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ORIGINAL RESEARCH

On the non-substantiality of logic: a case study

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Abstract
One of the goals of the natural sciences– for example biology– is to provide new information about certain phenomena with previously unknown nature. Their contribution to our knowledge is substantial. From this perspective, logic is seemingly not substantial. Sometimes, logic's insubstantiality is taken for granted while explaining the alleged insubstantiality of other notions. For example, according to truth deflationism, truth is a non-substantial notion in the sense of being a logical property. However, it is not fully clear to what such an insubstantiality amounts. It is also debatable whether logic really is insubstantial. In this paper, we aim to clarify this issue by proposing a formal way of looking at it. In particular, we used the notion of *conservativity*, which has already been used by truth deflationism, for a similar aim. We show that if insubstantiality is read in terms of conservativity, then classical logic is substantial. We then argue that such a verdict of substantiality can be resisted if precise stances on certain *prima facie* unrelated issues of philosophy of logic are taken, or an anti-exceptionalist view is adopted.

Keywords Insubstantiality · Anti-exceptionalism · Conservativity · Deflationism

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1 Introduction

One of the goals of the natural sciences, e.g., biology, is to provide us with information and explanations about certain phenomena whose nature we did not know before. In general, scientific discoveries and experiences reveal new truths about the world to us. They provide substantive claims by excluding previously admissible genuine possibilities and explaining why certain worldly facts hold. Although empirical investigations are typical examples of substantial epistemic sources, a priori reflections could also be considered substantial in this sense; for example, if rational intuition (keeping its controversial status aside) offers access to new truths. Another example is mathematics, at least in some ways of understanding it. However, not all epistemic sources and disciplines seem substantial. Allegedly, logic alone does not provide substantial truths or explain worldly facts, at most, it makes the consequences of a certain theory or view explicit. At least *prima facie,* and unlike other forms of enquiry, logic says nothing about the world. On its own, it is unable to explain extra-logical facts. As Bencivenga puts it, logicians should pretend to be locked in a dark room without any access to and knowledge about the outside world, trying to evaluate sentences using only their linguistic competence (possibly limited to logical constants).[1] This peculiar insubstantiality makes the use of logic suitable as an impartial tool to assess mundane disputes in a neutral way because, supposedly, logic is neutral on substantial issues. The alleged insubstantiality of logic is also apparently connected to and reflected by various ideas voiced by prominent philosophers, including Wittgenstein's view that logical truths are *sui generis*, or the Kantian or Carnapian view that logic is the realm of analyticity. The conception of logic as non-substantial can also provide motivation for certain demarcations of logical terms from non-logical terms, such as invariance under arbitrary permutation. Finally, insubstantiality directly makes logic an exceptional form of enquiry, which has a special status not shared by other fields.

Admittedly, although the aforementioned sketch of logic probably sounds familiar and, at least at first sight, appealing, it is objectionably vague in many key points.[2] The very characterisation of logic as insubstantial needs more clarification before proper understanding and assessment. One may even wonder whether it really is onto something since the cluster of ideas surrounding insubstantiality prompts in different directions. Moreover, not all philosophers agree on the putative insubstantiality of logic and its discrepancy with other sciences. Opponents include Quine, who held that logic is not dissimilar from natural science; Russell, who claimed, 'Logic is concerned with the real world just as truly as zoology [is]',[3] and, in general, any realist who considers that logic is about the fundamental structure of the world.[4] In addition,

[1] Bencivenga (1999). See also Varzi (2014). The problem of how to demarcate logical constants is beyond the scope of this paper. In, on logical neutrality, see Carrara and Stollo (forth.).

[2] The very characterization and status of logical anti-exceptionalism is problematic, though. See Rossberg and Shapiro (2021).

[3] Russell (1993).

[4] Today, different versions of this view can be found in Maddy (2002), Sher (1991) and Williamson (2013, 2014).

anti-exceptionalism about logic, which is often discussed nowadays, includes various ideas that have apparently lent support to the substantiality of logic.[5]

In this paper, we intend to fix this unclear situation by proposing an account of the insubstantiality of logic, which may be precise enough to be formally evaluated and the discussion of which can help advance the dispute. To achieve this aim, we use the notion of *conservativity*, as already used by truth deflationism, for a similar aim. We argue that conservativity can also be used to model the insubstantiality of logic. Subsequently, we consider a well-known result in proof-theoretic semantics– the non-conservativity of classical negation– as a case study. We show that if insubstantiality is read in terms of conservativity, classical logic, under seemingly unproblematic assumptions, turns out to be substantial against the intuitive characterisation of logic given at the beginning of the paper. We then argue that such a verdict of substantiality can be resisted if precise stances on specific, apparently unrelated issues of the philosophy of logic are considered (such as the role of formalisation and controversies about multiple/single conclusions and semantic atomism/holism). Hence, whether logic is substantial depends on which other theoretical views on the nature of logic are embraced.

2 Insubstantiality and conservativity

Despite the resistance of some philosophers, the insubstantiality of logic is sometimes taken for granted to the point of being invoked for explaining the insubstantiality of other notions. For example, to mark the deflationary conception of truth, deflationists claimed that truth is a *sui generis*, non-substantial property, because deflationary truth is a *logical* notion.[6] To further clarify the insubstantiality at stake, deflationists soon resorted to the formal notion of *conservativity*.[7] Intuitively, a theory T is conservative over a base theory B if it does not prove, in the language of B, anything that was not already provable by B alone. More formally, a theory T in the language L_T is (proof-theoretically) conservative over a base theory B in the language L_B if for every sentence φ in L_B, if $B \bigcup T \vdash \varphi$, then $B \vdash \varphi$.[8] According to the proposal, a theory of deflationary truth should be conservative over any (non-semantic) base theory if deflationary truth has to be considered insubstantial.[9] Conservativity has also been used for similar purposes to support deflationist views in metaphysics and ontology.[10]

[5] Hjortland (2017) introduced logical anti-exceptionalism in a recent debate.

[6] Instances of these claims can be found in Field (1999, p. 76) and Horwich (1998, p. 2–5).

[7] Horsten (1995), Ketland (1999) and Shapiro (1998) are the starting points. Many other authors contributed to the debate until today, as witnessed by Waxman (2017) or Fujimoto (2022).

[8] Conservativity comes in two non-equivalent versions, a proof-theoretic and a model-theoretic one. The two notions have different features, and, in particular, model-theoretic conservativity is stronger than proof-theoretic conservativity. A typical example of this comes from the field of axiomatic truth theories, where it is well known that the axiomatic truth theory of compositional truth without full induction (CT-) is proof-theoretic conservative but not model-theoretic conservative over PA. See Halbach (2011).

[9] For a critical and comprehensive survey, see Cieśliński (2017).

[10] Prominent examples are Schiffer (2003) and Thomasson (2015). See also the classic Field (1980) for a further use of conservativity.

