notion of "most" in terms of frequency (simple majority) or set cardinality.

Under this explanation of "most", when an assertion represents most individuals in a universe (a large set of individuals), the assertion expressing its complement represents the minority of individuals (a small set of individuals). Also, when a sentence φ represents most individuals and it forms a subset of an assertion expressed by ψ, then of course most individuals satisfy ψ too. Moreover, if φ and ψ represent most individuals of a universe, then there exists at least one individual of this universe which size of the set.

Taking into account the above exposition, we present some criteria for subsets of a set U to be considered large (in the sense of majority). Let X, Y be arbitrary subsets of U:

1. if X is large then its complement X^c with respect to U is not large;
2. if X is large and $X \subseteq Y$ then Y is large too;
3. if X and Y are large, then $X \cap Y \neq \varnothing$;
4. U is large.

These criteria are naturally based on the concept of the cardinal number $|X|$ associated with a set X. In this way, we say that *most* individuals satisfy X if and only if $|X| > |X^c|$.

Other approaches to this notion in the literature are [Pet79] (which analyzes the relationship between "few", "many", "most", "all" and "some" by means of Aristotelian squares of opposition), while [Res62] and [Sla88] treat the notion of "most", in the sense of majority, by means of a new quantifier.

As an illustration of the application of this account of "most" for uncertain reasoning, we recall the famous example of birds given in [Rei80]. Let B be the universe of birds and $F(x)$ be any property applicable to birds, as "flying" for example. We say "the property F applies to most birds" if for the set $F = \{x \in B : F(x)\}$ we have $|F| > |F^c|$.

Syntactically, we will include a new quantifier $\sharp$ in the syntax and denote the fact that the property R applies to most individuals by means of the sentence $\sharp x R(x)$.

The semantics and syntax of this modulated quantifier $\sharp$, which captures the notion of "most", were firstly introduced in [Res62]. Although some papers (as those mentioned above) have discussed this notion, its formal properties have not been investigated.

In the next section we present the formal system $L^{\tau}_{\omega\omega}(\sharp)$ to represent this notion and investigate some of its basic syntactical and semantical properties.

3.1. Soundness and incompleteness of $L^{\tau}_{\omega\omega}(\sharp)$

The system $L^{\tau}_{\omega\omega}(\sharp)$ constitutes a particular modulated logic intended to capture the notion of "most" in the sense of "'simple majority". In formal terms, the language of $L^{\tau}_{\omega\omega}(\sharp)$ is obtained by particularizing the quantifier Q in $L^{\tau}_{\omega\omega}(Q)$ to the quantifier $\sharp$ intended to express *simple majority* .

The semantics of the formulas in $L^{\tau}_{\omega\omega}(\sharp)$ is defined by means of modulated structures, identifying, in this case, the class q with the following subsets:

$$q = \{B \subseteq A : |B| > |B^c|\}.$$

For a sequence $\underline{a} = (a_1, \dots, a_n)$ in A, the *satisfaction* of a formula of the form $\sharp x\varphi$, whose free variables are contained in $\{y_1, \dots, y_n\}$, is defined by

$$\mathcal{A} \vDash \sharp x\varphi[\underline{a}] \text{ iff } \{b \in A : \mathcal{A} \vDash \varphi[b; \underline{a}]\} \in q,$$

i.e.,

$$\mathcal{A} \vDash \sharp x\varphi[\underline{a}] \text{ iff } |\{b \in A : \mathcal{A} \vDash \varphi[b; \underline{a}]\}| > |\{b \in A : \mathcal{A} \vDash \neg\varphi[b; \underline{a}]\}|. \qquad \text{(I)}$$

Intuitively, $\sharp x\varphi(x)$ is true in $\mathcal{A}$ iff most individuals (in the sense of simple majority, or more than half) in A satisfy $\varphi(x)$.

This kind of logic certainly has a great interest for voting and social choice theory (specially for infinite population, cf. [Fey04] see also [PS04] and [Tay05]), as we can obviously formalize sentences as "most voters prefer A" by $\sharp x A(x)$, but examples can be given in several scenarios.

Example 3.7. Let $I(x)$, $R(x)$ and $E(x)$ be predicates in $L^{\tau}_{\omega\omega}(\sharp)$ representing the properties of being an irrational number, a rational number, an even number, and $G(x, y)$ be "$x \leq y$ and x is odd", respectively.

(a) "most numbers are irrational" by $\sharp x I(x)$;

But also, considering the universe of natural numbers, one can formalize:

(b) "most numbers are not even" by $\sharp x \neg E(x)$;

(c) "for any odd natural number y, most of its predecessors are also odd" by $\forall y \sharp x G(x, y)$.

The axioms of $L^{\tau}_{\omega\omega}(\sharp)$ are the ones of $L^{\tau}_{\omega\omega}(Q)$ augmented by the following specific axioms for the quantifier $\sharp$:

(Ax5$_\sharp$) $\quad \forall x(\varphi(x) \to \psi(x)) \to (\sharp x(\varphi(x)) \to \sharp x(\psi(x)))$;

(Ax6$_\sharp$) $\quad \sharp x(\varphi(x)) \to \neg\sharp x(\neg\varphi(x))$;

(Ax7$_\sharp$) $\quad (\sharp x(\varphi(x)) \wedge \sharp x(\psi(x))) \to \exists x(\varphi(x) \wedge \psi(x))$.

Intuitively, given an interpretation with a universe A and formulas φ and ψ with exactly one free variable, then for sets $[\varphi] = \{a \in A : \varphi[a]\}$ and $[\psi] = \{a \in A : \psi[a]\}$ axioms (Ax5$_\sharp$) to (Ax7$_\sharp$) assert that:

- (Ax5$_\sharp$) if $[\varphi] \subset [\psi]$ and most (in the sense of majority of) individuals belong to $[\varphi]$, then most individuals belong to $[\psi]$;
- (Ax6$_\sharp$) if $[\varphi]$ is formed by most (in the sense of majority of) individuals, then $[\neg\varphi]$ is not formed by most individuals;
- (Ax7$_\sharp$) if $[\varphi]$ and $[\psi]$ are formed by most (in the sense of majority of) individuals, then their intersection is not empty.

We note here that such conditions seem to be indeed acceptable in terms of voting and social choice; this justifies some interpretations of this logic in terms of certain problems of judgement aggregation as we discuss at the end of this subsection.

The basic logic rules and usual syntactical concepts such as a proof, a theorem, logical consequence, consistency, etc., for $L^{\tau}_{\omega\omega}(\sharp)$ are the same as those for $L^{\tau}_{\omega\omega}(Q)$.

Returning to the Example 3.7, one can deduce, for example, by means of the axioms (Ax3) and (Ax6$_\sharp$) and the sentences (a) and $\forall x(\neg I(x) \leftrightarrow R(x))$ that "rational numbers do not constitute most of the numbers", that is: $\neg\sharp x R(x)$. By means of the established axioms for $L^{\tau}_{\omega\omega}(\sharp)$ it is easy to prove the following general theorems.

THEOREM 3.8. *The following formulas are theorems of* $L^{\tau}_{\omega\omega}(\sharp)$*:*

(a) $\sharp x(\varphi(x) \wedge \psi(x)) \to (\sharp x(\varphi(x)) \wedge \sharp x(\psi(x)))$;

(b) $\sharp x(\varphi(x)) \to \sharp x(\varphi(x) \vee \psi(x))$;

(c) $\sharp x(\neg\varphi(x)) \to \neg\sharp x(\varphi(x))$;

(d) $(\forall x(\varphi(x)) \wedge \sharp x(\varphi(x) \to \psi(x))) \to \sharp x(\psi(x))$.

PROOF. Routine. $\square$

It can also be easily verified that theorems 2.2, 2.3, 2.4 and 2.5 for general modulated logics remain valid in the "Logic of Simple Majority".

It can be proven, however, that the "Logic of Simple Majority", although preserving soundness, is not complete with respect to the defined models, that is, not every valid formula can be derived in $L^{\tau}_{\omega\omega}(\sharp)$. This justifies the interest in purely qualitative approaches to this type of quantifiers, as opposed to such a cardinality-based approach.

