

On the philosophical motivations for the logics of formal consistency and inconsistency

Walter Carnielli¹, Abílio Rodrigues²

¹CLE and Department of Philosophy
State University of Campinas, Campinas, SP, Brazil
walter.carnielli@cle.unicamp.br

²Department of Philosophy
Federal University of Minas Gerais, Belo Horizonte, MG, Brazil
abilio@ufmg.br

The aim of this paper is twofold. Firstly, we want to present a philosophical motivation for the logics of formal inconsistency (LFIs), a family of paraconsistent logics whose distinctive feature is that of having resources for expressing the notion of consistency (and inconsistency as well) within the object language. We shall defend the view according to which logics of formal inconsistency are theories of logical consequence of normative and epistemic character, that tell us how to make inferences in the presence of contradictions.

Secondly, we want to investigate to what extent an intuitive reading of the bivalued semantics for LFIs, based on classical logic as presented in Carnielli *et al.* (2007), can be maintained. In particular, we will analyze the logic *mbC*. The idea is to intuitively interpret paraconsistent negation in the following way. The acceptance of $\neg A$ means that there is some evidence that A is not the case. If such evidence is non-conclusive, it may be that there is simultaneously some evidence that A is the case. Therefore, the acceptance of a pair of contradictory sentences A and $\neg A$ need not to be taken in the strong sense that both are true.

The paper is structured as follows. In section 1, we shall present a general view of the problem in order to make clear for the reader the distinction between explosiveness and contradictoriness, essential for grasping the distinction between classical and paraconsistent

logics. In section 2, we present an axiomatic system for *mbC* with a correct and complete semantics, and also discuss to what extent this semantics is suitable for the interpretation of the acceptance of contradictions mentioned above.

In section 3, we shall examine the problem of the nature of logic, namely, whether logic as a theory of logical consequence has primarily an ontological or epistemological character. We defend the view that logics of formal inconsistency find their place on the epistemological side of logic.¹

1. Contradictions and explosions

Classical logic does not accept contradictions. This is not only because it endorses the validity of the principle of non-contradiction, $\sim(A \wedge \sim A)$, but an even stronger reason is that, classically, everything follows from a contradiction. This is the inference rule called *ex falso quodlibet*, or law of explosion,

$$(1) A, \sim A \vdash B.$$

From a pair of propositions A and $\sim A$, we can prove any proposition through logic, from ‘ $2+2=5$ ’ to ‘snow is black’.

Before the rise of modern logic (i.e., that which emerged in the late nineteenth century with the works of Boole and Frege), the validity of the principle of explosion was a contentious issue. However, in Frege’s *Begriffsschrift*, where we find for the first time a complete system of what later would be called first order logic, the principle of explosion holds (proposition 36 of part II). It is worth noting in passing that Frege’s logicist project failed precisely because the system was inconsistent, and therefore trivial. Frege strongly attacked the influence of psychologism in logic (we will return to this point in section 3), and in his writings we find a realist conception of mathematics and logic that contributed to emphasizing the ontological and realist vein of classical logic.

¹ One of the intentions of this paper is to present the logics of formal inconsistency to the non-technical minded reader. We presuppose only a basic knowledge of classical logic. The paper, we hope, is as intuitive as it can be without wasting space, while at the same time dealing with aspects of logics of formal inconsistency yet unexplored.

The account of logical consequence that was established as standard in the first half of twentieth century through the works of Russell, Tarski, and Quine, among others, is classical. Invariably, classical logic is the logic we first study in introductory logic books, and the law of explosion holds in this system. Let us put these things a little bit more precisely. Let T be a theory formulated in some language, call it L , whose underlying logic is classical. If T proves a pair of sentences A and $\sim A$, ' \sim ' being classical negation, T proves all sentences of L . In this case, we say that T is trivial. For this reason contradictions must be avoided at all costs in classical logic. The most serious consequence of a contradiction in a classical theory is not the violation of the principle of non-contradiction, but rather the trivialization of the theory.

Although classical logic has become the standard approach, several alternative accounts of logical consequence have been proposed and studied. At the beginning of the twentieth century, Brouwer (1907), motivated by considerations regarding the nature of mathematical knowledge, established the basis for a different account of logical consequence, intuitionistic logic. The principle of excluded middle, another so-called fundamental law of thought, is not valid in intuitionistic logic.

Intuitionistic logic, later formalized by Arend Heyting (1956), a former student of Brouwer, has an epistemological character in clear opposition to Frege's realism. In brief, starting from the assumption that mathematical objects are not discovered but rather created by the human mind, the intuitionists' motivation for rejecting the excluded middle is the existence of mathematical problems for which there are no known solutions. The usual example is the Goldbach conjecture: every even number greater than 2 is the sum of two prime numbers. Let us call this proposition G . Until now, there is no proof of G , nor is there a counterexample of an even number greater than 2 that is not the sum of two primes. The latter would be a proof of *not* G . For this reason, the intuitionist maintains that we cannot assert the relevant instance of excluded middle, G or *not* G . Doing so would commit us to a platonic, supersensible realm of previously given mathematical objects.

