SOME LOGICAL NOTATIONS FOR PRAGMATIC ASSERTIONS

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Abstract

The pragmatic notion of assertion has an important inferential role in logic. There are also many notational forms to express assertions in logical systems. This paper reviews, compares and analyses languages with signs for assertions, including explicit signs such as Frege's and Dalla Pozza's logical systems and implicit signs with no specific sign for assertion, such as Peirce's algebraic and graphical logics and the recent modification of the latter termed Assertive Graphs. We identify and discuss the main 'points' of these notations on the logical representation of assertions, and evaluate their systems from the perspective of the philosophy of logical notations. Pragmatic assertions turn out to be useful in providing intended interpretations of a variety of logical systems.

Keywords: Logical notations · Assertion · Pragmatic Logic · Existential Graphs · Assertive Graphs

1. Introduction

The pragmatic notion of assertion plays a key role in different logical systems both explicitly and implicitly. Frege's *Begriffsschrift*, Peirce's Existential Graphs and Heyting's explication of intuitionistic logical constants are examples of such pragmatically motivated approaches. The aim of the present paper is to consider the key pragmatic and notational aspects of a couple of logical systems: (i) Those that explicitly express logical assertions by introducing a specific sign for assertion, in particular (a) Frege's logical system and (b) Dalla Pozza's pragmatic logic (LP); and (ii) those in which no specific sign for assertion is introduced, in particular (c) Peirce's Existential Graphs (EGs) and (d) its recent variant of Assertive Graphs (AGs), in which assertions play a crucial inferential role even though the languages themselves exhibit no explicit signs of assertion.

From historical, notational and logical points of view, one encounters the following 'points' regarding logical facets of assertions and assertive signs (see Pietarinen and Bellucci 2017 for more details on the first three):

(i) *The Geach Point:*[1] One and the same proposition can occur both asserted and unasserted in different contexts. (Geach 1965)

In particular, Geach (1965, p. 449) pointed out that "[a] thought may have just the same content whether you assent to its truth or not; a proposition may occur in a discourse now asserted, now unasserted, and yet be recognizably the same proposition". A standard example is the justification of the inferential rule of *modus ponens*: assuming $\alpha \to \beta$ and α, one infers β. In the minor premise, α is usually considered asserted, while in the major premise, α is not asserted, because it is the conditional $\alpha \to \beta$ that is asserted (on this, see Russell 1903). This means that the very same proposition may be used both in its asserted and unasserted forms.[2]

Second, we can identify

(ii) *The Dudman Point*: The distinction between asserted and unasserted propositions should be notationally distinguished, *whatever form* the notational expression of this difference happens to take. (Dudman 1950)

This point has a long history. In a letter to Frege, Peano wrote that the various positions that a formula can have determines whether it occurs asserted or unasserted in some truth-functional context that is asserted. In particular, in Peano's words, "the several positions that a proposition can have in a formula completely determine what is asserted of it" (Peano 1958 [1895], p. 191). In reference to Peano's logical approach, Frege observed that this is because "the principal relation sign invariably carries assertoric force" (Frege 1991 [1897], p. 248) without any specific sign for assertion being present in the notation. This means that in Peano's notation it is impossible to write down a complex formula that would show the difference between asserted and unasserted uses of the formula. And this, in turn, leads one to

(iii) *The Frege Point*: Assertion must be represented by a specific logical sign (Frege 1879).

According to the Frege Point, an *ad hoc* sign of assertion is a notational requirement of the logical language. Frege indicates it with "$\vdash$", which stands for the sign of assertion (see Section 2). (Nowadays, the sign "$\vdash$", is commonly termed the turnstile and it expresses the concept of derivability or provability.)

A further perspective to the logic of assertions was given in terms of

(iv) *The Reichenbach Point*: Assertions cannot be iterated and they cannot be connected by truth-functional connectives (Reichenbach 1947).

