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On assertion and denial in the logic for pragmatics

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ABSTRACT

The aim of this paper is twofold: First, we present and develop a system of logic for pragmatics including the act of *denial*. Second, we analyse in our framework the so-called paradox of assertability. We show that it is possible to yield sentences that are not assertable. Moreover, under certain conditions, a symmetric result can be obtained: There is a specular paradox of deniability. However, this paradox is based on the problematic principle of classical denial equivalence.

1. Introduction

In the *classical theory for denial*, to deny A is equivalent to asserting $\neg A$:

Classical denial. A is correctly denied *iff* $\neg A$ is correctly asserted.

In Ripley [21] the above equivalence is called *denial equivalence*.[1] *Classical denial* implies what we call the *Frege's Thesis, i.e.* that there is just one speech act, *i.e.* assertion, and that denial is reducible to it. The aim of the paper is to defend the thesis that, from the assertion of the negation of A, it is possible to infer the denial of A, but not *vice versa*. Frege's Thesis does not hold. We will argue that, if the act of asserting is embedded in a general framework of pragmatic logic, then it is plausible to consider assertion and denial as relatively independent speech acts governed by different conditions of justification. We show that this view provides a suitable framework for dealing with some pragmatic paradoxes similar to the liar, the *assertability paradox* and the *deniability paradox*. We first introduce Dalla Pozza and Garola's *pragmatic logic* (LP) (section 2) where a *logic of assertion* is *formulated*. Then, we argue (section 3) against Frege's Thesis. We propose a basic, intuitive extension of LP with denial, considering *assertion* and *denial*

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[1] For general background on denial in non-classical theories, see (Ripley [21], §3). On the same topic see also Restall [20].

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(or rejection) as two relatively independent speech acts. Within such an extension, the *assertability paradox* and the *deniability paradox* (section 5) are analyzed.

2. A pragmatic logic for assertion (LP)

In the logical system called *Logic for Pragmatics* (LP), Dalla Pozza and Garola [9] provided a formal treatment of assertion by introducing some pragmatic connectives and the sign of *force*, which are required to formulate a pragmatic interpretation of intuitionistic propositional logic as a logic of assertions.[2]

Assertions are intended as "purely logical entities ... without making reference to the speaker's intention or beliefs" (Dalla Pozza and Garola [9], 83). LP is composed of two sets of formulas: *radical* and *sentential*. Every sentential formula contains at least a radical formula as a proper sub-formula.

Radical formulas are semantically interpreted by assigning them (classical) truth-values. *Sentential formulas* (briefly, *assertions*), on the other hand, are pragmatically evaluated by assigning them justification values defined in terms of the intuitive notion of proof. In other words, LP propositions can be either true or false, whereas the judgements expressed as assertions can be justified (J) or unjustified (U). The pragmatic language of LP is described below.

Alphabet.

The vocabulary of LP contains the following sets of signs.

Descriptive signs: the propositional letters p, q, r, ...
Logical signs for radical formulas: $\wedge, \vee, \neg, \rightarrow, \leftrightarrow$
Logical signs for sentential formulas: the assertion sign $\vdash$ and the pragmatic connectives $\sim$ (negation), $\cap$ (conjunction), $\cup$ (disjunction), $\supset$ (implication), $\equiv$ (equivalence).

Formation rules (FRs)
Radical formulas (*rfs*) are recursively defined by the following FRs.
FR1 (atomic formulas): Every propositional letter is an *rf*.
FR2 (molecular formulas):
(i) Let γ be an *rf*; then $\neg\gamma$ is a *rf*;
(ii) Let γ_1 and γ_2 be *rfs*; then $\gamma_1 \wedge \gamma_2$, $\gamma_1 \vee \gamma_2$, $\gamma_1 \rightarrow \gamma_2$, $\gamma_1 \leftrightarrow \gamma_2$ are *rfs*.