So-called classical logic is based on principles that, when rejected, may give rise to alternative accounts of logical consequence. Two accepted principles of classical logic are the aforementioned laws of excluded middle and non-contradiction. The principle of non-contradiction can be interpreted from an ontological point of view, as a principle about

reality according to which facts, events, and mathematical objects cannot be in contradiction with each other. Its formulation in first order logic, $\forall x \neg(Px \wedge \neg Px)$, says that it cannot be the case that an object simultaneously has and does not have a given property P . Note, and this is a point we want to emphasize, that the principle of explosion can be understood as a still more incisive way of saying that there can be no contradiction in reality – otherwise, everything is the case, and we know this cannot be. Explosion is thus a stronger way of expressing an idea usually attributed to non-contradiction.

However, it is a fact that contradictions appear in a number of contexts of reasoning, among which three are worth mentioning: (i) computational databases; (ii) semantic and set theoretic paradoxes; (iii) scientific theories. In the first case, contradictions invariably have a provisional character, as an indication of an error to be corrected. We will make only a few remarks with regard to the second case, as an analysis of this problem requires a technical exposition that cannot be gone into here. The third case will be discussed below.

Naïve set theory, which yields set theoretic paradoxes, has been revised and corrected. Up to the present time, there is no indication that ZFC (and its variants) is not consistent – in fact, all indications are to the contrary. The moral we can draw from this is that our intuitive conception of set is defective, a ‘product of thought’, so to speak, that yields contradictions from which nothing more should be concluded. With respect to semantic paradoxes, they are results about languages with certain characteristics. It is worth noting that the diagonal lemma, essential in the formulation of both the Liar and Curry’s paradoxes, is a result about language itself. Thus we think that the step from paradoxes to the claim that there are true contradictions in the sense that reality is contradictory, is too speculative to be taken seriously.

Here we will concentrate on the third case, the occurrence of contradictions in scientific theories. We are not going to throw away these theories if they are successful in predicting results and describing phenomena. Although mathematicians work based on the assumption that mathematics is free of contradictions, in empirical sciences contradictions seem to be unavoidable, and the presence of contradictions is not a sufficient condition for throwing away an interesting theory. It is a fact that contradictory theories exist; the question is how to deal with them. The problem we have on our hands is that of how to formulate an account of logical consequence capable of identifying, in such contexts, the

inferences that must be blocked. In what follows, we will present a logic of formal inconsistency designed to accomplish this task.

2. mbC: a logic of formal inconsistency

To begin, let us recall some useful definitions. A theory is a set of propositions closed under logical consequence. This means that everything that is a logical consequence of the theory is also part of the theory. For instance, everything that is a consequence of the basic principles of arithmetic, for instance, that $5 + 7 = 12$, is also part of arithmetic. We say that a theory T is:

- (i) *contradictory* if and only if there is a sentence A in the language of T such that T proves A and $\sim A$ (i.e., T proves a contradiction);
- (ii) *trivial* if and only if for any sentence A in the language of T , T proves A (i.e., T proves everything);
- (iii) *explosive* if and only if T is trivial in the presence of a contradiction (i.e., the principle of explosion holds in T).

There is a difference between a theory being contradictory and being trivial. Contradictoriness entails triviality only when the principle of explosion holds. Without explosion, we may have contradictions without triviality.

In books on logic, we find two different but classically equivalent notions of consistency. Hunter (1973, p. 78ff) calls them *simple* and *absolute* consistency. A deductive system S with a negation \sim is simply consistent if and only if there is no formula A such that $S \vdash A$ and $S \vdash \sim A$. The other notion of consistency, absolute consistency, says that a system S is consistent if and only if there is a formula B such that $S \not\vdash B$. In other words, S does not prove everything. The latter notion is tantamount to saying that S is not trivial, while the former is tantamount to saying that S is non-contradictory. From the point of view of classical logic, both notions are equivalent because the principle of explosion holds. The proof is simple and easy. Suppose S is trivial. Hence, it proves everything, including a pair of sentences A and $\sim A$. Now suppose S is contradictory, that is, it proves a pair of sentences A and $\sim A$. If the principle of explosion holds in S , as is the case in classical logic, then S

proves everything, hence S is trivial. The central point for paraconsistent logics is to separate triviality from inconsistency, restricting the principle of explosion.

In logics of formal inconsistency the principle of explosion is not valid in general, that is, it is not the case that from *any* pair of contradictory sentences everything follows. There is a non-explosive negation, represented here by ‘ \neg ’, such that $A, \neg A \not\vdash B$. In addition, there is a unary consistency connective called ‘ball’: ‘ $\circ A$ ’ means that A is consistent. We can thus isolate contradictions in such a way that the application of the law of explosion is restricted to consistent sentences only, thus avoiding triviality even in the presence of one or more contradictions. Therefore, a contradictory theory may be non-trivial. However, as we will see, we can go further and separate the concept of inconsistency from that of contradictoriness. Consistency may be taken as a primitive notion, its meaning being elucidated from outside the formal system. We will return to this point later on.

