Recovery operators, paraconsistency and duality

Walter Carnielli* Marcelo Esteban Coniglio*
Abilio Rodrigues**

*Department of Philosophy and
Centre for Logic, Epistemology and the History of Science
University of Campinas, Brazil
Email: {walter.carnielli, coniglio}@cle.unicamp.br

**Departament of Philosophy
Federal University of Minas Gerais, Brazil
Email: abilio@ufmg.br

Abstract

There are two foundational, but not fully developed, ideas in paraconsistency, namely, the duality between paraconsistent and intuitionistic paradigms, and the introduction of logical operators that express meta-logical notions in the object language. The aim of this paper is to show how these two ideas can be adequately accomplished by the Logics of Formal Inconsistency (**LFI**s) and by the Logics of Formal Undeterminedness (**LFU**s). **LFI**s recover the validity of the principle of explosion in a paraconsistent scenario, while **LFU**s recover the validity of the principle of excluded middle in a paracomplete scenario. We introduce definitions of duality between inference rules and connectives that allow comparing rules and connectives that belong to different logics. Two formal systems are studied, the logics **mbC** and **mbD**, that display the duality between paraconsistency and paracompleteness as a duality between inference rules added to a common core – in the case studied here, this common core is classical positive propositional logic ($\mathbf{CPL}^{+}$). The logics **mbC** and **mbD** are equipped with *recovery operators* that restore classical logic for, respectively, consistent and determined propositions. These two logics are then combined obtaining a pair of logics of formal inconsistency and undeterminedness (**LFIU**s), namely, **mbCD** and **mbCDE**. The logic **mbCDE** exhibits some nice duality properties. Besides, it is simultaneously paraconsistent and paracomplete, and able to recover the principles of excluded middle and explosion at once. The last sections offer an algebraic account for such logics by adapting the swap-structures semantics framework of the **LFI**s the **LFU**s. This semantics highlights some subtle aspects of these logics, and allows us to prove decidability by means of finite non-deterministic matrices.

1 Introduction

Although paraconsistent logics have not been invented by da Costa, it is fairly certain that in 1963 da Costa [17] not only presented the broadest formal study of paraconsistency proposed up to that time but also established a fruitful research program in logic and philosophy of logic.[1] The role of da Costa's work in establishing paraconsistency as an area of study is undisputed.

There are two foundational ideas in da Costa's approach to paraconsistency. The first is the division of propositions into two groups: those for which explosion does not hold and those for which explosion holds. The latter are called 'well-behaved', which means that the principle of non-contradiction holds for them. It is safe to employ classical logic only for well-behaved propositions, while the others demand a non-explosive logic. The second idea is the duality between da Costa's logic C_1 and intuitionistic logic. It is clear that some kind of duality between paraconsistent and intuitionistic logic has had an important role as a motivation for the axioms da Costa chose for C_1. However, we argue here that da Costa not only missed the point regarding the duality but also mistakenly emphasized the invalidity of non-contradiction instead of explosion as the central feature of paraconsistent logics. The aim of this paper is to show how these two ideas can be developed by Logics of Formal Inconsistency (**LFI**s) and Logics of Formal

[1] The concept of a logic without an explosive negation can be traced back to 1910, in the work of Vasiliev (see [26] pp. 307ff), while the first non-explosive logic has been presented in 1948 by Jaśkowski in [28]. For a comprehensive account of the history of paraconsistency, see Gomes [26]

Undeterminedness (**LFU**s). The former recovers the validity of the principle of explosion in a paraconsistent scenario, while the latter recovers the validity of the excluded middle in a paracomplete scenario.

The remainder of this paper is organized as follows. Section 2 examines how da Costa presented the well-behavedness operator and the 'duality' between C_1 and intuitionistic logic. In Section 3 we present the consistency and determinedness connectives $\circ$ and $\star$ as *recovery operators* that restore, respectively, explosion and excluded middle. Section 4 presents two formal systems, the logics **mbC** and **mbD**, that display the duality between paraconsistency and paracompleteness as a duality between inference rules added to a common core – in the case studied here, classical positive propositional logic (**CPL**$^+$). **mbC** and **mbD** are equipped with recovery operators that restore classical logic for, respectively, consistent and determined propositions. These two logics are then combined obtaining the logics of formal inconsistency and undeterminedness (**LFIU**s), **mbCD** and **mbCDE**. Finally, the swap structures semantics framework for **LFI**s, introduced by Carnielli and Coniglio in [7, chapter 6], is adapted here for **LFU**s and **LFIU**s. This semantics allows us to prove the decidability of the proposed systems by means of finite non-deterministic matrices.

2 Well-behavedness and 'duality' in da Costa's C_n hierarchy

2.1 da Costa's well-behavedness operator

We begin by defining da Costa's C_n hierarchy.

Definition 1 (Intuitionistic Positive Logic) *The* intuitionistic positive logic **IPL**$^+$ *is defined over the signature* $\Sigma_+ = \{\wedge, \vee, \to\}$ *by the following Hilbert calculus:*

Axiom schemas:

$$\alpha \to \big(\beta \to \alpha\big) \tag{AX1}$$

$$\big(\alpha \to (\beta \to \gamma)\big) \to \big((\alpha \to \beta) \to (\alpha \to \gamma)\big) \tag{AX2}$$

$$\alpha \to \big(\beta \to (\alpha \wedge \beta)\big) \tag{AX3}$$

$$\big(\alpha \wedge \beta\big) \to \alpha \tag{AX4}$$

$$\big(\alpha \wedge \beta\big) \to \beta \tag{AX5}$$

$$\alpha \to \big(\alpha \vee \beta\big) \tag{AX6}$$

$$\beta \to \big(\alpha \vee \beta\big) \tag{AX7}$$

$$\big(\alpha \to \gamma\big) \to \big((\beta \to \gamma) \to ((\alpha \vee \beta) \to \gamma)\big) \tag{AX8}$$

Inference rule:

$$\frac{\alpha \quad \alpha \to \beta}{\beta} \tag{MP}$$

Definition 2 (da Costa's C_n hierarchy) *Let $1 \leq n < \omega$. The system C_n is defined over the signature $\Sigma_C = \{\wedge, \vee, \to, \neg\}$ by adding to the axioms of intuitionistic positive logic* **IPL**$^+$ *the following axiom schemas (see [17]):*

$$\alpha \vee \neg\alpha \tag{AxPEM}$$

$$\neg\neg\alpha \to \alpha \tag{AxDN}$$

$$\beta^{(n)} \to ((\alpha \to \beta) \to ((\alpha \to \neg\beta) \to \neg\alpha)) \tag{AxCn1}$$

$$\big(\alpha^{(n)} \wedge \beta^{(n)}\big) \to \big((\alpha \wedge \beta)^{(n)} \wedge (\alpha \vee \beta)^{(n)} \wedge (\alpha \to \beta)^{(n)}\big) \tag{AxCn2}$$

Notice that the C_n hierarchy is defined as an extension of **IPL**$^+$ but it is also an extension of Classical Positive Logic **CPL**$^+$, since $(\alpha \to \beta) \vee \alpha$ (AX9 in Definition 14) can be proved in it. Each calculus of da Costa's C_n hierarchy has its own 'well-behavedness' operator, defined inductively such that, for each n, $1 \leq n < \omega$,

$$\alpha, \neg\alpha \nvdash_{C_n} \beta, \text{ while } \alpha^{(n)}, \alpha, \neg\alpha \vdash_{C_n} \beta.$$

In formal terms:

Definition 3 *Let α be a formula in the signature Σ_C and consider the following abbreviations:*

(1) $\alpha^\circ \stackrel{def}{=} \neg(\alpha \wedge \neg\alpha)$*;*

(2) $\alpha^0 \stackrel{def}{=} \alpha$, *and* $\alpha^{n+1} \stackrel{def}{=} (\alpha^n)^\circ$, *for* $0 \leq n < \omega$*;*

(3) $\alpha^{(1)} \stackrel{def}{=} \alpha^\circ$, *and* $\alpha^{(n+1)} \stackrel{def}{=} \alpha^{(n)} \wedge \alpha^{n+1}$, *for* $1 \leq n < \omega$*.*

For instance, $\alpha^1 = \alpha^{(1)} = \alpha^\circ$; $\alpha^2 = \alpha^{\circ\circ}$ and $\alpha^{(2)} = \alpha^\circ \wedge \alpha^{\circ\circ}$; while $\alpha^3 = \alpha^{\circ\circ\circ}$ and $\alpha^{(3)} = (\alpha^\circ \wedge \alpha^{\circ\circ}) \wedge \alpha^{\circ\circ\circ}$. Thus, a proposition α behaves classically:

in C_1, when $\alpha^{(1)} = \alpha^\circ = \neg(\alpha \wedge \neg\alpha)$ holds,

in C_2, when $\alpha^{(2)} = \alpha^\circ \wedge \alpha^{\circ\circ}$ holds,

in C_3, when $\alpha^{(3)} = (\alpha^\circ \wedge \alpha^{\circ\circ}) \wedge \alpha^{\circ\circ\circ}$ holds,

and so on. As the value of n grows, the negation gets weaker, and a monotonic hierarchy of logics is obtained. However, until now, this idea of increasingly weaker logics has not been as successful as the introduction of an operator capable of expressing metalogical notions in the object language.[2]

A not well-behaved proposition α does not cause any harm in C_1, if it is contradictory. On the other hand, $\alpha \wedge \neg\alpha$ and $\neg(\alpha \wedge \neg\alpha)$ cannot hold simultaneously: the latter by definition is α°, and so, together with $\alpha \wedge \neg\alpha$, triviality follows. Thus, $\alpha \wedge \neg\alpha$ is 'axiomatically well-behaved' in C_1. This seems strange: the point is not that in a paraconsistent logic $\alpha \wedge \neg\alpha$ and $\neg(\alpha \wedge \neg\alpha)$ must *always* be allowed to hold simultaneously. The point, rather, is that $\alpha \wedge \neg\alpha$ and $\neg(\alpha \wedge \neg\alpha)$ should not be prohibited to hold simultaneously. So, it should be possible to devise paraconsistent logics such that the consistency of α is logically independent of $\neg(\alpha \wedge \neg\alpha)$.

The Logics of Formal Inconsistency (**LFI**s, see [7], [8] and [9]) are paraconsistent logics that develop da Costa's approach further by internalizing the concept of consistency within the object language using the connective $\circ$. In **LFI**s $\circ\alpha$ means that α is consistent, but $\circ$ is introduced in such a way that $\circ\alpha$ is logically independent of $\neg(\alpha \wedge \neg\alpha)$. Analogously to the C_n hierarchy,

$$\alpha, \neg\alpha \nvdash_{LFI} \beta, \text{ while } \circ\alpha, \alpha, \neg\alpha \vdash_{LFI} \beta.$$

Splitting propositions into two classes, consistent and the inconsistent, is in accordance with the fact that in a paraconsistent logic it cannot be that all contradictions are logically equivalent, otherwise the principle of explosion holds. The proof of this fact is straightforward. If all contradictions are

[2] Instead of a hierarchy in which negations get weaker, a hierarchy of logics in which *consistency* gets stronger will be presented in Section 4.2.

equivalent, then $\alpha \wedge \neg\alpha \vdash \beta \wedge \neg\beta$, for any α and β. Hence, by elimination of conjunction, $\alpha \wedge \neg\alpha \vdash \beta$. So, if a logic is paraconsistent, then it has some pairs of non-equivalent contradictions. This fact fits the idea that in real-life contexts of reasoning some contradictions are more relevant than others. Thus, it is natural to devise a connective that is able to distinguish between different kinds of contradictions – and this is precisely the feature of da Costa's approach that has led to the introduction of **LFI**s.

At first glance, it may seem that the consistency operator of **LFI**s and the well-behavedness operator of da Costa's C_n hierarchy (recall that α° means that α is well-behaved) are the same thing when applied to a proposition α. This view, however, is mistaken. **LFI**s are a generalization of da Costa's idea of expressing the meta-logical notion of consistency inside the object language. Even though the logics of C_n hierarchy (for $1 \leq n < \omega$) end up being a special case of **LFI**s, an important point distinguishes **LFI**s from da Costa's C_n. In the latter, as we have just seen, α° is an abbreviation of $\neg(\alpha \wedge \neg\alpha)$, while in **LFI**s the unary connective $\circ$ may be primitive and logically independent from non-contradiction. So, in some **LFI**s, the equivalence between $\circ\alpha$ and $\neg(\alpha \wedge \neg\alpha)$ does not hold.[3]

2.2 'Duality' in da Costa's logics

We now turn to the role of the duality between paraconsistent and intuitionistic logics in da Costa's C_n hierarchy. Although the central feature of paraconsistent logics is the invalidity of the principle of explosion, da Costa in [17] and [18], emphasizes the invalidity of the principle of non-contradiction and takes a path longer than would be necessary to recover classical logic. Let us restrict ourselves to C_1, what is enough to establish our point. In C_1 (see Definition 2), classical logic is recovered for well-behaved formulas by means of the following axiom:

$$\alpha^\circ \rightarrow ((\beta \rightarrow \alpha) \rightarrow ((\beta \rightarrow \neg\alpha) \rightarrow \neg\beta))$$

[3]The operator $\circ$ as a primitive operator, not definable in terms of non-contradiction, appears for the first time in Carnielli and Marcos [9], where the logics of formal inconsistency (**LFI**s) have been introduced. For more precise historical details, **LFI**s appeared for the first time in the *II World Congress on Paraconsistency*, held in Juquehy, SP, Brazil, in May, 2000, dedicated to the 70^{th} birthday of Newton da Costa. 'A taxonomy of C-systems' [9], was published in the proceedings of this event.

where α° is defined as $\neg(\alpha \wedge \neg\alpha)$ and means that α is 'well-behaved'. Since the emphasis is put on the invalidity of non-contradiction, explosion is recovered through an unnecessary roundabout.[4] On the other hand, in **LFI**s the principle of explosion is recovered directly, by the axiom

$$\circ\alpha \to (\alpha \to (\neg\alpha \to \beta))$$

or an equivalent inference rule. Thus, the emphasis is not on the principle of non-contradiction, but rather on the principle of explosion, i.e. on an *inference* that concludes anything from a contradiction.

It seems to us that what impelled da Costa in placing the emphasis on non-contradiction was a misunderstanding of the nature of the duality between paraconsistent and intuitionistic logics. Indeed, if we take a look at how da Costa devises C_1, the first logic of his C_n hierarchy (see Definition 2), it is not difficult to see that there is a sort of 'duality' between C_1 and intuitionistic logic. Let us consider the formulas below:

(i) $\neg(\alpha \wedge \neg\alpha)$,

(ii) $\alpha \to \neg\neg\alpha$,

(iii) $\alpha \vee \neg\alpha$,

(iv) $\neg\neg\alpha \to \alpha$.

Formulas (i) and (ii), non-contradiction and double negation introduction, hold in intuitionistic logic but do not hold in C_1. On the other hand, formulas (iii) and (iv), excluded middle and double negation elimination, thought by da Costa to be a kind of 'dual' to (i) and (ii), hold in C_1 but do not hold in intuitionistic logic.[5] In [17, p. 9], he presents an argument to justify the validity of (iv) as an axiom of C_1 that runs as follows.[6]

[4]Suppose α°, α and $\neg\alpha$. So, $\neg\beta \to \alpha$ and $\neg\beta \to \neg\alpha$. From the axiom above we get $\neg\neg\beta$. Since double negation elimination holds, we obtain β.

[5]Actually, the invalidity of the principle of non-contradiction is not an essential feature of paraconsistent logics. An example of a paraconsistent logic where explosion does not hold but non-contradiction is a valid formula is the logic of paradox (see [33]).

[6]In the original [17, p. 9]: "ou A é 'bem comportada', no sentido de que não são simultaneamente verdadeiras A e $\neg A$, sendo, então, de se esperar que se aplique a lógica clássica, donde $\neg\neg A \to A$; ou A é 'mal comportada' e tem-se A e $\neg A$, advindo que qualquer proposição deve implicar A e, em particular, que $\neg\neg A \to A$".

> Either α is well-behaved, in the sense that α and $\neg\alpha$ do not hold simultaneously, or α is not well-behaved. (i) Suppose α is well-behaved. In this case, da Costa claims that 'classical logic may be applied', which means, as far as we can see, that classical reasoning holds for α. So, $\neg\neg\alpha \to \alpha$. Let us make clear what is going on in this step of the argument: classical reasoning holds for α; in classical reasoning, $\neg\neg\alpha$ implies α; therefore, $\neg\neg\alpha \to \alpha$ holds. Notice that this argument holds in the metatheory. (ii) Now, suppose α is not well-behaved and both α and $\neg\alpha$ hold. So, anything implies α, in particular $\neg\neg\alpha \to \alpha$. This step of the argument is not metatheorical but holds in the object language, since $\alpha \to (\beta \to \alpha)$ is an axiom in C_1.