Theorem 3.9 (Soundness). *If φ is a theorem in $L^{\tau}_{\omega\omega}(\sharp)$, then φ is valid.*

Proof. Since the inference rules (**MP**) and (**Gen**) in $L^{\tau}_{\omega\omega}(\sharp)$ preserve validity of formulas and the axioms are valid in such structures, soundness of $L^{\tau}_{\omega\omega}(\sharp)$ is clear. □

However, since the completeness problem for a language containing quantifiers $Q^1, \ldots, Q^s$ relies on whether the set of true formulas is recursively denumerable ([Mos57]), the converse of this theorem (i.e. Completeness) cannot hold, as it can be seen from the following argument.

Let $S = \{(\kappa, \theta) : \text{exist } R \subseteq \mathbf{A} \text{ such that } \kappa = |R| \text{ and } \theta = |R^c|\}$. Given $\mathcal{A}$ (a classical first-order structure), for each function $T : S \to \{0, 1\}, R \subseteq A^{n+1}$ and $\underline{b} = (b_1, \ldots, b_n) \in A^n$, [Mos57] defines:

$$Q_T(R) = T(|\{a \in \mathbf{A} : (a, \underline{b}) \in R\}|, |\{a \in \mathbf{A} : (a, \underline{b}) \in R\}^c|).$$

Therefore, for a formula φ in $L^{\tau}_{\omega\omega}(\sharp)$ with exactly one free variable, the definition in (I) is equivalent to:

$$\mathcal{A} \vDash \sharp x \varphi(x) \text{ iff } T^M(\kappa, \theta) = 1,$$

in which $T^M(\kappa, \theta) = 1$ iff $\kappa > \theta$.

The following definition, also from [Mos57], provides the basis for our proof.

Definition 3.10. Let m, n be non-negative integers and T', T'' functions such that $\{T'(\kappa, \theta) = 1\} \equiv (\kappa = m)$ and $\{T''(\kappa, \theta) = 1\} \equiv (\theta = n)$. Quantifiers $Q_{T'}$ and $Q_{T''}$ will be denoted by $\Sigma^{(m)}$ and $\Pi^{(n)}$, respectively. Boolean polynomials of quantifiers $\Sigma^{(m)}, \Pi^{(n)} (m, n = 0, 1, 2, \cdots)$ are called *numerical quantifiers*.

The quantifier $\sharp$ is clearly not a numeric quantifier, since it cannot be expressed by a Boolean polynomial of $\Sigma^{(m)}$ and $\Pi^{(n)}$ quantifiers. However, the following theorem directly implies that the completeness problem in $L^{\tau}_{\omega\omega}(\sharp)$ has a negative solution.

THEOREM 3.11. *Let the quantifiers $\exists$ and $\forall$ occur among $Q^1, \ldots, Q^s$ and let* $\mathbf{A}$ *be a denumerable universe. A necessary and sufficient condition for the completeness problem for quantifiers $Q^1, \ldots, Q^s$ to have a positive solution is that the quantifiers $Q^1, \ldots, Q^s$ be numerical.*

PROOF. A proof is found in [Mos57]. $\square$

Thus from the above theorem we conclude that there are valid formulas which cannot be proved in $L^{\tau}_{\omega\omega}(\sharp)$. This is an interesting case of elementary incompleteness having in its genetic constitution a numerical approach, which is, on the other hand, the essence of the celebrated incompleteness arguments of [Göd31]. It is well known that his first theorem is general, as much as it can be applied to any axiomatic theory which is ω consistent, whose proof procedure is effective and which is strong enough to represent basic arithmetic. However, *pace* Gödel, the theorem loses force as much as we should be able to avoid reasoning in numerical terms. This can of course be regarded as a motivation to think in other than numerical terms. Of course, big questions remain, as for example how effective or interesting this deviation could be. We do not try to answer this, but at least sketch on the essentials of what can be done from the qualitative side.

However, a very appealing interpretation of our incompleteness results concerns its relationship with the *discursive dilemma* (also called *doctrinal paradox*) of choice social theory and political theory. A clear account of this problem and a proposed solution by means of new aggregation procedures is given in [Pig06]; what the above incompleteness result shows is that there is no logical way to always grant consistent collective judgements under the rule of simple majority voting. In a sense, our result no only justifies a modal outlook to the question as in [PS04], but also represents one more of the "impossibility results" so popular after [Arr70].

4. The Logic of Many

This section considers a qualitative form of modulated logic intended to formalize inductive assertions of the kind "many...". The vague notion of "many" approached here is associated with the concept of a *large evidence set*, but not necessarily linked to the notion of majority (in terms of cardinality) nor to the invariant thresholds (in the sense of something not changing from one model to another).

So, for example, considering the universe of Brazilians, when we assert that "many Brazilians wear skirts", we have associated with the assertion an evidence set (Brazilians) considered large. On the other hand, when we assert "many Brazilians love soccer", we have associated with this statement another set of Brazilians, considered large too. However, not necessarily the two sets of Brazilians are of the same size. Besides, not necessarily the evidence observed in favor of these assertions represent the majority of the universe, nor their intersection is necessarily large.

In this way the notion of "many" treated here and expressing "a great amount of evidence" is more abstract than the one treated in the Logic of Simple Majority.

The notion of "many" can be considered, as well as some accounts of "most" and "almost all", as an intermediary notion between the maniqueist vision of the universe conveyed by mere "there exists" and "all". However, it distinguishes itself from those ("most" and "almost all") by the fact that the set of instances that do not satisfy the assertions within the scope of "many" is not necessarily small. In the notions of "most" and "almost all" the set of individuals that do not satisfy the assertions is necessarily small (in the particular case of "almost all", they are termed exceptions). There are situations, however, in which the evidence set in favor of the assertion as well as of its complement (in relation to a particular universe) can be considered large. For example, "many Brazilians wear skirts" is large, but it seems intuitive that "many Brazilians don't wear skirts" is (hopefully) a large set too.

As another example of statements formalized under the notion of "many", consider the universe of natural numbers. It seems intuitive that one can assert that "many natural numbers are odd" and "many natural numbers are even". In the Logic of Simple Majority, however, we cannot assert either of them and, in the Ultrafilter Logic (that treats the notion of "almost all"), stating one of them impeaches stating the other.

Besides the properties presented for general modulated logics, the following property is clearly identified with the notion of "many": if many individuals satisfy a sentence φ, and φ is contained in ψ, then ψ is also satisfied by many individuals of the universe.

The notion of "many", exposed above, can be captured by the mathematical concept of *upper closed families* (or *families of principal filters*). An *upper closed family* F over a universe A is a collection of subsets of A such that

(i) if $B \in F$ and $B \subseteq C$, then $C \in F$;

(ii) $A \in F$;

(iii) $\varnothing \notin F$.

The notion of a large set in a universe is thus identified with the concept of upper closed families over this universe. In this way, a property is true for many individuals in a universe (i.e., the evidence set is large) if it belongs to the family of principal filters associated with the model.

By the well-known finite intersection property for filters (see [BM77]) if B is a set of subsets of A then B can be extended to a family of principal filters ([Grá99]).

Syntactically, we define a new modulated quantifier, $\heartsuit$ (in the language of modulated logics) called *quantifier of many*, by

$$\heartsuit x \varphi(x)$$

meaning "for many x, $\varphi(x)$".

In the following section we present the formal system $(L^{\tau}_{\omega\omega}(\heartsuit))$ and its semantics, which formalize the notion of "many" interpreted by the concept of upper closed families (or families of principal filters).