We believe that the deflationist strategies of using conservativity can be taken on board with similar motivations to handle the very case of logic, which was the initial trigger of those attempts. The connection with logic can indeed be rendered straightforward to the point of looking like a mere reformulation, in more technical terms, of the intuitive opening characterisation. The train of thought goes as follows: Logic says nothing about the world, and it does not explain extra-logical facts. Logic, if insubstantial, can help articulate a worldly view, but it does not add constraints to what such a view is about. For its extra-logical part, any theory is conserved the way it is. Extralogical claims, in turn, can be identified, for the sake of simplicity, with those made by means of atomic sentences, for they do not contain logical vocabulary. This is a drastic choice; however, it allows us to avoid difficult questions, such as 'Are compounds only about the world or also about logic?'.[11] Given such an identification, insubstantial logic does not prove new atomic sentences. (Let us call such a property 'atomic conservativity'.)[12] Steinberger expressed the very same idea while speaking of a principle of innocence according to which 'logic alone should not be a source of new information. That is, it should not be possible, solely by engaging in deductive reasoning, to discover hitherto unknown (atomic) truths about the world that we would have been incapable of discovering (at least in principle) independently of logic.'[13]

Similar to the case of deflationary truth, the connection between conservativity and insubstantiality can be reinforced by the following considerations[14]: Suppose that the addition of the logic T (call such an extending logic 'the *target* logic') to a base theory B in the base language L_B implies new atomic sentences in the language L_B. Let P be this newly derived atomic sentence. Before the extension, B was compatible with both being the case that P and not being the case that P. After the extension, instead, P is established, and a commitment to a new fact is incurred. The newly obtained claim that P constrains reality forcing the admission of an extra-logical fact that did not have to obtain otherwise. T is then capable of extra-logical implications, because claims about non-logical B facts would then be derived. T enriches the view of B by requiring further L_B truths to hold. In this sense, a non-conservative logic is non-neutral with respect to theory B because it extends the body of information on substantial issues. In contrast, an insubstantial logic should be neutral with respect to such facts and not prove any new atomic sentence in the language L_B of the base theory. Conservativity can also be connected to a lack of explanatory power, which is another chief way of clarifying insubstantiality in the deflationist camp. If a target logic is atomic conservative, then it can be dispensed in the derivation of any atomic

[11] If we did not restrict our attention to atomic sentences, our case for insubstantiality would be easier, since we could work with the standard notion of conservativity, rather than with the stronger notion of *atomic* conservativity.

[12] Depending on whether identity is treated as a logical notion, atomic identity sentences can be excluded. In this paper we focus on a propositional language, though, so that the problem does not arise.

[13] Steinberger (2009, p. 60). Note that the apparently obvious complication for which a worldly theory usually already involves logic– so that a base theory and a logic might not be separable– can be easily handled by a proper formulation of the requirement and it will be dealt with below.

[14] The following argument is close in spirit to the one offered by Shapiro (1998) with respect to the substantiality of the property of truth when its theory is not conservative over PA.

sentence. Atomic sentences can only be provable in base theory.[15] Accordingly, such logic does not contribute to any essential explanation of extra-logical claims.

Admittedly, conservativity has already been considered a requirement for logic, but for different purposes. The clearest example is provided in debates on proof-theoretic semantics, in which conservativity was used in a first attempt at elucidating the notion of *harmony* to demarcate logical constants.[16] Intuitively, harmony consists of a desirable balance between the introduction and elimination rules of logical constants. In other words, what can be inferred from a logically complex sentence by the elimination rules for its main connective is no more and no less than what needs to be established to infer that logically complex using the introduction rules for its main connective. While the informal characterisation of harmony is quite intuitive, its formal characterisation proves more difficult. In a first attempt, harmony was clarified exactly in terms of conservativity; however, in general, conservativity is now commonly recognised as neither a sufficient nor a necessary condition for harmony.[17] However, this initial resort to conservativity to capture the nature of logicality is particularly natural if understood on the implicit assumption of the insubstantiality of logic. In this spirit, for example, Murzi writes that, ‘One motivating thought behind the requirement of harmony is that logic is innocent: it shouldn’t allow one to prove atomic sentences that we couldn’t otherwise prove’.[18] This shows that, despite its limits in the context of proof-theoretic semantics, the link between conservativity and insubstantiality should not be dropped. Although conservativity can be ill-suited to clarify the notion of logicality, it can still be well-suited to clarify the insubstantiality of logic. Accordingly, in this paper, we make use of the notion of conservativity in a way that is inspired by, but is different from, the one originally made in the proof-theoretic semantics camp. Indeed, once the issue of logicality is separated from that of insubstantiality, failure to satisfy conservativity does not necessarily exclude a logic, it only makes it substantial.

We acknowledge that such a connection between conservativity and the substantiality of logic has already been drawn and briefly discussed by some deflationists and logicians working on formal theories of truth.[19] In particular, they suggested that con-

[15] It should be kept in mind that the base theory also has a logic (the *base* logic). Thus, in the base theory, atomic sentences can already be derived if they are axioms of the theory, or by means of the base logic. *New* atomic sentences, in contrast, would require use also of the target logic extending the base.

[16] For extensive critical surveys, see Hjortland (2009) and Steinberger (2009). On harmony, seminal works include Gentzen (1934), Popper (1947), Belnap (1962), Prawitz (1974), Hacking (1979), Došen (1989) and Dummett (1991). See Carrara and Murzi (2014) for an overview. Harmony often goes with some sort of inferentialism, as defended, for example, by Gentzen (1934), Dummett (1991), Tennant (1997), Brandom (2000), Boghossian (2012) and Prawitz (2012). See also Jacinto and Read (2016) and Dicher (2016).

[17] There are apparently harmonious rules that are not conservative, and conservative rules that are not harmonious. Another discrepancy between harmony and conservativity is that the former is a local requirement whereas the latter is a global one (Prawitz, 1994).

[18] Murzi (2020, p. 394). See also Steinberger (2009) for similar considerations.

[19] Horsten (2009), Galinon (2015), Picollo and Schindler (2018), Fujimoto (2019). Some of these authors mention various facts to support the non conservativity of logic. Beside the one related to classical negation, on which we focus in this paper, they also mention higher order logic (Galinon, 2015; Picollo & Schindler, 2018) and Peano Arithmetics (PA) with induction restricted to atomic formulas (Galinon, 2015). The case for higher order logic can be taken on board corroborating our project, since the logicality and

servativity could not be a good indicator of the insubstantiality of deflationary truth because, otherwise, proof-theoretic results would show that logic itself is substantial, which, they claim, is not the case. However, remarkably, these authors neither offer arguments to support the non-substantiality of logic, which is just taken for granted, nor do they put forward a systematic analysis of the mentioned results. In this paper, we fill these gaps and move in the opposite direction. On the one hand, we present a more articulated and regimented discussion of the topic, formulating it in the context of a possible general account. On the other hand, we question and explore the consequences of rejecting the alleged non-substantiality of logic. By eventually arguing, in an anti-exceptionalist stance, that logicality and insubstantiality need not go together, we contend that substantiality does not necessarily exclude logicality. Contrary to what these authors claim, logic can be substantial.

3 Formalising insubstantiality

Given the previously discussed general motivations, we now elaborate on a formal framework to implement the idea that a non-(atomic) conservative logic is substantial. In other words, if a logic is insubstantial, it must be atomic conservative for all base theories (suitable to a chosen target logic).[20] After this technical intermezzo providing various formal details, we return to a more philosophically oriented discussion in Sect. 7.

In general, we assume a single-conclusion natural deduction framework with a self-contained formulation of the rules of each logical constant. This choice has various motivations. On the one hand, it conforms to a quite standard textbook presentation and, as such, can be taken as a well-established and not particularly contentious approach, making it suitable as a starting point for drawing at least *prima facie* philosophical morals. On the other hand, it allows for a sensible formulation of the criterion (as illustrated subsequently). However, the choice is not without consequences. Choosing a particular formulation and an underlying conception of logic have a notable bearing on substantiality. Enlightening these implications is a task for the second, more philosophical part of the paper. From this perspective, the initial assumptions can also be taken as conditions that qualify the range of the results. As shown in the second part of the paper, opting for different frameworks would lead to different but equally limited results. A final general upshot of the paper is that the issue of logic substantiality is deeply entrenched in several other seemingly independent and non-trivial issues, which appears only as a part of a more complex view in the philosophy of logic.

substantiality status of higher-order logic is already very contentious. The example of PA is more problematic, since it is not a case of a theory, with a poorer language, that has a non conservative extension in a richer language. PA, with and without full induction, are formulated in the same language and only have different axioms. Thus, its role in the present debate is not very clear. However, if the case of restricted PA could be properly developed, it would also provide further arguments for our claim that logic is arguably substantial.

[20] Subsequently, we lay down several constraints that the base theory must meet to be a sensible base relatively to a certain target logic.