Reichenbach (1947, p. 346) argues for this from the fact that the term "assertion" is used in three different ways. Namely, "it denotes, first, the act of asserting; second, the result of this act, i.e., an expression of the form '$\vdash$ p'; third, a statement which is asserted, i.e. a statement 'p' occurring within an expression '$\vdash$ p'". Regarding the result of an assertion, Reichenbach claims that "since assertive expressions are not propositions, they cannot be combined by propositional operators" (Reichenbach 1947, § 57, p. 337). The assertion sign works, according to

[1] In the literature this is also known as the Frege-Geach Point or just as the Frege Point.

[2] Bell (1979) comments Russell's views on modus ponens in the following way: "Now this would imply that either all inferences of the form modus ponens (to take but one example) are invalid, or, at least, that all those with either a true antecedent or a true consequent in the conditional premiss are invalid. This is, of course, quite unacceptable" (pp. 87-88). On the possibility of reconciling Russell's views on modus ponens without facing Bell's untoward consequences, see the justification of modus ponens in pragmatic logic provided by Chiffi and Di Giorgio (2017) based on the Bridge Principle (d) given in Section 2.2 of the present paper.

Reichenbach, in its "pragmatic capacity", since it cannot be, for instance, negated with a propositional connective. And if so, then inferences can be understood as processes that allow us to derive justified asserted conclusions once the asserted premises are also justified.

This means that there can be no nested or iterated occurrences of the assertion sign, because the truth-functional connectives only operate on propositions and never on judgments. Furthermore, inferences operate only on assertions and never on propositions. So even though we have named this point "The Reichenbach Point", it also has manifest Fregean roots.[3]

Fifth, we can identify the following point, which we term

(v) *The Elementary-restriction Point*: No one connective (of any sort) is ever applied to two different assertions.

This is to say that an asserted formula is elementary if it cannot be connected by any other (non-truth-functional) connective. This means that an elementary assertion is composed by a unique assertion sign prefixed to the asserted content. This is, for instance, the case with Frege's Begriffsschrift. A similar restriction is presented in Reichenbach's treatise, in which it underlies the distinction between assertions and (propositional) content:

> [T]wo assertive expressions in sequence do not represent a conjunction of these expressions. Rather we must say that juxtaposition of two assertive expressions amounts to the same as assertion of the conjunction of the two respective statements. Thus the sentences in a book, each asserted by the period sign, follow one another; this arrangement amounts to the same as asserting the conjunction of all these sentences, each taken without the assertion sign. Juxtaposition, therefore, constitutes the pragmatic analogue of conjunction. The other binary operations do not have such analogues. (Reichenbach 1947, p. 338)

The conjunction of assertions is thus expressed as the assertion of the conjunction, since there is in Reichenbach's picture no way to connect genuine assertions with one another, even with non-truth functional connectives.

There are two marked implications from this. On the one hand, Reichenbach is able to clarify why juxtaposition is a pragmatic analogue to the propositional conjunction (as a kind of pragmatic connective among assertions). On the other hand, he assumes that the pragmatic conjunction can be formalised as the assertion of a conjunction (of propositions), not as a conjunction of assertions. The latter is only possible for conjunctive connectives, as there the equivalence between the assertion of conjunction and the conjunction of assertions holds. Indeed, Reichenbach cannot directly express the conjunction of assertion (by means of the juxtaposition of assertions). He can only express the equivalent assertion of a conjunction. Moreover, he cannot express, for instance, the disjunction of assertions, which is usually assumed to be a different case from the assertion of a disjunction (of propositions).

Given such conundrums, a number of other, non-classical and variant systems of logic have been proposed which might well be more appropriate for the

[3] Frege holds this point only since the Begriffsschrift (see the next section).

representation of assertions. Let us highlight the following general point from one of them next, termed

(vi) *The Dalla Pozza Point*: Complex assertions may be logically combined by an application of intuitionistic-like connectives. (Dalla Pozza 1991)

This states that complex asserted formulas may be expressed by means of connectives that explicate intuitionistic meanings of logical constants, the behaviour of which is not truth-functional. Moreover, intuitionistic connectives can indicate the (pragmatic) justification-conditions for (acts of) assertions.

This paper is structured in the following way. Section 2 provides an analysis of Frege's assertion sign and its use in pragmatic logic. Section 3 is devoted to the investigation of the role of pragmatic assertions in two diagrammatical systems, namely Peirce's Existential Graphs (EGs) and its recent modification termed Assertive Graphs (AGs). Section 4 proposes a critical comparison regarding the inferential role of assertions in the aforementioned logical systems, and Section 5 concludes the paper.