It seems, however, that an analogous argument would also justify $\alpha \to \neg\neg\alpha$.

> Either $\neg\neg\alpha$ is well-behaved or it is not. (i) Suppose $\neg\neg\alpha$ is well-behaved and classical reasoning holds for $\neg\neg\alpha$. So, since α implies $\neg\neg\alpha$ in classical logic, $\alpha \to \neg\neg\alpha$ should hold. (ii) Suppose, on the other hand, that $\neg\neg\alpha$ is not well-behaved, and both $\neg\neg\alpha$ and $\neg\neg\neg\alpha$ hold. As above, any proposition implies $\neg\neg\alpha$, in particular $\alpha \to \neg\neg\alpha$.

The central point here is step (i) of both arguments. If classical logic holds for α, and the fact that $\neg\neg\alpha$ implies α in classical logic is sufficient to conclude that $\neg\neg\alpha \to \alpha$ holds, then, when classical logic holds for $\neg\neg\alpha$, the fact that α implies $\neg\neg\alpha$ in classical logic should be sufficient to conclude that $\alpha \to \neg\neg\alpha$ holds. Our conclusion, therefore, is that da Costa's argument does not justify the validity of double negation elimination in C_1.

It is worth noting that, moreover, rejecting (ii) is strange because its invalidity does not fit with da Costa's claim in [17] that in C_1 as many schemas and rules of classical logic as possible should be valid. In fact, double negation introduction can be added to C_1 without affecting its paraconsistent properties. Let us call the system so obtained C_1'. An adequate semantics for C_1' is obtained just by adding the clause

$$v(\alpha) = 1 \quad \Longrightarrow \quad v(\neg\neg\alpha) = 1$$

to the semantics presented in [20] and [30]. Clearly, such semantics does not validate explosion, and it can easily be proved that C_1' has no trivial models. The paraconsistent logic $C_1^{\neg\neg}$, stronger than C_1', has been presented in [4]. $C_1^{\neg\neg}$ is obtained by adding to C_1 double negation introduction plus the axiom $\neg(\neg\alpha \wedge \alpha) \rightarrow \neg(\alpha \wedge \neg\alpha)$. The paraconsistent negation of $C_1^{\neg\neg}$ is still closer to classical negation. So, da Costa's claim that C_1 should contain 'the maximum possible number of schemes and deduction rules of the classical calculus' [17, p. 7] is not really pursued by him.

In the original presentation of C_1, it already seems clear that the main motivation for adopting the formulas (iii) and (iv), and rejecting (i) and (ii), was to establish a 'duality' with intuitionistic logic. But a conclusive piece of evidence for the above claim can be found in [21] where da Costa and Marconi present the hierarchy of propositional paracomplete logics P_n. There, we read:

> [in this paper] we describe a hierarchy of paracomplete logics and mention the possibility of extending it to others [i.e. to some paracomplete predicate calculi, [21, p. 508]] which are, in a certain sense, "dual" of the hierarchies [presented in [17], [18] *et al.* – i.e. C_n].

The first logic of the P_n hierarchy is P_1, obtained by adding to classical positive logic $\mathbf{CPL}^+$ (see Definition 14 below) the following axioms (α^* is defined as $\alpha \vee \neg\alpha$):

1. $\alpha^* \rightarrow ((\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow \neg\beta) \rightarrow \neg\alpha))$,
2. $(\alpha^* \wedge \beta^*) \rightarrow [(\alpha \wedge \beta)^* \wedge (\alpha \vee \beta)^* \wedge (\alpha \rightarrow \beta)^* \wedge \neg\alpha^*]$,
3. $\neg(\alpha \wedge \neg\alpha)$,
4. $\alpha \rightarrow (\neg\alpha \rightarrow \beta)$,
5. $\alpha \rightarrow \neg\neg\alpha$.

Marconi and da Costa do not explain exactly why C_n and P_n are "in a certain sense dual" to each other. There is no precise characterization of duality between logics in that paper, nor in [17], [18] and [19].[7] P_1 has axioms 3 and 5 above, precisely the formulas (i) and (ii) whose 'dual' formulas have been adopted in C_1. So, it is clear that da Costa erroneously conceived the duality

[7]Although da Costa says in [19] p. 29 that "a hierarchy of paracomplete logics was introduced [in [21]], that are «dual», in a precise sense, of some paraconsistent logics studied in [see [17] and [18]]", we have not found an explanation of the duality between the logics of C_n and P_n hierarchies.

between paraconsistency and paracompleteness as a duality between non-contradiction and excluded middle as formulas, and not between explosion and excluded middle as rules of inference. Notice also that axiom 4, the principle of explosion, had to be added to the system, together with 3 and 5, which is not surprising, since non-contradiction and explosion are logically independent (regarding P_n hierarchy, see also Remark 19 below).

Indeed, there is a duality between paraconsistent and paracomplete (so, intuitionistic) logics that gives some interesting insights and provides philosophical motivations for both (see [12]). But the central point is not that the *logics* are dual, nor that the *formulas* excluded middle and non-contradiction are dual. The point is that excluded middle and explosion are dual *inferences*. In the next section we will take a closer look at this point and show, based on the duality between paraconsistency and paracompleteness, how the idea of internalizing metatheoretical notions in the object language may be further developed.

3 Duality and recovery operators in LFIs and LFUs

We begin by defining duality between connectives in classical logic.

Definition 4 *Two n-ary logical connectives κ_1 and κ_2 are said to be dual if $\sim\kappa_1(\alpha_1, ..., \alpha_n)$ and $\kappa_2(\sim\alpha_1, ..., \sim\alpha_n)$ are materially equivalent, where $\sim$ is classical negation.*

Thus, classical negation $\sim$ is the dual of itself and $\wedge$ and $\vee$ are dual of each other. The idea of duality may also be applied to inference rules. But in order to do that we have to move to sequent calculus and multiple-conclusion logic.

The symmetry displayed by the rules of Gentzen's sequent calculus LK [23] is well known. Gentzen remarks that:

> If [the rules] $\rightarrow$-*IS* and $\rightarrow$-*IA* are excluded, the calculus LK is dual in the following sense: If we reverse all sequents of an LK-derivation (in which the $\rightarrow$-symbol does not occur), i.e., if for $\alpha_1...\alpha_m \Rightarrow \beta_1...\beta_n$ we put $\beta_1...\beta_n \Rightarrow \alpha_1...\alpha_m$, and if we exchange, in inference figures with two upper sequents, the right- and left-hand upper sequents, including their derivations, and also replace every occurrence of $\wedge$ by $\vee$, $\forall$ by $\exists$, $\vee$ by $\wedge$, and $\exists$ by $\forall$ (in the case

of $\wedge$ and $\vee$ we also have to interchange the respective scopes of the symbols, e.g., for $\beta \vee \alpha$ we have to put $\alpha \wedge \beta$, then another LK-derivation results. This can be seen at once from the schemata. (Special care was taken to arrange them in such a way as to bring out their symmetry.) (Cf. H.-A.'s duality principle, p. 62.) [23, p. 86].

Except for the implication rules $R\rightarrow$ and $L\rightarrow$, all the other rules, including the structural ones, have dual in LK. So, we may define:

Definition 5 *Two sequent calculi rules R_1 and R_2 are dual if one is obtained from the other as follows:*

1. *For each one of the premises, and for the conclusion, put the succedent in the place of the antecedent and vice-versa – i.e. change each $\alpha_1 ... \alpha_m \Rightarrow \beta_1 ... \beta_n$ with $\beta_1 ... \beta_n \Rightarrow \alpha_1 ... \alpha_m$;*

2. *Change the connective of the main formula of the rule with its dual.*

Gentzen is also concerned with the *order* of the formulas, but this does not matter when we work with multisets. An $R*$ rule, yields an $L*^d$ rule (and vice-versa), with $*^d$ being the connective dual of $*$. The rules $R\vee$ and $L\wedge$ are dual to each other, as well as the rules $L\vee$ and $R\wedge$. $R\neg$ is the dual of $L\neg$, and vice-versa. The right rules of weakening, contraction and interchange are dual of the corresponding left rules (and vice-versa), and cut is the dual of itself. Even the axiom, $\alpha \Rightarrow \alpha$, considered as a rule from no premise, is the dual of itself. So, the sense in which Gentzen says that the system LK is 'dual' is that it has *dual inference rules.*

Remark 6 *The basic idea of multiple-conclusion logic appeared for the first time in 1935 (see Gentzen [23]). Indeed, multiple-conclusion framework is suitable for expressing duality between both rules and connectives, but this is because multiple-conclusion* is already *duality. Dealing with premises and conclusions in an uniform way, as multiple-conclusion does, also allows us to deal uniformly with truth and falsehood: from the point of view of the preservation of truth, an argument goes from premises to conclusion, but from the point of view of the preservation of falsity, an argument goes the other way, from conclusion to premises. Since multiple-conclusion considers sets of premises and sets of conclusions, it acquires a symmetry that is missing in* *the single-conclusion logic. The intuitive, pretheoretical idea of logical consequence is expressed, in multiple-conclusion, in terms of sets:* $\Gamma \Vdash \Delta$ *holds when it is not possible that the propositions in* Γ *are true and the propositions in* Δ *are false. So, if all propositions in* Γ *are true, some proposition in* Δ *is true, and if all propositions in* Δ *are false, some proposition in* Γ *is false. But in the latter formulation the duality between the quantifiers all and some, and also between conjunction and disjunction, is already present. It is worth noting that Gentzen in 1935 [23, see quotation above] says that "Special care was taken to arrange them [the sequent rules of LK] in such a way as to bring out their symmetry" and immediately mentions the principle of duality presented in the first edition (1928) of Hilbert and Ackermann's book* Grundzüge der Theoretischen Logik*. We do not have access to the first edition of this book, but in [27, §5, p. 16] (translation of the second edition, 1938) such principle reads: "From a formula* $\alpha \leftrightarrow \beta$ *which is logically true, and both of whose sides are formed from elementary sentences and their negations by conjunction and disjunction only, there results another true equation by the interchange of* $\wedge$ *and* $\vee$*". Such principle shows, for example, that the formulas* $\alpha \vee (\beta \wedge \gamma) \leftrightarrow (\alpha \vee \beta) \wedge (\alpha \vee \gamma)$ *and* $\alpha \wedge (\beta \vee \gamma) \leftrightarrow (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$ *are dual to each other and logically true. It is clear then, that Hilbert and Ackermann's principle of duality (1928) already had the basic idea of Gentzen's sequent calculus LK (1935) and, moreover, this fact is acknowledged by Gentzen.*

Now, consider the negation rules of LK:

$$\frac{\Gamma \Rightarrow \Delta, \alpha}{\neg\alpha, \Gamma \Rightarrow \Delta}\ L\neg, \qquad \frac{\Gamma, \alpha \Rightarrow \Delta}{\Gamma \Rightarrow \neg\alpha, \Delta}\ R\neg.$$

Together, they characterize classical negation: in just one step from the axiom, $L\neg$ and $R\neg$ yield the following sequents

$$\alpha, \neg\alpha \Rightarrow \quad \text{and} \quad \Rightarrow \alpha, \neg\alpha$$

which means that one and at most one between α and $\neg\alpha$ holds. The rules $L\neg$ and $R\neg$, respectively, are equivalent to explosion and excluded middle in the sense that a system equivalent to LK can be defined by adding to the positive fragment of LK the rules below,

$$\frac{}{\Gamma, \alpha, \neg\alpha \Rightarrow \Delta}\ EXP \qquad \frac{}{\Gamma \Rightarrow \alpha, \neg\alpha, \Delta}\ PEM.$$

Let us call LK' the positive fragment of LK plus *PEM* and *EXP*. In LK', $L\neg$ is obtained by *EXP* and one application of *cut*, and $R\neg$ is obtained by

PEM and one application of *cut*. *EXP* and *PEM* are proved in *LK* by one application, respectively, of $L\neg$ and $R\neg$. We prefer to call *EXP* and *PEM* rules with zero premises, rather than axioms, in order to emphasize that *EXP* works like a negation-left rule and *PEM* like a negation-right rule. These rules are 'mirror images' of each other and express the fact that, classically, anything follows from $\alpha \wedge \neg\alpha$, and $\alpha \vee \neg\alpha$ follows from anything.

There are two points about Definition 5 that we want to call attention to. First, given a certain connective, it provides the dual of that connective and the rule that governs it. Second, it provides a general criterion that establishes the duality between connectives and rules that may belong to *different logics*.

In a paraconsistent logic, the principle of explosion does not hold in general. In an **LFI**, *EXP* holds only for 'consistent' propositions, i.e.

$$\frac{}{\Gamma, \circ\alpha, \alpha, \neg\alpha \Rightarrow \Delta}\ EXP^{\circ}.$$

EXP° says that if a proposition α is marked as consistent, there can be no contradiction w.r.t. α, on pain of triviality. Now, according to Definition 5, EXP° can be dualized obtaining the rule:

$$\frac{}{\Gamma \Rightarrow \star\alpha, \alpha, \neg\alpha, \Delta}\ PEM^{\bigstar}.$$

Notice that by dualizing the rule EXP° we obtain not only the connective $\star$, the dual of $\circ$, but also a paracomplete negation that is the dual of the original paraconsistent negation (see Proposition 40 below). This is because $\circ$ and $\star$ 'work together' with their respective negations (see Section 4.5).

Let us take a look at the undeterminedness connective $\star$. If both α and $\neg\alpha$ do not hold, then $\star\alpha$ holds. $\star\alpha$ thus means that α is *undetermined*. Now, a *recovery operator of determinedness* $\star$ may be obtained from $\star$ if we look at $\star\alpha$ as the classical negation of $\star\alpha$, and this is very plausible, since from the metatheoretical viewpoint, a proposition α is either determined or undetermined, and not both (we return to this point below), that is:

$$\frac{}{\Gamma, ☆\alpha \Rightarrow \alpha, \neg\alpha, \Delta}\ PEM^{☆}.$$

The rule $PEM^{☆}$ recovers the validity of excluded middle for formulas we call determined. Notice that the operator ☆, in turn, is the dual of an inconsistency operator $\bullet$, given by the following rule:

$$\frac{}{\Gamma, \alpha, \neg\alpha \Rightarrow \bullet\alpha, \Delta}\ EXP^{\bullet}.$$

The four rules $EXP^{\bullet}$, $PEM^{\star}$, EXP° and $PEM^{\bigstar}$ above (together with the positive fragment of LK) are provably equivalent to the more convenient rules (proofs left to the reader), as follows:

$$\frac{\circ\alpha, \Gamma \Rightarrow \Delta, \alpha}{\circ\alpha, \neg\alpha, \Gamma \Rightarrow \Delta}\ L\neg^{\circ} \qquad \frac{\Gamma, \alpha \Rightarrow \star\alpha, \Delta}{\Gamma \Rightarrow \neg\alpha, \star\alpha, \Delta}\ R\neg^{\bigstar}$$

$$\frac{\Gamma \Rightarrow \Delta, \alpha, \bullet\alpha}{\neg\alpha, \Gamma \Rightarrow \bullet\alpha, \Delta}\ L\neg^{\bullet} \qquad \frac{\Gamma, \star\alpha, \alpha \Rightarrow \Delta}{\star\alpha, \Gamma \Rightarrow \neg\alpha, \Delta}\ R\neg^{\star}$$

The operators $\star$ and $\circ$ are *recovery operators* in the sense that they recover a logical property (respectively excluded middle and explosion) for a proposition in their scope.

From an intuitive and metatheorical viewpoint, the connectives $\circ$ and $\bullet$, that represent respectively consistency and inconsistency, behave classically w.r.t. each other in the sense that

'α is inconsistent *iff* it is not the case that α is consistent'.

Analogous reasoning applies to the connectives $\star$ and $\bigstar$:

'α is undetermined *iff* it is not the case that α is determined'.

Of course, it is presupposed that the metalogical notions that are being expressed in the object language are such that, given a proposition α, one and at most one among $\circ\alpha$ and $\bullet\alpha$ holds (*mutatis mutandis* for $\star\alpha$ and $\bigstar\alpha$). So, if classical negation is available, we may define $\bullet\alpha$ as the classical negation of $\circ\alpha$, and $\bigstar\alpha$ as the classical negation of $\star\alpha$ (see Section 4.3.1 below).