It is worth mention here a common misunderstanding about paraconsistent logics in general. It is true that in the majority of paraconsistent logics the principle of non-contradiction is not valid. But it does not mean that *systems* of paraconsistent logic have contradictions as theorems. This is as wrong as saying that intuitionistic logic proves the negation of excluded middle.²

The fundamental distinction between classical logic and paraconsistent logics occurs at the sentential level. In what follows, we will present an axiomatic system of a sentential logic of formal inconsistency, *mbC*.

Let L be a language with sentential letters, the set of logical connectives $\{\vee, \wedge, \rightarrow, \neg, \circ\}$, and parentheses. Notice that the consistency operator ‘ \circ ’, mentioned above, is a primitive symbol. The set of formulas of L is obtained recursively in the usual way. Consider the following axiom-schemas:

Ax. 1. $A \rightarrow (B \rightarrow A)$

Ax. 2. $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$

Ax. 3. $A \rightarrow (B \rightarrow (A \wedge B))$

² A *theory* whose underlying logic is paraconsistent may have a contradiction as a theorem, since the idea is precisely to be able to handle contradictory but non trivial theories.

- Ax. 4. $(A \wedge B) \rightarrow A$
 Ax. 5. $(A \wedge B) \rightarrow B$
 Ax. 6. $A \rightarrow (A \vee B)$
 Ax. 7. $B \rightarrow (A \vee B)$
 Ax. 8. $(A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow ((A \vee B) \rightarrow C))$
 Ax. 9. $A \vee (A \rightarrow B)$
 Ax. 10. $A \vee \neg A$
 Ax. bc1. $\circ A \rightarrow (A \rightarrow (\neg A \rightarrow B))$

Inference rule: *modus ponens*

The definition of a proof of A from premises Γ ($\Gamma \vdash_{\text{mbC}} A$) is the usual one: a sequence of formulas $B_1 \dots B_n$ such that A is B_n and each B_i (that is, each line of the proof) is an axiom, a formula that belongs to Γ , or the result of modus ponens. Thus, monotonicity holds:

(2) if $\Gamma \vdash_{\text{mbC}} B$, then $\Gamma, A \vdash_{\text{mbC}} B$, for any A .

A theorem is a formula proved from the empty set of premises. Ax.1 and Ax.2, plus modus ponens, give the deduction theorem:

(3) if $\Gamma, A \vdash_{\text{mbC}} B$, then $\Gamma \vdash_{\text{mbC}} A \rightarrow B$.

We thus have inferences that correspond to introduction and elimination rules of implication in a natural deduction system. Since monotonicity holds, we have also the converse of (3), and the deduction theorem holds in both directions.

(4) $\Gamma, A \vdash_{\text{mbC}} B$ if and only if $\Gamma \vdash_{\text{mbC}} A \rightarrow B$.

Axioms 3-5 and 6-8 correspond, respectively, to the introduction and elimination rules of conjunction and disjunction in a natural deduction system. The presence of axiom 9 fits the criterion established by da Costa (1974, p. 498), according to which a paraconsistent system must have everything that could be added without validating explosion and non-contradiction. Furthermore, as we will see, with the help of Ax.9 we can define a classical negation within *mbC*.

Axioms 1-9 form a complete system of positive classical logic, i.e., a system that proves all tautologies that can be formed with \rightarrow , \vee and \wedge . Negation shows up in Ax.10 and

Ax.11. Non-contradiction is not a theorem, and excluded middle holds. Explosion is restricted to consistent formulas. Consistency is a primitive notion; it is not definable in terms of non contradiction, that is, $\neg(A \wedge \neg A)$ and $\circ A$ are not equivalent. It is clear that in a system whose consequence relation \vdash enjoys the deduction theorem in both directions, the law of explosion

$$(1) A, \neg A \vdash B$$

and the axiom schema

$$(5) \vdash A \rightarrow (\neg A \rightarrow B)$$

are equivalent. To make things simpler, we call both the law of explosion.

With axioms 1-8, axiom 10 (excluded middle), and the law of explosion (5), we get a complete system of classical logic, that is, a system that proves all classical tautologies. Notice that (5) is nothing but Axiom bc1 without the mention of consistent formulas. Therefore it is easy to see that we get classical logic for consistent formulas.

The system is not explosive with respect to *non-consistent formulas*. For this reason, we say that the logic is *gently explosive*. If we simultaneously have contradiction and consistency, the system explodes, becoming trivial. But the point is precisely this: *there cannot be consistency and contradiction, simultaneously and with respect to the same formula*. If explosion in classical logic means that there can be no contradiction at all, the restricted principle of explosion may be understood as a more refined way of stating the same basic idea.

A remarkable feature of *mbC* is that it can be seen as an extension of classical logic. We can define a bottom particle as follows:

$$(6) \perp \stackrel{\text{def}}{=} \circ A \wedge A \wedge \neg A.$$

Now, as an instance of Ax.9, we get

$$(7) A \vee (A \rightarrow \perp).$$

We also have that

$$(8) \vdash \perp \rightarrow B,$$

since, by Ax.bc1,

$$(9) \circ A \wedge A \wedge \neg A \vdash B.$$

It is well known that Axioms 1-8 plus (7) and (8) give us classical logic.