Sentential formulas (*sfs*) are recursively defined by the following FRs.
FR3 (elementary formulas): Let γ be an *rf*, then $\vdash \gamma$ is an *sf*.
FR4 (complex formulas):
(i) Let δ be an *sf*; then $\sim \delta$ is an *sf*;
(ii) Let δ_1 and δ_2 be *sfs*; then $\delta_1 \cap \delta_2$, $\delta_1 \cup \delta_2$, $\delta_1 \supset \delta_2$, $\delta_1 \equiv \delta_2$ are *sfs*.

Every radical formula of LP has a truth-value. Every sentential formula has a justification value, which is defined in terms of the intuitive notion of proof and depends on the truth-value of its radical sub-formulas. The semantics of LP is the same as for classical logic, and it provides only the interpretation of the radical formulas by assigning them truth-values and taking propositional connectives as truth functions in a standard way.

To be precise, the semantic rules are the usual classical Tarskian ones and specify the truth-conditions (only for radical formulas) through assignment function σ, thus regulating the semantic interpretation of LP. Let γ_1, γ_2 be radical formulas and $1 =$ true and $0 =$ false; then:

[2] For an extension of LP see Carrara et al. [6,7], Bellin et al. [2,3].

1. $\sigma(\neg\gamma_1) = 1$ iff $\sigma(\gamma_1) = 0$
2. $\sigma(\gamma_1 \wedge \gamma_2) = 1$ iff $\sigma(\gamma_1) = 1$ and $\sigma(\gamma_2) = 1$
3. $\sigma(\gamma_1 \vee \gamma_2) = 1$ iff $\sigma(\gamma_1) = 1$ or $\sigma(\gamma_2) = 1$
4. $\sigma(\gamma_1 \rightarrow \gamma_2) = 1$ iff $\sigma(\gamma_1) = 0$ or $\sigma(\gamma_2) = 1$
5. $\sigma(\gamma_1 \leftrightarrow \gamma_2) = 1$ iff $\sigma(\gamma_1) = \sigma(\gamma_2)$

Whenever only classical metalinguistic procedures of proof are admitted in LP, the pragmatic connectives have a meaning that is explicated by the *BHK* (Brouwer, Heyting, Kolmogorov) interpretation of intuitionistic logical constants. Thus, the meaning of an elementary formula has two components: the *pragmatic* component is grounded on the illocutionary force of the assertion whilst the *semantic* component is given by the interpretation of γ within a classical Tarskian framework.

Justification rules regulate pragmatic evaluation π, specifying the justification conditions for the sentential formulas in the function of the σ-assignments of truth-values for their radical sub-formulas. A pragmatic interpretation of LP is given by the ordered pair $\langle\{J, U\}, \pi\rangle$, where $\{J, U\}$ is the set of justification values and π is a function of pragmatic evaluation in accordance with the following justification rules.

JR1 – Let γ be a radical formula. $\pi(\vdash \gamma) = J$ iff a proof exists that γ is true, *i.e.* that σ assigns the value 1 to γ. $\pi(\vdash \gamma) = U$ iff no proof exists that γ is true.

JR2 – Let δ be a sentential formula; then, $\pi(\sim \delta) = J$ iff a proof exists that δ is unjustified, *i.e.*, that $\pi(\delta) = U$.

JR3 – Let δ_1 and δ_2 be sentential formulas. Then:
1. $\pi(\delta_1 \cap \delta_2) = J$ iff $\pi(\delta_1) = J$ and $\pi(\delta_2) = J$;
2. $\pi(\delta_1 \cup \delta_2) = J$ iff $\pi(\delta_1) = J$ or $\pi(\delta_2) = J$;
3. $\pi(\delta_1 \supset \delta_2) = J$ iff a proof exists that $\pi(\delta_2) = J$ whenever $\pi(\delta_1) = J$;
4. $\pi(\delta_1 \equiv \delta_2) = J$ iff $\pi(\delta_1 \supset \delta_2) = J$ and $\pi(\delta_2 \supset \delta_1) = J$.

The soundness criterion (SC) is the following:

(SC) Let γ be a radical formula; then $\pi(\vdash \gamma) = J$ implies that $\sigma(\gamma) = 1$.