For the sake of simplicity, we limit our treatment to propositional logic. We also assume that every base theory is equipped with a standard Tarskian structural consequence relation $\vdash$, holding between a finite (possibly empty) set of sentences (the premises) and a sentence (the conclusion) of a certain propositional language L. $\vdash$ embodies a multiple premisses-single conclusion relation. More formally, given the set $Sent_L$ of sentences of L, $\vdash$ is a relation $\vdash \subseteq P(Sent_L) \times Sent_L$, satisfying, for every $X, Y \subseteq Sent_L$ and $\varphi, \psi \in Sent_L$, the following constraints: reflexivity (if $\varphi \in X$, then $X \vdash \varphi$), monotonicity (if $X \vdash \varphi$ and $Y \subseteq X$, then $X \vdash \varphi$) and transitivity (if $X \vdash \varphi$ and, for every $\psi \in X$, $Y \vdash \psi$, then $Y \vdash \varphi$). In sequent calculus, if the calculus is formulated with ordered sequences instead of sets of sentences, we have the standard structural rules. In particular, we have identity, left weakening, left contraction, left exchange, and cut. Where multiple conclusions are admitted, the definition should be modified accordingly and extended with these structural rules for the right side as well.[21] We omit a full presentation of the calculus for space reasons.

Let us define a *base theory*. Intuitively, a base theory is given by closing, according to a certain logic $\vdash_i$, a set of axioms in a propositional language. Thus, a base theory is just a formal theory in standard form. In particular, the logic $\vdash_i$ can extend the Tarskian relation by axioms or rules for the set of logical constants, and the set of axioms can include sentences of any complexity. More formally, a base theory is formulated in a base propositional language L_B inductively defined using a set of propositional variables {P, Q, R…} and a (possibly empty) set of logical constants C_B. The *base logic* $\vdash_B$ is a logic resulting from a consequence relations together with rules for the logical constants in C_B as usual. Thus, when admitted, the logical constants, together with the rule governing them, yield the extension of $\vdash$ to $\vdash_B$. If the set of logical constants C_B is empty, $\vdash_B$ is just $\vdash$. Let then A_B be a (possibly empty) set of sentences in L_B. A_B are the *axioms* of the base theory. Note that the axioms can be logically complex sentences in L_B. The *base theory* B is the result of the closure of A_B under $\vdash_B$. Thus, B can be conveniently identified with the pair $<A_B, \vdash_B>$. If the set of axioms is empty, a logical theory $<\emptyset, \vdash_B>$ is obtained, namely $\vdash_B$. If the set of logical constants is also empty, B is $<\emptyset, \vdash>$, namely $\vdash$.

It is noteworthy that the base theory already involves a logic, $\vdash_B$, which can also include logical constants with their own rules or axioms. This base logic is not the logic we want to assess with respect to conservativity (the target logic); however, it is part of the theory over which conservativity is assessed. A base theory already having a logic is not a problem, provided that the base logic and the target logic are different. The target logic will always be taken to be different from the base logic, as the target logic can only involve logical constants absent in the base logic, without overlap.[22] Thus, the logical constants of the base logic cannot be modified by axioms or rules added to the target logic. The case in which a target logic modifies or extends rules or axioms for logical constants in the base language is ruled out. This is important to

[21] In natural deduction, structural rules are usually embodied in the derivation rules. For more details on structural rules, see Negri and Von Plato (2001) and Paoli (2002).

[22] The only exception is when both the base logic and the target logic are the mere Tarskian relation, or empty logic, $\vdash$, since in this case the requirement that the the intersection of the sets of logical constants in the two logic be empty is met. However, this is not a problem and indeed a convenient trait.

keep in mind to resist the temptation of dismissing the proposal under the light of too hasty counterexamples. We discuss one of them below.[23] To follow these points and make the base logic suitably separable from the target logic, rules or axioms governing logical constants should be self-contained. The base theory must have an independent formulation from the rest of the target system. Such separability constrains the formulation of formal systems eligible for the proposed treatment.

Let us now turn to the *target theory*, that is, the logic that will be assessed for conservativity. Similar to the base logic $\vdash_B$, the target logic $\vdash_T$ is obtained by specifying a set C_T of logical constants, such that $C_T \cap C_B = \varnothing$, together with axioms or rules governing them, extending $\vdash$. Then $\vdash_T$ is formulated in the language L_T. Since our attention is limited to logics– not to conservativity of theories in general– the set of extra-logical axioms for the target theory can be assumed to always be empty. Given that the target theory T coincides with the pair $< \varnothing, \vdash_T >$, it is just a logic and can be indicated directly with $\vdash_T$. Note that the requirement that the constants in the target logic and the constants in the base theory be disjoint, namely $C_T \cap C_B = \varnothing$, is due to the requirement that the rules of the base logic and those of the target logic must govern different connectives.[24] Because we consider extensions of the base logic with the target logic to assess conservativity, it is often convenient to refer to the target logic as the one resulting from $C_B \bigcup C_T$, namely $\vdash_{B \bigcup T}$, rather than just $\vdash_T$. As long as $C_T \cap C_B = \varnothing$ holds, this abuse of terminology is harmless.

Given these specifications and reading insubstantiality in terms of conservativity relative to atomic sentences, we say that a (target) logic $\vdash_T$ is insubstantial for all base theories B in L_B and all *atomic* sentences φ in L_B, and if B $\vdash_{B \bigcup T}$ φ then B $\vdash_B$ φ. Intuitively, this means that the extension of any base theory to target logic does not prove new atomic sentences. With respect to atomic sentences, the extension of the target logic is powerless. This reflects the idea that an insubstantial logic does not explain any worldly fact. In line with the previous informal characterisation, we call this *atomic conservativity* in short. It is important to keep in mind that a base theory may have non-logical axioms as well; hence, it may be able, on its own, to prove atomic sentences. The point is that enriching its logic does not prove *more* atomic sentences.

To help grasp how the proposal works, let us sketch some examples of right and wrong applications of the approach. First, suppose we introduce a base theory whose base logic includes a connective ∧ with no rules, and whose theory only includes as axiom (P ∧ Q). Then, B is $< \{(P \wedge Q)\}, \vdash_\wedge >$. Now, extend this theory by the target logic $\vdash_{\wedge\text{-elim}}$ obtained by adding a new rule for ∧-elimination, mimicking conjunction elimination. In the extended theory, one can easily obtain P, which was not provable in the base theory. Thus, the target logic is not conservative. In such a situation, a question arises: Is that a case of non-atomic conservativity as we defined it? The answer is: No. The problem is that ∧ belongs to both the base theory and the target theory, that is, the extending logic. The target logic does not add new logical con-

[23] As Belnap (1962) did, we also assume a uniqueness requirement to avoid mere notational variants for the same inferential role.

[24] Being completely disjunct is not really necessary, since the target logic could also be formulated as an extension of the base logic. However, this is irrelevant for conservativity.

stants with their rules, but rather rules/axioms for old ones (∧). This is a violation of the requirement that the set of connectives in the base and in the target logic be disjoint. This clause is exactly intended to prevent the target logic from tampering with the base logic. In general, it is important to stress that a case in which a base logic has a certain connective and the target logic adds a rule for it is blocked as not permissible.

Here is another example: one can easily check that the target logic of classic conjunction (namely a logic in which conjunction is the only logical constant and is governed by the classical rules of introduction and elimination), call it $\vdash_{\&}$, is atomic conservative over a base theory with only one propositional constant as axiom and the mere Tarskian consequence relation as logic. Hence, $\vdash_{\&}$ is atomic conservative over $<\{P\}, \vdash>$. In contrast, let *Tonk* be a connective characterised by the following rules:

Tonk introduction: If $\Gamma \vdash \varphi$ or $\Gamma \vdash \psi$, then $\Gamma \vdash \varphi$ *Tonk* ψ.

Tonk elimination: If $\Gamma \vdash \varphi$ *Tonk* ψ, then $\Gamma \vdash \varphi$ and $\Gamma \vdash \psi$.