4 The systems mbC, mbD, mbCD and mbCDE

This section reviews the logics **mbC** and **mbCD** proposed respectively in [9] and [11], and introduces their new variants **mbD** and **mbCDE**. We begin by defining the *languages* to be used in the remainder of this paper. Besides the signatures Σ_+ and Σ_C (Definitions 1 and 2), the following propositional signatures will be employed:

Definition 7 (Additional signatures)

$\Sigma_\circ = \{\wedge, \vee, \rightarrow, \neg, \circ\}$

$\Sigma_\star = \{\wedge, \vee, \rightarrow, \neg, \star\}$

$\Sigma_\circledast = \{\wedge, \vee, \rightarrow, \neg, \circledast\}$

$\Sigma_{\circ\star} = \{\wedge, \vee, \rightarrow, \neg, \circ, \star\}$
where $\circ$, $\star$ *and* $\circledast$ *are unary connectives.*

If Θ is a propositional signature, then $For(\Theta)$ will denote the (absolutely free) algebra of formulas over Θ generated by a given denumerable set $\mathcal{V} = \{p_n \ : \ n \in \mathbb{N}\}$ of propositional variables.

Let us recall from [35] the following useful notions:

Definition 8 (Tarskian logics) *Let* $\mathbf{L} = \langle For, \vdash \rangle$ *be a logic defined over a set of formulas* For*, which has a consequence relation* $\vdash$*.*

(1) $\mathbf{L}$ *is said to be* Tarskian *if it satisfies the following properties, for every* $\Gamma \cup \Delta \cup \{\alpha\} \subseteq For$*:*

(P1) if $\alpha \in \Gamma$ *then* $\Gamma \vdash \alpha$ *(Reflexivity);*

(P2) if $\Gamma \vdash \alpha$ *and* $\Gamma \subseteq \Delta$ *then* $\Delta \vdash \alpha$ *(Monotonicity);*

(P3) if $\Delta \vdash \alpha$ *and* $\Gamma \vdash \beta$ *for every* $\beta \in \Delta$ *then* $\Gamma \vdash \alpha$ *(Cut).*

(2) $\mathbf{L}$ *is said to be* finitary *if it satisfies the following:*

(P4) if $\Gamma \vdash \alpha$ *then there exists a finite subset* Γ_0 *of* Γ *such that* $\Gamma_0 \vdash \alpha$*.*

(3) $\mathbf{L}$ *is said to be* structural *if* $For = For(\Theta)$ *for a propositional signature* Θ *such that the following property holds:*

(P5) if $\Gamma \vdash \alpha$ *then* $\sigma[\Gamma] \vdash \sigma(\alpha)$*, for every substitution* σ *of formulas for variables.*

As mentioned above, **LFI**s are paraconsistent logics enriched with a primitive or defined *consistency connective* $\circ$ which allows recovering from the explosion in a 'controlled way'. Formally:

Definition 9 *Let* $\mathbf{L} = \langle \Theta, \vdash \rangle$ *be a Tarskian, finitary and structural logic defined over a propositional signature* Θ*, which contains a negation* $\neg$*, and let* $\circ$ *be a (primitive or defined) unary connective. Then,* $\mathbf{L}$ *is a* Logic of Formal Inconsistency *(***LFI***) with respect to* $\neg$ *and* $\circ$ *if the following holds:*

(i) $\alpha, \neg\alpha \nvdash \beta$ *for some* α *and* β*;*

(ii) there are two formulas α *and* β *such that*

(ii.a) $\circ\alpha, \alpha \nvdash \beta$*;*

(ii.b) $\circ\alpha, \neg\alpha \nvdash \beta$*;*

(iii) $\circ\alpha, \alpha, \neg\alpha \vdash \beta$ *for every* α *and* β*.*

Note that condition (ii) of the definition of **LFI**s is required in order to satisfy condition (iii) in a non-trivial way.

Remark 10 *The definition of an* **LFI** *presented above, in which consistency is defined by means of a single connective (Definition 9 above), is a simplified version of the general definition of* **LFI***s. In the general case, consistency can be defined by means of a nonempty set of formulas (see [8, p. 21] and [7, p. 31-33]). However, Definition 9, although characterizing a particular case of* **LFI***s as defined in [9] (see also [8]), comprises all the logics studied in [8] and [7], and also the logics called* ***C****-systems [9]. Actually, our definition of* **LFI** *here is closer to the definition of* ***C****-system [8, p. 23]. It is worth noting that the* **LFU***s defined below, analogously, could be called* ***D****-systems.*

The basic idea of **LFI**s may be extended: excluded middle may be recovered in paracomplete logics analogously to the way in which explosion is recovered in **LFI**s.

Definition 11 *Let* $\mathbf{L} = \langle \Theta, \vdash \rangle$ *be a Tarskian, finitary and structural logic defined over a propositional signature* Θ*, which contains a negation* $\neg$*. Assume that* $\mathbf{L}$ *has a (primitive or defined) disjunction* $\vee$ *which enjoys the standard property, namely: for every set of formulas* $\Gamma \cup \{\alpha, \beta\}$*,* $Cn(\Gamma \cup \{\alpha\}) \cap Cn(\Gamma \cup \{\beta\}) = Cn(\Gamma \cup \{\alpha \vee \beta\})$*, where* $Cn(\Delta) \stackrel{def}{=} \{\gamma : \Delta \vdash \gamma\}$*, for every* Δ*. Let* $\star$ *be a (primitive or defined) unary connective in* Θ*. Then,* $\mathbf{L}$ *is said to be a* Logic of Formal Undeterminedness *(***LFU***) with respect to* $\neg$ *and* $\star$ *if the following holds:*

(i) $\nvdash \alpha \vee \neg\alpha$ *for some* α*;*

(ii) there is a formula α *such that*

(ii.a) $\star\alpha \nvdash \alpha$*;*

(ii.b) $\star\alpha \nvdash \neg\alpha$*;*

(iii) $\star\alpha \vdash \alpha \vee \neg\alpha$ *for every* α*.*

If $\vee$ and $\vee'$ are two disjunctions in **L** then $\alpha \vee \beta$ and $\alpha \vee' \beta$ are interderivable, for every α and β. Thus, the definition of **LFU**s does not depend on a particular choice of a (standard) disjunction in **L**. On the other hand, condition (ii) is required in order to satisfy condition (iii) in a non-trivial way. Notice, however, that disjunctions in Definition 11 could be completely dispensed with, had we defined **LFU**s in the framework of multiple-conclusion logics.

Remark 12 *The concept of Logics of Formal Undeterminedness has been introduced by Marcos in [31], but the idea of recovering excluded middle analogously to how non-contradiction and explosion are recovered in* C_n *and* **LFI***s, as we have seen in Section 2.2, can be traced back to da Costa and Marconi in [21]. Carnielli and Rodrigues in [12] presented a conceptual approach to the duality, arguing that from an epistemic viewpoint paracomplete and paraconsistent logics may be understood, respectively, as dealing with a notion stronger and weaker than truth. Some ideas presented by Marcos in [31] matches our interest in the duality between paraconsistency and paracompleteness, and we have tried to develop these ideas further in this paper However, Marcos approaches the duality from a different viewpoint to ours: he is concerned with paracomplete and paraconsistent negations defined in modal terms, respectively, as* $\frown\alpha \stackrel{def}{=} \Box{\sim}\alpha$ *and* $\smile\alpha \stackrel{def}{=} \Diamond{\sim}\alpha$ *[31, p. 292]. We have adopted the same symbols for consistency, inconsistency, determinedness and undeterminedness connectives (respectively,* $\circ$, $\bullet$, $\star$ *and* $\bigstar$*) [31, p. 290], but we do not define them in terms of other connectives because we are interested mainly in* $\circ$ *and* $\star$ *as primitive recovery operators. As an aside, it seems strange to us, though, why Marcos defines undeterminedness,* $\bigstar\alpha$ *as* $\alpha \vee \frown\alpha$*, since determinedness,* $\star\alpha$*, should be so defined.*

Now, we combine the features of **LFI**s and **LFU**s to define a class of paracomplete and paraconsistent logics in which explosion and excluded middle may be recovered, at once or one at a time.

Definition 13 *Let* $\mathbf{L} = \langle \Theta, \vdash \rangle$ *be a Tarskian, finitary and structural logic defined over a propositional signature* Θ*, which contains a negation* $\neg$*. Assume that* $\mathbf{L}$ *has a (primitive or defined) disjunction* $\vee$ *which is standard in the sense of Definition 11. Let* $\star$ *and* $\circ$ *be two (primitive or defined, possibly equal) unary connectives. Then,* $\mathbf{L}$ *is said to be a* Logic of Formal Inconsistency and Undeterminedness *(***LFIU***) with respect to* $\neg$*,* $\star$ *and* $\circ$ *if the following holds:*[8]

(i) $\alpha, \neg\alpha \nvdash \beta$ *for some* α *and* β*;*

(ii) $\nvdash \alpha \vee \neg\alpha$ *for some* α*;*

(iii) there is a formula α *such that*

- *(iii.a)* $\star\alpha \nvdash \alpha$*;*
- *(iii.b)* $\star\alpha \nvdash \neg\alpha$*;*

(iv) there are two formulas α *and* β *such that*

- *(iv.a)* $\circ\alpha, \alpha \nvdash \beta$*;*
- *(iv.b)* $\circ\alpha, \neg\alpha \nvdash \beta$*;*

(v) For every formula α *and* β*:*

- *(v.a)* $\star\alpha \vdash \alpha \vee \neg\alpha$*;*
- *(v.b)* $\circ\alpha, \alpha, \neg\alpha \vdash \beta$*.*

If $\mathbf{L}$ *is an* **LFIU** *such that* $\star\alpha$ *and* $\circ\alpha$ *are interderivable for every formula* α *(and in particular if* $\star = \circ$*), then* $\mathbf{L}$ *is said to be a* strict **LFIU***.*

As in the case of **LFU**s, the definition of **LFIU**s does not depend on a particular choice of standard disjunction in $\mathbf{L}$. On the other hand, conditions (iii) and (iv) are required in order to satisfy condition (v) in a non-trivial way.

Now, we define the basic **LFI**s, **LFU**s and **LFIU**s as extensions of classical positive logic. Following the usual presentation of **LFI**s, these systems will be introduced by means of Hilbert calculi (below each one of these logics $\mathbf{L}$ will be reintroduced by means of a sequent calculus $\mathbf{L}_S$).

[8] Logic systems with a negation simultaneously paraconsistent and paracomplete are called *paranormal* by some authors.

Definition 14 (Classical Positive Logic) *The* classical positive logic $\mathbf{CPL}^+$, *defined over the language* $For(\Sigma_+)$, *is obtained from intuitionistic positive logic* $\mathbf{IPL}^+$ *(Definition 1) by adding the following axiom:*

$$\big(\alpha \to \beta\big) \vee \alpha \qquad (AX9)$$

Definition 15 (Paraconsistent logic mbC) *The logic* **mbC**, *defined over the language* $For(\Sigma_\circ)$, *is obtained from* $\mathbf{CPL}^+$ *by adding the following (see [9]):*

$$\alpha \vee \neg\alpha \qquad (AxPEM)$$

$$\circ\alpha \to \big(\alpha \to \big(\neg\alpha \to \beta\big)\big) \qquad (GEXP)$$

As is well-known, the logic **mbC** is an **LFI**. It is paraconsistent and the unary operator $\circ$ recovers explosion by means of the axiom *GEXP* (also called 'gentle explosion principle', tantamount here to EXP°). We now define a paracomplete logic where the operator $\star$ recovers the excluded middle by means of the axiom *GPEM* ('gentle excluded middle', tantamount here to $PEM^\star$).

Definition 16 (The paracomplete logic mbD)
The logic **mbD**, *defined over the language* $For(\Sigma_\star)$, *is obtained from* $\mathbf{CPL}^+$ *by adding the following:*

$$\star\alpha \to \big(\alpha \vee \neg\alpha\big) \qquad (GPEM)$$

$$\alpha \to \big(\neg\alpha \to \beta\big) \qquad (AxEXP)$$

Both properties of negation can be recovered simultaneously in a paracomplete and paraconsistent system which combines the previous ones:

Definition 17 (Paraconsistent and paracomplete logic mbCD)
The logic **mbCD**, *defined over the language* $For(\Sigma_\circledast)$, *is obtained from* $\mathbf{CPL}^+$ *by adding the following (see [11]):*

$$\circledast\alpha \to \big(\alpha \vee \neg\alpha\big) \qquad (GPEM)$$

$$\circledast\alpha \to \big(\alpha \to \big(\neg\alpha \to \beta\big)\big) \qquad (GEXP)$$

The logic **mbCD** is a strict **LFIU** based on $\mathbf{CPL}^+$, in which $\circledast$ can be seen as a *classicality operator*, where $\circledast\alpha$ indicates that α behaves classically, and so α obeys the laws of classical logic. Of course, the two properties of classical negation mentioned in Definition 17 could be recovered separately by means of a specific connective, that is, by means of a non-strict **LFIU**. This motivates the following:

Definition 18 (Paraconsistent and paracomplete logic mbCDE) *The logic* **mbCDE**, *defined over the language* $For(\Sigma_{\circ\star})$, *is obtained from* $\mathbf{CPL}^+$ *by adding the following:*

$$\star\alpha \to \big(\alpha \vee \neg\alpha\big) \qquad (GPEM)$$

$$\circ\alpha \to \big(\alpha \to \big(\neg\alpha \to \beta\big)\big) \qquad (GEXP)$$

Remark 19 *In 1986 da Costa and Marconi introduced in [21] a hierarchy P_n (for $1 \leq n < \omega$) of paracomplete logics intended to be 'dual' to the hierarchy C_n (for $1 \leq n < \omega$) of paraconsistent logics (see Section 2, p. 9 above). Notice that this approach is analogous to* **mbD**, *and P_1 and* **mbD** *have the same relationship that holds between C_1 and* **mbC**. *Additionally, in 1989 da Costa proposed in [19] a hierarchy N_n (for $1 \leq n < \omega$) of paraconsistent and paracomplete logics based on* $\mathbf{CPL}^+$, *called 'nonalethic', intended to simultaneously generalize the hierarchies C_n and P_n. The system N_1, the first logic of the N_n hierarchy, has some analogy with the system* **mbCDE**. *In [19], the system obtained from N_1 by adding the axiom schema α° is P_1; the system obtained from N_1 by adding the axiom schema α^* is C_1; and the system obtained from N_1 by simultaneously adding the axiom schemas α° and α^* is* **CPL**. *These features are analogous to the ones described for* **mbCDE** *in Remark 37 below. We could say that the relationship between N_1 and* **mbCDE** *is the same as the one between C_1 and* **mbC** *and the one between P_1 and* **mbD**.

Now, in order to emphasize the duality between their connectives and rules, the same logics will be presented by means of sequent calculi. The equivalence between both presentations will be obtained in Corollary 32 below.

Definition 20 (Sequent calculus for Classical Positive Logic) *Let* $\mathbf{CPL}_S^+$ *be the sequent calculus for classical positive propositional logic* $\mathbf{CPL}^+$ *defined over the language* $For(\Sigma_+)$ *by the following rules:*

$$\alpha \Rightarrow \alpha \quad \textit{Axiom}$$

$$\frac{\Gamma \Rightarrow \Delta}{\alpha, \Gamma \Rightarrow \Delta}\ \textit{L-Weak} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \alpha}\ \textit{R-Weak}$$

$$\frac{\alpha, \alpha, \Gamma \Rightarrow \Delta}{\alpha, \Gamma \Rightarrow \Delta}\ \textit{L-Cont} \qquad \frac{\Gamma \Rightarrow \alpha, \alpha, \Delta}{\Gamma \Rightarrow \Delta, \alpha}\ \textit{R-Cont}$$

$$\frac{\alpha, \beta, \Gamma \Rightarrow \Delta}{\alpha \wedge \beta, \Gamma \Rightarrow \Delta}\ L\wedge \qquad \frac{\Gamma \Rightarrow \alpha, \Delta \quad \Gamma \Rightarrow \beta, \Delta}{\Gamma \Rightarrow \alpha \wedge \beta, \Delta}\ R\wedge$$

$$\frac{\Gamma, \alpha \Rightarrow \Delta \quad \Gamma, \beta \Rightarrow \Delta}{\Gamma, \alpha \vee \beta \Rightarrow \Delta}\ L\vee \qquad \frac{\Gamma \Rightarrow \alpha, \beta, \Delta}{\Gamma \Rightarrow \alpha \vee \beta, \Delta}\ R\vee$$

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \beta, \Gamma \Rightarrow \Delta}{\alpha \to \beta, \Gamma \Rightarrow \Delta}\ L \to \qquad \frac{\Gamma, \alpha \Rightarrow \beta, \Delta}{\Gamma \Rightarrow \alpha \to \beta, \Delta}\ R \to$$

$$\frac{\Gamma \Rightarrow A, \Delta \quad \Gamma', A \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta', \Delta}\ \textit{Cut}$$

Concerning **mbC** the following calculus was proposed by T. Rodrigues in [34]:

Definition 21 (Sequent calculus for mbC) *Let* $\mathbf{mbC}_S$ *be the sequent calculus for* **mbC** *over the language* $For(\Sigma_\circ)$ *defined by adding to* $\mathbf{CPL}_S^+$ *the following rules:*

$$\frac{\circ\alpha, \Gamma \Rightarrow \Delta, \alpha}{\circ\alpha, \neg\alpha, \Gamma \Rightarrow \Delta}\ L\neg^\circ \qquad \frac{\Gamma, \alpha \Rightarrow \Delta}{\Gamma \Rightarrow \neg\alpha, \Delta}\ R\neg$$

Note that the rules $L\neg^\circ$ and $R\neg$ correspond to the axioms *GEXP* and *AxPEM* of Definition 15, respectively.