4.1. Syntax, semantics and axiomatics for $L^{\tau}_{\omega\omega}(\heartsuit)$

The language of $L^{\tau}_{\omega\omega}(\heartsuit)$ is obtained by particularizing in $L^{\tau}_{\omega\omega}(Q)$ the quantifier Q to the quantifier $\heartsuit$ for *many*. The semantical interpretation of formulas in $L^{\tau}_{\omega\omega}(\heartsuit)$ is carried out in modulated structures where the subsets q are identified with families of principal filters. Formally, a modulated structure for $L^{\tau}_{\omega\omega}(\heartsuit)$, called a *structure of principal filters*, is constructed by endowing $\mathcal{A}$ with a family of principal filters F^A over A and given by

$$\mathcal{A}^F = \langle \mathcal{A}, F^A \rangle = \langle A, \{R_i^{\mathcal{A}}\}_{i \in I}, \{f_j^{\mathcal{A}}\}_{j \in J}, \{c_k^{\mathcal{A}}\}_{k \in K}, F^A \rangle.$$

The notion of satisfaction of a formula of the form $\heartsuit x \varphi$, whose free variables belong to $\{y_1, \ldots, y_n\}$, by a sequence $\underline{a} = (a_1, \ldots, a_n)$ in A, is defined by

$$\mathcal{A}^F \vDash \heartsuit x \varphi[\underline{a}] \text{ iff } \{b \in A : \mathcal{A}^F \vDash \varphi[b; \underline{a}]\} \in F^A$$

for F^A a family of principal filters in A.

Intuitively $\heartsuit x \varphi(x)$ is true in $\mathcal{A}^F$, i.e., the set of individuals in A satisfying $\varphi(x)$ belongs to F^A if and only if many individuals of $\mathbf{A}$ satisfy $\varphi(x)$.

Again, the usual semantical notions like a model, validity and semantical consequence for this system are the same as the general ones for modulated logics.

The following examples illustrate assertions that can be naturally expressed in $L^{\tau}_{\omega\omega}(\heartsuit)$.

Example 4.12. Let $E(x)$, $O(x)$ and $G(x, y)$ be predicates in $L^{\tau}_{\omega\omega}(\heartsuit)$, standing for "$x$ is even", "x is odd" and "x is greater than y", respectively.

Considering the universe of natural numbers, we represent the assertions:

(a) "many natural numbers are even" by: $\heartsuit x E(x)$;

(b) "many natural numbers are odd" by: $\heartsuit x O(x)$;

(c) "for any natural number, many natural numbers are greater than it" by: $\forall y \heartsuit x G(x, y)$.

The axioms for $L^{\tau}_{\omega\omega}(\heartsuit)$ are the ones of $L^{\tau}_{\omega\omega}(Q)$, augmented by the following specific axiom for quantifier $\heartsuit$:

$$(\text{Ax5}_{\heartsuit}) \qquad \forall x(\varphi(x) \to \psi(x)) \to (\heartsuit x(\varphi(x)) \to \heartsuit x(\psi(x))).$$

Given an interpretation with a universe A and φ, ψ formulas, with exactly one free variable, axiom (Ax5$_{\heartsuit}$) intuitively asserts, for sets $[\varphi] = \{a \in A : \varphi[a]\}$ and $[\psi] = \{a \in A : \psi[a]\}$, that if $[\varphi]$ contains many individuals and $[\varphi]$ is a subset of $[\psi]$, then $[\psi]$ also contains many individuals.

The inference rules, usual syntactical notions like a sentence, a proof, a theorem, a logical consequence, consistence, etc., for $L^{\tau}_{\omega\omega}(\heartsuit)$ are the same as those for general modulated logics.

Looking at the definition of satisfaction and the axiomatic system in $L^{\tau}_{\omega\omega}(\heartsuit)$, we see that, syntactically, the quantifiers $\heartsuit$ and $\exists$ have the same logical consequences. However, the modulated quantifier $\heartsuit$ differs semantically from $\exists$ by offering a free choice of the sets representing large sets, i.e., sets that represent many individuals. In this way, under the notion of "many", we establish for each situation (or model) a measure of largeness. Under the notion of "there exists" such a measure is invariant for all structures (and models), i.e., the existential quantifier is interpreted in all structures by $q \subseteq \wp(A) - \{\varnothing\}$, for a universe A .

The following logical consequence of $L^{\tau}_{\omega\omega}(\heartsuit)$ can be easily proven.

Theorem 4.13. *The following formulas are theorems in* $L^{\tau}_{\omega\omega}(\heartsuit)$*:*

(a) $\heartsuit x\varphi(x) \wedge \heartsuit x\psi(x) \to \heartsuit x(\varphi(x) \vee \psi(x))$;

(b) $\neg\heartsuit x(\varphi(x) \wedge \neg\varphi(x))$.

Proof. Routine. $\square$

Example 4.14. Let $B(x, y)$, $S(x)$ and $D(x)$ be predicates in $L^{\tau}_{\omega\omega}(\heartsuit)$, standing for "$x$ likes drinks of a type y", "x wears shoes" and "x wears dresses". Examples of formalized assertions in $L^{\tau}_{\omega\omega}(\heartsuit)$ are:

(a) "many people like some type of drink" by: $\heartsuit x \exists y B(x, y)$;

(b) "many people like many species of drink" by: $\heartsuit x \heartsuit y B(x, y)$;

(c) "many people wear dresses" by: $\heartsuit x D(x)$.

So, on the basis of the axiomatic system of the Logic of Many, we can deduce for example the following from (c) above :

(d) "many people wear dresses or wear shoes", or

(e) "many people like some kind of drink or wear dresses"

But the intersection of assertions (d) and (e) in this example is not a logical consequence in $L^{\tau}_{\omega\omega}(\heartsuit)$, i.e., the intersection of those assertions does not constitute a collection of "many" individuals. This fact agrees with our intuition that, although assertions (d) and (e) represent large sets, the set interpreting their intersection is not necessarily large. Besides, as already cited in previous examples, we may not deduce in this case the negation of assertion (d), i.e., $\neg \heartsuit x(D(x) \vee S(x))$, neither the negation of assertion (e). Again, this seems to go along our intuition, since apparently, both negations (of (d) and (e)) fail to constitute small sets.

We can easily verify that theorems 2.2, 2.3, 2.4 and 2.5 for modulated logics remain valid in the Logic of Many.

4.2. Soundness and Completeness for $L^{\tau}_{\omega\omega}(\heartsuit)$

This section shows that the Logic of Many is sound and complete with respect to structures of principal filters.

Theorem 4.15 (Soundness). *If φ is a theorem of $L^{\tau}_{\omega\omega}(\heartsuit)$, then φ is valid.*

Proof. This is immediate, since rules (**MP**) and (**Gen**) preserve the validity of formulas, and the axioms of $L^{\tau}_{\omega\omega}(\heartsuit)$ are true in every structure of principal filters. □

The proof of the Completeness Theorem for $L^{\tau}_{\omega\omega}(\heartsuit)$ uses an analogous construction of the well-known method of building models by adding witnesses (known as Henkin's method) to modulated models.

Theorem 4.16 (Extended Completeness Theorem). *Let T be a set of sentences of $L^{\tau}_{\omega\omega}(\heartsuit)$. Then, T is consistent if and only if T has a model.*

Proof. We sketch here only the crucial steps of the proof. Given a set T of sentences in $L^{\tau}_{\omega\omega}(\heartsuit)$ and C a set of new constants, we extend T to a consistent set T^* such that T^* has C as the witness set. We define a canonical model of a family of principal filters $\mathcal{A}^F = \langle A, \{R_i^{\mathcal{A}}\}_{i\in I}, \{f_j^{\mathcal{A}}\}_{j\in J}, \{c_k^{\mathcal{A}}\}_{k\in K}, F^{\mathcal{A}}\rangle$ in a similar way to the completeness proof for classical logic, including the definition of an appropriate family of principal filters F^A over A as follows:

(1) We define for each formula $\varphi(\underline{v})$, with free variables $\underline{v} = \{v_1, \ldots, v_n\}$,

$$\varphi(v)^T = \{(c_1^{\sim}, \ldots, c_n^{\sim}) \in A^n : T \vdash \varphi(c_1, \ldots, c_n)\}$$

and considering the formulas $\theta(x)$, with only one free variable,

$$\mathcal{B}^T = \{\theta(x)^T \subseteq A : T \vdash \heartsuit x\theta(x)\}.$$

(2) In view of the axiom (Ax5$_\heartsuit$) in $L^{\tau}_{\omega\omega}(\heartsuit)$, and using the finite intersection theorem for filters, $\mathcal{B}^T \subseteq \wp(A)$ can be extended to a family of principal filters F^A. Let $F^A \subseteq \wp(A)$ be the family of principal filters generated by $\mathcal{B}^T$.