SC states that, if an assertion is justified, then the content of assertion is true. It is evident from the justification rules that sentential formulas have an intuitionistic-like formal behaviour and can be translated into modal system S4, where '$\Box\gamma$' means there is an (intuitive) proof (conclusive evidence) for γ.[3]

The classical fragment of LP, CLP, is made up of all of the *sfs* that do not contain pragmatic connectives. Axioms for CLP are the following:

A1 $\vdash (\gamma_1 \rightarrow (\gamma_2 \rightarrow \gamma_1))$
A2 $\vdash ((\gamma_1 \rightarrow (\gamma_2 \rightarrow \gamma_3)) \rightarrow ((\gamma_1 \rightarrow \gamma_2) \rightarrow (\gamma_1 \rightarrow \gamma_3)))$
A3 $\vdash ((\neg\gamma_2 \rightarrow \neg\gamma_1) \rightarrow ((\neg\gamma_2 \rightarrow \gamma_1) \rightarrow \gamma_2))$

The *modus ponens* rule in CLP is the following:

[MPP] If $\vdash \gamma_1$ and $\vdash (\gamma_1 \rightarrow \gamma_2)$, then $\vdash \gamma_2$.

[3] Of course, there is a lot to say about the concept of *proof* or *conclusive evidence*. In the original framework of LP, the authors are rather vague on the characterisation of the notion of proof.

The intuitionistic fragment of LP, ILP, is made up of all of the *sfs* containing only atomic radicals. The axioms of the intuitionistic fragment of ILP are the following (where $\delta_1, \delta_2, \delta_3$ contain only atomic radicals):

A1* $\delta_1 \supset (\delta_2 \supset \delta_1)$
A2* $(\delta_1 \supset \delta_2) \supset ((\delta_1 \supset (\delta_2 \supset \delta_3)) \supset (\delta_1 \supset \delta_3))$
A3* $\delta_1 \supset (\delta_2 \supset (\delta_1 \cap \delta_2))$
A4* $(\delta_1 \cap \delta_2) \supset \delta_1; (\delta_1 \cap \delta_2) \supset \delta_2$
A5* $\delta_1 \supset (\delta_1 \cup \delta_2); \delta_2 \supset (\delta_1 \cup \delta_2)$
A6* $(\delta_1 \supset \delta_3) \supset ((\delta_2 \supset \delta_3) \supset ((\delta_1 \cup \delta_2) \supset \delta_3))$
A7* $(\delta_1 \supset \delta_2) \supset ((\delta_1 \supset (\sim \delta_2)) \supset (\sim \delta_1))$
A8* $\delta_1 \supset ((\sim \delta_1) \supset \delta_2)$

The *modus ponens* rule for ILP is the following:

[MPP′] If δ_1 and $\delta_1 \supset \delta_2$, then δ_2

It is worth noting that the justification rules do not allow for the determination of the justification value of a complex sentential formula even when all of the justification values of its components are known: no principle analogous to the truth-functionality principle for classical connectives holds for the pragmatic connectives in LP, given that pragmatic connectives are partial functions of justification. For instance:

NR1 $\pi(\delta) = J$ implies $\pi(\sim \delta) = U$;
NR2 $\pi(\delta) = U$ does not necessarily imply $\pi(\sim \delta) = J$;
NR3 $\pi(\sim \delta) = J$ implies $\pi(\delta) = U$;
NR4 $\pi(\sim \delta) = U$ does not necessarily imply $\pi(\delta) = J$.