2. Some Citation Examples

2.1. Frege's Assertion Sign

Frege's assertion sign "$\vdash$" is a pivotal element of his logical notation. This sign is placed before a content Δ to express the fact that a complete judgement is the case: $\vdash \Delta$. A judgement is understood as the internal counterpart of the act of assertion. The assertion sign is not intended by Frege to be a primitive one, since it is assumed to be composed by the vertical "|" and the horizontal "—" strokes. The vertical stroke is also known as the judgement stroke[5]. The interpretation of the assertion sign, its two constituents and the asserted content is not something that came to be fixed in Frege's writings (Bell 1979).

In the *Begriffsschrift*, Δ stands for a conceptual content that can be judged, — Δ represents a thought, and $\vdash \Delta$ indicates a full judgement which Frege believed to be the common predicate of all judgements saying "it is a fact". The idea is that a possible content (with a propositional nature) Δ is converted by the horizontal stroke into a nominalisation (e.g., converting a proposition α into an expression of the form "that α"). By the vertical stroke one may predicate that the judgement expresses a fact (e.g., "it is a fact that α"). As noted by Geach, given this Fregean notation, it is now possible to differentiate asserted from non-asserted occurrences of one and the same formula.

Frege's views on the assertion sign in the *Begriffsschrift* suffer from some problems, as discussed in Dudman (1970) and Bell (1979). His proposal does not explain well the use of formulas in the context of indirect proofs that are *assumed* rather than asserted as true. Also, the interpretation of the assertion sign as a predicate critically violates what we have called the "Reichenbach Point": assertion, which is a pragmatic notion, cannot be reduced to the semantic notion of a predicate that designates a fact.

[5] On the Fregean notion of judgement, see (Schaar 2018; Smith 2009).

Subsequently, then, Frege in the *Grundgesetze* and in “Function and Concept” modified his views on the assertion sign, holding a more coherent perspective. In a footnote to “Function and Concept”, Frege indeed noticed that:

> [T]he judgement stroke cannot be used to construct a functional expression; for it does not serve, in conjunction with other signs, to designate an object. ‘2 + 3 = 5’ does not designate [*bezeichnet*] anything; it asserts something. (Frege 1997, p. 142 n).

Now, the horizontal stroke is a function-name denoting a concept under which the True falls, while the judgement stroke is the acknowledgement of the truth of the Thought. So, $—\Delta$ means that Δ can be presented without asserting its being true. In this way, also the premise of an indirect proof need not be put forth assertively, but for instance, can merely be assumed and put forth in a hypothetical form.

Even if it is a well-known fact that in the beginning of “Der Gedanke” Frege classically defines logic as the science that is specifically concerned with the notion of truth, in his *Posthumous Writings* he pointed out that the notion of assertion is intimately associated with the nature of logic:

> [W]hat logic is really concerned with is not contained in the word ‘true’ at all but in the assertoric force with which a sentence is uttered. (Frege 1915, p. 323)

Indeed, the pragmatic notion of assertoric force ends up as being a crucial element of logicality in Frege’s mature reflections.

2.2 Assertion in Pragmatic Logic

Wittgenstein (1961; 1979) famously dismissed Frege’s assertion sign as a psychological hocus-pocus, which is “logically quite meaningless”. Other authors, including Reichenbach (1947) and Dummett (1993), by contrast, have defended the appeals to Frege’s assertion sign. The pragmatic sign of assertion certainly has some well-known expressive limitations, as it cannot be iterated and it cannot occur under the scope of a truth-conditional operator (Reichenbach 1947; Dummett 1981). We termed this limitation the Reichenbach Point. The result is that in Frege’s logical system, the assertion sign is always fixed at the beginning of any formula, followed by the content that is asserted. Consequently, in Frege’s system one may talk about the assertion of a disjunction or the assertion of a conjunction, for instance, but one cannot express the conjunction or disjunction of assertions. Connectives operating on assertions cannot semantically function in a truth-conditional fashion.