It can be shown by induction on the length of φ that for every sentence φ of $T, \mathcal{A}^F \vDash \varphi$ iff $T \vdash \varphi$. The only interesting step is where φ is a sentence of the form $\heartsuit x\varphi(x)$. Let $\theta(x) \equiv \heartsuit x\varphi(\underline{x})$.

$\mathcal{A}^F \vDash \heartsuit x\varphi(x)$ iff $\{c^{\sim} \in A : \mathcal{A}^F \vDash \varphi(c^{\sim})\} \in F^A$ iff, by inductive hypothesis, $\{c^{\sim} \in A : T \vdash \varphi(c^{\sim})\} \in F^A$ iff $(\varphi(x))^T \in \mathcal{B}^T$ iff $T \vdash \heartsuit x\varphi(x)$. □

Since $L^{\tau}_{\omega\omega}(\heartsuit)$ preserves the Completeness Theorem, then Compactness and Löwenheim-Skolem Theorems can be adapted to hold for $L^{\tau}_{\omega\omega}(\heartsuit)$. It is worth noting that though there may be an apparent conflict involving such results and the well-known Lindström's theorems which characterize classical logics (see, e.g., [Flu85]), this can be easily explained taking into consideration that our notion of a model is not a standard one, due to the presence of families of principal filters in the models.

5. The Logic of Plausibility

The purpose of this section is to formalize inductive statements of the kind "a 'good' number of...". The expression "'good' number" is used here to mean a significant set of positive evidence, but not necessarily large with respect to a universe. The sense of the notion of a significant set employed here is that, although it can be small, it represents a characteristic which is present almost everywhere in the universe.

For example, consider again the universe of Brazilians and the assertion "a 'good' number of Brazilians are unemployed". What we mean in this situation is that, even considering that the set formed by unemployed Brazilians may not be large (in relation to the universe size), it represents a property present almost everywhere in the country. In other words, even if we know a single Brazilian who is not unemployed, we can find someone in a close neighborhood who is. We will see further that we can also consider this logic as formalizing the notion of "φ is ubiquitous" or "φ is valid almost everywhere", for a sentence φ.

The concept we want to formalize is independent of the notion of a large evidence set, but akin to the notion of "significant evidence" ascribed to an assertion. A smaller set may be more significant than a larger one, or just as significant.

On the one hand, those statements express a more "vague" form of inductive reasoning. On the other, sentences of this kind (" significant positive evidence") represent assertions close to those used in statistical inference, in which the evidence set (sample) considered sufficient to establish inferences, although significant, is generally small in relation to the universe size. In this theory, in general, a small non-biased (random) set is more significant than another one, perhaps larger but biased.

This suggests a connection between modulated logics and Bayesian inference, which permits interpretations of probabilities as degrees of belief, contrary to strict frequentism, and also with Bayesian epistemology, a contemporary theory aiming to create a formal apparatus for inductive logic[5].

Thinking about the features of the notion of "a 'good' number of", it seems natural that if there exists a 'good' number of individuals for which φ is true, and for a 'good' number of individuals ψ is also satisfied, then for a 'good' number of individuals, φ or ψ is satisfied too. For instance if "a 'good' number of Brazilians love soccer" and "a 'good' number of Brazilians love samba", then it is natural that we can assert that "a 'good' number of Brazilians love soccer or love samba".

It also seems completely intuitive that if we know that the whole universe satisfies an assertion φ, then a 'good' number of individuals of this universe also satisfies φ. Conversely, if no individual in the universe satisfies an assertion φ, then certainly there does not exist a 'good' number of individuals that satisfy φ.

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[5]Both Bayesian inference and Bayesian epistemology are based upon certain presuppositions inherited from Thomas Bayes in the 18th Century.

Another apparently intuitive property of this notion is that if two sets are not significant, i.e., do not represent assertions of the type "a 'good' number of ...", then their union is also not significant. For example, if the set of Brazilians that love baseball is not significant, i.e., we do not have that "a 'good' number of Brazilians love baseball", and the set of Brazilians that love golf is not significant either, then it seems intuitive that the set of Brazilians that love baseball or love golf is not significant.

Dually we can assert that if two sets are both significant, i.e., they represent assertions of the type "a 'good' number of...", then their intersection is significant too. [SCV99] uses a similar argument to justify that the intersection of (qualitative) large sets is large.

The notion of plausibility sketched above has the following structural properties:

1. if two assertions are plausible, then so is their conjunction and disjunction;
2. if every individual of the universe satisfies the assertion, then it is plausible;
3. if no individual of the universe satisfies the assertion, then it is not plausible.

Such properties lead us to the idea of topology.

However, the usual clause in the definition of topology which asserts that "the union of an arbitrary families of open sets is an open set" has no counterpart in flexible reasoning, since reasonings and arguments (as deductions) are assumed to be of finite character. Moreover, the empty set is open in every topology, but no form of inductive reasoning can infer assertions without supporting evidence (cf. axiom (Ax2) of general modulated logics).

For such reasons, in the formalization of inductive assertions of the type "a 'good' number of $x, \varphi(x)$" we employ a more abstract notion of topology called *reduced topology*.

Definition 5.17. A *reduced topology* is a family $\Im$ of subsets of a set X, called *reduced open subsets* which satisfies the following conditions:

(a) the intersection of two reduced open subsets is a reduced open subset;

(b) the union of two reduced open subsets is a reduced open subset;

(c) X is a reduced open subset;

(d) the empty set $\varnothing$ is not a reduced open subset.

The notions of reduced topological space and reduced closed subset are defined analogously to those for the usual topology. Also, by analogy with corresponding notions in topology, we can introduce a reduced topology in a set by describing just the 'basic' reduced open sets as follows.

A reduced *open basis*, or simply a basis in a reduced topological space $(X, \Im)$, is a collection B of reduced open subsets of X, called basic *reduced opens*, with the following property: every reduced open subset $A \subseteq X$ is expressed as a non-void finite union $A = \cup \beta_\lambda$ of reduced open subsets which belong to B.

The following theorem can be proven:

THEOREM 5.18. *A family B of sets is a basis for some reduced topology over $X = \bigcup\{\beta : \beta \in B\}$ if and only if all members of B are pairwise non-disjoint and, for any two members U, V of B and each point x in $U \cap V$, there is W in B such that $x \in W$ and $W \subseteq U \cap V$.*

PROOF. Analogous to the corresponding theorem for topological spaces (see [Kel55]). □

This last condition is fulfilled, in particular, when $U \cap V \in B$.

It can be easily noted that the notion of a reduced topology does not refer to a new concept of neighborhood, but rather to the concept of a dense neighborhood, since, as it will be shown, every reduced topology defines a basis of dense opens of a topological space (in another space). This fact is a consequence of the following proposition:

THEOREM 5.19. *Every reduced topology defines a basis of dense opens of a topological space.*

PROOF. Let $\Im$ be a reduced topology. Given $U \in \Im$, we define

$$\widehat{U} = \{V \in \Im : \ U \subseteq V\}, \text{ where } \widehat{U} \subseteq \Im.$$

- The set $B = \{\widehat{U} : \ U \in \Im\}$ is a basis of a topology in $\Im$.

In fact, if $\widehat{U}_1, \ldots, \widehat{U}_n \in B$, then $\widehat{U}_1 \cap \cdots \cap \widehat{U}_n = (\widehat{\bigcup_{i=1}^n U_i}) \in B$. So

$$\Im^* = \{\bigcup_{i \in I} \widehat{U}_i : \ \widehat{U}_i \in B\} \text{ is a topology over } \Im.$$

Moreover, $\widehat{U}$ is dense. In fact, let $V = \bigcup_{i \in I} \widehat{U}_i \neq \varnothing$ be an open subset in $\Im^*, \widehat{U} \cap \bigcup_{i \in I} \widehat{U}_i = \bigcup_{i \in I} \widehat{U} \cap \widehat{U}_i = \bigcup_{i \in I} (\widehat{U \cup U_i})$. But, $U \cup U_i \in (\widehat{U \cup U_i})$, then $U \cup U_i \in \bigcup_{i \in I} (\widehat{U \cup U_i})$, whence $\widehat{U} \cap \bigcup_{i \in I} \widehat{U}_i \neq \varnothing$. Then, $\widehat{U}$ is dense.