In addition, δ is pragmatically valid, or *p*-valid (respectively, invalid or *p*-invalid), if, for every π and σ, δ is justified (respectively, δ is unjustified).[4] Function $(\ )^*$ mapping the set of *sfs* into an extension of the set of *rfs* obtained by means of modal operator $\Box$ (*proved*), *i.e.* a modal translation of pragmatic assertive formulas, is (recursively) induced by the following correspondence:

$(\vdash \gamma)^* = \Box\gamma$
$(\sim \delta)^* = \Box\neg(\delta)^*$
$(\delta_1 \cap \delta_2)^* = (\delta_1)^* \wedge (\delta_2)^*$
$(\delta_1 \cup \delta_2)^* = (\delta_1)^* \vee (\delta_2)^*$
$(\delta_1 \supset \delta_2)^* = \Box((\delta_1)^* \to (\delta_2)^*)$
$(\delta_1 \equiv \delta_2)^* = \Box((\delta_1)^* \leftrightarrow (\delta_2)^*)$

Radical and sentential formulas are related by means of the following 'bridge principles'[5]:

(a) $(\vdash \neg\gamma) \supset (\sim\vdash \gamma)$
(b) $(\vdash \gamma_1 \cap \vdash \gamma_2) \equiv \vdash (\gamma_1 \wedge \gamma_2)$
(c) $(\vdash \gamma_1 \cup \vdash \gamma_2) \supset \vdash (\gamma_1 \vee \gamma_2)$

4 Pragmatic criteria of validity are presented in Dalla Pozza and Garola [9].
5 Again, on this see Dalla Pozza and Garola [9].

(d) $(\vdash (\gamma_1 \to \gamma_2)) \supset (\vdash \gamma_1 \supset \vdash \gamma_2)$
(e) $(\vdash (\gamma_1 \leftrightarrow \gamma_2)) \supset (\vdash \gamma_1 \equiv \vdash \gamma_2)$

It is worth observing that **(a)**–**(e)** show the formal relations between classical truth-functional connectives and pragmatic ones. Formula **(a)** states that, from the assertion of not-γ, one can infer the fact that γ cannot be asserted. **(b)** states that the conjunction of two assertions is equivalent to the assertion of a conjunction. **(c)** states that, from the disjunction of two assertions, one can infer the assertion of a disjunction. **(d)** expresses the idea that the pragmatic implication between two assertions follows from the assertion of a classical material. Finally, **(e)** indicates that from the assertion of a biconditional follows the pragmatic equivalence of assertions.

It is evident that a key element in the formulation of the justification (viz. unjustification) conditions of LP is that of *proof*. As said before, Dalla Pozza and Garola employ the concept of *intuitive proof*, which is assumed to be *factive*: The assertion of γ is justified if and only if there is a proof that γ is true.[6] But, of course, one could relax the requirement of the soundness criterion; in the following, it is possible to assume a more general point of view by speaking about *reasons* to assert content, where the assertion of γ is pragmatically justified if and only if there are (good) reasons for γ even if it is possible that γ is false.

Now, we go on as follows: Following Frege's thesis, we are going to try to define a pragmatic denial ($\dashv$) through:

- (i) the negation of the content and
- (ii) the negation of the act of assertion.

We anticipate that every attempt will not be made without problems.

3. Assertion and denial

Ley us start by an intuitive characterisation of the justification conditions for the denial.[7] The denial of A is justified if and only if A is, in a sense to be specified, *incompatible* with the accepted epistemic framework. We will follow two options.

(1) Denying Content

The first option is to define the denial of A in LP as:

1 $\pi(\dashv A) = J$ iff $\pi(\vdash \neg A) = J$.

That is, the denial of A is justified if and only if the assertion of $\neg A$ is justified. It is easy to realize that 1 does not work. Observe that 1 can be detached as:

1′ If $\pi(\dashv A) = J$, then $\pi(\vdash \neg A) = J$

and:

1″ If $\pi(\vdash \neg A) = J$, then $\pi(\dashv A) = J$.

[6] Usually, the idea of *factivity* connects a proposition under the scope of an operator with the truth of that proposition. Probably in our context it is requested something more than that but we can use the notion of factivity in this sloppy way. Thanks to an anonymous referee for making explicit this point.
[7] We expand here some sparse observations from Carrara et al. [5].