Then, over the same base theory, the logic of Tonk, $\vdash_{\text{Tonk}}$ is not atomic conservative.[25] In addition, note also that this formulation respects the aforementioned constraints because the target logics, $\vdash_{\&}$ and $\vdash_{\text{Tonk}}$, do not extend the rules/axioms for logical constants in the base theory. Apart from indicating how the approach works, these examples indicate the *prima facie* correctness of the verdicts. We arguably expect classical conjunction to be insubstantial and *Tonk* to be substantial because of such anticipated outcomes.

4 Non-conservativity: a case study

We can now proceed with our case study in terms of the non-conservativity of classical logic. Because of debates in proof-theoretic semantics, classical negation is known not to be conservative over the logic generated by the other connectives (call it the 'negation-free fragment'). Namely, the extension of *classical logic without negation* with *classical negation* produces a non-conservative extension of the former. In the present context, this leads us to deem classical logic substantial. Before commenting on this result, let us express it in terms of the present framework and extend it to non-atomic conservativity.

The non-conservativity of classical logic can be rendered as follows: Let the base theory *NF* (for *Negation-Free)* be the logic generated by the classical connectives except negation. Thus, the language of *NF* is a propositional language with an infinite stock of propositional variables and the logical constants $C_{NF} = \{\wedge, \vee, \rightarrow, \bot\}$. Since we just consider a logic, $\vdash_{NF}$, the set of (extra-logical) axioms A_{NF} is empty, and the logic $\vdash_{NF}$ is obtained by associating rules with the logical constants in C_{NF} (with $\bot$ be a 0-ary constant with no rule governing it). The rules for these connectives (which do not include negation) are the usual intuitionistic and classically acceptable rules. The base logic, $NF = <\varnothing, \vdash_{NF}>$ or $\vdash_{NF}$, is the negation-free logic obtained by the

[25] Suppose that the axiom of the base theory is P, then by Tonk-introduction P *Tonk* Q is obtained, and by Tonk elimination Q is derived. On *Tonk*, see Prior (1960).

logical constants governed by standard rules, minus negation.[26] Note that $\bot$ is treated minimally; it is not governed by any rule, and in particular, it does not imply everything. Thus, the negation-free fragment obtained is minimally, intuitionistically and classically acceptable. Having a common negation-free fragment is convenient for assessing different extensions of the same base. We discuss some of the implications of choosing different rules in Sect. 7.[27]

The target logic, $C_\neg$ or $\vdash_{C\neg}$, is instead the logic of classical negation, which yields full classical logic when conjoined with the base logic $\vdash_{NF}$. $C_\neg$ or $\vdash_{C\neg}$ is obtained by posing the set of logical constants $C_{C\neg} = \{\neg\}$ with the classical rules of negation (see the Appendix). $C_\neg$ (namely $\vdash_{C\neg}$) can be proved to not be conservative over *NF* (namely $\vdash_{NF}$), because the addition of classical negation to the negation-free fragment allows proving sentences in the negation-free fragment language that are not derivable by the base logic $\vdash_P$ alone.[28] In particular, Peirce's law: $((\varphi \rightarrow \psi) \rightarrow \varphi) \rightarrow \varphi)$, while formulated in the language of the base theory, is only provable once classical negation is available, given that the derivation involves the application of *reductio ad absurdum*, which is enabled by classical negation.

Because the instances of Peirce's law are not atomic, the previous result does not indicate a failure of atomic conservativity. However, such a failure is immediate once a base theory L, formulated again in the negation-free fragment, is given (an instance of) the antecedent of Peirce's law as an axiom, that is, $L = < \{((R \rightarrow Q) \rightarrow R)\}, \vdash_{NF} >$. The extension of L with the logic of classical negation $\vdash_{C\neg}$ proves the relevant instance of Peirce's law $((R \rightarrow Q) \rightarrow R) \rightarrow R$, (not provable in $\vdash_{NF}$), and by *modus ponens*, R is derived.

Since the previous result seems to show that the logic *par excellence*– classical logic– is not always atomic conservative, and, as argued, atomic conservativity is necessary for insubstantiality, a puzzle emerges. The atomic non-conservativity just displayed is also enough for our purpose. The example is clear and striking enough to provide a solid basis and precise material for philosophical elaboration. This is the main case study on which we intend to focus to assess the general claim of the insubstantiality of logic. Since the result is well known and widely discussed in the context of proof-theoretic semantics, we skip the formal details, postpone some of them to the final Appendix and keep focusing on the philosophical point that concerns us.[29]

Some preliminary clarifications are in order. Although classical logic is enough to make our point significant, it can be useful to briefly gesture towards possible

[26] These rules are those given in standard textbooks, for example, in Chiswell and Hodges (2007).

[27] One might object that since negation could be introduced in terms of implication and $\bot$, it is somehow present in the language. This is certainly true if a new symbol for negation were introduced accordingly, but that would yield an extension of the language as well. If no new symbol is introduced, however, the implicit presence in the language should be clarified. Indeed, given that in minimal logic $\bot$ is not governed by any rule and is akin to a mere propositional variable, the objection would look puzzling, since it would make a negation-free fragment seemingly impossible. Hence, to go through, the objection should be turned into a substantial thesis about the meaning of connectives, their relation with how logics are formulated and their implicit presence in a language. The essence of our paper, however, is precisely to argue that resisting non-conservativity results demand adopting stances that are far from trivial.

[28] This result can be found, for example, in Dummett (1991, p. 271). See also Read (2000) for further discussion.

[29] The result is mentioned, for example, in the introductory textbook by Chiswell and Hodges (2007).

extensions of the treatment of logics besides classical logic. Let us consider intuitionistic logic. A conservative extension of the negation-free fragment *NF* is obtained if intuitionistic rather than classical negation is added. In other words, let the target logic $I_{\neg}$, or $\vdash_{I\neg}$, be obtained by the intuitionistic rules of negation (see Appendix). $\vdash_{I\neg}$ conservatively extends (and thus also atomically extends) the negation-free fragment introduced previously, that is *NF*. In particular, the resulting conservative extension $\vdash_{NF \bigcup I\neg}$ just yields intuitionistic logic. The discrepancy between classical and intuitionistic negations led Dummett to dismiss classical logic in favour of intuitionistic logic. However, in the present context, in which atomic conservativity over extra-logical base theories is also considered, such considerations on behalf of intuitionistic logic may be too quick, as shown by the following case. Consider the theory *F* obtained by posing falsity ⊥ as axiom and closed under $\vdash_{NF}$. In other words, $F = < \{\bot\}, \vdash_P >$. Remember that ⊥ is a logical constant with no rule governing it. Let us then extend this theory with the logic of intuitionistic negation $\vdash_{I\neg}$. The result is the theory with ⊥ as axiom closed under intuitionistic logic. Since *ex falso quodlibet* is intuitionistically, but not $\vdash_{NF}$ (or minimally) valid, any propositional constant can be derived and atomic conservativity is lost. It follows that intuitionistic logic is also substantial. The culprit can now be spotted in the *ex falso quodlibet* principle.[30] This principle allows for extracting more than has been initially accepted, allowing one to start with certain premises to obtain a completely unrelated sentence. Interestingly, the substantiality results obtained so far emerge from principles such as *excluded middle* (embodied by classical negation) and *ex falso quodlibet* (embodied by intuitionistic negation), two logical laws that originated several debates for their philosophical role and dubious metaphysical status. Their non-conservativity sheds new light on them.

Since in minimal logic neither *excluded middle* nor *ex falso quodlibet* hold, the two cases leading to non-conservativity of classical and intuitionistic logic cannot be replicated in minimal logic. Thus, minimal logic seems to be a better candidate for insubstantiality.[31] However, if tampering with structural rules is also permitted, minimal logic may be non-conservative.[32] The reason is that once one goes sub-structural, further non-conservativity results emerge and the number of substantial logics arguably increases. This would drag into a situation in which only some very weak substructural logics, if any, may be insubstantial.[33] The status of structural rules and whether structural rules should be modified or dropped are critical issues that we do

[30] In both our formulations of minimal logic and intuitionistic logic, ⊥ does not imply everything directly. In contrast, in intuitionistic logic, but not in minimal logic, we have that a contradiction implies everything. However, in both, the link with negation is retained by adopting the usual abbreviation of ‘$(\varphi \rightarrow \bot)$’ as ‘$\neg\varphi$’. In the presence of negation in the language, this allows to turn ⊥ into a contradiction, and thus to have that ⊥ indirectly implies everything in intuitionistic logic (via contradiction), but not in minimal logic. For a precise presentation of the rules, see the Appendix.