Therefore, every reduced topology defines a basis of dense opens of a topological space. □

In this way, every reduced topology $\Im$ defines a topology $\Im^*$ (in another space) such that the reduced open sets of $\Im$ are identified with elements of a basis of dense subsets of $\Im^*$.

The previous result justifies an alternative interpretation of $\blacktriangledown x\varphi(x)$ as "$\varphi(x)$ is ubiquitous", i.e., although the evidence set may not be large (in relation to the universe), individuals satisfying that property are densely widespread in the universe.

As an example of this assertion, consider the universe of real numbers and let $R(x)$ be the unary predicate standing for "x is rational". We can assert "the rational numbers are 'ubiquitous"', since in any open neighborhood of a real number we find a rational. We remind, however, that the set of rational numbers is not large (in relation to the size of real numbers).

We can still easily verify the following property about reduced topology.

Theorem 5.20. *Every family of dense opens in a topological space is a reduced topology.*

Proof. Immediate from Definition 5.17. □

Syntactically, we identify the modulated quantifier Q with a new quantifier $\blacktriangledown$, called *quantifier of plausibility*, in the language of modulated logic, given by

$$\blacktriangledown x\varphi(x)$$

representing the assertion "a 'good' number of $x, \varphi(x)$" or "there are sufficient x such that $\varphi(x)$ is 'ubiquitous"'.

We call this particularization of modulated logics the *Logic of Plausibility*.

The following section presents the formal system and semantics for the Logic of Plausibility, intended to formalize the notion of plausibility of an assertion supported by pieces of evidence.

5.1. Syntax, semantics and axiomatics of $L^{\tau}_{\omega\omega}(\blacktriangledown)$

Taking into account that the system $L^{\tau}_{\omega\omega}(\blacktriangledown)$ constitutes another particularization of general modulated logics designed to capture the notion of "for a 'good' number of", it will be defined extending $L^{\tau}_{\omega\omega}(Q)$ by interpreting the quantifier Q as a subset q with a reduced topology and including specifics axioms for this kind of inductive assertions.

In this way, the language of $L^{\tau}_{\omega\omega}(\blacktriangledown)$ is obtained by identifying the quantifier Q in $L^{\tau}_{\omega\omega}(Q)$ with the *plausibility quantifier* $\blacktriangledown$. The semantical interpretation of formulas in $L^{\tau}_{\omega\omega}(\blacktriangledown)$ is defined through modulated models where the subset q is identified with a reduced topology. Formally, a structure for $L^{\tau}_{\omega\omega}(\blacktriangledown)$, called a *reduced topological structure*, is defined by endowing $\mathcal{A}$ with a reduced topology $\Im^{A}$ over A, given by:

$$\mathcal{A}^{\Im} = \langle \mathcal{A}, \Im^{A} \rangle = \langle A, \{R_j^{\mathcal{A}}\}_{j\in J}, \{f_j^{\mathcal{A}}\}_{j\in J}, \{c_k^{\mathcal{A}}\}_{k\in K}, \Im^{A} \rangle.$$

For a sequence $\underline{a} = (a_1, \ldots, a_n)$ in $\mathbf{A}$, the notion of *satisfaction* of a formula of the form $\blacktriangledown x\varphi$ whose set of variables is contained in $\{y_1, \ldots, y_n\}$ is defined by:

$$\mathcal{A}^{\Im} \vDash \blacktriangledown x\varphi[\underline{a}] \text{ iff } \{b \in A : \mathcal{A}^{\Im} \vDash \varphi[b; \underline{a}]\} \in \Im^{A}.$$

In intuitive terms, $\blacktriangledown x\varphi(x)$ is true in $\mathcal{A}^{\Im}$ iff a certain 'good' number of individuals of $\mathbf{A}$ satisfy $\varphi(x)$. Furthermore, a 'good' number of individuals of A satisfy $\varphi(x)$ iff the set of individuals satisfying φ belongs to $\Im^{A}$, i.e., $\varphi(x)$ is ubiquitous.

The usual semantic notions such as a model, validity, semantical consequence, etc., are defined for this system in a similar way as those for general modulated logics.

The following examples illustrate assertions expressed in $L^{\tau}_{\omega\omega}(\blacktriangledown)$.

Example 5.21. Let $C(x)$ and $S(x)$ be unary predicates in $L^{\tau}_{\omega\omega}(\blacktriangledown)$ standing for "x likes coffee" and "x likes samba", respectively.

Considering the universe of Brazilians, the following sentences can be formalized in $L^{\tau}_{\omega\omega}(\blacktriangledown)$:

(a) "a 'good' number of people like coffee" by: $\blacktriangledown xC(x)$;

(b) "a 'good' number of people like samba" by: $\blacktriangledown xS(x)$.

whose intuitive meaning is the following: even if you know that a single individual does not like samba, for example, you will find in a closer neighborhood someone who does, i.e., $S(x)$ is ubiquitous.

Example 5.22. Take the universe of real numbers and let $R(x)$ be a unary predicate in $L^{\tau}_{\omega\omega}(\blacktriangledown)$ standing for "x is rational". The following sentences can be formalized in $L^{\tau}_{\omega\omega}(\blacktriangledown)$:

(a) "a 'good' number of real numbers is rational" by: $\blacktriangledown xR(x)$; or

(a′) "The set of rational numbers is ubiquitous among real numbers" by: $\blacktriangledown xR(x)$.

The axioms of $L^{\tau}_{\omega\omega}(\blacktriangledown)$ are those of $L^{\tau}_{\omega\omega}(Q)$, augmented by the following specific axioms for quantifier $\blacktriangledown$:

(Ax5$_{\blacktriangledown}$) $(\blacktriangledown x\varphi(x) \wedge \blacktriangledown x\psi(x)) \to \blacktriangledown x(\varphi(x) \wedge \psi(x))$;

(Ax6$_{\blacktriangledown}$) $\blacktriangledown x\varphi(x) \wedge \blacktriangledown x\psi(x) \to \blacktriangledown x(\varphi(x) \vee \psi(x))$.

Considering an interpretation with universe A, formulas φ and ψ with exactly one free variable and sets $[\varphi] = \{a \in A : \varphi[a]\}$ and $[\psi] = \{a \in A : \psi(a)\}$, axioms (Ax5$_{\blacktriangledown}$) and (Ax6$_{\blacktriangledown}$), intuitively, assert that:

- (Ax5$_{\blacktriangledown}$) if a 'good' number of individuals belong to $[\varphi]$ and $[\psi]$, then a 'good' number of individuals belong to their conjunction;
- (Ax6$_{\blacktriangledown}$) if a 'good' number of individuals satisfy conditions $[\varphi]$ and $[\psi]$, then a 'good' number of individuals satisfy their disjunction.

Usual syntactical notions like a sentence, a proof, a theorem, logical consequence, consistency, etc., are, again, defined for $L^{\tau}_{\omega\omega}(\blacktriangledown)$ in an analogous way as those defined in the general modulated logics.

Theorem 5.23. *The following formulas are theorems in $L^{\tau}_{\omega\omega}(\blacktriangledown)$:*

(a) $\blacktriangledown x\varphi(x) \wedge \blacktriangledown x\psi(x) \to \exists x(\varphi(x) \wedge \psi(x))$;

(b) $\blacktriangledown x\varphi(x) \to \neg\blacktriangledown x\neg\varphi(x)$.

Proof. Routine. □

5.2. Soundness and completeness for $L^{\tau}_{\omega\omega}(\blacktriangledown)$

This subsection shows that, as well as in the Logic of Many, a theory in $L^{\tau}_{\omega\omega}(\blacktriangledown)$ is sound and complete with respect to reduced topological structures.