A point that certainly has already occurred to the reader regards the interpretation of the system above. Three questions that pose themselves are:

- (i) How can a semantics be given for the formal system above?
- (ii) What is the intuitive meaning of the consistency operator?
- (iii) How can an intuitive interpretation for contradictory formulas be provided?

Paraconsistent logics were initially introduced in proof-theoretical terms, a procedure that fits the epistemological character that we claim here is a central feature of them. In Carnielli et al. (2007, pp. 38ff), we find a bivalued non-truth-functional semantics that is complete and correct for *mbC*. An *mbC-valuation* is a function that attributes values 0 and 1 to formulas of L , according to the following clauses:

- (i) $v(A \wedge B) = 1$ if and only if $v(A) = 1$ and $v(B) = 1$
- (ii) $v(A \vee B) = 1$ if and only if $v(A) = 1$ or $v(B) = 1$
- (iii) $v(A \rightarrow B) = 1$ if and only if $v(A) = 0$ or $v(B) = 1$
- (iv) $v(\neg A) = 0$ implies $v(A) = 1$
- (v) $v(oA) = 1$ implies $v(A) = 0$ or $v(\neg A) = 0$

We say that a valuation v is a model of Γ if and only if every sentence of Γ receives the value 1 in v . The notion of logical consequence is defined as usual: $\Gamma \models_{mbC} A$ if and only if for every valuation v , if v is a model of Γ , $v(A) = 1$.

The values 0 and 1 should not be taken, in this interpretation as, respectively, false and true *simpliciter*. Rather, they express the *existence of evidence*: $v(A) = 1$ means that there is evidence that A is the case, and $v(A) = 0$ means that there is no evidence that A is the case. Non-conclusive evidence can be changed, although truth (that is, conclusive evidence) cannot. The following passage from da Costa (1982 pp. 9-10) helps to elucidate what we mean by evidence.

Let us suppose that we want to define an operational concept of negation, at least for the negation of some atomic sentences. $\neg A$, where A is atomic, is to be true if, and only if, the clauses of an appropriate criterion c are fulfilled, clauses that must be empirically testable; i.e., we have an empirical criterion for the truth of the negation of A . Naturally, the same must be valid for the atomic proposition A , for the sake of coherence. Hence, there exists a

criterion d for the truth of A . But clearly it may happen that the criteria c and d be such that they entail, under certain critical circumstances, the truth of both A and $\neg A$.

It seems to us that it is much more reasonable, in a situation like the one described above, to not draw the conclusion that A and $\neg A$ are *true*.³ It is better to be more careful and to take the contradictory data only as a provisional state.

Since excluded middle holds, we have the following three possible scenarios, together with the respective values attributed to A and $\neg A$:

A1.

There is evidence that A is the case; $v(A) = 1$

There is evidence that A is not the case. $v(\neg A) = 1$

A2.

There is evidence that A is the case; $v(A) = 1$

There is no evidence that A is not the case. $v(\neg A) = 0$

A3.

There is no evidence that A is the case; $v(A) = 0$

There is evidence that A is not the case. $v(\neg A) = 1$

Accordingly, we suggest the following meaning for negation:

$v(\neg A) = 1$ means that *there is some evidence that A is not the case*;

$v(\neg A) = 0$ means that *there is no evidence that A is not the case*.

Now, if the evidence is conclusive, we have only two possible (classical) scenarios:

B1. There is conclusive evidence that A is the case.

B2. There is conclusive evidence that A is not the case.

In other words, we are now talking about *truth* and not just evidence. We may have conflicting evidence, but not conflicting truth values. Accordingly,

$v(\circ A) = 1$ means that *the truth value of A has been conclusively established*.

³ We do not endorse dialetheism, the doctrine that there are true contradictions, sentences A and $\neg A$ such that both are *true* in the sense that reality is contradictory and makes them true (Priest & Berto 2013). According to the intuitive reading of the semantics presented here, the system does not tolerate true contradictions.

In scenarios B1 and B2 we have $v(\circ A) = 1$. In A1, we must have $v(\circ A) = 0$. But in A2 and A3 we have $v(\circ A) = 0$. In these scenarios, when $v(\circ A)$ turns out to be 1, we get scenarios B1 or B2. A remarkable feature of the above intuitive interpretation is that the simultaneous *truth* of A and $\neg A$ is not allowed, on pain of triviality. In other words, the system is *not neutral* with respect to true contradictions.

Note that paraconsistent negation is weaker than classical negation; it does not have all the properties that classical negation has. Regarding the consistency operator, we want to call attention to the fact that in *mbC* it is not definable from other connectives. This is in accord with a central point of the notion of consistency, namely, that it is polysemic and need not be defined from negation. The notion of consistency in natural language and informal reasoning has a number of senses, not always directly related to negation (cf. Carnielli 2011, p. 84). A proposition is often said to be consistent when it is coherent or compatible with previous data, or when it refers to an unchanging or constant situation.