[31] For a defence of minimal negation, see Kürbis (2019).

[32] For non-conservativity results about relevant logic, see Mares (2000), Øgaard (2020a,b,c). See also Core logic by Tennant (2017), which is intended to be both relevant and constructive. Note that in Core logic, the structural rule of transitivity is restricted. For general connection between natural deduction and sequent calculus, see Negri and Von Plato (2001).

[33] For example, without the structural rule of contraction (equivalent to multiple discharge of assumptions in natural deduction), one cannot derive sentences like $(R \rightarrow (R \rightarrow Q)) \rightarrow (R \rightarrow Q)$, which would be

not probe here. For the sake of the argument, we assume that the slippery slope can be blocked by not going sub-structural and that at least a logic, say minimal logic, is an atomic conservative and insubstantial logic. Of course, even keeping substructurality aside, a demonstration of the insubstantiality of minimal logic would require proof of its conservativity over all (suitable) base theories. Due to a lack of such proof, the issue remains unsettled. However, we leave the issue of insubstantiality of minimal logic open, put the question aside and assume it to be insubstantial. We can do that for the sake of debate, because, for our purposes, we do not need minimal logic to be insubstantial (or substantial). In this paper, in fact, we neither aim to give a complete classification of various logics nor to provide a systematic enquiry of the conditions under which most logics are conservative or not. Such a study would be beyond the scope of this paper. Rather, our purpose is to focus on the philosophical significance of an approach based on conservativity to investigate the alleged insubstantiality of logic. To achieve this goal, regimenting a single quite well-known case study (the non-conservativity of classical negation), which has already been the topic of similar but underdeveloped considerations, seems to be sufficient and an efficient way to proceed. That the logic *par excellence*, classical logic, comes out as substantial in a specific case is, for us, a striking fact worthy of philosophical attention. If the approach is already insignificant in this case study, more systematic studies might be worthless. Thus, extending the analysis to further logics is an important task left for future research.

Related to this point, one might nonetheless worry that having a range of variously classified logics is crucial for displaying a promising project. After all, a useful criterion should arguably be met by some, but not all, logics. One may even be worried that if nothing satisfies the criterion and no logic is atomic conservative, then nothing counts as logical. However, these natural concerns are misplaced in the present context. To dispel them, we should remind ourselves and emphasise that we are using conservativity not to mark logicality (as done in proof-theoretic semantics), but to characterise the possible substantiality of logic. The two coincide only if the idea that logic must be insubstantial is accepted. If such a thesis is not assumed from the beginning and theoretical space for logic to be substantial is made, then the possibility that all logics are substantial is no longer absurd. From a neutral standpoint, all, none, or some logics might turn out to be substantial. The criterion would then not be undermined by any of these outcomes. At any rate, since the most interesting case is probably one in which some, but not all, logics are substantial, we hypothesise that at least one logic is insubstantial. (This is why the insubstantiality of minimal logic is assumed just for the sake of debate.) However, the main relevant point is what to do with a verdict according to which a venerable logic, such as classical logic, is substantial. To discuss this, the non-conservativity of classical negation over the negation-free fragment is a sufficient case study, and in the rest of the paper, we restrict our attention to it. The implications of separating logicality and substantiality are explored further in the sections that follow, especially in relation to logical anti-exceptionalism.

derivable once extensions with structural logics are considered. See Hjortland and Standefer (2018) for discussion of similar results.

5 Philosophical consequences

If substantiality is modelled and formally clarified using conservativity, a prominent logic, such as classical logic, turns out to be substantial. This outcome can be resisted in various ways. However, and noticeably, rejecting or accepting the insubstantiality of logic revealed by conservativity requires taking certain stances on specific, *prima facie* unrelated issues in the philosophy of logic. Overall, the following discussion concerns the interplay of the (in)substantiality of logic with other topics, witnessing the fertility of an analysis moving from conservativity.

Before exploring the philosophical implications of the non-conservativity of logic, let us briefly compare the issue regarding logic with that regarding truth. As mentioned earlier, conservativity has been routinely employed in a formal context to shed light on the alleged insubstantiality of deflationary truth. In the case of the conservativity of truth, a quite standard and shared approach is as follows: usually, a base theory including enough theory of syntax (typically Peano Arithmetic) is extended by adding various axioms for a truth predicate, and conservativity is assessed for such extensions. In all of this, a certain logic, usually classical logic, is assumed and left untouched. In contrast, as the various remarks in this section will also make clear, a similarly fixed approach to the conservativity of logic does not seem possible. Rather, there is a variety of systems whose conservativity depends on subtle choices regarding both the base and the target theory and in which the underlying logic is repeatedly modified. Given this discrepancy, the analogy between truth and logic may seem superficial. The situation is quite complex, though, since, also in the case of truth, different base theories and logical consequences have been considered, including, for example, set theory, second-order logical consequences and free logic.[34] Moreover, also in the case of truth, the interplay between a base theory and a target theory gives rise to several complications, depending on how the systems and the extensions are formulated. Examples are provided by whether disquotational axioms are taken in a 'uniform' form (namely, with universal quantification binding variables across truth ascriptions). In both cases, conservativity is affected. For example, the theory yielded by (positive type free) biconditionals, PTB, is conservative over PA, but the uniform version, PUTB, is not.[35] Indeed, it is not surprising that when logic is assessed, logical principles are modified, whereas when truth is investigated, truth principles are mostly modified. In the case of logic, the critical issue is how logic is formulated, whereas in the case of truth, the main issue is how a truth theory is formulated. Thus, the apparent differences between the debates on truth and logic seem to exist because different phenomena are under scrutiny.

Let us then focus on the philosophical consequences of logic and consider possible strategies that can be adopted to resist the substantiality result. We begin by discussing a radical criticism addressing the very explanation of substantiality in

[34] Shapiro (1998); Hyttinen and Sandu (2004); Fujimoto (2010, 2012); Heylen and Horsten (2017).

[35] See Cieśliński (2011) for the result about PTB, and Halbach (2009) for the result about PUTB. Something analogous, but with respect to model theoretic conservativity, also holds in the simpler case of non iterated truth ascriptions (TB and UTB), since in the presence of induction, uniform disquotational biconditionals (UTB) restrict the class of models of PA in a different and stronger way than 'non-uniform' biconditionals (TB) do, as illustrated in Strollo (2013) and Cieśliński (2017, Ch. 6).

terms of conservativity. To safeguard the innocence of classical logic, one's most direct reaction is that of attacking the proposed formal account, since, after all, it is not mandatory to model insubstantiality in terms of conservativity. Accordingly, the verdicts of substantiality can be dismissed by rejecting the characterisation in terms of conservativity. However, to adopt this strategy, reasons supporting the rejection of the proposed criterion should be offered. Such a rejection cannot be motivated by the mere desire to keep the insubstantiality of logic, since whether logic is substantial is exactly the issue, and pre-supposing insubstantiality begs the question. Moreover, since conservativity has been adopted in terms of positive considerations, one should show that a logic can be insubstantial, despite giving rise to non-conservative extensions in certain cases. The question is: By rephrasing Shapiro, how can certain logics be insubstantial if they enable the derivation of new extra-logical claims? Due to a lack of such motivations, we move on to a second possible objection.