Although $1''$ is plausible, $1'$ is not. The reason $1'$ does not hold is that there are many situations where it is perfectly legitimate to deny a certain content, say A, with no conclusive proof to assert its negation. They are indeed cases in which it is extremely difficult to provide good reasons or a proof for $\neg A$ but it is reasonable to deny A on the basis of some contextual reasons. Let us consider the following example. Imagine that we are in the presence of a healing phenomenon with no available explanation from a medical point of view. B states that the healing happened in virtue of a supernatural (miraculous) divine action. Now, a hypothetical Humean sceptic about miracles could argue as follows. I have at my disposal no proof (in a technical sense) of the falsity of B, as, more generally, I am not able to provide a *refutation* of theism. However, I find it quite plausible to reject a miraculous explanation: It is much more rational to suppose that there are biological causes explaining the healing phenomenon that are unknown to our scientific knowledge at the moment. I can rationally reject B without *proving* $\neg B$. In other words: logically speaking, the miraculous explanation is still open; however, it is rational to exclude this hypothetical explanation even without its refutation.

For $1'$, notice that there is a limit case in which it turns out to be valid; if the reasons for denying A become *logical*, then the denial of A is equivalent to the assertion of $\neg A$. If, for instance, A is logically inconsistent, then the logical reasons for rejecting it are the *very same* logical reasons for asserting its negation.

Summing up, we accept just one direction of the *denial equivalence*, from the assertion of the negation of A to the denial of A.

(2) Denying Act

The second option exploits the expressive resources of pragmatic negation:

2 $\pi(\dashv A) = J$ iff $\pi(\sim\vdash A) = J$.

It means that the denial of A is justified if and only if it is justified that A is not justified. Even in this case, we can detach the double conditional:

$2'$ If $\pi(\dashv A) = J$, then $\pi(\sim\vdash A) = J$ and
$2''$ If $\pi(\sim\vdash A) = J$, then $\pi(\dashv A) = J$.

$2''$ is not a general principle: If the assertion of A cannot be conclusively justified, this may be not a sufficient condition for denying A, which may still be consistent with an accepted epistemic framework. Moreover, $2'$ is also problematic for the same reason previously expressed: The reasons supporting the denial of A could not be sufficient for guaranteeing that A is not provable.

It seems that neither (1) nor (2) succeeds in defining the justification conditions for the act of denial. The reason remains in the same justification conditions for LP:

$$\boxed{\begin{array}{ccc} & & \pi(\dashv A) = J \\ & \nearrow & \\ \pi(\vdash \neg A) = J & & \\ & \searrow & \\ & & \pi(\sim\vdash A) = J \end{array}}$$

Indeed, if $\pi(\sim\vdash A) = J$, then it does not necessarily follow that $\pi(\dashv A) = J$, as, if there are reasons for not asserting A, it is still possible that A may not be denied because it is coherent with a specific accepted framework. Conversely, from $\pi(\dashv A) = J$, it does not follow that $\pi(\sim\vdash A) = J$, *i.e.*, the mere denial of A

does not imply that there are reasons preventing, in principle, a possible assertion of A. This means that the pragmatic negation of an assertion ($\sim$) and the act of denying ($\dashv$) cannot be confused. Moreover, if we accept the criticism of 1 and 2, we need to extend the logical system of LP for assertions. Let us recall our previous intuition about the justification condition for denial:

A can be denied when it is incompatible with an *accepted framework.*

In other words, we are justified in denying A when A is incompatible with the set of accepted assumptions. We consider assertion and denial to be the expressions of two fundamental epistemic attitudes, that is, *acceptance* and *rejection.*[8] On the basis of this pre-theoretical distinction, we define a pragmatic contradiction as

(PC) $\vdash A \cap \dashv A$.

There is no way of asserting and denying the same content. Observe that, in LP, we already have a possible way of expressing a pragmatic contradiction as:

(PAC) $\vdash A \cap \vdash \neg A$.

However, in our framework the assertion of a negation implies a denial (but not *vice versa*), then PAC entails PC, but not *vice versa.*

Consider now:

(PACC) $\dashv A \cap \dashv \neg A$.