Definition 22 (Sequent calculus for mbD) *Let* $\mathbf{mbD}_S$ *be the sequent calculus for* **mbD** *over the language* $For(\Sigma_{\star})$ *defined by adding to* $\mathbf{CPL}_S^+$ *the following rules:*

$$\frac{\Gamma \Rightarrow \Delta, \alpha}{\neg\alpha, \Gamma \Rightarrow \Delta}\ L\neg \qquad \frac{\Gamma, \star\alpha, \alpha \Rightarrow \Delta}{\star\alpha, \Gamma \Rightarrow \neg\alpha, \Delta}\ R\neg^{\star}$$

Note that the rules $L\neg$ and $R\neg^{\star}$ correspond to the axioms *AxEXP* and *GPEM* of Definition 16, respectively.

Definition 23 (Sequent calculus for mbCD) *Let* $\mathbf{mbCD}_S$ *be the sequent calculus for* **mbCD** *defined over the language* $For(\Sigma_{\circledast})$ *defined by adding to* $\mathbf{CPL}_S^+$ *the following rules:*

$$\frac{\circledast\alpha, \Gamma \Rightarrow \Delta, \alpha}{\circledast\alpha, \neg\alpha, \Gamma \Rightarrow \Delta}\ L\neg^{\circledast} \qquad \frac{\Gamma, \circledast\alpha, \alpha \Rightarrow \Delta}{\circledast\alpha, \Gamma \Rightarrow \neg\alpha, \Delta}\ R\neg^{\circledast}$$

Definition 24 (Sequent calculus for mbCDE) *Let* $\mathbf{mbCDE}_S$ *be the sequent calculus for* **mbCDE** *defined over the language* $For(\Sigma_{\circ\star})$ *defined by adding to* $\mathbf{CPL}_S^+$ *the following rules:*

$$\frac{\circ\alpha, \Gamma \Rightarrow \Delta, \alpha}{\circ\alpha, \neg\alpha, \Gamma \Rightarrow \Delta}\ L\neg^{\circ} \qquad \frac{\Gamma, \star\alpha, \alpha \Rightarrow \Delta}{\star\alpha, \Gamma \Rightarrow \neg\alpha, \Delta}\ R\neg^{\star}$$

In order to deal with similar logics in an homogeneous way, we now define the following collections of formal systems:

Definition 25

$$\mathcal{L} \stackrel{def}{=} \{\mathbf{mbC}, \mathbf{mbD}, \mathbf{mbCD}, \mathbf{mbCDE}\}$$

$$\mathcal{L}_S \stackrel{def}{=} \{\mathbf{mbC}_S, \mathbf{mbD}_S, \mathbf{mbCD}_S, \mathbf{mbCDE}_S\}$$

$$\mathcal{L}^+ \stackrel{def}{=} \{\mathbf{CPL}^+, \mathbf{mbC}, \mathbf{mbD}, \mathbf{mbCD}, \mathbf{mbCDE}\}$$

$$\mathcal{L}_S^+ \stackrel{def}{=} \{\mathbf{CPL}_S^+, \mathbf{mbC}_S, \mathbf{mbD}_S, \mathbf{mbCD}_S, \mathbf{mbCDE}_S\}.$$

Note that each $\mathbf{L}$ in $\mathcal{L}$ (resp. in $\mathcal{L}^+$) has a corresponding element $\mathbf{L}_S$ in $\mathcal{L}_S$ (resp. in $\mathcal{L}_S^+$).

4.1 Valuation semantics

As was done with several **LFI**s, in particular with **mbC** (see [8, 7]), valuation semantics over $\{0,1\}$ are now defined so as to characterize the formal systems presented in the previous section.

Definition 26 (Valuations)
(1) A function $v : For(\Sigma_+) \to \{0,1\}$ *is a* valuation for $\mathbf{CPL}^+$ and $\mathbf{CPL}^+_S$ *if it satisfies the following:*

(*vAnd*) $v(\alpha \wedge \beta) = 1 \iff v(\alpha) = 1$ *and* $v(\beta) = 1$

(*vOr*) $v(\alpha \vee \beta) = 1 \iff v(\alpha) = 1$ *or* $v(\beta) = 1$

(*vImp*) $v(\alpha \to \beta) = 1 \iff v(\alpha) = 0$ *or* $v(\beta) = 1$

(2) A function $v : For(\Sigma_\circ) \to \{0,1\}$ *is a* valuation for **mbC** and $\mathbf{mbC}_S$ *if it satisfies the clauses for a* $\mathbf{CPL}^+$*-valuation, plus the following:*

(*vNeg*) $v(\neg\alpha) = 0 \implies v(\alpha) = 1$

(*vCon*) $v(\circ\alpha) = 1 \implies \big(v(\alpha) = 0$ *or* $v(\neg\alpha) = 0\big)$.

(3) A function $v : For(\Sigma_\star) \to \{0,1\}$ *is a* valuation for **mbD** and $\mathbf{mbD}_S$*, if it satisfies the clauses for a* $\mathbf{CPL}^+$*-valuation, plus the following:*

(*vNeg*D) $v(\neg\alpha) = 1 \implies v(\alpha) = 0$

(*vCon*D) $v(\star\alpha) = 1 \implies \big(v(\alpha) = 1$ *or* $v(\neg\alpha) = 1\big)$.

(4) A function $v : For(\Sigma_\circledast) \to \{0,1\}$ *is a* valuation for **mbCD** and $\mathbf{mbCD}_S$*, if it satisfies the clauses for a* $\mathbf{CPL}^+$*-valuation, plus the following:*

(*vCon*CD) $v(\circledast\alpha) = 1 \implies \big(v(\alpha) = 1$ *iff* $v(\neg\alpha) = 0\big)$.

(5) A function $v : For(\Sigma_{\circ\star}) \to \{0,1\}$ *is a* valuation for **mbCDE** and $\mathbf{mbCDE}_S$*, if it satisfies the clauses for a* $\mathbf{CPL}^+$*-valuation, plus the following:*

(*vCon*D) $v(\star\alpha) = 1 \implies \big(v(\alpha) = 1$ *or* $v(\neg\alpha) = 1\big)$

(*vCon*) $v(\circ\alpha) = 1 \implies \big(v(\alpha) = 0$ *or* $v(\neg\alpha) = 0\big)$.

For every logic $\mathbf{L} \in \mathcal{L}^+ \cup \mathcal{L}^+_S$ (recall Definition 25) let $\vdash_\mathbf{L}$ and $\vDash_\mathbf{L}$ be the consequence relation of **L** w.r.t. derivations (in the corresponding calculus) and w.r.t. its valuations, respectively. As specified in Definition 26, $\vDash_\mathbf{L} = \vDash_{\mathbf{L}_S}$

for every $\mathbf{L} \in \mathcal{L}^+$. To give a uniform treatment in the framework of Tarskian logics, we define: $\Gamma \vdash_{\mathbf{L}_S} \varphi$ iff the sequent $\Gamma \Rightarrow \varphi$ is derivable in the sequent calculus $\mathbf{L}_S$.

Theorem 27 (Soundness) *Let* $\mathbf{L} \in \mathcal{L}^+ \cup \mathcal{L}^+_S$. *For every set* $\Gamma \cup \{\varphi\}$ *of formulas of* $\mathbf{L}$*:* $\Gamma \vdash_{\mathbf{L}} \varphi$ *implies* $\Gamma \vDash_{\mathbf{L}} \varphi$.

Proof. Straightforward. □

As an immediate application of the soundness for each $\mathbf{L}$, it is easy to prove the following:

Proposition 28
(1) The logic **mbD** *(resp.* $\mathbf{mbD}_S$*) is an* **LFU**.
(2) The logic **mbCD** *(resp.* $\mathbf{mbCD}_S$*) is a strict* **LFIU**.
(3) The logic **mbCDE** *(resp.* $\mathbf{mbCDE}_S$*) is a non-strict* **LFIU**.

Proof. (1) Let p be a propositional variable, and consider a valuation v for **mbD** and $\mathbf{mbD}_S$ such that $v(p) = v(\neg p) = 0$. From this, $\nvDash_{\mathbf{mbD}} p \vee \neg p$ and so $\nvdash_{\mathbf{mbD}} p \vee \neg p$ and $\nvdash_{\mathbf{mbD}_S} p \vee \neg p$, by soundness. Now, let v' be a valuation for **mbD** such that $v'(p) = 0$ and $v'(\star p) = v'(\neg p) = 1$. Then $\star p \nvDash_{\mathbf{mbD}} p$ and so $\star p \nvdash_{\mathbf{mbD}} p$ and $\star p \nvdash_{\mathbf{mbD}_S} p$, by soundness. Analogously it is shown that $\star p \nvdash_{\mathbf{mbD}} \neg p$ and $\star p \nvdash_{\mathbf{mbD}_D} \neg p$. Finally, condition (iii) of Definition 11 follows by axiom (*GPEM*) and (*MP*) (in the case of **mbD**) and from $PEM^{\star}$ that, as we have seen in Section 3, holds in $\mathbf{mbD}_S$. Items (2) and (3): the proof is analogous, and is left to the reader. □

In order to prove completeness, first it is necessary to recall some notions. A set of formulas Γ of a (Tarskian) logic $\mathbf{L}$ is *maximal relative to a formula* α *in* $\mathbf{L}$ if $\Gamma \nvdash_{\mathbf{L}} \alpha$, but $\Gamma, \beta \vdash_{\mathbf{L}} \alpha$ whenever $\beta \notin \Gamma$. If Γ is maximal relative to α in $\mathbf{L}$ then it is a closed theory in $\mathbf{L}$, that is: $\Gamma \vdash_{\mathbf{L}} \beta$ iff $\beta \in \Gamma$, for every formula β. Recall the following classical result:

Theorem 29 (Lindenbaum-Łos) *Let* $\mathbf{L}$ *be a Tarskian and finitary logic over a language* For*, and let* $\Gamma \cup \{\alpha\} \subseteq For$ *such that* $\Gamma \nvdash_{\mathbf{L}} \alpha$. *Then there exists a set* Δ *such that* $\Gamma \subseteq \Delta \subseteq For$ *and* Δ *is maximal relative to* α *in* $\mathbf{L}$.

Proof. See [35, Theorem 22.2]. □

The proof of the following result is straightforward:

Proposition 30 *Let $\mathbf{L} \in \mathcal{L}^+ \cup \mathcal{L}_S^+$, and let Γ be a set of formulas of $\mathbf{L}$ which is maximal relative to a formula φ of $\mathbf{L}$. Let α and β be formulas of $\mathbf{L}$. Then:*
(1) $\alpha \wedge \beta \in \Gamma$ iff $\alpha \in \Gamma$ and $\beta \in \Gamma$.
(2) $\alpha \vee \beta \in \Gamma$ iff $\alpha \in \Gamma$ or $\beta \in \Gamma$.
(3) $\alpha \to \beta \in \Gamma$ iff $\alpha \notin \Gamma$ or $\beta \in \Gamma$.
(4) If $\mathbf{L} = \mathbf{mbC}$ then:
(4.1) If $\neg\alpha \notin \Gamma$ then $\alpha \in \Gamma$.
(4.2) If $\circ\alpha \in \Gamma$ then $\neg\alpha \notin \Gamma$ or $\alpha \notin \Gamma$.
(5) If $\mathbf{L} = \mathbf{mbD}$ then:
(5.1) If $\neg\alpha \in \Gamma$ then $\alpha \notin \Gamma$.
(5.2) If $\star\alpha \in \Gamma$ then $\neg\alpha \in \Gamma$ or $\alpha \in \Gamma$.
(6) If $\mathbf{L} = \mathbf{mbCD}$ and $\circledast\alpha \in \Gamma$ then: $\neg\alpha \in \Gamma$ iff $\alpha \notin \Gamma$.
(7) If $\mathbf{L} = \mathbf{mbCDE}$ then:
(7.1) If $\star\alpha \in \Gamma$ then $\neg\alpha \in \Gamma$ or $\alpha \in \Gamma$.
(7.2) If $\circ\alpha \in \Gamma$ then $\neg\alpha \notin \Gamma$ or $\alpha \notin \Gamma$.

It is easy to see that every logic $\mathbf{L} \in \mathcal{L}^+ \cup \mathcal{L}_S^+$ is Tarskian, finitary and structural (recall Definition 8). Thus, Theorem 29 holds for all of them and so completeness follows easily:

Theorem 31 (Completeness) *Let $\mathbf{L} \in \mathcal{L}^+ \cup \mathcal{L}_S^+$. For every set $\Gamma \cup \{\varphi\}$ of formulas of $\mathbf{L}$: $\Gamma \vDash_{\mathbf{L}} \varphi$ implies $\Gamma \vdash_{\mathbf{L}} \varphi$.*

Proof. Suppose that $\Gamma \nvdash_{\mathbf{L}} \varphi$, and let Δ be a set of formulas of $\mathbf{L}$ such that $\Gamma \subseteq \Delta$ and Δ is maximal relative to φ in $\mathbf{L}$ (Theorem 29). Let v be the mapping from the set of formulas of $\mathbf{L}$ to $\{0, 1\}$ defined as follows: $v(\varphi) = 1$ iff $\varphi \in \Delta$, for every φ. By using Proposition 30, it is easy to see that v is a valuation for $\mathbf{L}$ such that $v[\Gamma] \subseteq \{1\}$ but $v(\varphi) = 0$. Thus $\Gamma \nvDash_{\mathbf{L}} \varphi$. □

Corollary 32 *Let $\mathbf{L} \in \mathcal{L}^+$. Then $\vdash_{\mathbf{L}} = \vdash_{\mathbf{L}_S}$. That is, for every set $\Gamma \cup \{\varphi\}$ of formulas of $\mathbf{L}$: $\Gamma \vdash_{\mathbf{L}} \varphi$ iff $\Gamma \vdash_{\mathbf{L}_S} \varphi$.*

4.2 A hierarchy based on stronger notions of consistency

As we have said in Section 2, up to now the idea of a hierarchy of increasingly weaker logics, proposed by da Costa, has not been as successful as the introduction of an operator capable of expressing meta-logical notions in the object language. This section proposes a hierarchy of logics based on **mbC**, called $\mathbf{mbC}_n$ $(1 \leq n < \omega)$, in which consistency (i.e. the condition for recovering explosion in $\mathbf{mbC}_n$) gets stronger as n grows up. It might be instructive to make an analogy with tribunal systems in many countries, with their own structure for dealing with cases and appeals. As we made clear, the decision about consistency of a judgment is always performed outside the formal system, and thus it becomes determined from outside whether $\circ\alpha$ is true or not. This procedure may be regarded as being produced by a trial court (or a district court). However, a second, higher level court, may decide whether that mechanism is itself consistent, in the sense that it does not produce $\circ\alpha$ and $\neg\circ\alpha$. If so, this court establishes that $\circ(\circ\alpha)$ is true. So it may be that

$$\circ\alpha, \alpha, \neg\alpha \nvdash \beta, \text{ while } \circ\circ\alpha, \circ\alpha, \alpha, \neg\alpha \vdash \beta,$$

or in general, for $\circ_n\alpha = \circ(\circ_{n-1}\alpha)$,

$$\circ_n\alpha, \circ_{n-1}\alpha, \cdots, \circ_1\alpha, \alpha, \neg\alpha \nvdash \beta, \text{ while } \circ_{n+1}\alpha, \circ_n\alpha, \cdots, \circ_1\alpha, \alpha, \neg\alpha \vdash \beta.$$

The general case would correspond to a hierarchy of higher and higher level courts, which might end up in a Kafkian chain of appellation courts in the limit case – but of course, one may envisage practical situations in which two or three levels could suffice. The idea is that it should be possible to express degrees of consistency and to establish a point in which classical reasoning is restored.