Lacking positive reasons to reject the role of conservativity in accounting for insubstantiality, one may accept it while trying to undermine its significance. A first radical option can be that of de-classifying conservativity results as mere artefacts. This can be implemented by appealing to the fact that conservativity results are highly dependent on how a logical theory is formulated, that is, what the formal system is like. This is clear in the case of the non-conservativity of classical negation on the negation-free fragment, because, in the context of proof-theoretic semantics, it is well known that such a result vanishes once a multiple-conclusion consequence relation, rather than a single-conclusion one, is adopted.[36] Basically, by taking advantage of multiple conclusions and their structural rules (to speak in a sequent setting vocabulary), it is possible to prove Peirce's law directly in the negation-free fragment, even without negation.[37] This countermove is naturally appealing and powerful; however, its strength should not be overestimated. First, one can reverse the accusation, insisting that what is an artefact of the presentation is the conservativity result delivered by certain versions. To this, the opponent may rejoin by holding that, depending on formulations, all results are artefacts: both conservativity and non-conservativity should be deemed mere by-products that do not signal anything philosophically deep. However, such a rejoinder is questionable. On the one hand, some formulations can be standard, independently better motivated and arguably more natural than others. Hence, it is at least questionable that all formulations are on a par. On the other hand, certain properties of a logic seem to be revealed only under some formal presentations, and insubstantiality can be one of these. For example, consider the usability of a logic in actual reasoning. Natural deduction is usually regarded as a good choice for finding proofs and deriving conclusions, whereas axiomatic systems can be cumbersome for this task. However, the practicality of classical logic can hardly be declassified as a mere artefact of the presentation, because it is revealed only under some formulations. This way, the issue of logical substantiality leads to the general themes of how to define a logic, the relation between a logic and its formalism and, more in general, between a theory, its presentation and its subject matter. We cannot enter

[36] Since a multiple-conclusions relation is more easily modelled in a sequent calculus than in natural deduction, the point is usually made in terms of sequents.

[37] The result relies on the application of right weakening and contraction. See Read (2000).

these subtle issues here, but for our purposes, it is enough to note that there is an intimate connection between the (in)substantiality of logic and other specific problems in the philosophy of logic.

The dependence of conservativity resulting from specific formulations may also suggest another strategy. Since conservativity, and thus insubstantiality, is lost because of having single-conclusions, one can avoid the outcome simply by adopting a multiple-conclusions formulation. However, this natural strategy is problematic. First, it can be pointed out that, at least in natural deduction, a system with a single-conclusion consequence relation is arguably the default and standard choice. Second, since insubstantiality requires conservativity over every base theory, the crucial point is not really whether multiple conclusions are legitimate, but whether a base logic formulated in a single-conclusion consequence relation is also permissible. As long as such a base theory is acceptable, the non-conservativity result undermines the insubstantiality of classical negation. To avoid this outcome, the acceptance of multiple conclusions is irrelevant, because what is needed is the rejection of single-conclusion consequence relations. Clearly, rejecting single-conclusion systems is a different and harder task than admitting multiple-conclusions systems. This situation becomes even more vivid from the perspective of a logical pluralist accepting both classical and intuitionistic logic. Since intuitionistic logic may fit better with a single-conclusion formulation, a pluralist admitting both classical and intuitionistic logic may have little room to reject the legitimacy of the problematic base theory. Third, even if these strategies were successful, the possibility of their extension to other cases of non-conservativity is not obvious, for example those possibly involving sub-structural logics. Fourth, positive defences of single-conclusion formulations can be and have been provided.[38] Engagement with such arguments would then be necessary. Fifth, to resist our replies, opponents can observe that similar remarks also hold in reverse, and the same criticisms apply to us. After all, the substantiality of classical logic crucially relies on single-conclusion formulations, so that it also requires precise stances on a specific and debatable issue. Since this kind of countermove can be adopted for most of the present objections, we postponed the explicit discussion at the end of the section. For the moment, let us just note that this objection confirms our general thesis: to have or avoid logic substantiality, precise stances on other, disputed issues in the philosophy of logic are required.

Another attempt to neutralise the non-conservativity result, but without touching on the number of admissible conclusions, can be that of turning to alternate formulations of classical logics in which Peirce's law is provable without invoking negation. For example, Peirce's law can be assumed in rule form from the beginning.[39] In such a formulation, Peirce's law can be easily obtained without resorting to negation so that conservativity can be secured.[40] However, once again, to be successful, the

[38] For such a defence see, for instance, Steinberger (2011).

[39] In the context of proof-theoretic semantics this move could be blocked because a Peirce's law rule might be not harmonious.

[40] In this way, a general recipe to neutralise all non-conservativity results could be proposed, because if the base logic includes all sentences with certain connectives derivable in the target logic, conservativity is assured.

objection should show that the base theory not including Peirce's law is illegitimate; otherwise, the non-conservativity result can still be produced over the initial base theory. Since the formulation of such a base theory is quite standard, the prospects for this counter-strategy look bleak.[41] Moreover, in this case, a precise stance on an independent topic of the philosophy of logic would be required, namely, the issue of what is the right set of axioms and rules for a given logic.

In a final attempt to resist the verdict of substantiality, one may turn back, for inspiration, to the conservativity debate on deflationary truth. In that debate, one of the early results showed that a theory of deflationary truth is not conservative over the empty base theory, which is identified with pure logic.[42] However, this fact was not taken to indicate the substantiality of truth, but rather the wrong choice of the base theory. In particular, it has been argued that deflationary truth has a logical and syntactical nature, which requires it to be assessed against the background of a theory containing sufficient syntactic information. Non-conservativity over empty theory has been deemed irrelevant. The supporter of the non-substantiality of logic can try to adopt an analogous move by undermining the general significance of base theories. One way to do this might be to object to any separation of connectives, blocking the very formulation of a base logic distinct from the target logic.[43] In this way, the conservativity approach would not even get off the ground, and the problematic results would be blocked from the beginning. This view can be defended by embracing holism about meaning and, in particular, the meaning of the logical constants, possibly taken as originally given in our natural language. The meaning of every connective would be determined by its relations with all other connectives, so that all theorems would contribute to determining meaning, regardless of the connectives they involve. If logical constants and the logical laws governing them should be holistically considered, then they cannot be separated from one another. In this case, no separation is possible, and no distinct base logic is forthcoming. Extending the logic with a target logic would holistically change the language, neutralising the possibility of extensions. This counter-strategy crucially depends on a specific view of the nature of the meaning of logical constants, namely, semantic holism. Although semantic holism is a legitimate view, it is not the only view about the meaning of logical constants, since semantic atomism and molecularism are alternative and possibly even more attractive options. Once again, to block the non-conservativity result and defend the non-substantiality of logic, a commitment towards a particular position in the philosophy of language and logic is required. Clearly, supporters of insubstantiality could rejoin by blaming us for doing the same. Also our verdict of substantiality relies on a particular semantic thesis, namely, a kind of atomism according to which the meaning of logical constants can be given in separation from the other constants.

[41] However, one way to go in that direction may be to embrace meaning holism and reject the separability of logical constants. Rejecting, in particular, the separability of the base and the target theory. This manoeuvre crucially depends on semantic holism, and, once again, involves non trivial issues in philosophy of logic.

[42] The result is proved and discussed in Halbach (2001).

[43] This objection is due to Fujimoto's (2019, fn. 26). Thanks to an anonymous referee for suggesting it.

Thus, unless we insist that semantic holism is arguably more contentious, neither view has the upper hand.