What does (PACC) mean? Is it a kind of contradiction? First, observe that, in many cases, if A is incompatible with the accepted framework, then $\neg A$ is compatible. Let us recall our previous example of the miraculous explanation of the healing phenomenon. If we think, reasonably, that miracles are incompatible with our naturalistic framework, then we cannot for the same reasons say that the absence of miracles is incompatible with our framework. All we can do is, as we said before, to say that, although implausible, the miracles are possible, in a very *wide* sense of 'possible'. However, It seems there is a conclusive argument to state that (PACC) *is* a contradiction.[9] Indeed, from $\vdash \neg A$ we have $\dashv A$ and from $\vdash \neg\neg A$ we have $\dashv \neg A$. But since it is inconsistent $\vdash \neg A$ *and* $\vdash \neg\neg A$ we have that (PACC) is a contradiction. Still, one could observe that there are cases in which $\dashv A \cap \dashv \neg A$ does not seem, at least *prima facie*, contradictory: Let us imagine moral dilemmas. One rationally wants to reject option A, *e.g.* kill one man, and option $\neg A$, *e.g.* not-kill one man but kill five men, provided that these are the only possibilities at issue. How to account this intuition? Following the previous proof, we have to conclude that in case, of moral dilemmas, the principles governing the assertions (and the denials) have to be revised.

4. Assertability paradox and deniability paradox

Does our proposal incur in paradoxical results?

Let *exclusive negation* be a propositional connective $\sim$ such that, in virtue of its very meaning, A and $\sim A$ are incompatible. In other words A and $\sim A$ cannot be both true, *i.e.* it is excluded that A and $\sim A$

[8] Here we are following Priest's intuition. Cf. Priest [17].
[9] Thank to an anonymous referee for this point.

are both true.[10] Such a kind of negation is standardly called *Boolean negation*; Priest's dialetheism argues for its inexistence (for example, see Priest [14,17,18]).

Dialetheism holds that there are *dialetheias*, *i.e. glut* propositions that are both true and false.[11] It is a glut theory postulating the existence of gluts on the face of semantical, and perhaps soritical, paradoxes (see e.g. Priest [17–19], Beall [1], Colyvan [8], Weber [22]). And for a glut theorist the Liar Paradox teaches us that negation fails to be *exhaustive*: there is overlap between truth and falsity, and the *law of Ex Contraditione Quodlibet* ECQ must be given up. In standard natural deduction, ECQ can be derived using *reductio ad absurdum* RAA and other apparently non-problematic rules. However, *reductio* is dialetheically invalid. For, if, for example, $A \wedge \neg A$ is a *glut* or a *dialetheia*, it may be compatible with the truth of all assumptions it depends on. So the derivation of ECQ – using RAA – is blocked.

A standard criticism against those arguing for the inexistence of *Boolean negation* concerns the difficulty of expressing the notion of exclusivity – *exclusively* (or *only*) *true* and *exclusively* (or *only*) *false* – dialetheically. Take, for example, *only true*: It is usually understood as 'true and not false', where the 'not' is the exclusive one, which is not available for a dialetheist.

However, a dialetheist *needs* the possibility of expressing such a notion. For instance, having held that a sentence may be true, false or both, the dialetheist should be able to *reason by cases*, distinguishing three possible cases: *true only*, *false only*, and *true and false*. In order to make up for the lack of *exclusive negation*, Priest introduced the notion of *rejection* of a sentence A, to be clearly distinguished from the *acceptance* of the negation of A.

So, rejection and assertion are central speech acts because they are considered as *mutually incompatible speech acts* in the dialetheist frame: While it is possible to accept both a sentence and its negation one cannot accept and reject the same sentence; assertion and rejection are mutually incompatible speech acts.[12] The exclusivity of negation is lost (or perhaps was never had) in the realm of logic, but it is regained at the pragmatic level. Question: If giving up the exclusivity of negation is the key to solving the semantic paradoxes, doesn't the exclusivity of assertion or denial land us back in paradox?