Since consistency is a primitive notion, its meaning is elucidated from outside the formal system. Our suggestion is nothing but one way of interpreting it. It is worth noting that the above interpretation fits well with the old philosophical idea that truth implies consistency, but consistency does not imply truth.⁴ Indeed, $\circ A \not\equiv_{\text{mbC}} A$. The consistency of A does not imply the truth of A ; but if the truth-value of A has been conclusively established as true, then A is consistent.

Clauses (i)-(iii) are exactly as in classical logic, and therefore all classical tautologies with \wedge , \vee and \rightarrow are valid in the *mbC*-valuation. An important feature of the valuation above is that both the negation and the consistency connectives *are not truth-functional*, that is, the value attributed to $\neg A$ and $\circ A$ does not depend only on the value attributed to A . Clause (iv), negation, expresses only a necessary condition. If $v(\neg A) = 0$, we cannot also have $v(A) = 0$. This was expected, as excluded middle is valid and at least one formula among A and $\neg A$ must receive the value 1. On the other hand, since we do not have a sufficient condition for $\neg A = 0$, it is possible that $v(A) = 1$ and $v(\neg A) = 1$, since $v(A) = 1$ does not make $v(\neg A) = 0$. Since truth-functionality is a special case of

compositionality, it is also remarkable that this semantics is not compositional in the sense that the semantic value of the whole expression is determined by the semantic values of its parts and the way they are combined.

Non-contradiction, as expected, is not valid. Intuitively, it can be the case that there is some non-conclusive evidence both for A and $\neg A$, and this makes it possible to have $v(\neg(A \wedge \neg A))=0$. In addition, it is easy to see that explosion is also not valid: if $v(A) = v(\neg A) = 1$ and $v(B) = 0$, $v(A \rightarrow (\neg A \rightarrow B)) = 0$. We may have evidence for A and $\neg A$, but have no evidence for B .

Clause (v) also expresses only a necessary condition for attributing the value 1 to $\circ A$: we must have exactly one among A and $\neg A$ with value 0 (by clause (iv) they cannot be both 0). If we have both, $v(A) = v(\neg A) = 1$, we do not have $v(\circ A) = 1$. It is also worth noting that according to clause (v), $\circ A \models_{\text{mbC}} \neg(A \wedge \neg A)$. Intuitively, if A is consistent, its truth-value has been conclusively established. Hence, it cannot be that we still have evidence for both A and $\neg A$. On the other hand, the converse does not hold: $\neg(A \wedge \neg A) \not\models_{\text{mbC}} \circ A$. We may well have non-conclusive evidence for A and no evidence at all for $\neg A$. In this case $v(\neg(A \wedge \neg A))=1$, but since the evidence for A is non-conclusive, $v(\circ A) = 0$. Consistency is independent of contradiction.

Excluded middle is valid. Suppose $v(A \vee \neg A) = 0$. Then we should have both $v(A) = 0$ and $v(\neg A) = 0$, but this is impossible by the clause of negation. Now this means that we always have either some evidence for A , or some evidence for $\neg A$. If we know nothing about proposition A , by default we consider that it is evidence for $\neg A$.

There are some inferences that are not allowed once explosion is not valid in general. One is disjunctive syllogism. Actually, disjunctive syllogism and explosion are equivalent in the sense that, added to positive sentential logic (Ax.1-9 plus modus ponens), each one implies the other. But it is easy to see that according to the interpretation proposed here, disjunctive syllogism is not valid. It can be the case that we have some evidence both for A and $\neg A$, so we may have $v(A \vee B) = v(\neg A) = 1$, but $v(B) = 0$.

⁴ This is the basic argument against truth as coherence. See, for example, Russell (1913), ch. XII.

Modus ponens holds, but modus tollens (as well as all versions of contraposition) does not hold.

$$(10) A \rightarrow B, \neg B \not\models_{\text{mbC}} \neg A$$

However, for B consistent,

$$(11) \circ B, A \rightarrow B, \neg B \models_{\text{mbC}} \neg A.$$

Is there an intuitive justification for the fact that modus tollens is not valid in mbC? The answer is affirmative. In classical logic modus tollens is valid because the truth of $\neg B$ implies the falsity of B . Hence, as well as truth being preserved by modus ponens, falsity is also preserved by modus tollens: given the truth of $A \rightarrow B$, if B is false, A is false too. This is not the case, however, when we do not interpret 1 and 0 as true and false. $v(\neg B) = 1$ means that we have some evidence that B is not the case. This does not imply, however, that we do not have simultaneous evidence for B , that is, $v(\neg B) = 1$ does not imply $v(B) = 0$. Suppose that we have evidence both for A and B . In this case, $v(A \rightarrow B) = 1$ (notice that there is not required any kind of connection between the meanings of A and B in order to have $v(A \rightarrow B) = 1$). Now, in order to see that modus tollens is not valid, make $v(B) = v(\neg B) = 1$, and $v(A) = 1$, $v(\neg A) = 0$. On the other hand, if $v(\circ B)=1$, we cannot have $v(B) = v(\neg B) = 1$. Hence, (11) is valid.