The simplest hierarchy may be obtained just by iterating the unary consistency operator $\circ$. The hierarchy $\mathbf{mbC}_n$ is thus defined by replacing $GEXP$ (see Definition 15) with the axiom $GEXP^n$ (*iterated gentle explosion principle*),

$$\circ_n\alpha \to (\circ_{n-1}\alpha \to \cdots \to (\circ_1\alpha \to (\alpha \to (\neg\alpha \to \beta)\cdots) \qquad (GEXP^n)$$

for each n, where $\circ_n\alpha = \circ(\circ_{n-1}\alpha)$. In this case, there is only one connective $\circ$, that may be primitive or defined. As much as for **mbC**, each $\mathbf{mbC}_n$ is an **LFI**, and explosion is recovered by means of the axiom $GEXP^n$. Note that we need all the premises $\circ_n\alpha$, $\circ_{n-1}\alpha, \cdots, \circ_1\alpha$ because each statement of the form $\circ_i\beta$ is a guarantee that β will never be contradictory, but not that β is asserted. In other words, a statement of the form $\circ\beta$ is a negative stipulation, or a clause which expressly prevents contradictions, not a positive utterance.

Simple iteration, although illustrative, is not the only possible way to formulate an axiomatization for $\mathbf{mbC}_n$. Alternatively, primitive or defined connectives $\circ_1, \circ_2, \circ_3, \cdots, \circ_n$ may be conceived independently of each other. They may express, for example, different and independent criteria that together constitute conclusive evidence for a proposition α. So, $\circ_1\alpha, \circ_2\alpha, \circ_3\alpha, \cdots,$ $\circ_n\alpha$ are, together, a sufficient condition and each $\circ_i\alpha$ is a necessary condition for establishing α conclusively, and the fact that the truth of α has been conclusively established is expressed by the validity of explosion w.r.t. α. In this case, the premises could be represented by a set, $\circledcirc\alpha = \{\circ_1\alpha, \circ_2\alpha, \circ_3\alpha, \cdots \circ_n\alpha\}$, such that for any $\circledcirc\alpha'$ proper subset of $\circledcirc\alpha$,

$$\circledcirc\alpha', \alpha, \neg\alpha \nvdash \beta, \text{ while } \circledcirc \alpha, \alpha, \neg\alpha \vdash \beta.$$

Analogous approaches may be applied to **mbD** and **mbCD**, in order to produce hierarchies of determinedness and classicality, and the swap structures semantics framework can also be adapted for those logics.

Remark 33 *In [15], Ciuciura presents a hierarchy of* **LFI***s called* $\mathbf{mbC}^n$ *($1 \leq n < \omega$), in which the consistency operator $\circ_n$ of* $\mathbf{mbC}^n$ *is given by* $\neg^2\alpha \wedge \neg^3\alpha \wedge \cdots \wedge \neg^{n+1}\alpha$. *Thus, $\circ\alpha = \circ_1\alpha \overset{def}{=} \neg\neg\alpha$ expresses the consistency operator in* $\mathbf{mbC}^1$. *Since the consistency operators $\circ_n$ are expressed in terms of the others (namely, negation $\neg$ and conjunction $\wedge$), these systems are in fact* **dC***-systems, a sub-class of* **LFI***s as defined in [9, Subsection 3.8] (see also [8, Definition 32] and [7, Section 3.3]). The author claims that* $\mathbf{mbC}^1$ *"essentially coincides with* **mbC***" ([15, pag. 174]). However, this is not the case: just note that adding $\neg\neg\alpha \to \alpha$ to* $\mathbf{mbC}^1$ *yields classical logic* **CPL***, while adding $\alpha \to \neg\neg\alpha$ to* $\mathbf{mbC}^1$ *yields a logic which is not really paraconsistent, that is, a logic controllably explosive w.r.t. the formula schema $\neg p_0$, i.e. $\neg\alpha, \neg\neg\alpha \vdash \beta$ for every α and β (see [8, Definition 9]). On the other hand, it is well-known that* **mbC** *can be expanded either by $\neg\neg\alpha \to \alpha$, by $\alpha \to \neg\neg\alpha$ or by both without crashing into* **CPL** *or into a controllably explosive logic. Moreover, in the abstract of [15] it is claimed that the construction of the hierarchy* $\mathbf{mbC}^n$ *"makes the connective of consistency redundant". Indeed, the consistency operator in* $\mathbf{mbC}^n$ *is innocuous, since it is a defined notion (as in da Costa's C_n), but in general* **LFI***s consistency operators are by no means redundant.*

4.3 Recovering classical logic

Classical logic may be recovered in the logics of the family $\mathcal{L}$ (and so in the family $\mathcal{L}_S$) in two ways: by defining a classical negation, and by means of a derivability adjustment theorem (DAT).

4.3.1 Defining classical negation

Recall, from the discussion after Remark 6, that a classical negation, in a given sequent calculus, is a unary connective $\sim$ (primitive or defined) satisfying the rules *EXP* and *PEM* (resp. $L\sim$ and $R\sim$, see Section 3). In a Hilbert calculus, this is equivalent to saying that $\sim$ satisfies, respectively, the schemas *AxPEM* and *AxEXP* (see Definitions 15 and 16).

Proposition 34 *For* $\mathbf{L} \in \mathcal{L}_S$*, a classical negation is definable in* $\mathbf{L}$*.*

Proof. In $\mathbf{mbC}_S$ and $\mathbf{mbCDE}_S$, define: $\bot \stackrel{\text{def}}{=} \circ\alpha \wedge (\alpha \wedge \neg\alpha)$ for any formula α. In $\mathbf{mbCD}_S$ define: $\bot \stackrel{\text{def}}{=} \circledast\alpha \wedge (\alpha \wedge \neg\alpha)$ for any formula α. In $\mathbf{mbD}_S$, define: $\bot \stackrel{\text{def}}{=} \alpha \wedge \neg\alpha$ for any formula α. In $\mathbf{L} \in \mathcal{L}_S$, define $\sim\alpha \stackrel{\text{def}}{=} \alpha \to \bot$. Now, we prove that

$$\frac{\Gamma \Rightarrow \Delta, \alpha}{\sim\alpha, \Gamma \Rightarrow \Delta}\ L\sim \qquad \text{and} \qquad \frac{\Gamma, \alpha \Rightarrow \Delta}{\Gamma \Rightarrow \sim\alpha, \Delta}\ R\sim$$

hold in $\mathbf{L} \in \mathcal{L}_S$.

(i) $L\sim$ holds in $\mathbf{mbC}_S$ and $\mathbf{mbCDE}_S$:

$$\dfrac{\Gamma \Rightarrow \Delta, \alpha \qquad \dfrac{\dfrac{\dfrac{\Gamma, \circ\alpha, \alpha, \Rightarrow \alpha, \Delta}{\Gamma, \circ\alpha, \alpha, \neg\alpha \Rightarrow \Delta}\ L\neg^{\circ}}{\Gamma, \circ\alpha \wedge \alpha \wedge \neg\alpha \Rightarrow \Delta}\ L\wedge}{\Gamma, \bot \Rightarrow \Delta}\ Def\bot}{\alpha \to \bot, \Gamma \Rightarrow \Delta}\ L\to$$

(ii) $L\sim$ holds in $\mathbf{mbCD}_S$: the proof is analogous to that given in item (i), but now using the rules for $\circledast$.

(iii) $L\sim$ holds in $\mathbf{mbD}_S$:

$$\dfrac{\Gamma \Rightarrow \Delta, \alpha \qquad \dfrac{\dfrac{\dfrac{\Gamma, \alpha, \Rightarrow \alpha, \Delta}{\Gamma, \alpha, \neg\alpha \Rightarrow \Delta}\ L\neg}{\Gamma, \alpha \wedge \neg\alpha \Rightarrow \Delta}\ L\wedge}{\Gamma, \bot \Rightarrow \Delta}\ Def\bot}{\alpha \to \bot, \Gamma \Rightarrow \Delta}\ L\to$$

(iv) $R{\sim}$ holds in $\mathbf{L} \in \mathcal{L}_S$.

$$\dfrac{\dfrac{\Gamma, \alpha \Rightarrow \Delta}{\Gamma, \alpha \Rightarrow \bot, \Delta}\ R\text{-}Weak}{\Gamma \Rightarrow \alpha \to \bot, \Delta}\ R\to$$

So, $\sim$ is a classical negation in $\mathbf{L} \in \mathcal{L}_S$ by the considerations given in Section 3. □

Remark 35 *The logics here presented are said to be 'minimal' in the sense that they have minimum resources to define a classical negation inside them. This is the meaning of the 'm' in the names* **mbC**, **mbD** *etc. 'bC' and 'bD' mean, respectively, 'basic property of consistency' and 'basic property of determinedness'. Proposition 34 shows that all the logics studied here are able to express every classical inference, as well as having additional resources to deal with contradictory and incomplete scenarios. So, they may be seen as extensions of classical logic. From this point of view, what is accomplished by* **mbC**, **mbD**, **mbCD** *and* **mbCDE** *is nothing but adding resources to classical logic in order to deal with paraconsistent and paracomplete scenarios. Thus, although they reject some classical inferences w.r.t. inconsistent and/or undetermined propositions, in fact, they are not weaker than classical logic.*

4.3.2 A derivability adjustment theorem – DAT

The basic idea of Derivability Adjustment Theorems (DATs) is that we have to 'add some information' to the premises in order to restore the inferences that are lacking. *DAT*s are especially interesting because they show what is needed in order to restore classical consequence in non-classical contexts.

It will now be shown that **CPL**, classical propositional logic defined over the signature Σ_C (see Definition 2), can be recovered from **mbD**, **mbCD** and **mbCDE** by adding a suitable set of hypothesis of the form $\star\alpha$, $\circledast\alpha$, or

$\circ\alpha$ and $\star\beta$ in the case, respectively, of **mbD**, **mbCD** and **mbCDE**. This result holds for the **LFI**s studied in [9, 8, 7] (including **mbC**).

Given a set of formulas Δ let $\#\Delta \stackrel{\text{def}}{=} \{\#\alpha \,:\, \alpha \in \Delta\}$, for $\# \in \{\circ, \star, \circledast\}$.

Theorem 36 (DAT)
(1) Let $\Gamma \cup \{\alpha\}$ be a set of formulas in $For(\Sigma_C)$. Then:

$$\Gamma \vdash_{\mathbf{CPL}} \alpha \;\; \textit{iff} \;\; \star\Delta, \Gamma \vdash_{\mathbf{mbD}} \alpha \textit{ for some } \Delta \subseteq For(\Sigma_C).$$

(2) Let $\Gamma \cup \{\alpha\}$ be a set of formulas in $For(\Sigma_C)$. Then:

$$\Gamma \vdash_{\mathbf{CPL}} \alpha \;\; \textit{iff} \;\; \circledast\Delta, \Gamma \vdash_{\mathbf{mbCD}} \alpha \textit{ for some } \Delta \subseteq For(\Sigma_C).$$

(3) Let $\Gamma \cup \{\alpha\}$ be a set of formulas in $For(\Sigma_C)$. Then:

$$\Gamma \vdash_{\mathbf{CPL}} \alpha \;\; \textit{iff} \;\; \circ\Delta, \star\Delta', \Gamma \vdash_{\mathbf{mbCDE}} \alpha \textit{ for some } \Delta \cup \Delta' \subseteq For(\Sigma_C).$$

Proof. (1) Suppose that $\Gamma \vdash_{\mathbf{CPL}} \alpha$. Observe that any derivation of α from Γ in **CPL** can be seen as a derivation in $\mathbf{CPL}^+$ in which some instances of axioms (*AxPEM*) and (*AxEXP*) are (possibly) used as additional hypothesis. Given a derivation $\alpha_1 \ldots \alpha_n = \alpha$ of α from Γ in **CPL**, replace any instance $\alpha_i = \big(\beta_i \vee \neg\beta_i\big)$ of (*AxPEM*) by the sequence $\big(\star\beta_i \to \big(\beta_i \vee \neg\beta_i\big)\big)\star\beta_i\big(\beta_i \vee \neg\beta_i\big)$. The resulting sequence of formulas is a derivation in **mbD** of α from $\Gamma \cup \star\Delta$, where $\star\Delta$ is the set of formulas of the form $\star\beta_i$ added by the process described above. Observe that every β_i is in $For(\Sigma_C)$.

Now, assume that $\star\Delta, \Gamma \vdash_{\mathbf{mbD}} \alpha$ for some $\Delta \subseteq For(\Sigma_C)$. Then, by Theorem 27 it follows that $\star\Delta, \Gamma \models_{\mathbf{mbD}} \alpha$. Let v be a valuation for **CPL** over Σ_C such that $v[\Gamma] \subseteq \{1\}$, and extend v to a maping $v' : For(\Sigma_\star) \to \big\{0, 1\big\}$ by defining $v'(\star\beta) = 1$ for every $\beta \in For(\Sigma_\star)$. Then, v' is a valuation for **mbD** such that $v'[\Gamma \cup \star\Delta] \subseteq \{1\}$ and so $v'(\alpha) = 1$. But v' extends v, thus $v(\alpha) = 1$. Hence, $\Gamma \models_{\mathbf{CPL}} \alpha$. By completeness of **CPL** w.r.t. valuations it follows that $\Gamma \vdash_{\mathbf{CPL}} \alpha$.

(2) 'Only if' part: it is proven analogously to item (1). However, besides processing the instances $\alpha_i = \big(\beta_i \vee \neg\beta_i\big)$ of (*AxPEM*) as described in item (1) (but now using the connective $\circledast$), any instance $\alpha_k = \big(\delta_k \to \big(\neg\delta_k \to \gamma_k\big)\big)$ of (*AxEXP*) must be replaced by $\big(\circledast\delta_k \to \big(\delta_k \to \big(\neg\delta_k \to \gamma_k\big)\big)\big)\circledast\delta_k\big(\delta_k \to \big(\neg\delta_k \to \gamma_k\big)\big)$, and the set $\circledast\Delta$ must also include occurrences of formulas of the form $\circledast\delta_k$ introduced in this way. Once again, observe that every δ_k is in $For(\Sigma_C)$. The 'If' part is proved analogously to item (1).

(3) It is proved in a similar way. The details are left to the reader.

□

Remark 37 *Theorem 36 above shows that* **CPL** *can be recovered inside any of the systems* **mbD**, **mbCD** *and* **mbCDE** *by adding an appropriate set of premises. Moreover, by adding* $\star\alpha$ *(resp.* $\circledast\alpha$*) as a schema axiom in the case of* **mbD** *(resp.* **mbCD***), the logic collapses to* CPL_e*, the presentation of* **CPL** *over signature* $\Sigma_\star$ *(resp.* $\Sigma_\circledast$*) obtained by adding* $\star\alpha$ *(resp.* $\circledast\alpha$*) as a schema axiom. In the case of* **mbCDE**, *there are three possibilities: by adding* $\star\alpha$ *as a schema axiom* mbC_e *is obtained, a presentation of* **mbC** *over* $\Sigma_{\circ\star}$*; by adding* $\circ\alpha$ *as a schema axiom* mbD_e *is obtained, a presentation of* **mbD** *over* $\Sigma_{\circ\star}$*; and finally, by adding both* $\circ\alpha$ *and* $\star\alpha$ *as schema axioms* $CPL_e^{\circ\star}$ *is obtained, a presentation of* **CPL** *over signature* $\Sigma_{\circ\star}$ *in which both* $\circ\alpha$ *and* $\star\alpha$ *are top particles. Compare these features of* **mbCDE** *with the ones enjoyed by da Costa's paraconsistent and paracomplete logic* N_1 *described briefly in Remark 19.*

4.4 The inconsistency and the undeterminedness operators

As remarked in Section 3, the inconsistency operator $\bullet$ and the undeterminedness operator $\bigstar$ may be defined from $\circ$ and $\star$ when a classical negation is available. So, $\bullet$ and $\bigstar$ can be defined in **mbC**, **mbD** and **mbCDE**, since a classical negation is definable in these systems (see Proposition 34):

$$\bullet\alpha \stackrel{\text{def}}{=} {\sim}\circ\alpha,$$

$$\bigstar\alpha \stackrel{\text{def}}{=} {\sim}\star\alpha.$$

Proposition 38 *The rules below hold in* **mbC**, **mbD** *and* **mbCDE***:*

$$\frac{\Gamma, \circ\alpha \Rightarrow \Delta}{\Gamma \Rightarrow \bullet\alpha, \Delta} \qquad \frac{\Gamma \Rightarrow \bullet\alpha, \Delta}{\Gamma, \circ\alpha \Rightarrow \Delta}$$

$$\frac{\Gamma, \star\alpha \Rightarrow \Delta}{\Gamma \Rightarrow \bigstar\alpha, \Delta} \qquad \frac{\Gamma \Rightarrow \bigstar\alpha, \Delta}{\Gamma, \star\alpha \Rightarrow \Delta}$$

Proof. Directly from the definitions of $\bigstar$, $\bullet$, and the rules $L{\sim}$ and $R{\sim}$. □

Now, rules for $\bullet$ and $\bigstar$ can be obtained from $L\neg^\circ$ and $R\neg^\star$:

$$\frac{\Gamma \Rightarrow \Delta, \alpha, \bullet\alpha}{\neg\alpha, \Gamma \Rightarrow \bullet\alpha, \Delta}\ L\neg^\bullet, \qquad \frac{\Gamma, \alpha \Rightarrow \bigstar\alpha, \Delta}{\Gamma \Rightarrow \neg\alpha, \bigstar\alpha, \Delta}\ R\neg^\bigstar,$$

and semantic clauses for $\bullet$ and $\star$ are as follows:

(vInc) $\big(v(\alpha) = 1 \text{ and } v(\neg\alpha) = 1\big) \implies v(\bullet\alpha) = 1,$

(vUnd) $\big(v(\alpha) = 0 \text{ and } v(\neg\alpha) = 0\big) \implies v(\star\alpha) = 1.$

The clause ***(vInc)*** says that $v(\bullet\alpha) = 1$ is only a necessary condition for $v(\alpha) = v(\neg\alpha) = 1$: if the latter holds, the former has to hold. On the other hand, it may be that α is not contradictory (i.e. it is not the case that $v(\alpha) = v(\neg\alpha) = 1$) but $\bullet\alpha$ holds. Thus, w.r.t. α, we may say that $\bullet\alpha$ means that a contradiction is *permitted*, while $\circ\alpha$ means that a contradiction is *prohibited* (i.e. not permited). This reading is in accordance with the fact that $\bullet\alpha$ is the classical negation of $\circ\alpha$, and the clause ***(vInc)*** is the contrapositive of ***(vCon)***.