Theorem 5.24 (Soundness). *If φ is a theorem in $L^{\tau}_{\omega\omega}(\blacktriangledown)$, then φ is valid.*

Proof. This is a routine proof, since the rules (**MP**) and (**Gen**) preserve validity of formulas, and axioms of $L^{\tau}_{\omega\omega}(\blacktriangledown)$ are true in every reduced topological structure. □

In order to prove the completeness theorem for $L^{\tau}_{\omega\omega}(\blacktriangledown)$ it is convenient to adapt the arguments of the completeness for $L^{\tau}_{\omega\omega}(\blacktriangledown)$.

Theorem 5.25 (Extended Completeness Theorem). *Let T be a set of sentences in $L^{\tau}_{\omega\omega}(\blacktriangledown)$. Then, T is consistent if and only if T has a reduced topological model.*

Proof. We define a canonical reduced topological model $\mathcal{A}^{\Im} = \langle A, \{R_i^{\mathcal{A}}\}_{i\in I}, \{f_j^{\mathcal{A}}\}_{j\in J}, \{c_k^{\mathcal{A}}\}_{k\in K}, \Im^{A}\rangle$ in a similar way as for the completeness proof of the Logic of Many, changing the definition in order to obtain an appropriate reduced topology $\Im^{A}$ over A in the following way: (1) define first, for each formula $\varphi(\underline{v})$ with free variables, $\underline{v} = \{v_1, \ldots, v_n\}$,

$$\varphi(v)^T = \{(c_1^{\sim}, \ldots, c_n^{\sim}) \in A^n : T \vdash \varphi(c_1, \ldots, c_n)\} \text{ and}$$

for formulas $\theta(x)$ with only one free variable,

$$\mathcal{B}^T = \{\theta(x)^T \subseteq A : T \vdash \blacktriangledown\theta(x)\}.$$

In view of the axiom (Ax5$_\blacktriangledown$) and by Theorem 5.18, $\mathcal{B}^T \subseteq \wp(A)$ is a basis for some reduced topology. It is clear that $\mathcal{B}^T$ is a reduced topology, considering axioms (Ax1), (Ax6$_\blacktriangledown$) and (Ax2). In fact, if $\theta(x)^T \in \mathcal{B}^T$, then $T \vdash \blacktriangledown(x)\theta(x)$. But by (Ax2), $T \vdash \blacktriangledown x\theta(x) \to \exists x\theta(x)$ so, by (**MP**), $T \vdash \exists x\theta(x)$, from where we have that $T \vdash \exists x\theta(x) \to \theta(c)$ for some $c \in C$. Applying (**MP**), we have that $T \vdash \theta(c)$ which means that $\theta(x)^T \neq \varnothing$. Therefore, $\varnothing$ does not belong to $\mathcal{B}^T$. On the other hand, by axiom (Ax1), $T \vdash \forall x(x = x) \to \blacktriangledown x(x = x)$, and $T \vdash \forall x(x = x)$, hence we obtain, by (**MP**), that $T \vdash \blacktriangledown x(x = x)$ and, then, $(x = x)^T = A \in \mathcal{B}^T$. Also, if $\theta(x)^T$, $\varphi(x)^T \in \mathcal{B}^T$ we have that $T \vdash \blacktriangledown x\theta(x)$ and $T \vdash \blacktriangledown x\varphi(x)$, from which follows that $T \vdash \blacktriangledown x\theta(x) \wedge \blacktriangledown x\varphi(x)$. But, by the axiom (Ax6$_\blacktriangledown$), $T \vdash \blacktriangledown x\theta(x) \wedge \blacktriangledown x\varphi(x) \to \blacktriangledown x(\theta(x) \vee \varphi(x))$ and then, by (**MP**), $T \vdash \blacktriangledown x(\theta(x) \vee \varphi(x))$. Then, $(\theta(x) \vee \varphi(x))^T \in \mathcal{B}^T$ or $\theta(x)^T \cup \varphi(x)^T \in \mathcal{B}^T$.

Therefore $\mathcal{B}^T$ is a reduced topology.

Now we show, by induction on the length of φ, that for every sentence φ of T:

$$\mathcal{A}^{\Im} \vDash \varphi \text{ iff } T \vdash \varphi.$$

The only interesting case remaining to be proven is when φ is a sentence of the form $\blacktriangledown x\psi(x)$. Let φ be a sentence of the form $\blacktriangledown x\psi(x)$.

$\mathcal{A}^{\Im} \vDash \blacktriangledown x\psi(x)$ iff $\{c^{\sim} \in A : \mathcal{A}^{\Im} \vDash \psi(c^{\sim})\} \in \mathcal{B}^T$ iff, by inductive hypothesis, $\{c^{\sim} \in A : T \vdash \psi(c^{\sim})\} \in \mathcal{B}^T$ iff $(\psi(x))^T \in \mathcal{B}^T$ iff $T \vdash \blacktriangledown x\psi(x)$. □

Since $L^{\tau}_{\omega\omega}(\blacktriangledown)$ is complete, it is not difficult to see that Compactness and Löwenheim-Skolem Theorems can be also proved for $L^{\tau}_{\omega\omega}(\blacktriangledown)$.

6. The Logic of Almost All: Ultrafilter Logic

The development of Modulated Logics has had as its first motivation the Ultrafilter Logic ($L^{\tau}_{\omega\omega}(\nabla)$), proposed in ([CS94]) in an endeavor to propose a monotonic substitute for the default logic of [Rei80] based on the concept of ultrafilter. Since then further work about Ultrafilter Logic has been developed (e.g., [CV97], [SCV99], [Vel99b] and [VC04]).

This system aims to formalize an intuition for "almost all" or "generally", by means of including a generalized quantifier in the classical first order language. The central idea in this approach is the semantical interpretation of the quantifier "almost all" by a proper ultrafilter structure.

However, although Ultrafilter Logic[6] has represented the initial motivation for Modulated Logics, this logical system may be characterized as particular cases of Modulated Logics, as we will see below.

The language of ($L^{\tau}_{\omega\omega}(\nabla)$) is obtained by identifying the quantifier Q in $L^{\tau}_{\omega\omega}(Q)$ with the quantifier ∇. The semantic interpretation of formulas in ($L^{\tau}_{\omega\omega}(\nabla)$) is carried out in modulated structures where the subsets q are identified with a proper ultrafilter. Formally, a structure for ($L^{\tau}_{\omega\omega}(\nabla)$) can be defined endowing a classical first order structure $\mathcal{A}$, with an ultrafilter $\mathcal{U}^A$ over A and given by

$$\mathcal{A}^{\mathcal{U}} = \langle \mathcal{A}, \mathcal{U}^A \rangle = \langle A, \{R_i^{\mathcal{A}}\}_{i \in I}, \{f_j^{\mathcal{A}}\}_{j \in J}, \{c_k^{\mathcal{A}}\}_{k \in K}, \mathcal{U}^A \rangle.$$

The notion of satisfaction of a formula of the form $\nabla x\varphi$, whose free variables belong to $\{y_1, \ldots, y_n\}$, by a sequence $\underline{a} = (a_1, \ldots, a_n)$ in A, is defined by

$$\mathcal{A}^{\mathcal{U}} \vDash \nabla x\varphi[\underline{a}] \text{ iff } \{b \in A : \mathcal{A}^{\mathcal{U}} \vDash \varphi[b; \underline{a}]\} \in \mathcal{U}^A$$

for $\mathcal{U}^A$ a proper ultrafilter on A.

In intuitive terms, $\nabla x\varphi(x)$ is true in $\mathcal{U}^A$ iff almost all individuals of A satisfy $\varphi(x)$. Furthermore, almost all individuals in A satisfy $\varphi(x)$ iff the set of individuals satisfying $\varphi(x)$ belongs to a proper ultrafilter $\mathcal{U}^A$.

The usual semantical notions such as a model, validity, semantical consequence, etc. are defined for Ultrafilter Logic in a similar way as those for general modulated logics.