In mbC , as with the majority of logics of formal inconsistency, once the consistency of certain formulas is established, we get classical logic. The schemas below are valid:

$$\circ B \models_{\text{mbC}} (A \rightarrow B) \rightarrow ((A \rightarrow \neg B) \rightarrow \neg A)$$

$$\circ \neg A \models_{\text{mbC}} \neg \neg A \rightarrow A$$

$$\circ A \models_{\text{mbC}} A \rightarrow \neg \neg A$$

$$\circ A, \circ B \models_{\text{mbC}} (B \rightarrow A) \leftrightarrow (\neg A \rightarrow \neg B)$$

$$\circ A, \circ B \models_{\text{mbC}} (\neg B \rightarrow A) \leftrightarrow (\neg A \rightarrow B)$$

However, as usual, not everything works perfectly when we try to give an intuitive interpretation to a formal system. Classical logic has several counterintuitive results, and some of them have been the motivation for developing non-classical logics. As might be expected, the situation is not different with non-classical logics in general, and in particular

with *mbC*. A problem to be faced by the above interpretation is the failure of the replacement property with respect to negation and the consistent operator, that is:⁵

$$(12) \not\models *(A \# B) \leftrightarrow *(B \# A), \text{ where } * \in \{\neg, \circ\} \text{ and } \# \in \{\wedge, \vee\},$$

although the following hold:

$$(13) \models (A \# B) \leftrightarrow (B \# A), \# \in \{\wedge, \vee\}.$$

It is difficult to accommodate the invalidities of (12) above within the proposed intuitive interpretation, for if we have evidence for $\neg(A \wedge B)$, of course it should also be evidence for $\neg(B \wedge A)$ (*mutatis mutandis* for disjunction and consistency). An alternative, suggested and discussed in Carnielli et al.(2007, p. 31), is simply to add the schemas below to *mbC*:

$$(14) A \dashv\vdash B \text{ implies } \neg A \dashv\vdash \neg B,$$

$$(15) A \dashv\vdash B \text{ implies } \circ A \dashv\vdash \circ B.$$

Although from the formal point of view this may be a solution, it is not clear if it works for the intuitive interpretation we have in mind here. Besides, doing this seems to be an *ad hoc* solution, and surely there is a philosophical price to be paid. What is beyond doubt is that there is still a lot of work to be done with respect to the philosophical aspects of logics of formal inconsistency. As has been shown in Carnielli et al. (2007) and in Carnielli & Marcos (2002), several systems may be formulated, with very subtle differences among them. Aside from the technical aspects of finding complete and correct semantics, and proving metatheorems, there are still lots of open questions related to their intuitive interpretation, or, in other words, to the philosophical concepts expressed by them.

3. On the nature of logic

In building formal systems we deal with several logical principles, and it may justly be asked what these are principles about. This is a central issue in philosophy of logic. Here we follow Popper (1963, pp. 206ff), who presents the problem in a very clear way. The question is whether the rules of logic are:

(I.a) laws of thought in the sense that they describe how we actually think;

⁵ For details, see Carnielli et al. (2007) pp. 30ff.

(I.b) laws of thought in the sense that they are normative laws, i.e., laws that tell us how we should think;

(II) the most general laws of nature, i.e., laws that apply to any kind of object;

(III) laws of certain descriptive languages.

We thus have three basic options, which are not mutually exclusive: the laws of logic have (I) epistemological, (II) ontological, or (III) linguistic character. With respect to (I), they may be (I.a) normative or (I.b) descriptive. Let us illustrate the issue with some examples.

Aristotle, defending the principle of non-contradiction, makes it clear that it is a principle about reality, “the most certain principle of all things” (*Metaphysics* 1005b11). Worth mentioning also is the well-known passage, “the same attribute cannot at the same time belong and not belong to the same subject in the same respect” (*Metaphysics* 1005b19-21), which is a claim about objects and their properties.

On the other hand, a very illustrative example of the epistemological side of logic can be found in the so-called logic of Port-Royal, where we read:

Logic is the art of conducting reasoning well in knowing things, as much to instruct ourselves about them as to instruct others.

This art consists in reflections that have been made on the four principal operations of mind: *conceiving, judging, reasoning, and ordering*.

(...) [T]his art does not consist in finding the means to perform these operations, since nature alone furnishes them in giving us reason, but in reflecting on what nature makes us do, which serves three purposes.

The first is to assure us that we are using reason well ...

The second is to reveal and explain more easily the errors or defects that can occur in mental operations.