As already hinted at before, this kind of countermove can be generalised to counter most of the replies considered here. To this, we may react by suggesting that one formulation leading to non-conservativity is enough or trying to argue that the one we chose is somehow privileged. However, we neither intend to and nor need to do that. We agree that most philosophical remarks hold in both directions. Our general point is that, to determine whether logic is substantial, several other issues in the philosophy of logic must be settled. The discussion of the resisting strategies showed that it is possible to retain conservativity and reject the substantiality of classical logic, provided that certain theoretical stances are adopted.[44] However, they indirectly show that also the view presented in this paper so far (holding non-conservativity and substantiality) relies on certain theoretical choices. It follows that both substantiality and non-substantiality require additional assumptions regarding subtle issues related to the nature of logic. In general, with respect to substantiality, the status of logic depends on several other specific views on seemingly independent and non-trivial issues. This is an upshot of the paper: if substantiality is read in terms of conservativity, whether a logic is substantial is an issue crucially connected to other philosophical topics about logic. Accordingly, a claim about substantiality is secured or rejected only as part of a larger theoretical package. To arrive at this outcome, we initially adopted certain views (single conclusion, natural deduction, meaning atomism, and so on) as entry points. For the most part, these theoretical choices are apparently reasonable, so they support the earlier claim that if insubstantiality is read in terms of conservativity, under seemingly reasonable assumptions, classical logic is substantial. However, such assumptions eventually reveal that they play a pivotal and not neutral role. Such a verdict can be resisted by adopting different but legitimate views. At the same time, such alternative views are not less disputable than the initially assumed views. To emphasise the dialectical point, we should note that the paper could have been written in reverse. We could have moved from the alternative views (adopting multiple conclusions, semantic holism,…) to arrive at a conservativity and insubstantiality result. Then, we could have shown how to resist such a different verdict by adopting alternative but legitimate views about various logical issues (single conclusion, meaning atomism,…).

6 Anti-exceptionalism and logical pluralism

Perplexed by the apparent verdict of the substantiality of classical logic and problems of the counter-strategies, a different reading is possible. As mentioned at the beginning of the paper, support for the insubstantiality of logic is not universally shared. Accordingly, the lack of conservativity of classical logic, displayed in certain settings and leading to substantiality, can be accepted and taken on board. The idea that logic is insubstantial can be rejected as a wrong prejudice, possibly due to an exceptional-

[44] Other options serving the same goal could also be found (e.g. by admitting only one connective in the language, such as the Sheffer stroke).

ist bias about the nature of logic. If an anti-exceptionalist perspective is adopted, the result immediately stops being problematic. Logic, being continuous and analogous to the rest of science, can contribute to settling substantial issues and provide new information about the world. This substantiality would emerge as non-atomic conservativity in certain contexts. While adopting an anti-exceptionalist perspective is a radical move, a perk of this strategy is that it does not require engaging with issues such as the admissible number of conclusions or the role of formalisation.

Notably, such an anti-exceptionalist option is available only because conservativity is used to model the insubstantiality of logic rather than to capture which rules governing logical constants are well defined. In the latter case, the lack of conservativity would force the conclusion that something goes wrong with non-conservative logical expressions. Biting the bullet would be off the table, since a lack of conservativity would indicate an anomaly in the rules governing classical connectives. However, once conservativity is used to capture insubstantiality rather than logicality, an alternative option becomes available. Well-defined rules and the demarcation of logical constants can be accounted for in different ways. Hence, one can maintain that a non-conservative logic is substantial while being a logic nonetheless. In other words, although insubstantiality is often embraced as a characterising trait of logic; whether it is really so is an open question. It follows that, in the present context, room remains for taking non-conservativity on board, adopting a view according to which logic (classical logic in particular) is both legitimate and substantial. This goes in the direction of an anti-exceptionalist conception that many consider plausible in any case. If the anti-exceptionalist idea that logic can be substantial is accepted, conservativity results do not discriminate between legitimate and illegitimate logics.

An anti-exceptionalist perspective can also be combined, in an interesting way, with a pluralist stance. Logical pluralism holds that more than one correct logic exists. It has been articulated in different ways, but the proposal of Beall and Restall is often considered typical.[45] The starting point of Beall and Restall is the Generalised Tarski Thesis (GTT):

GTT: An argument is valid_x if and only if, in every case_x in which the premises are true, so is the conclusion.

Superficially, this definition is the familiar one in terms of truth preservation. What gives rise to pluralism is the alleged under-determination of the notion of *case*, indicated by the subscript x in 'valid_x' and 'case_x'. Beall and Restall argued that different specifications of case lead to different logics and, in particular, to classical, intuitionistic and relevant logic. Other prominent forms of logical pluralism include Carnap's tolerant pluralism and pluralism stemming from logic as modelling, among others.[46] Once a minor position in the philosophy of logic, when monism was implicitly accepted as the default option, logical pluralism has recently gained momentum and popularity. If a pluralist view is adopted, for example, it is possible that both classical logic and minimal logic are admitted. This option is interesting because, in this case, a pluralist can hold that not all (correct) logics are insubstantial. Some, such as classical logic, may be substantial, while others, such as minimal logic, may

[45] Beall and Restall (2006).

[46] Carnap (1937) and Cook (2010).

not be. Accordingly, in combination with the present discussion, logical pluralism opens the door to a view in which there is a plurality of legitimate logics with varying substantiality. The combination with logical pluralism helps emphasise that substantiality need not be in contrast with logicality, and that insubstantiality need not render a logic privileged. Indeed, once a plurality of logics is accepted, their possible statuses with respect to substantiality can contribute to shedding light on their differences and similarities. Besides, an even deeper implication for logical pluralism can arise from the fact that claims of substantiality must be accompanied by several other views on logic (admissible number of conclusions, role of the formulation, meaning of logical constant and so on). A particularly deep form of logical pluralism can countenance the legitimacy of all such conceptions, admitting that substantiality does not only vary with different logics but also with how logic as such is understood. It is clear, and we are aware, that much more could be written on this topic. However, this would require a separate discussion that could only be adequately developed in another paper.

Once pluralism is introduced, however, one could have the impression that, if monism is instead adopted, our discussion becomes pointless. If there is only one true logic, then there are no other logics over which the true logic could be conservative or not. For example, if a monist subscribes to classical logic, then its non-conservativity over intuitionistic logic is insignificant, and if intuitionistic logic is right, then considering classical negation has no point. Thus, it seems that the entire project needs logical pluralism to be formulated in a sensible way. However, the situation is more complex. It is to avoid this situation that we assess the conservativity of a target logic over fragments of the target logic itself. For example, classical logic is not assessed merely over intuitionistic logic but over a negation-free fragment that is also acceptable (indeed standard) to the classical logician. Thus, to neutralise the result, the monist classical logician should not reject intuitionistic logic, rather the acceptability of a certain classical fragment. Since the resulting issue is internal to classical logic, appealing to monism is mostly in vain. To be sure, there are strategies that classical logicians can attempt in that direction, such as those discussed in Sect. 7. However, such strategies have their costs, as already shown.

7 Conclusion

Following similar strategies in other debates, especially on truth deflationism, in this paper, we propose to understand the substantiality of logic in terms of the formal notion of conservativity. A first upshot of our investigation is that there are, at least prima facie, reasons to consider some logics– classical logic in particular– to be substantial in a precise sense. This is a noticeable result, aligning with other views that deem logical investigations to be part of substantial inquiries.[47] Further shadows are cast on the idea that logic should be considered a merely neutral tool to be impartially used in the construction and assessment of scientific or philosophical theories.

[47] Williamson (2014, 2023) are notable examples.

At the same time, however, the resulting substantiality of logic could be resisted in various ways. We explored some of the chief counter-strategies, showing that both their adoption and rejection require taking a stand on subtle issues and adopting specific philosophical positions that should be independently supported. This leads to a second significant upshot: substantiality and insubstantiality only come as part of a more complex combination of philosophical and logical views. Overall, we take our paper to indicate that an account of insubstantiality in terms of conservativity can shed new light on the nature of logic and its place in philosophical theorising.

Appendix

Since we employ standard textbook formulations,[48] we limit the presentation of the employed systems to a minimum. In particular, we work assuming a (single-conclusion) natural deduction setting. The logics we consider all include conjunction (∧), disjunction (∨) and conditional (→). The rules, which for the most part are standard textbook ones, as given, for example, in Chiswell and Hodges (2007), are as follows:

Conjunction introduction: If $\Gamma \vdash \varphi$ and $\Delta \vdash \psi,$ then $\Gamma,\ \Delta \vdash (\varphi \wedge \psi).$

Conjunction elimination: If $\Gamma \vdash (\varphi \wedge \psi),$ then $\Gamma \vdash \varphi$ and $\Gamma \vdash \psi.$

Disjunction introduction:If $\Gamma \vdash \varphi$ or $\Gamma \vdash \psi,$ then $\Gamma \vdash (\varphi \vee \psi).$

Disjunction elimination:

If $(\Gamma \cup \{\varphi\} \vdash \chi)$ and $(\Delta \cup \{\psi\} \vdash \chi)$, then $(\Gamma \cup \Delta \cup \{(\varphi \vee \psi)\} \vdash \chi)$

Conditional introduction:If $(\Gamma \cup \{\varphi\} \vdash \psi)$, then $\Gamma \vdash (\varphi \rightarrow \psi).$

Conditional elimination:If $(\Gamma \vdash \varphi)$ and $(\Delta \vdash (\varphi \rightarrow \psi)$, then $\Gamma \cup \Delta \vdash \psi.$

We also add and a 0-ary constant for absurdity (⊥) with no rule.