The axioms of $L^{\tau}_{\omega\omega}(\nabla)$ can be defined as those of $L^{\tau}_{\omega\omega}(Q)$, augmented by the following specific axioms for the quantifier ∇:

[6] For a detailed exposition concerning Ultrafilter Logic see [SCV99], [CV97] and also [VC04]

(Ax5$_\nabla$) $\forall x(\varphi(x) \to \psi(x)) \to ((\nabla x\varphi(x)) \to (\nabla x\psi(x)))$;

(Ax6$_\nabla$) $((\nabla x\varphi(x)) \wedge (\nabla x\psi(x))) \to \nabla x(\varphi(x) \wedge \psi(x))$;

(Ax7$_\nabla$) $\nabla x\varphi(x) \vee \nabla x\neg\varphi(x)$.

Given an interpretation with a universe A and formulas φ and ψ with exactly one free variable, then for sets $[\varphi] = \{a \in A: \varphi[a]\}$ and $[\psi] = \{a \in A: \psi[a]\}$, intuitively, axioms (Ax5$_\nabla$) to (Ax7$_\nabla$) assert that (cf. [SCV99]):

- (Ax5$_\nabla$) if $[\varphi] \subset [\psi]$ and $[\varphi]$ is large (almost all individuals of A), then so is $[\psi]$;
- (Ax6$_\nabla$) if $[\varphi]$ and $[\psi]$ are large, then $[\varphi] \cap [\psi]$ is large too;
- (Ax7$_\nabla$) either φ or its complement $[\neg\varphi]$ is large.

[Vel99b] admits that axiom (Ax7$_\nabla$) is probably the least intuitively acceptable one. He justifies its adoption by means of the notion of importance of sets, asserting that"the universe is so important (i.e. carries so much weight) that any attempt to cover it by finitely many subsets must employ a very important subset (one carrying considerable weight, or equivalently, almost as important as the entire universe)" ([Vel99b], p. 480). In that paper, Veloso prefers to use the notion of "important" instead of "large" to approach questions concerning the notion of "almost all" or "generally".

Usual syntactical notions as a sentence, a proof, a theorem, logical consequence, consistency, etc., can be defined for $L^{\tau}_{\omega\omega}(\nabla)$ in the same way as those defined in the general modulated logics.

Several properties of Ultrafilter Logic were already proven, as mentioned before. Some remarkable features that this logic shares with classical logic include the deduction theorem, soundness and completeness, compactness and Löwenheim-Skolem (e.g. [CV97]), existence of prenex normal forms (cf. [SCV99]), many-sorted versions (e.g. [CV97]) and the development of a natural deduction system ([RHV03]).

In the next section we discuss some fundamental issues about the relationships among particular forms of Modulated Logics, presenting examples of situations in which some are better suited than the others.

7. Comparing systems of modulated logic

This section compares the particular forms of Modulated Logic introduced here (including Ultrafilter Logic), trying to clarify the syntactical relations among them. Denoting by $\mathrm{Th}(L(Q))$ the theory of the basic system L[7], and denoting by Q one of the quantifiers $\sharp, \heartsuit, \blacktriangledown, \nabla$, we can easily prove the following syntactical relations among those particular forms of Modulated Logics ([Grá99]):

$$\mathrm{Th}(L) \subseteq \mathrm{Th}(L(\blacktriangledown)) \subseteq \mathrm{Th}(L(\nabla)) \text{ and}$$
$$\mathrm{Th}(L) \subseteq \mathrm{Th}(L(\heartsuit)) \subseteq \mathrm{Th}(L(\sharp)) \subseteq \mathrm{Th}(L(\nabla)).$$

Illustratively, taking $\mathrm{Th}(L(Q)) \to \mathrm{Th}(L(Q'))$ to mean $\mathrm{Th}(L(Q)) \subseteq \mathrm{Th}(L(Q'))$, we can express the relationship above in the following way:

$$\begin{tikzcd}
\mathrm{Th}(L(\nabla)) & & \mathrm{Th}(L(\sharp)) \arrow[ll] \\
\mathrm{Th}(L(\blacktriangledown)) \arrow[u] & & \mathrm{Th}(L(\heartsuit)) \arrow[u] \\
& \mathrm{Th}(L) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

We will present some statements that will help to clarify the distinctive features of those systems.

Let Q be ∇, $\heartsuit$, $\blacktriangledown$ or $\sharp$; then the formulas of the form $\neg Qx\varphi(x) \to Qx\neg\varphi(x)$ are theorems of Ultrafilter Logic, but they are neither theorems in the Logic of Simple Majority, nor in the Logic of Many, nor in the Logic of Plausibility. So this gives to Ultrafilter Logic a maximal status with respect to inclusion and provides to this system a decisive criterion for 'large' subsets.

In the same way, formulas of the form

$$(Qx\varphi(x) \wedge Qx\psi(x)) \to Qx(\varphi(x) \wedge \psi(x))$$

are theorems of the Ultrafilter Logic and of the Logic of Plausibility, but they are not theorems of the Logic of Simple Majority and of the Logic of Many. So, a characteristic of the Logic of Plausibility is that it formalizes a notion of a significant set, such that the intersection of two significant sets is also significant. Similarly, the Ultrafilter Logic treats the notion of 'large' set, so that it makes sense to assert that the intersection of 'large' sets is a large set too. These systems are only concerned with the treatment of inductive assertions that are coherent with such principles. If this is not

[7] By a question of simplicity, we will adopt in this section the symbol L instead of $L^{\tau}_{\omega\omega}$.

observed in the situation to be formalized, probably the assertions will be better formalized by other forms of inductive assertion, for example, as those proposed by the Logic of Many or the Logic of Simple Majority.

It is still to be observed that formulas of the form

$$(Qx\varphi(x) \wedge Qx\psi(x)) \to \exists x(\varphi(x) \wedge \psi(x))$$

are theorems of the Ultrafilter Logic, the Logic of Plausible and the Logic of Simple Majority, but they are not theorems of the Logic of Many; thus intuitively, in the last system, the inherent concept of 'large' is such that the intersection of large sets is not necessarily nonempty.

Besides, it should be clear from the axiomatics (details can be found in [Grá99]) that the quantifiers $\nabla, \heartsuit, \blacktriangledown, \sharp$ are intermediary between $\forall$ and $\exists$, and that their relative position is the following (take $Q \to Q'$ to mean $Qx\varphi \to Q'x\varphi$):

$$\begin{tikzcd}
\exists & & \\
\nabla \arrow[u] & & \sharp \arrow[ll] \\
\blacktriangledown \arrow[u] & & \heartsuit \arrow[u] \\
& \forall \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Semantically, taking into account that the Logic of Simple Majority shows itself to be incomplete with respect to the intended models, we will restrict our analysis to the relationship among the Logic of Many, the Logic of Plausibility and the Ultrafilter Logic.

For $\mathcal{A}$, a classical first-order structure with a universe A, let $[\forall] = \{A\}$, $[\exists] = \wp(A) - \{\varnothing\}$, $\Im$ be a reduced topology over A, and F a family of principal filters over A.

It can be easily seen (see [Grá99]) that we can construct an ultrafilter U on the basis of $\Im$ and an ultrafilter U' on the basis of F. Considering that $[\forall]$ is, by definition, a subset of F and $\Im$ and that U and U', by definition, belong to $[\exists]$, we can express the following relations:

$$[\forall] \subseteq \Im \subseteq U \subseteq [\exists] \text{ and}$$
$$[\forall] \subseteq F \subseteq U' \subseteq [\exists].$$

By availing ourselves of the semantical relationship observed above, the following picture (where $C \to C'$ means $C \subseteq C'$) illustrates the relations among $[\forall]$ and $[\exists]$ sets, ultrafilters U, reduced topologies $\Im$ and families of principal filters F on A:

$$\begin{tikzcd}
 & {[\exists]} & \\
 & U \arrow[u] & \\
 \Im \arrow[ur] & & F \arrow[ul] \\
 & {[\forall]} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Reflecting on the possibilities of applying these systems, we can conclude that there exists no universally applicable inductive reasoning pattern. In agreement with [McC86], each system seems to give rise to a different form of inductive reasoning. So, also in agreement with [DW91], to regard one of these systems as preferable in relation to the others in all questions of inductive reasoning does not seem to be a reasonable position.