Littman and Simmons [11] give a positive answer to the above question; in particular, they have introduced a paradox on assertion called the *assertability paradox*. Take sentence α as having the form:

(α) α is not assertable.

They argue that (α) is a dialetheia. The argument is the following:

> Suppose (α) is true. Then what it says is the case. So (α) is not assertable. But we have just asserted (α). So (α) is assertable – and we have a contradiction. Suppose, on the other hand, that (α) is false. Then what (α) says is not the case. So (α) is assertable. Thus we may assert: (α) is not assertable. Again, we have a contradiction. (Littman and Simmons [11, 320])

Before discussing the *pros* and *cons* of the assertability paradox for LP, observe that the argument is not correct in the above formulation: The mere supposition that α is true *does not imply its assertability.* Indeed, assertability implies the recognition, not just the mere supposition, of the truth of (α). To amend the argument, let us prove dialetheically that α is true by distinguishing the following two cases:

[10] Of course this explanation is circular, because exclusive negation is embedded in the word "cannot", as well as it is involved in the notion of *exclusion* and *incompatibility*. But this circularity is unavoidable in any explanation of a primitive notion. Observe, however, that exclusive negation must be grasped by any competent speaker of the natural language, since it is essential for human verbal communication.

[11] Priest uses the terms 'dialetheias' and 'true contradictions' to indicate 'gluts', propositions both true and false, a term coined by K. Fine in [10]. For an introduction to dialetheism, see e.g. Berto [4]. Among the dialetheists, Priest (for example, in Priest [13,15–19]).

[12] On denial and paradoxes see Murzi and Carrara [12].

(1) Assume that (α) is false. Then its negation is true, so (α) is assertable and thus true.
(2) Assume that (α) is true. Then it is true.

According to the Law of the Excluded Middle (LEM) and *disjunction elimination*, (α) is true. In this way, we have proof – not just a supposition – of the truth of (α). And we can assert it. So (α) is assertable, in opposition to what (α) claims and thus is false. Therefore, it is a dialetheia, a sentence (statement/proposition) that is both true and false. According to Littman and Simmons, (α) would be more problematic than the strengthened liar. If α is a dialetheia, then it is both assertable and not assertable. But how is it possible to both assert and not assert a sentence?

Does LP incur in the same or in similar difficulties? What we have, up to now, are two operators, $\vdash$ and $\dashv$, respectively, whose intended meanings express the illocutionary force of assertion and denial. This means that, in our pragmatic framework, we cannot *talk* about assertion or denial in general but we can only assert or deny contents expressed in the radical part of the pragmatic formula. We have no resources for formalising a sentence such as α. In light of the above fact, we propose adding operator $\mathbf{A}$ in radical formulas, whose intended meaning is "it is assertable that", *i.e.* a description of a possible assertion. The logical behaviour of $\mathbf{A}$ should be governed by the following principles:

IA $\vdash p \supset \vdash \mathbf{A}(p)$
EA $\vdash \mathbf{A}(p) \supset \vdash p$

The two principles are quite plausible; if there are good reasons to assert p, this is a good reason to assert that p is assertable. On the other hand, if there are good reasons to assert that p is assertable, then there are good reasons to assert p.

Let us consider K, a self-referential proposition that says of itself that it is non-assertable:

$$(K \leftrightarrow \neg\mathbf{A}(K)).$$

It is possible to derive that K is not assertable.

First horn:

(1)	$\vdash \mathbf{A}(K)$	Assumption
(2)	$\vdash K$	1, principle EA
(3)	$\vdash \neg\mathbf{A}(K)$	2, definition of K
(4)	$\dashv \mathbf{A}(K)$	3, property of denial
(5)	$\bot$	1, 4, pragmatic contradiction

Second horn:

(1)	$\vdash \neg\mathbf{A}(K)$	Assumption
(2)	$\vdash K$	1, definition of K
(3)	$\vdash \mathbf{A}(K)$	2, IA
(4)	$\dashv \mathbf{A}(K)$	1, property of denial
(5)	$\bot$	3, 4, pragmatic contradiction

In LP, augmented in the manner previously specified, a pragmatic assertability paradox can be formulated.