With the *negation-free fragment*, we mean the logic $\vdash_{NF}$ generated by such logical constants and their rules. Namely, *NF* is formulated in a propositional language L_{NF} with an infinite stock of propositional variables, the logical constants $C_{NF} = \{\wedge, \vee, \rightarrow, \bot\}$, an empty set of (extra-logical) axioms A_{NF} and $\vdash_{NF}$ obtained by the rules above. Thus, the negation-free fragment *NF*, or $\vdash_{NF}$, is $< \varnothing, \vdash_{NF} >$.

The difference between classical $\vdash_C$, intuitionistic $\vdash_I$ and minimal logic $\vdash_M$, as is well known, can be ascribed to different rules governing negation. Minimal logic is obtained by adding negation to the language and by letting '¬φ' abbreviate '(φ → ⊥)', with no rule for negation or absurdity. Intuitionistic logic adds rules for negation that are usually given in terms of absurdity. However, because we need self-contained rules for connectives, negation should be given a rule that does not rely on absurdity. This separated treatment of negation and absurdity allows us to disentangle the connectives and have a uniform progressive path from minimal logic to intuitionistic and, eventually, classical logic in the present context. It also allows defining base theories with a base logic that includes absurdity but not negation, in a way that is also acceptable to minimal logic. We thus adopt a rule for negation introduction (NI) formulated in a sentential parameter A, rather than ⊥. Note that if we focused exclusively on the relationship between classical and intuitionistic logic (which is the core of the paper), without gesturing towards minimal logic, such a disentanglement

[48] See, for example, Chiswell and Hodges (2007).

would be unnecessary. Rules for negation could also be given in terms of absurdity, fully adhering to a textbook standard. In particular, intuitionistic negation is governed by the following rules of negation introduction and elimination:

Negation introduction (NI):If, for every sentence A, $\Gamma, \varphi \vdash A$, then $\Gamma \vdash \neg\varphi$.

Negation elimination (NE):If $\Gamma \vdash \varphi$ and $\Delta \vdash \neg\varphi$, then $\Gamma, \Delta \vdash \psi$.

Again, to keep the intended link between absurdity and negation in intuitionistic logic, we have that '$\neg\varphi$' abbreviates '$(\varphi \rightarrow \bot)$'. Classical negation is obtained by adding the further rule of double negation (DN) elimination.

Double negation elimination (DN):If $\Gamma \vdash \neg\neg\varphi$, then $\Gamma \vdash \varphi$.

Non-conservativity of classical negation over the negation-free fragment

Let the base theory *NF* be as above. Let the target logic $C_\neg$ be yielded by posing the set of logical constants $C_{C\neg} = \{\neg\}$ and $\vdash_{C\neg}$ be obtained by classical rules of negation (NI, NE, DN). $C_\neg = <\varnothing, \vdash_{C\neg}>$ is the logic of classical negation, so that $\vdash_{NF \cup C\neg}$ is full classical logic.

Theorem 1

$C_\neg$ is non-conservative over NF. (Namely, classical negation is not conservative over the negation-free fragment.)

Proof (sketch)

Peirce's law $(((\varphi \rightarrow \psi) \rightarrow \varphi) \rightarrow \varphi)$ is derivable in $\vdash_{NF \cup C\neg}$ but not in $\vdash_{NF}$. We refer the reader to Negri and Von Plato (2001, p. 44), where it is proof-theoretically proved that Peirce's law is not provable in intuitionistic logic, thus implying that it cannot be proved in the negation-free fragment either.[49] □

Non-atomic conservativity of classical negation over a theory in the negation-free fragment

Let L be the theory obtained by the negation-free base theory *NF* by posing the axiom $A_L = \{((R \rightarrow Q) \rightarrow R)\}$, namely the antecedent of an instance of Peirce's law, obtaining $L = <\{((R \rightarrow Q) \rightarrow R)\}, \vdash_{NF}>$. Let $C_\neg$ be as above.

[49] Note also that Peirce's law implies the intuitionistically invalid law of excluded middle. Peirce's law and excluded middle are equivalent, in the sense that adding either of them to intuitionistic logic yields classical logic.

Theorem 2

$C_{\neg}$ is non-atomic conservative over L.

Proof

From ((R → Q) → R) and Peirce's law (derivable in $\vdash_{NF \cup C\neg}$ by Theorem 1), by *modus ponens*, R is derived. However, R is not derivable in *L*. Suppose it is, then, by → introduction, (((R → Q) → R) → R) is derived with no assumption. However, this is an instance of Peirce's law, which is not derivable in $\vdash_{NF}$.□

Non-atomic conservativity of intuitionistic logic over a theory in the negation-free fragment

Let F be the theory obtained by the negation-free fragment *NF* by posing the axiom $A_F = \{\bot\}$. F = < {⊥}, $\vdash_{NF}$>, that is, the theory with absurdity as an axiom, and closed under the logic of the negation-free fragment.

Let the target logic $I_{\neg}$ be yielded by posing the set of logical constants $C_{I\neg} = \{\neg\}$, and $\vdash_{I\neg}$ be obtained by intuitionistic rules of negation (NI, NE). $I_{\neg}$ = < ∅, $\vdash_{I\neg}$ > is the logic of intuitionistic negation, so that $\vdash_{NF \cup I\neg}$ is full intuitionistic logic.

Theorem 3

$I_{\neg}$ is non-atomic conservative over F.

Proof

From $A_F \vdash_{NF \cup I\neg} \bot$, by → introduction we obtain $A_F \vdash_{NF \cup I\neg} \bot \rightarrow \bot$, namely $A_F \vdash_{NF \cup I\neg} \neg\bot$. (Note that the last step cannot be performed in the base theory F, since the base logic $\vdash_{NF}$ does not include negation in the language.) Then, by NE (on $A_F \vdash_{NF \cup I\neg} \bot$ and $A_F \vdash_{NF \cup I\neg} \neg\bot$), we have Aintroduction we obtain $A_F \vdash_{NF \cup I\neg} \varphi$, for any formula φ, and any atomic α in particular. (Note that since minimal logic does not have rules for ⊥ and negation, in particular NE, this step cannot be performed in minimal logic.)

Now, if $A_F \vdash_{NF} \alpha$ for any atomic formula α (namely, if already the base theory F without negation proved any atomic α), then $A_F \vdash_{NF} \varphi$ for any formula φ in the negation-free fragment language L_{NF}, just by iterations of constants introduction.[50] (Namely, if the base theory formulated in the negation-free language and governed by the negation-free logic proves any atomic α, then it proves any formula.)[51] If so,

[50] We omit the straightforward corresponding proof by induction.

[51] We could also have that $A_F \vdash_{NF} \varphi'$ for any formula φ' in the expanded language $L_{NF} \cup \{\neg\}$. However, since minimal logic validates a restricted form of *ex falso quodlibet*, limited to negated conclusions, such

since $A_F = \{\bot\}$, by → introduction, we have that $\vdash_{NF} \bot \rightarrow \varphi$, for any formula φ in L_{NF}. However, since $\vdash_{NF}$ is a fragment of minimal logic $\vdash_M$, and $\nvdash_M \bot \rightarrow \varphi$ for all φ in L_{NF}, then $A_F \nvdash_{NF} \alpha$.□

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an extension would be irrelevant for the proof.

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