The third purpose is to make us better acquainted with the nature of the mind by reflecting on its actions. (Arnauld, A. & Nicole, 1996, p. 23)

Logic is conceived as having a normative character. So far so good. But logic is also conceived as a tool for analyzing mental processes of reasoning. This analysis, when further extended by different approaches to logical consequence, as is done now by some non-classical logics, shows that there can be different standards of correct reasoning in

different situations. This aspect of logic, however, has been put in a secondary place by Frege's attack on psychologism. Frege wanted to eliminate everything subjective from logic. For Frege, laws of logic cannot be obtained from concrete reasoning practices. Basically, the argument is the following. From the assumption that truth is not relative, it follows that the basic criterion for an inference to be correct, namely, the preservation of truth, should be the same for everyone. When different people make different inferences, we must have a criterion for deciding which one is correct. Combined with Frege's well-known Platonism, the result is a conception of logic that emphasizes the ontological (and realist) aspect of classical logic.

Our conception of the laws of logic is necessarily decisive for our treatment of the science of logic, and that conception in turn is connected with our understanding of the word 'true'. It will be granted by all at the outset that the laws of logic ought to be guiding principles for thought in the attainment of truth, yet this is only too easily forgotten, and here what is fatal is the double meaning of the word 'law'. In one sense a law asserts what is; in the other it prescribes what ought to be. Only in the latter sense can the laws of logic be called 'laws of thought'(...) If being true is thus independent of being acknowledged by somebody or other, then the laws of truth are not psychological laws: they are boundary stones set in an eternal foundation, which our thought can overflow, but never displace. (Frege 1893, (1964) p. 13).

(...) [O]ne can very well speak of laws of thought too. But there is an imminent danger here of mixing different things up. Perhaps the expression "law of thought" is interpreted by analogy with "law of nature" and the generalization of thinking as a mental occurrence is meant by it. A law of thought in this sense would be a psychological law. And so one might come to believe that logic deals with the mental process of thinking and the psychological laws in accordance with which it takes place. This would be a misunderstanding of the task of logic, for truth has not been given the place which is its due here. (Frege 1918, (1997) p. 325).

For Frege, logic is normative, but in a secondary sense. As well as truths of arithmetic, the logical relations between propositions are already given, eternal. This is not surprising at

all. Since he wanted to prove that arithmetic could be obtained from purely logical principles, truths of arithmetic would inherit, so to speak, the realistic character of the logical principles from which they were obtained. Logic thus has an ontological character; it is part of reality, as are mathematical objects.

It is very interesting to contrast Frege's realism to Brouwer's intuitionism, whose basic ideas can be found for the first time in his doctoral thesis, written at the very beginning of twentieth century. The approaches are quite opposed.

Mathematics can deal with no other matter than that which it has itself constructed.

In the preceding pages it has been shown for the fundamental parts of mathematics how they can be built up from units of perception (Brouwer 1907, (1975) p. 51)

The words of your mathematical demonstration merely accompany a mathematical *construction* that is effected without words ...

While thus mathematics is independent of logic, logic does depend upon mathematics: in the first place *intuitive logical reasoning* is that special kind of mathematical reasoning which remains if, considering mathematical structures, one restricts oneself to relations of whole and part (Brouwer 1907, (1975) p. 73-74).

It is remarkable that Brouwer's doctoral thesis (1907) was written between the two above-quoted works by Frege (1893 and 1919). Brouwer, like Frege himself, is primarily interested in mathematics. For Brouwer, however, the truths of mathematics are not discovered but rather constructed. Mathematics is not a part of logic, as Frege wanted to prove. Quite the contrary, logic is abstracted from mathematics. It is, so to speak, a description of human reasoning in constructing correct mathematical proofs. Mathematics is a product of the human mind, mental constructions that do not depend on language or logic. The raw material for these constructions is the intuition of time (this is the meaning of the phrase 'built up from units of perception').

This epistemological motivation is reflected in intuitionistic logic, formalized by Heyting (1956). Excluded middle is rejected precisely because mathematical objects are considered mental constructions. Accordingly, to assert an instance of excluded middle related to an unsolved mathematical problem (for instance, Goldbach's conjecture), would

be a commitment to a platonic realm of abstract objects, an idea rejected by Brouwer and his followers.

With respect to the linguistic aspects of logic, we shall make just a few comments. According to a widespread opinion, a linguistic conception of logic prevailed during the last century. From this perspective, logic has to do above all with the structure and functioning of certain languages. Indeed, sometimes logic is defined as a mathematical study of formal languages. There is no consensus about this view, however, and it is likely that it is not prevalent today.⁶ Even though we cannot completely separate the linguistic from the epistemological aspects – i.e., separate language from thought –, we endorse the view that logic is primarily a theory about reality and thought, and that the linguistic aspect is secondary.

At first sight, it might seem that Frege's conception according to which there is only one logic, that is, only one account of logical consequence, is correct. Indeed, for Frege, Russell, and Quine, *the* logic is classical logic. From a realist point of view, this fits well with the perspective of the empirical sciences: excluded middle and bivalence have a strong appeal. Ultimately, reality will decide between *A* and *not A*, which is the same as deciding between the truth and falsity of *A*.

The identification of an intuitionistic notion of provability with truth was not successful. As is shown by Raatikainen (2004), in the works of Brouwer and Heyting we find some attempts to formulate an explanation of the notion of truth in terms of provability, but all of them produce counterintuitive results.