Let us consider whether a deniability paradox can also be formulated in our pragmatic framework. Consider operator $\mathbf{D}$ as we have done with operator $\mathbf{A}$, describing deniability in radical formulas. Let us consider the following principles:

ID $\dashv p \supset \vdash \mathbf{D}(p)$
ED $\vdash \mathbf{D}(p) \supset \dashv p$

Let M be the proposition expressing its deniability, that is, $(M \leftrightarrow \mathbf{D}(M))$.

First horn:

(1)	$\vdash M$	Assumption
(2)	$\vdash \mathbf{D}(M)$	1, definition of deniability
(3)	$\dashv M$	2, principle ED

Second horn:

(1)	$\dashv M$	Assumption
(2)	$\vdash \mathbf{D}(M)$	1, principle ID
(3)	$\vdash M$	2, definition of deniability

Horns are perfectly symmetrical; if we assume both ID and ED, we can prove that there are sentences not assertable and not deniable. However, let us notice that, in light of the failure of Frege's Thesis, the ID principle is problematic: ID allows for deriving an assertion from a denial. But, as we said previously, in our entire framework there is no symmetry between assertion and denial, as it is possible to derive the denial from the assertion but not *vice versa*. If we maintain this intuition, we cannot assume ID. But without it, it is easy to observe that there is no pragmatic contradiction within the second horn:

(1)	$\dashv M$	Assumption
(2)	$\dashv \mathbf{D}(M)$	1, definition of M

Thus, it seems that there is no paradox in our framework when denying M.

5. Conclusion

In this work, we have tried to logically analyse the illocutionary acts of assertion and denial. As we have seen, there are good reasons to think that they are relatively independent acts: Denial cannot be reduced to (a form of) assertion. The conceptual ground of this lies in the *asymmetry* that shapes the justification conditions of the pragmatic acts. Although the assertion of the negation of a proposition is sufficient for denying the proposition, the converse seems to be invalid. In other words, given the conditions of justification, in our framework, Frege's Thesis does not hold.

So, we have two illocutionary acts to which two fundamental epistemic attitudes can correspond: *acceptance* and *rejection*. In this way, we are close to Graham Priest's suggestion according to which, even if there may be *dialetheias*, it is impossible to accept and deny content. Therefore, we followed Priest's train of thought and defined a pragmatic contradiction as the conjunction of the assertion and denial of A ($\vdash A$ and $\dashv A$).

Equipped with these conceptual tools, we have explored Littman and Simmons's *assertability paradox*. In the original paper, the authors wanted to show a difficulty for the dialetheist: It is easy to construct a sentence that is both assertable and not assertable. But that would be impossible, even for a generous dialetheist. To catch in our framework the argument of the paradox, we needed to amend the underlying logic. In fact, we added a special operator $\mathbf{A}$, which allows for reflecting the act of asserting into the content of the assertion. It was easy, then, to yield sentence K stating its non-assertability: A pragmatic contradiction follows from the assertion of both K and $\neg K$.

Does the same hold for the denial? We have taken into account this by introducing, in a similar vein, a *deniability* operator ($\mathbf{D}$), which is supposed to work as its assertability counterpart. However, things are not so easy. To ensure the emergence of the paradox in that case, we had to assume principle:

(ID) $\dashv p \supset \vdash \mathbf{D}(p)$.

But ID *does* violate the asymmetry from which we started. From the denial of p, we can assert the deniability of p. No matter how this could appear reasonable, this is in accord with the intuition underlying Frege's Thesis.

So, it seems that, working on the radicals of the illocutionary acts paradoxes arise. But if we observe the structure of the logical proof of the assertability paradox, we can conclude that the contradiction occurs because, from an assertion, it is always possible to conclude a denial. The converse is not generally possible for less than assuming the controversial ID principle. So we can conjecture that the assumption of the symmetry between the justification conditions of assertions and denials explains the rising of paradoxes.

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