Analogously, the clause ***(vUnd)*** says that $v(\star\alpha) = 1$ is only a necessary condition for $v(\alpha) = v(\neg\alpha) = 0$: it cannot be that the latter holds but $v(\star\alpha) = 0$. On the other hand, it may be that $v(\alpha) = 1$ or $v(\neg\alpha) = 1$ (i.e. $\alpha \vee \neg\alpha$ holds) but $v(\star\alpha) = 1$ (i.e $\star\alpha$ still holds). Thus, w.r.t. α, we may understand $\star\alpha$ as meaning that undeterminedness is *permitted*, while $\star\alpha$ means that undeterminedness is *prohibited* (i.e. not permitted). This reading, in its turn, is in accordance with the fact that $\star\alpha$ is the classical negation of $\star\alpha$, and the clause ***(vUnd)*** is the contrapositive of ***(vConD)***.

4.5 Back to duality

The basic idea of duality for classical propositional connectives, expressed by Definition 4, is that the connectives are functions from $\{0,1\}$ to $\{0,1\}$ such that, when the inputs are inverted, the outputs are also inverted. Thus classical $\wedge$ and $\vee$ are dual, because they correspond, respectively, to the following functions:

$$\wedge = \{\langle 1,1,1\rangle, \langle 1,0,0\rangle, \langle 0,1,0\rangle, \langle 0,0,0\rangle\}$$

$$\vee = \{\langle 0,0,0\rangle, \langle 0,1,1\rangle, \langle 1,0,1\rangle, \langle 1,1,1\rangle\}.$$

This idea can be extended to the non-truth-functional connectives of **mbC** and **mbD**. Let us use the symbol $\neg_c$ to refer to the negation of **mbC**. The paraconsistent negation $\neg_c$ is not functional, in the sense that the semantic value of $\neg_c\alpha$ is not functionally determined by the semantic value of α: when $v(\alpha) = 1$, $v(\neg_c\alpha)$ may be 0 or 1. So, $\neg_c$ is represented by the relation below:

$$\neg_c = \{\langle 0,1\rangle, \langle 1,0\rangle, \langle 1,1\rangle\}.$$

The paracomplete negation of **mbD**, referred to by $\neg_d$, in turn, is represented by the relation

$$\neg_d = \{\langle 1,0\rangle, \langle 0,1\rangle, \langle 0,0\rangle\}.$$

The idea that inverted inputs yield inverted outputs is maintained, we just do not have 'truth-functionality' any more, but the connectives are represented by *non-functional relations*. The idea of considering truth-relations instead of truth-functions for dealing with non-truth-functional connectives can be traced back to Fidel. Indeed, Fidel, in 1977, introduced an algebraic-relational kind of structure for da Costa's systems C_n in [22] in which the paraconsistent negation is interpreted by means of relations. As far as we know, for the first time Fidel proved the decidability of the logics of C_n hierarchy. Such structures are now called *Fidel structures* or **F**-*structures*, after Odintsov (see [32]). As proven in [7, Chapter 6], there is a close relationship between **F**-structures and a semantics of multialgebras called *swap structures*, which will be analyzed in Section 5.

The next definition formalizes these intuitions:

Definition 39 *Let κ_1 and κ_2 be n-ary connectives semantically characterized by valuation semantics over $\{0,1\}$ expressed by the (non-necessarily functional) relations R_1 and R_2, respectively. Let* Inv *be the operation over $\{0,1\}$ such that $Inv(1) = 0$ and $Inv(0) = 1$. We say that κ_1 and κ_2 are dual just in case:*

$$\langle x_1, x_2, ..., x_n, y\rangle \in R_1 \text{ iff } \langle Inv(x_1), Inv(x_2), ..., Inv(x_n), Inv(y)\rangle \in R_2.$$

Given convenient valuation semantics, the definition above allows the comparison of connectives from different logics.[9] Let us take a look at the connectives $\circ$, $\star$, $\bullet$, $\star$, in **mbC** and **mbD**. Although from the syntactic viewpoint

[9] As far as we know, the notion of dual connectives defined here by means of non-functional relations on $\{0,1\}$ has not yet been regarded in the literature. Note that the operation *Inv* is not applied to the relations, but rather to the elements of the n-tuples. We could define a notion of dual relation giving birth to different notions of duality between connectives. Of course, general cases of *triality*, *quaternality* or in general *k-ality*, can be also defined, even for n-valued logics, by using cyclic groups as in [5]. This is being investigated elsewhere.

these connectives are unary, they have to be represented by ternary relations, since the value of $*\alpha$, $* \in \{\circ, \star, \bullet, \bigstar\}$, depends on the values of α and $\neg\alpha$. So, for instance, in **mbC**,

$$\circ = \{\langle 1,1,0\rangle, \langle 1,0,0\rangle, \langle 1,0,1\rangle, \langle 0,1,0\rangle, \langle 0,1,1\rangle\},$$

and in **mbD**,

$$\bigstar = \{\langle 0,0,1\rangle, \langle 0,1,1\rangle, \langle 0,1,0\rangle, \langle 1,0,1\rangle, \langle 1,0,0\rangle\}.$$

The idea of considering triples (called *snapshots*) (z_1, z_2, z_3) in which z_1, z_2 and z_3 represent the truth-value of α, $\neg\alpha$ and $\circ\alpha$, respectively, is the starting point of the swap-structures semantics for **LFI**s, to be analyzed in Section 5.

Proposition 40 *The following connectives are dual to each other:* $\circ$ *of* **mbC** *and* $\bigstar$ *of* **mbD***;* $\bullet$ *of* **mbC** *and* $\star$ *of* **mbD***;* $\neg$ *of* **mbC** *and* $\neg$ *of* **mbD**.
Proof. Straightforward, from Definition 39. □

Remark 41

I. *An important feature of the notion of duality as defined by Definition 39, and differently from Definition 4, is that given a connective* $*$ *and its dual* $*^d$*, being* $\sim$ *classical negation, it may be that* $*\alpha$ *and* $\sim *^d \sim\alpha$ *are not materially equivalent. Indeed, neither* $\circ\alpha$ *and* $\sim\bigstar\sim\alpha$*, nor* $\star\alpha$ *and* $\sim\bullet\sim\alpha$*, are equivalent in* **mbCDE** *– in order to see this, consider the* **mbCDE***-valuation* $v(\alpha) = 1$*,* $v(\neg\alpha) = 0$*,* $v(\sim\alpha) = 0$*,* $v(\neg\sim\alpha) = 1$*. In this valuation, it may be that* $v(\circ\alpha) = 1$ *and* $v(\bigstar\sim\alpha) = 1$*. In this case* $v(\sim\bigstar\sim\alpha) = 0$*, and the equivalence between* $\circ\alpha$ *and* $\sim\bigstar\sim\alpha$ *does not hold (*mutatis mutandis *for* $\star\alpha$ *and* $\sim\bullet\sim\alpha$*). This happens because, as we have just seen above, these connectives are not functions but rather* relations*. So, the connectives* $\circ$*,* $\bullet$*,* $\star$ *and* $\bigstar$ *are not interdefinable in the sense that, for example,* $\star\alpha$ *cannot be defined as* $\circ\sim\alpha$ *(the pairs* $\circ$ *and* $\bullet$*, as well as* $\star$ *and* $\bigstar$ *are, of course, interdefinable).*

II. *The* Square of Oppositions *proposed by Marcos in [31] for the connectives* $\circ$*,* $\bullet$*,* $\star$ *and* $\bigstar$ *defined in modal terms (pp. 291-292) does not hold for these connectives in any of the logics studied here (note that in [31, p. 291], if* $\bigstar$ *means undeterminedness,* $\star p$ *and* $\bigstar p$ *should have their positions exchanged). In* **mbCDE***, for example, the pairs* $\bigstar\alpha$ *and* $\bullet\alpha$

are not contraries, nor subcontraries, since they may simultaneously receive value 1 *(or true), as well as value* 0 *(or false), for example, when* $v(\alpha) = 1$ *and* $v(\neg\alpha) = 0$ *(*ditto *for* $\star\alpha$ *and* $\circ\alpha$*). This is in accordance with the intuitive reading of the connectives proposed in page 32 above:* $\circ\alpha$, $\bullet\alpha$, $\star\alpha$ *and* $\bigstar\alpha$ *mean, respectively, that, w.r.t. to* α*, a contradiction is prohibited, a contradiction is permitted, undeterminedness is prohibited and undeterminedness is permitted.*

4.5.1 Duality in mbCD and mbCDE

In the logic **mbCDE** that contains the four connectives, all the dualities mentioned above hold, and the negation $\neg$ is dual to itself.

The logic **mbCD** collapses the connectives $\circ$ and $\star$ into $\circledast$ in order to recover excluded middle and explosion at once. The classical negation of $\circledast$ in **mbCD** produces a connective $✪\alpha \stackrel{\text{def}}{=} \sim\circledast\alpha$ governed by the rules

$$\frac{\Gamma \Rightarrow \Delta, ✪\alpha, \alpha}{\neg\alpha, \Gamma \Rightarrow ✪\alpha, \Delta}\ L\neg^{✪}, \qquad \frac{\Gamma, \alpha \Rightarrow ✪\alpha, \Delta}{\Gamma \Rightarrow ✪\alpha, \neg\alpha, \Delta}\ R\neg^{✪},$$

and the associated semantic clause

(vNcla) $\big(v(\alpha) = 1$ and $v(\neg\alpha) = 1\big)$ or $\big(v(\alpha) = 0$ and $v(\neg\alpha) = 0\big) \Longrightarrow v(✪\alpha) = 1$.

As mentioned after Definition 17, in **mbCD** the connective $\circledast$ may be understood as a *classicality operator*. So, in **mbCD**, $✪$ may be interpreted as a *non-classicality operator* in the sense that $✪\alpha$ is a consequence of α being either contradictory or undetermined: according to the clause *vNcla*, $✪\alpha$ is a necessary condition for either $v(\alpha) = v(\neg\alpha) = 1$ or $v(\alpha) = v(\neg\alpha) = 0$.

The connectives $\circledast$, $✪$ and the negation $\neg$ are represented in **mbCD** by the relations below:

$$\circledast = \{\langle 1,1,0\rangle, \langle 0,0,0\rangle, \langle 1,0,0\rangle, \langle 1,0,1\rangle, \langle 0,1,0\rangle, \langle 0,1,1\rangle\},$$

$$✪ = \{\langle 0,0,1\rangle, \langle 1,1,1\rangle, \langle 0,1,1\rangle, \langle 0,1,0\rangle, \langle 1,0,1\rangle, \langle 1,0,0\rangle\},$$

$$\neg = \{\langle 0,1\rangle, \langle 0,0\rangle, \langle 1,0\rangle, \langle 1,1\rangle\}.$$

So, in **mbCD** the negation $\neg$ is the dual of itself, and $\circledast$ and $✪$ are dual of each other.

The examples presented in this section about dual connectives, represented as relations, suggest an interesting topic for future research. Moreover, the framework of Fidel structures seems to be suitable for dealing with such notions.

5 Swap structures

As it is well-known, most of the **LFI**s studied in the literature are not algebraizable by means of the usual techniques such as the general framework of Blok and Pigozzi (see [3]). Moreover, most **LFI**s are not even characterizable by a single logical matrix. This justifies the search of alternative semantics for these logics, such as possible-translation semantics, Fidel structures and Nmatrices.

The notion of swap structures for **mbC**, as well as for some **LFI**s axiomatically extending **mbC**, was introduced in [7, Chapter 6]. Swap structures for **LFI**s are multialgebras $\mathcal{B}$ formed by triples (called *snapshots*) over a given Boolean algebra $\mathcal{A}$, where each triple (z_1, z_2, z_3) corresponds to a (complex) truth-value in which z_1 represents the truth-value of a formula α, while z_2 and z_3 represent a possible truth-value for $\neg\alpha$ and $\circ\alpha$, respectively. The possibilities of swap structures semantics lie beyond the scope of **LFI**s. For instance, in [14] and [24, Chapter 3] swap structures were defined as a semantical counterpart for some non-normal modal logics, where the snapshots are triples (z_1, z_2, z_3) in which z_1, z_2 and z_3 represent the truth-value of formulas α, $\Box\alpha$ and $\Box{\sim}\alpha$, respectively.

Given a swap structure $\mathcal{B}$ for a given logic **L**, it originates a non-deterministic matrix (in the sense of Avron and Lev, see for instance [1]) such that the class of such Nmatrices semantically characterizes **L**. In this section, this technique (which was additionally developed from the algebraic point of view in [13]) will be used in order to semantically characterize the logics **mbD**, **mbCD** and **mbCDE** (in the latter, snapshots will be quadruples instead of triples). Moreover, a decision procedure will be obtained for such logics from this semantics. Recall the following:

Definition 42 *An* implicative lattice *is an algebra $\mathcal{A} = \langle A, \wedge, \vee, \to\rangle$ for Σ_+ where $\langle A, \wedge, \vee\rangle$ is a lattice such that $\bigvee\{c \in A \ : \ a \wedge c \leq b\}$ exists for every $a, b \in A$, and $\to$ is an implication defined as follows: $a \to b \stackrel{def}{=} \bigvee\{c \in A \ : \ a \wedge c \leq b\}$ for every $a, b \in A$ (observe that $1 \stackrel{def}{=} a \to a$ is the top element of*

A, for any $a \in A$). If, additionally, $a \vee (a \to b) = 1$ for every a, b then $\mathcal{A}$ is said to be a classical implicative lattice.[10]

It is well-known that, if $\mathcal{A}$ is a classical implicative lattice and it has a bottom element 0, then it is a Boolean algebra. An algebraic semantics for **CPL**$^+$ is given by classical implicative lattices. That is, $\Gamma \vdash_{\mathbf{CPL}^+} \alpha$ iff, for every classical implicative lattice $\mathcal{A}$ and for every homomorphism v from $For(\Sigma_+)$ to $\mathcal{A}$, if $v(\gamma) = 1$ for every $\gamma \in \Gamma$ then $v(\alpha) = 1$.

Let $\mathcal{A} = \langle A, \wedge, \vee, \to, 0, 1\rangle$ be a Boolean algebra. Let $\pi_j : A^3 \to A$ be the canonical projections, for $1 \leq j \leq 3$. Hence, if $z \in A^3$ and $z_j = \pi_j(z)$ for $1 \leq j \leq 3$ then $z = (z_1, z_2, z_3)$. Analogously, if $z \in A^4$ then we write $z = (z_1, z_2, z_3, z_4)$, where z_j denotes the jth projection of z.