On the other hand, the basic intuitionistic argument that rejects a supersensible realm of abstract objects is philosophically motivated – it is remarkable that it usually seems rather convincing to students of philosophy. As Velleman & Alexander (2002, pp. 91ff) put it, realism seems to be compelling when we consider a proposition like *every star has at least one planet orbiting it*. However, when we pass from this example to Goldbach's conjecture, the situation changes quite a bit. In the former case, it is very reasonable to say that reality is one way or the other; but if we say with regard to the latter

⁶ A rejection of the linguistic conception of logic, and an argument that logic is above all a theory with ontological and epistemological aspects, can be found in the *Introduction* to Chateaubriand (2001).

case that ‘the world of mathematical numbers’ is one way or the other, there is a question to be faced: where is this world?

What is the moral to be taken from this? That classical and intuitionistic logic are not talking about the same thing. The former is connected to reality through a realist notion of truth; the latter is not about truth, but rather about reasoning. In our view, assertability based on the intuitionistic notion of constructive proof is what is expressed by intuitionistic logic.

Now, one may ask what all of this has to do with logics of formal inconsistency and paraconsistent logics in general. The question concerning the nature of logic is a perennial problem of philosophy. We believe that it has no solution in the sense of some conclusive argument in defense of one or the other view. This is so, first, because logic is simultaneously about language, thought, and reality, and, second, because different accounts of logical consequence may be more appropriate for expressing one view than another. We also claim that this is precisely the case with, on the one hand, classical logic and its ontological motivations and, on the other hand, with the epistemological approach of intuitionism and logics of formal inconsistency.

The reader may have already noticed a duality between intuitionistic and paraconsistent logics. If we stop to think about them for a moment, and put aside any realist bias, we see that we are facing two analogous situations. Let us take a look at this point.

At first sight, it really seems that an inference principle like *modus tollens* is valid whether we want it to be or not, that its validity is not a matter of any kind of choice or context whatever. This is indeed correct if we are talking about truth in the realist sense, a framework in which classical logic works well. However, we have just seen the reason why *modus tollens* is not valid if we are reasoning in a context with a non-explosive negation, in which *not A* does not mean *A is false*. Something analogous happens in intuitionistic logic. Once we have endorsed a constructive notion of proof, we cannot carry out a proof by cases using excluded middle because the result might not be a constructive proof. When we say that logics of formal inconsistency accept some contradictions without exploding the system, this does not mean that these contradictions are *true*. We may compare this with Kripke models for intuitionistic logic, where it can happen that a pair of propositions *A* and *not A* (intuitionistic negation) receive the value 0 in some stage (or possible world). Such a

stage would be a refutation of excluded middle. This does not mean that A and $\textit{not } A$ are *false*, however, but rather that neither has been proved yet. Analogously, when two propositions A and $\textit{not } A$ receive 1 in an mbC-valuation, as suggested above, it means that we have evidence for both, not that both are simultaneously true.

4. Final Remarks

As is well known, paraconsistent logics are non-explosive logics, that is, in these logics it is not the case that everything follows from a contradiction. For this reason, they can accommodate contradictions in a theory or in a body of knowledge. The problem is that to accept a contradiction seems to be a clear violation of the so-called ‘most basic of all principles of reasoning’, namely, the principle of non-contradiction. We have tried to show here that the task of devising a logic that disobeys the principle of non-contradiction can have good philosophical motivations.

Furthermore, we tried to show that to have available a logical formalism able to deal with contradictions is not the same as having some kind of philosophical sympathy for contradictions. We still believe that trying to avoid contradictions is an indispensable criterion of rationality. In order to do this, however, we need a logic that does not collapse in the face of a pair of propositions A and $\textit{not } A$. A contradiction may be taken as a provisional state, a kind of excess of information or excessively optimistic attitude that should, at least in principle, be eliminated by means of further investigation.

Finally, a few words about a semantics for logics of formal inconsistency. The task of finding intuitively acceptable semantics for non-classical logics is indeed a major philosophical challenge. In the case of intuitionistic logic, at first sight Kripke models seem to be elegant and faithful to the ideas of Brouwer that motivated intuitionistic logic, since they intend to represent the mind of an idealized mathematician that proves propositions as time goes on. However, it is well known that, if pressed, Kripke models for intuitionistic logic have some problems from the intuitionist point of view.

Although the bivalued semantics presented above seems to be appropriate for expressing the intuitive meanings of the consistency operator and paraconsistent negation when we are dealing with the notions of truth and evidence, and fits well the validity (or

invalidity) of several inferences, it fails to express some inferences that should be valid from the intuitive point of view, for example, those represented by schema (12) above. Perhaps we can live with this, something not unusual in non classical logics. Otherwise, there are two main options we might take: either (i) we keep the idea of a two-valued semantics but modify both the deductive system and the semantics clauses, or (ii) we try to find another kind of semantics, suitable to the idea of dealing simultaneously with truth and evidence. With respect to the latter, it is not unlikely that we could find a better way of expressing the relationship between truth and evidence by means of possible-translations semantics (Carnielli 2000). However, the task of investigating such options will be left for another time.

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