The account for the Logic of Simple Majority depends on the notion of large sets taken in the "counting", "non-vague" sense. In this case, the exact size of the set is relevant and the parameter (threshold) that establishes largeness of sets is just one in all models: a set is only considered large when its cardinal number is greater than the cardinal number of its complement. Thus the notion of "most" expressed through the concept of cardinality treated in this logic is reduced to a quantitative, measurable aspect, and presumably obtains conclusions less exposed to debate. The example about real numbers (in which we formalize the statement asserting that irrational numbers constitute the most of real numbers) is a typical example of application of the Logic of Simple Majority; on the other hand, this logic suffers from incompleteness, as seen in Subsection 3.1.

In the Logic of Many the notion of largeness is more vague. In this system, two sets can be large, without their intersection being nonempty, i.e., they not necessarily constitute the greater part of universe, as in the Logic of Simple Majority. This is the case of Example 4.12, presented in Section 4.1, in which we formalize the argument that many natural numbers are odd and that many natural numbers are even. Another example where the Logic of Many is very useful in situations as those below Example 4.14 at page 229 where the intersection of (d) and (e) is not a large set, even it can be non-empty. Such account of "many" has a purely qualitative character, since it is invariant from one model to another.

On the other hand, the Logic of Plausibility does not depend on the notion of a large set, but it is connected with the notion of a significant evidence set. Examples 5.21 and 5.22 appropriately illustrate the cases where this system works very well. This notion of significance of evidence has a pertinent relationship to the notion of Bayesian statistical inference, where we do not necessarily use large samples, but sufficiently relevant samples to carry out the inferences.

Still, the Logic of Plausibility can be considered as a subsystem of the Ultrafilter Logic in which the notion of plausibility is not inherited to supersets of plausible sets (recall that it does not deduce theorems of the form $\forall x(\varphi(x) \to \psi(x)) \to (Qx(\varphi(x)) \to Qx(\psi(x))))$.

As a brief conclusion concerning the essential distinction between the inductive arguments supported by the systems presented in sections 3, 4 and 5, we can say that the Logic of Simple Majority intends to formalize rigid quantitative inductive assertions, while other systems support flexible, qualitative inductive assertions.

8. Scope and significance of modulated logics

Modulated logics are conservative extensions of the classical first-order logic, and the purely qualitative cases studied here (which excludes the Logic of Simple Majority) enjoy sound and complete deductive system that share with the classical first-order logic some important features, such as compactness and Löwenheim-Skolem properties. The fact that we are using a non-standard notion of a model by including a mathematical structure in the models confers to modulated logics an independent model-theoretical interest.

The concept of modulated logics, which leads to the development of the systems treated in sections 3, 4 and 5, consists basically in the inclusion of a generalized quantifier in the syntax of the classical logic. This new quantifier is intended to represent intuitively a particular form of inductive assertions. Semantically, each generalized quantifier is interpreted by an appropriate mathematical structure within a modulated model. The axiom sets which characterize each particular form of Modulated Logic are divided into two groups of axioms, one formed by basic axioms ((Ax1) to (Ax4) presented in Section 2.1) and the other constituted by specific axioms to characterize the mathematical structure that is interpreting a particular modulated quantifier.

Other proposals which treat somehow similar problems with (in principle) comparable views are [Wey97] and [Sch95]. Even if our approach seems to drastically diverge from them on what concerns the role of monotonicity in logic, there may be more points of contact than noticeable at first sight. A principled comparison is yet to be done.

The different forms of inductive argumentation that modulated logics are capable of formalizing open interesting possibilities for the analysis of linguistics questions, since they offer a sharp tool that permits "logical syntax to correspond more closely to natural language syntax" ([BC81], p. 159).

Modulated quantifiers, regarded as a theory of generalized quantifiers break away the naive notion that "the meaning of the quantifiers must be built into the logic, and hence cannot vary from one model to another" ([BC81], p. 162).

The notion of truth and falsity associated with generalized quantifiers in the scope of modulated logics does not depend on any *a priori* logic, but depends on which underlying measure we are using, and that "must be included as part of the model before the sentences have any truth value whatsoever" ([BC81], p. 163).

Another area of obvious interest are the extensions of modulated logics in the direction of modal modulated logics. Logics of this kind seem to be naturally appropriate to certain inherently qualitative reasoning as reasoning with spatial relations, of practical interest for geographical information systems. We can only conclude that modulated quantifiers deserve further study, contributing to the understanding of the relationship between logic and language.

Another advantage of modulated logics is that they do not associate degrees of belief with the assertions supported by evidence, but work with an intrinsic qualitative notion given by the associate mathematical structure of their models, and thus free themselves from possible incoherences generated to confront degrees of belief associated with assertions and theories of probability.

A related and developing research direction is to employ modulated logics to provide alternative foundations to fuzzy concepts and fuzzy reasoning. The notion of a fuzzy set first introduced in [Zad65] (but see [Zim01] for the developments) has its intuitive interpretation of the concept of degress of membership, and their logic counterpart has been connected to the idea of many-valuedness (logics with more than two truth-values, see [Háj98]). Now, modulated structures offer a new approach to fuzziness which is totally independent of many-valuedness, permitting to built natural and elegant purely qualitative fuzzy logics. The simple idea is to attach partial ordering relations to sets which are interpretations of our new quantifiers (see also [VC04], Section 3.1).

If, as certain interpretations claim, fuzzy sets (and fuzzy logics) intend to cope with the vagueness and ambiguity of natural language and common reasoning and has to face more than one imprecision (as, for example, which chunk of the universe is to be measured, which parameters are actually to be measured, and so on) our purely qualitative approach seems to be much more appropriate.

As examples, suppose that $\mathcal{A}^q = \langle \mathcal{A}, q \rangle$ is a modulated structure where $\mathcal{A} = \langle A, \{R_i^{\mathcal{A}}\}_{i \in I}, \{f_j^{\mathcal{A}}\}_{j \in J}, \{c_k^{\mathcal{A}}\}_{k \in K}, q \rangle$ is a structure (in the usual sense) such that $R(x, y) \in \{R_i^{\mathcal{A}}\}_{i \in I}$ interprets a partial ordering $\leq$ on A (that is, $\leq$ is interpreted as a reflexive, transitive, and anti-symmetric relation).

If $p(x)$ is a function symbol in the language $(L^{\tau}_{\omega\omega}(\nabla x))$, we define a *fuzzy predicate* as a wff of the form:

$$\varphi(y) := (\nabla x)(p(x) \leq p(y)) \text{ or } \phi(y) := (\nabla x)(p(y) \leq p(x))$$

For instance, taking $p(x)$ to represent body weight, $\varphi(y)$ expresses, for an individual y, that "y is heavy if for most individuals x, the weight $p(x)$ is less than the weight $p(y)$ of y". This individual fuzziness can be contrasted to the population fuzziness: $(\nabla y)\varphi(y)$ expresses the information that most individuals of a certain population are fat.

Therefore $\varphi(y)$ and $\phi(y)$ represent vague or debatable predicates, and there is a natural qualitative fuzzy gauge attached to them which is independent of any degrees of membership. Suppose that the complex q contains another partial ordering $\leq_E$ on $\wp(A) \times \wp(A)$; $\leq_E$ is an *external* partial ordering, and in this case $q \subseteq \wp^3(A)$). We say that $\leq_E$ is a *plausibility measure* if the following conditions hold:

1. $\varnothing \leq_E B$ and $\varnothing \leq_E A$, for every $B \in \wp(A)$;
2. If $B \subseteq C$ then $B \leq_E C$, for every $B, C \in \wp(A)$.

Observe that plausibility measures are order homomorphisms on upper closed families (or families of principal filters) introduced in Section 4. Such internal and external partial orderings play respectively the role of the "fuzzy measures" and "measures of fuzziness" in the usual quantitative fuzzy set theory. Thus a new branch of fuzzy theory can be established, based on modulated structures.

On the other side, modulated logics have also a natural interpretation which permits one to express interesting questions related to classical problems of philosophy of science, specially those concerning induction. For example, the naive belief that "inductive argumentation" is necessarily "contrary-to-inductive argumentation" is clearly challenged by the modulated logics, as well as the view that flexible reasoning that formalizes uncertainty necessarily involves non-monotonicity. Some investigations in this direction are suggested in [VC04], Section 7, but there is still much to be done.

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