Definition 43 *Let $\mathcal{A}$ be a Boolean algebra with domain A.*
(1) The universe of the swap structures for **mbD** *over $\mathcal{A}$ is the set*

$$\mathrm{B}_{\mathcal{A}}^{\mathbf{mbD}} = \{z \in A^3 \ : \ z_1 \wedge z_2 = 0 \ \ \textit{and}\ z_3 \to (\ z_1 \vee z_2) = 1\}.$$

(2) The universe of the swap structures for **mbCD** *over $\mathcal{A}$ is the set*[11]

$$\begin{aligned}\mathrm{B}_{\mathcal{A}}^{\mathbf{mbCD}} &= \{z \in A^3 \ : \ z_3 \wedge (\ z_1 \wedge z_2) = 0 \ \ \textit{and}\ \ z_3 \to (\ z_1 \vee z_2) = 1\} \\ &= \{z \in A^3 \ : \ z_3 \leq (\ z_1 \vee z_2) \wedge {\sim}(\ z_1 \wedge z_2)\}.\end{aligned}$$

(3) The universe of the swap structures for **mbCDE** *over $\mathcal{A}$ is the set*

$$\mathrm{B}_{\mathcal{A}}^{\mathbf{mbCDE}} = \{z \in A^4 \ : \ z_3 \wedge (\ z_1 \wedge z_2) = 0 \ \ \textit{and}\ \ z_4 \to (\ z_1 \vee z_2) = 1\}.$$

Definition 44 *Let $\mathcal{A} = \langle A, \wedge, \vee, \to, 0, 1\rangle$ be a Boolean algebra.*
(1) A multialgebra $\mathcal{B} = \langle B, \wedge_{\mathcal{B}}, \vee_{\mathcal{B}}, \to_{\mathcal{B}}, \neg_{\mathcal{B}}, \star_{\mathcal{B}}\rangle$ over $\Sigma_\star$ is a swap structure for **mbD** over $\mathcal{A}$ *if $B \subseteq \mathrm{B}_{\mathcal{A}}^{\mathbf{mbD}}$ and the following holds, for every z and w in B:*

(i) $\emptyset \neq z \#_{\mathcal{B}} w \subseteq \{u \in B \ : \ u_1 = z_1 \# w_1\}$, *for each* $\# \in \{\wedge, \vee, \to\}$;

(ii) $\emptyset \neq \neg_{\mathcal{B}}(z) \subseteq \{u \in B \ : \ u_1 = z_2\}$;

(iii) $\emptyset \neq \star_{\mathcal{B}}(z) \subseteq \{u \in B \ : \ u_1 = z_3\}$.

[10] The name was taken from H. Curry, see [16].
[11] Here, $\sim$ denotes the Boolean complement in $\mathcal{A}$.

(2) A multialgebra $\mathcal{B} = \langle B, \wedge_{\mathcal{B}}, \vee_{\mathcal{B}}, \to_{\mathcal{B}}, \neg_{\mathcal{B}}, \circledast_{\mathcal{B}} \rangle$ *over* $\Sigma_{\circledast}$ *is a* swap structure for **mbCD** over $\mathcal{A}$ *if* $B \subseteq \mathrm{B}_{\mathcal{A}}^{\mathbf{mbCD}}$*, the multioperations* $\#_{\mathcal{B}}$*, are defined as in item (1) (for* $\# \in \{\wedge, \vee, \to, \neg\}$*) and, for every* z *in* B*:*

(iii) $\emptyset \neq \circledast_{\mathcal{B}}(z) \subseteq \{u \in B \ : \ u_1 = z_3\}$.

(3) A multialgebra $\mathcal{B} = \langle B, \wedge_{\mathcal{B}}, \vee_{\mathcal{B}}, \to_{\mathcal{B}}, \neg_{\mathcal{B}}, \circ_{\mathcal{B}}, \star_{\mathcal{B}} \rangle$ *over* $\Sigma_{\circ\star}$ *is a* swap structure for **mbCDE** over $\mathcal{A}$ *if* $B \subseteq \mathrm{B}_{\mathcal{A}}^{\mathbf{mbCDE}}$*, the multioperations* $\#_{\mathcal{B}}$*, are defined as in item (1) (for* $\# \in \{\wedge, \vee, \to, \neg, \circ\}$*) and, for every* z *in* B*:*

(iv) $\emptyset \neq \star_{\mathcal{B}}(z) \subseteq \{u \in B \ : \ u_1 = z_4\}$.

As mentioned at the beginning of this section, each snapshot (z_1, z_2, z_3) in a swap structure for **mbD** can be seen as a kind of complex truth-value such that z_1 encodes the truth-value of a formula α, while z_2 and z_3 encode a possible truth-value for $\neg\alpha$ and $\star\alpha$, respectively. The snapshots of swap structures for **mbCD** have a similar interpretation, but now z_3 represents a possible truth-value for $\circledast\alpha$. In the case of **mbCDE**, a snapshot (z_1, z_2, z_3, z_4) is such that z_1 represents the truth-value of a formula α, while z_2, z_3 and z_4 encode a truth-value for $\neg\alpha$, $\circ\alpha$ and $\star\alpha$, respectively.

From now on, the subscript '$\mathcal{B}$' will be omitted when referring to the multioperations of $\mathcal{B}$.

Definition 45 *Let* $\mathcal{A}$ *be a Boolean algebra and* $\mathbf{L} \in \big\{\mathbf{mbD}, \mathbf{mbCD}, \mathbf{mbCDE}\big\}$*. There is a unique swap structure* $\mathcal{B}_{\mathcal{A}}^{\mathbf{L}}$ *for* $\mathbf{L}$ *with domain* $\mathrm{B}_{\mathcal{A}}^{\mathbf{L}}$ *such that '*$\subseteq$*' is replaced by '*$=$*' in Definition 44.*

As a consequence of Definition 45, the multioperations in each swap structure $\mathcal{B}_{\mathcal{A}}^{\mathbf{L}}$ are defined as follows:

(i) $z\#w = \{u \in \mathrm{B}_{\mathcal{A}}^{\mathbf{L}} \ : \ u_1 = z_1\#w_1\}$, for each $\# \in \{\wedge, \vee, \to\}$ and each $\mathbf{L}$;

(ii) $\neg(z) = \{u \in \mathrm{B}_{\mathcal{A}}^{\mathbf{L}} \ : \ u_1 = z_2\}$, for each $\mathbf{L}$.

On the other hand:

(i) $\star(z) = \{u \in \mathrm{B}_{\mathcal{A}}^{\mathbf{mbD}} \ : \ u_1 = z_3\}$, for **mbD**;

(ii) $\circledast(z) = \{u \in \mathrm{B}_{\mathcal{A}}^{\mathbf{mbCD}} \ : \ u_1 = z_3\}$, for **mbCD**;

(iii) $\circ(z) = \{u \in \mathrm{B}_{\mathcal{A}}^{\mathbf{mbCDE}} \ : \ u_1 = z_3\}$, for **mbCDE**;

(iii) $\star(z) = \{u \in \mathrm{B}_{\mathcal{A}}^{\mathbf{mbCDE}} \ : \ u_1 = z_4\}$, for **mbCDE**.

6 From swap structures to Nmatrix semantics

In this section $\mathbf{L}$ will denote any logic in $\{\mathbf{mbD}, \mathbf{mbCD}, \mathbf{mbCDE}\}$. Recall the semantics associated to Nmatrices introduced by Avron and Lev [2]. For each $\mathbf{L}$ as above, let $\mathbb{K}_{\mathbf{L}}$ be the class of swap structures for $\mathbf{L}$. By adapting what was done in [7, Chapter 6] for several **LFI**s, as well as the techiques introduced in [13], it will be shown that each $\mathcal{B} \in \mathbb{K}_{\mathbf{L}}$ induces a non-deterministic matrix such that the class of such Nmatrices semantically characterizes $\mathbf{L}$.

Definition 46 *For each* $\mathcal{B} \in \mathbb{K}_{\mathbf{L}}$ *let* $D_{\mathcal{B}} = \{z \in |\mathcal{B}| \ : \ z_1 = 1\}$. *The* Nmatrix associated to $\mathcal{B}$ *is* $\mathcal{M}(\mathcal{B}) = (\mathcal{B}, D_{\mathcal{B}})$. *Let*

$$Mat(\mathbb{K}_{\mathbf{L}}) = \{\mathcal{M}(\mathcal{B}) \ : \ \mathcal{B} \in \mathbb{K}_{\mathbf{L}}\}.$$

Using the definition of valuation semantics over Nmatrices introduced in [2], the following valuation semantics can be associated to each class of Nmatrices considered above:

Definition 47 *Let* $\mathcal{B} \in \mathbb{K}_{\mathbf{L}}$ *and* $\mathcal{M}(\mathcal{B})$ *as above. A* valuation *over* $\mathcal{M}(\mathcal{B})$ *is a function* v *from the set of formulas of* $\mathbf{L}$ *to* $|\mathcal{B}|$ *such that, for every formula* α *and* β*:*

(i) $v(\alpha \# \beta) \in v(\alpha) \# v(\beta)$*, for every* $\# \in \{\wedge, \vee, \to\}$*;*

(ii) $v(\neg\alpha) \in \neg v(\alpha)$*;*

(iii) $v(\star\alpha) \in \star v(\alpha)$*, if* $\mathbf{L} \in \{\mathbf{mbD}, \mathbf{mbCDE}\}$*;*

(iii) $v(\circ\alpha) \in \circ v(\alpha)$*, if* $\mathbf{L} = \mathbf{mbCDE}$*;*

(iv) $v(\circledast\alpha) \in \circledast v(\alpha)$*, if* $\mathbf{L} = \mathbf{mbCD}$.

Definition 48 *Let* $\Gamma \cup \{\alpha\}$ *be a set of formulas of* $\mathbf{L}$.
(1) We say that α *is* a consequence of Γ in $\mathcal{M}(\mathcal{B}) \in Mat(\mathbb{K}_{\mathbf{L}})$*, denoted by* $\Gamma \models_{\mathcal{M}(\mathcal{B})} \alpha$*, if* $v(\alpha) \in D_{\mathcal{B}}$ *for every valuation* v *over* $\mathcal{M}(\mathcal{B})$ *such that* $v(\gamma) \in D_{\mathcal{B}}$ *for every* $\gamma \in \Gamma$.
(2) We say that α *is* a consequence of Γ in the class $Mat(\mathbb{K}_{\mathbf{L}})$ of Nmatrices, *denoted by* $\Gamma \models_{Mat(\mathbb{K}_{\mathbf{L}})} \alpha$*, if* $\Gamma \models_{\mathcal{M}(\mathcal{B})} \alpha$ *for every* $\mathcal{M}(\mathcal{B}) \in Mat(\mathbb{K}_{\mathbf{L}})$.

Theorem 49 (Soundness of L w.r.t. swap structures) *Let* $\Gamma \cup \{\alpha\}$ *be a set of formulas of* **L**. *Then:* $\Gamma \vdash_{\mathbf{L}} \alpha$ *implies* $\Gamma \models_{Mat(\mathbb{K}_{\mathbf{L}})} \alpha$.

Proof. It is an easy consequence of the definitions and of the fact that $\mathbf{CPL}^+$ is sound w.r.t. classical implicative lattices (and so w.r.t. Boolean algebras). Details are left to the reader (for swap structures for **LFI**s see [7, Chapter 6] and [13]). □

In order to prove completeness, the technique introduced in [14] (see also [24, 13]) for constructing a Lindenbaum-Tarski multialgebra together with a canonical valuation will be adapted here.

Let Γ be a non-trivial theory in **L**. An equivalence relation $\equiv_\Gamma^{\mathbf{L}}$ in the set $For_{\mathbf{L}}$ of formulas of **L** is defined as follows: $\alpha \equiv_\Gamma^{\mathbf{L}} \beta$ iff $\Gamma \vdash_{\mathbf{L}} \alpha \to \beta$ and $\Gamma \vdash_{\mathbf{L}} \beta \to \alpha$. Clearly, $\equiv_\Gamma^{\mathbf{L}}$ is a congruence w.r.t. the connectives of $\mathbf{CPL}^+$ and so the quotient set $For_{\mathbf{L}}/_{\equiv_\Gamma^{\mathbf{L}}}$ is a classical implicative lattice with top element $1_\Gamma^{\mathbf{L}} \stackrel{\text{def}}{=} [p_1 \to p_1]_\Gamma^{\mathbf{L}}$ (here, $[\alpha]_\Gamma^{\mathbf{L}}$ denotes the equivalence class of α w.r.t. $\equiv_\Gamma^{\mathbf{L}}$). Moreover, $0_\Gamma^{\mathbf{L}} \stackrel{\text{def}}{=} [p_1 \wedge \neg p_1]_\Gamma^{\mathbf{L}}$ (for $\mathbf{L} = \mathbf{mbD}$); $0_\Gamma^{\mathbf{L}} \stackrel{\text{def}}{=} [\circ p_1 \wedge (p_1 \wedge \neg p_1)]_\Gamma^{\mathbf{L}}$ (for $\mathbf{L} = \mathbf{mbCDE}$); and $0_\Gamma^{\mathbf{L}} \stackrel{\text{def}}{=} [\circledast p_1 \wedge (p_1 \wedge \neg p_1)]_\Gamma^{\mathbf{L}}$ (for $\mathbf{L} = \mathbf{mbCD}$) is the bottom element of $For_{\mathbf{L}}/_{\equiv_\Gamma^{\mathbf{L}}}$. Thus, $\mathcal{A}_\Gamma^{\mathbf{L}} \stackrel{\text{def}}{=} \langle For_{\mathbf{L}}/_{\equiv_\Gamma^{\mathbf{L}}}, \wedge, \vee, \to, 0_\Gamma^{\mathbf{L}}, 1_\Gamma^{\mathbf{L}} \rangle$ is a Boolean algebra (details are left to the reader).

Definition 50 *Let* Γ *be a non-trivial theory in* **L**. *The Lindenbaum-Tarski swap structure for* **L** *(over* Γ*) is the swap structure* $\mathcal{B}_{\mathcal{A}_\Gamma^{\mathbf{L}}}^{\mathbf{L}}$ *defined over the Boolean algebra* $\mathcal{A}_\Gamma^{\mathbf{L}}$ *(see Definition 45). The associated Nmatrix is denoted by* $\mathcal{M}_\Gamma^{\mathbf{L}}$.

Definition 51 *The* canonical valuation $v_\Gamma^{\mathbf{L}}$ *over* $\mathcal{M}_\Gamma^{\mathbf{L}}$ *is defined as follows:*

(i) $v_\Gamma^{\mathbf{L}}(\alpha) \stackrel{def}{=} \big([\alpha]_\Gamma^{\mathbf{L}}, [\neg\alpha]_\Gamma^{\mathbf{L}}, [\star\alpha]_\Gamma^{\mathbf{L}}\big)$, *for* $\mathbf{L} = \mathbf{mbD}$*;*

(ii) $v_\Gamma^{\mathbf{L}}(\alpha) \stackrel{def}{=} \big([\alpha]_\Gamma^{\mathbf{L}}, [\neg\alpha]_\Gamma^{\mathbf{L}}, [\circledast\alpha]_\Gamma^{\mathbf{L}}\big)$, *for* $\mathbf{L} = \mathbf{mbCD}$*;*

(ii) $v_\Gamma^{\mathbf{L}}(\alpha) \stackrel{def}{=} \big([\alpha]_\Gamma^{\mathbf{L}}, [\neg\alpha]_\Gamma^{\mathbf{L}}, [\circ\alpha]_\Gamma^{\mathbf{L}}, [\star\alpha]_\Gamma^{\mathbf{L}}\big)$ *for* $\mathbf{L} = \mathbf{mbCDE}$.

It can be proved that $v_\Gamma^{\mathbf{L}}$ is indeed a valuation over $\mathcal{M}_\Gamma^{\mathbf{L}}$ such that, by the very definitions, $v_\Gamma^{\mathbf{L}}(\alpha)$ is designated iff $\Gamma \vdash_{\mathbf{L}} \alpha$.

The Lindenbaum-Tarski swap structure together with the canonical valuation allows us to prove the completeness of **L** w.r.t. swap structures in a straightforward way:

Theorem 52 (Completeness of L w.r.t. swap structures) *Let* $\Gamma\cup\{\alpha\}$ *be a set of formulas of* $\mathbf{L}$. *Then:* $\Gamma \models_{Mat(\mathbb{K}_{\mathbf{L}})} \alpha$ *implies* $\Gamma \vdash_{\mathbf{L}} \alpha$.

Proof. Suppose that $\Gamma \nvdash_{\mathbf{L}} \alpha$. Then, $\mathcal{M}^{\mathbf{L}}_{\Gamma}$ (see Definition 50) is an Nmatrix for $\mathbf{L}$, and the canonical valuation $v^{\mathbf{L}}_{\Gamma}$ (see Definition 51) is a valuation over $\mathcal{M}^{\mathbf{L}}_{\Gamma}$ such that $v^{\mathbf{L}}_{\Gamma}(\gamma)$ is designated, for every $\gamma \in \Gamma$, but $v^{\mathbf{L}}_{\Gamma}(\alpha)$ is not designated. From this, $\Gamma \not\models_{Mat(\mathbb{K}_{\mathbf{L}})} \alpha$.

□

7 Decidability by finite Nmatrices

As it happens with several **LFI**s and other logics characterized by swap structures defined over Boolean algebras (see [7, 13, 14, 24]), the swap structure $\mathcal{B}^{\mathbf{L}}_{\mathcal{A}_2}$ with domain $\mathrm{B}^{\mathbf{L}}_{\mathcal{A}_2}$ over the 2-element Boolean algebra $\mathcal{A}_2$ (with domain $\{0,1\}$) is enough to characterize the logics $\mathbf{L} \in \{\mathbf{mbD}, \mathbf{mbCD}, \mathbf{mbCDE}\}$. This produces a decision procedure for each $\mathbf{L}$ by means of a finite Nmatrix, thanks to the semantical characterization of these logics through valuations (recall Section 4.1).

From definitions 43 and 44, the special case for $\mathcal{A}_2$ produces, for **mbD**, a universe $\mathrm{B}^{\mathbf{mbD}}_{\mathcal{A}_2} = \{T,\, t_0,\, F,\, f_0,\, f\}$ such that $T = (1,0,1)$, $t_0 = (1,0,0)$, $F = (0,1,1)$, $f_0 = (0,1,0)$, and $f = (0,0,0)$. The set of designated elements of the Nmatrix $\mathcal{M}^{\mathbf{mbD}}_{\mathcal{A}_2} = \mathcal{M}\big(\mathcal{B}^{\mathbf{mbD}}_{\mathcal{A}_2}\big)$ is $\mathrm{D} = \{T,\, t_0\}$, while $\mathrm{ND} = \{F,\, f_0,\, f\}$ is the set of non-designated truth-values. The multioperations are defined as follows:

$\wedge$	T	t_0	F	f_0	f
T	D	D	ND	ND	ND
t_0	D	D	ND	ND	ND
F	ND	ND	ND	ND	ND
f_0	ND	ND	ND	ND	ND
f	ND	ND	ND	ND	ND

$\vee$	T	t_0	F	f_0	f
T	D	D	D	D	D
t_0	D	D	D	D	D
F	D	D	ND	ND	ND
f_0	D	D	ND	ND	ND
f	D	D	ND	ND	ND

$\to$	T	t_0	F	f_0	f
T	D	D	ND	ND	ND
t_0	D	D	ND	ND	ND
F	D	D	D	D	D
f_0	D	D	D	D	D
f	D	D	D	D	D

	$\neg$
T	ND
t_0	ND
F	D
f_0	D
f	ND

	$\star$
T	D
t_0	ND
F	D
f_0	ND
f	ND

Theorem 53 (Characterization of mbD by a finite Nmatrix)
For every set of formulas $\Gamma \cup \{\alpha\} \subseteq For(\Sigma_\star)$*:* $\Gamma \vdash_{\mathbf{mbD}} \alpha$ *iff* $\Gamma \models_{\mathcal{M}^{\mathbf{mbD}}_{\mathcal{A}_2}} \alpha$.

Proof.
The 'only if' part (soundness) is an immediate consequence of Theorem 49. The 'if' part (completeness) follows by the following:
Fact: For every valuation v for **mbD** (see Definition 26) the mapping $v_{\mathbf{mbD}} : For(\Sigma_\star) \to \mathrm{B}^{\mathbf{mbD}}_{\mathcal{A}_2}$ given by $v_{\mathbf{mbD}}(\alpha) = (v(\alpha), v(\neg\alpha), v(\star\alpha))$ is a valuation over the Nmatrix $\mathcal{M}^{\mathbf{mbD}}_{\mathcal{A}_2}$ such that: $v_{\mathbf{mbD}}(\alpha) \in \mathrm{D}$ iff $v(\alpha) = 1$, for every formula α.

The proof of the **Fact** is analogous to the proof of Theorem 6.4.9 in [7], and is left to the reader. From this the result follows in a straightforward way. □

Clearly, the 5-valued Nmatrix $\mathcal{M}^{\mathbf{mbD}}_{\mathcal{A}_2}$ provides a decision procedure for **mbD**. Note that the negation of **mbD** is explosive and paracomplete, in the sense that excluded middle is not valid (because of the behavior of f in the table of $\neg$.).

Concerning the logic **mbCD**, the algebra $\mathcal{A}_2$ gives origin to a universe $\mathrm{B}^{\mathbf{mbCD}}_{\mathcal{A}_2} = \{T, t_0, t, F, f_0, f\}$ such that T, t_0, F, f_0 and f are as above, and $t = (1,1,0)$. The set of designated elements of the Nmatrix $\mathcal{M}^{\mathbf{mbCD}}_{\mathcal{A}_2} = \mathcal{M}(\mathcal{B}^{\mathbf{mbCD}}_{\mathcal{A}_2})$ is D' $= \{T, t_0, t\}$, while ND $= \{F, f_0, f\}$ is the set of non-designated truth-values. The multioperations are defined as follows:

$\wedge$	T	t_0	t	F	f_0	f
T	D'	D'	D'	ND	ND	ND
t_0	D'	D'	D'	ND	ND	ND
t	D'	D'	D'	ND	ND	ND
F	ND	ND	ND	ND	ND	ND
f_0	ND	ND	ND	ND	ND	ND
f	ND	ND	ND	ND	ND	ND

$\vee$	T	t_0	t	F	f_0	f
T	D'	D'	D'	D'	D'	D'
t_0	D'	D'	D'	D'	D'	D'
F	D'	D'	D'	ND	ND	ND
f_0	D'	D'	D'	ND	ND	ND
f	D'	D'	D'	ND	ND	ND

$\to$	T	t_0	t	F	f_0	f
T	D’	D’	D’	ND	ND	ND
t_0	D’	D’	D’	ND	ND	ND
t	D’	D’	D’	ND	ND	ND
F	D’	D’	D’	D’	D’	D’
f_0	D’	D’	D’	D’	D’	D’
f	D’	D’	D’	D’	D’	D’

	$\neg$
T	ND
t_0	ND
t	D’
F	D’
f_0	D’
f	ND

	$\circledast$
T	D’
t_0	ND
t	ND
F	D’
f_0	ND
f	ND

The following result can be proved in a similar way to the proof of Theorem 53:

Theorem 54 (Characterization of mbCD by a finite Nmatrix) *For every set of formulas* $\Gamma \cup \{\alpha\} \subseteq For(\Sigma_{\circledast})$*:* $\Gamma \vdash_{\mathbf{mbCD}} \alpha$ *iff* $\Gamma \models_{\mathcal{M}^{\mathbf{mbCD}}_{\mathcal{A}_2}} \alpha$.

The 6-valued Nmatrix $\mathcal{M}^{\mathbf{mbCD}}_{\mathcal{A}_2}$ provides a decision procedure for **mbCD**. Note that the negation of **mbCD** is both paraconsistent and paracomplete, respectively because of the behavior of t and the behavior of f in the table of $\neg$.

Finally the logic **mbCDE** will be analyzed. The algebra $\mathcal{A}_2$ produces a universe $\mathrm{B}^{\mathbf{mbCDE}}_{\mathcal{A}_2} = \{T^1, T^0, t^1_0, t^0_0, t^1, t^0, F^1, F^0, f^1_0, f^0_0, f^{01}, f^0\}$ such that $T^1 = (1,0,1,1)$, $T^0 = (1,0,1,0)$, $t^1_0 = (1,0,0,1)$, $t^0_0 = (1,0,0,0)$, $t^1 = (1,1,0,1)$, $t^0 = (1,1,0,0)$, $F^1 = (0,1,1,1)$, $F^0 = (0,1,1,0)$, $f^1_0 = (0,1,0,1)$, $f^0_0 = (0,1,0,0)$, $f^{01} = (0,0,1,0)$, $f^0 = (0,0,0,0)$. The set of designated truth-values of the Nmatrix $\mathcal{M}^{\mathbf{mbCDE}}_{\mathcal{A}_2} = \mathcal{M}(\mathcal{B}^{\mathbf{mbCDE}}_{\mathcal{A}_2})$ is D” $= \{T^1, T^0, t^1_0, t^0_0, t^1, t^0\}$, and ND’ $= \{F^1, F^0, f^1_0, f^0_0, f^{01}, f^0\}$ is the set of non-designated truth-values. The multioperations are defined as follows:

$\wedge$	T^1	T^0	t^1_0	t^0_0	t^1	t^0	F^1	F^0	f^1_0	f^0_0	f^{01}	f^0
T^1	D”	D”	D”	D”	D”	D”	ND’	ND’	ND’	ND’	ND’	ND’
T^0	D”	D”	D”	D”	D”	D”	ND’	ND’	ND’	ND’	ND’	ND’
t^1_0	D”	D”	D”	D”	D”	D”	ND’	ND’	ND’	ND’	ND’	ND’
t^0_0	D”	D”	D”	D”	D”	D”	ND’	ND’	ND’	ND’	ND’	ND’
t^1	D”	D”	D”	D”	D”	D”	ND’	ND’	ND’	ND’	ND’	ND’
t^0	D”	D”	D”	D”	D”	D”	ND’	ND’	ND’	ND’	ND’	ND’
F^1	ND’	ND’	ND’	ND’	ND’	ND’	ND’	ND’	ND’	ND’	ND’	ND’
F^0	ND’	ND’	ND’	ND’	ND’	ND’	ND’	ND’	ND’	ND’	ND’	ND’
f^1_0	ND’	ND’	ND’	ND’	ND’	ND’	ND’	ND’	ND’	ND’	ND’	ND’
f^0_0	ND’	ND’	ND’	ND’	ND’	ND’	ND’	ND’	ND’	ND’	ND’	ND’
f^{01}	ND’	ND’	ND’	ND’	ND’	ND’	ND’	ND’	ND’	ND’	ND’	ND’
f^0	ND’	ND’	ND’	ND’	ND’	ND’	ND’	ND’	ND’	ND’	ND’	ND’

$\vee$	T^1	T^0	t_0^1	t_0^0	t^1	t^0	F^1	F^0	f_0^1	f_0^0	f^{01}	f^0
T^1	D”	D”	D”	D”	D”	D”	D”	D”	D”	D”	D”	D”
T^0	D”	D”	D”	D”	D”	D”	D”	D”	D”	D”	D”	D”
t_0^1	D”	D”	D”	D”	D”	D”	D”	D”	D”	D”	D”	D”
t_0^0	D”	D”	D”	D”	D”	D”	D”	D”	D”	D”	D”	D”
t^1	D”	D”	D”	D”	D”	D”	D”	D”	D”	D”	D”	D”
t^0	D”	D”	D”	D”	D”	D”	D”	D”	D”	D”	D”	D”
F^1	D”	D”	D”	D”	D”	D”	ND’	ND’	ND’	ND’	ND’	ND’
F^0	D”	D”	D”	D”	D”	D”	ND’	ND’	ND’	ND’	ND’	ND’
f_0^1	D”	D”	D”	D”	D”	D”	ND’	ND’	ND’	ND’	ND’	ND’
f_0^0	D”	D”	D”	D”	D”	D”	ND’	ND’	ND’	ND’	ND’	ND’
f^{01}	D”	D”	D”	D”	D”	D”	ND’	ND’	ND’	ND’	ND’	ND’
f^0	D”	D”	D”	D”	D”	D”	ND’	ND’	ND’	ND’	ND’	ND’

$\rightarrow$	T^1	T^0	t_0^1	t_0^0	t^1	t^0	F^1	F^0	f_0^1	f_0^0	f^{01}	f^0
T^1	D”	D”	D”	D”	D”	D”	ND’	ND’	ND’	ND’	ND’	ND’
T^0	D”	D”	D”	D”	D”	D”	ND’	ND’	ND’	ND’	ND’	ND’
t_0^1	D”	D”	D”	D”	D”	D”	ND’	ND’	ND’	ND’	ND’	ND’
t_0^0	D”	D”	D”	D”	D”	D”	ND’	ND’	ND’	ND’	ND’	ND’
t^1	D”	D”	D”	D”	D”	D”	ND’	ND’	ND’	ND’	ND’	ND’
t^0	D”	D”	D”	D”	D”	D”	ND’	ND’	ND’	ND’	ND’	ND’
F^1	D”	D”	D”	D”	D”	D”	D”	D”	D”	D”	D”	D”
F^0	D”	D”	D”	D”	D”	D”	D”	D”	D”	D”	D”	D”
f_0^1	D”	D”	D”	D”	D”	D”	D”	D”	D”	D”	D”	D”
f_0^0	D”	D”	D”	D”	D”	D”	D”	D”	D”	D”	D”	D”
f^{01}	D”	D”	D”	D”	D”	D”	D”	D”	D”	D”	D”	D”
f^0	D”	D”	D”	D”	D”	D”	D”	D”	D”	D”	D”	D”

	$\neg$
T^1	ND’
T^0	ND’
t_0^1	ND’
t_0^0	ND’
t^1	D”
t^0	D”
F^1	D”
F^0	D”
f_0^1	D”
f_0^0	D”
f^{01}	ND’
f^0	ND’

	$\circ$
T^1	D”
T^0	D”
t_0^1	ND’
t_0^0	ND’
t^1	ND’
t^0	ND’
F^1	D”
F^0	D”
f_0^1	ND’
f_0^0	ND’
f^{01}	D”
f^0	ND’

	$\star$
T^1	D”
T^0	ND’
t_0^1	D”
t_0^0	ND’
t^1	D”
t^0	ND’
F^1	D”
F^0	ND’
f_0^1	D”
f_0^0	ND’
f^{01}	ND’
f^0	ND’

As in the previous cases, the following result can be easily proved:

Theorem 55 (Characterization of mbCDE by a finite Nmatrix) *For every set of formulas* $\Gamma \cup \{\alpha\} \subseteq For(\Sigma_{\circ\star})$*:*

$$\Gamma \vdash_{\mathbf{mbCDE}} \alpha \;\; \textit{iff} \;\; \Gamma \models_{\mathcal{M}_{\mathcal{A}_2}^{\mathbf{mbCDE}}} \alpha.$$

Proof. The proof is analogous to that for Theorem 53, with the following change: given a valuation v for **mbCDE** (see Definition 26), it induces a valuation $v_{\mathbf{mbCDE}}$ over the Nmatrix $\mathcal{M}_{\mathcal{A}_2}^{\mathbf{mbCDE}}$ as follows: $v_{\mathbf{mbCDE}}(\alpha) = (v(\alpha), v(\neg\alpha), v(\circ\alpha), v(\star\alpha))$. □

The 12-valued Nmatrix $\mathcal{M}_{\mathcal{A}_2}^{\mathbf{mbCDE}}$ constitutes a decision procedure for **mbCDE**. As in the case of **mbCD**, the negation of **mbCDE** is both paraconsistent (because of t^1 and t^0) and paracomplete (because of f^{01} and f^0).

8 Final remarks

We have seen here how two foundational ideas of paraconsistency may be developed further: the duality between paraconsistency and paracompleteness, and the introduction of logical operators that express meta-logical notions in the object language. The idea of da Costa's well-behavedness operator has been further developed by the consistency operator of **LFI**s. The later, in its turn, has given rise to the more general concept of *recovery operators*, represented here by the unary operators $\circ$, $\star$, and $\circledast$. Not only explosion but also excluded middle may be recovered inside systems in which they are not, in general, valid. The connectives $\circ$ and $\star$ may be combined in order to recover classical logic at once, and so the combined operator $\circledast$ may be called a *classicality operator.* Actually, it is fair to say that a whole path has been opened by the concept of '*logics of formal* $*$', where the wildcard symbol $*$ marks a space to be fulfilled by some logical property that we want to restrict and control inside a formal system.

Additionally, we have already seen how a hierarchy of logics may be constructed in order to represent the degrees whereby the logical properties represented in object language are controlled. We may, for instance, represent levels of consistency, and establish the point in which the consistency of a proposition α is enough to recover explosion. The same idea may be extended to determinedness, classicality, and in principle to any other logical property restricted/controlled by means of a recovery operator in an **LF**$*$.

As mentioned at the beginning of Section 5, most of the **LFI**s introduced in the literature are not algebraizable by the usual tecnhiques, and so several alternatives were proposed in the literature, as for instance possible translations semantics (see [7, Secs. 4.3 and 6.8]). Swap structures semantics constitutes a simple and fruitful approach to algebraizability in a broader sense, by considering non-deterministic algebras instead of ordinary algebras (see [6], and [13] for recent algebraic developments on swap structures). The question of algebraizability of the **LFU**s and **LFIU**s introduced here has not been studied yet.

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