However, nothing prevents connecting assertions in such a wise that is not truth-functional. Then it becomes possible to disambiguate, for instance, the assertion of a disjunction from the disjunction of assertions, among others.

In Dalla Pozza (1991) and Dalla Pozza & Garola (1995), the strategy of appealing to non-truth-functional connectives operating over assertions is adopted.[7] This is the distinguished feature of the kind of pragmatic logic that is known as Logic for Pragmatics or Pragmatic Logic (LP for short). The basic idea of LP is to

[7] This is a distinguished feature of pragmatic logic. Extensions of LP to other illocutionary acts and applications to philosophical problems have been provided in (Carrara, Chiffi, De Florio 2016; Carrara, Chiffi, De Florio 2019; Carrara, Chiffi, De Florio, Pietarinen 2019) among others. For a classical bilateral system equipped with the signs of assertion and denial, see (Rumfitt 2000). An overview on many logical systems for assertion and denial is (Buekens et al. 2017).

connect logical formulas expressing assertions with intuitionistic-like operators, whereas the formulas expressing the content of a single assertion can be connected by classical truth-functional connectives. In short, in LP there are *radical formulas* such as p, q, α, which express contents of assertions. In order to get an elementary *assertive*, also known as *sentential*, *formula*, it should be prefixed by the assertion sign. For instance, $\vdash \alpha$ is an elementary assertive formula stating that α is asserted. As in Frege's system, it is possible to express the assertion of a conjunction as $\vdash (\alpha \wedge \beta)$, or the assertion of a disjunction as $\vdash (\alpha \vee \beta)$, or the assertion of material implication as $\vdash(\alpha \rightarrow \beta)$. But unlike in Frege's case, in LP it is also possible to formulate the conjunction of assertions: $(\vdash \alpha) \cap (\vdash \beta)$; the disjunction of assertions: $(\vdash \alpha) \cup (\vdash \beta)$; and the implication of assertions: $(\vdash \alpha) \supset (\vdash \beta)$. All these can be done by introducing specific pragmatic connectives that apply to assertive formulas.

In order to differentiate pragmatic connectives for assertive formulas from truth-functional connectives for radical formulas, Dalla Pozza & Garola (1995) expressed pragmatic connectives by signs taken from Łukasiewicz's or Polish logical notation. According to that approach, pragmatic conjunction, disjunction, negation and implication are indicated by the upper-case Latin letters K, A, C, N, respectively.[8] However, the resulting notation is a normal infix notation with parentheses. Polish symbols are merely used in order to provide an explicit distinction between two kinds of connectives, those for radical and those for assertive formulas. (In the subsequent literature on pragmatic logic, the use of Polish symbols was dropped by convention.)

Radical formulas are evaluated in LP in the standard Tarskian way, while the meaning of pragmatic connectives is explicated by an assertion-based variant of the BHK (Brouwer-Heyting-Kolmogorov) interpretation of intuitionistic constants. According to Heyting (1956), intuitionistic constants can indeed be explicated by means of the notion of assertion.[9] An assertion expresses the construction (or the method of verification, demonstration, etc.) that yields a proof of a (propositional) content. Briefly put, P and Q being the contents, we have that:

(i) P and Q can be asserted iff both P and Q can be asserted.

(ii) P or Q can be asserted iff at least one of the propositions P, Q can be asserted.

(iii) $\neg$ P can be asserted iff one possesses a construction which, from the supposition that a construction for P were carried out, leads to a contradiction.

(iv) P $\rightarrow$ Q can be asserted iff one possesses a construction R which, joined to any construction proving P would automatically effect a construction proving Q; that is, a proof of P, together with R, would form a proof of Q.

Unlike in Heyting's proposal, in LP there exists an explicit reference to the notion of justification of an assertion. An assertion, being an act, is not true or false but

[8] An eminent friend of Łukasiewicz's notation was also Arthur Prior.

[9] Heyting (1930), translated in (Mancosu 1998), had pointed out that "to satisfy the intuitionistic demands, the assertion must be the realisation of the expectation expressed by the proposition p. Here, then, is the Brouwerian assertion of p: It is known how to prove p" (p. 308). For the role of assertion in contemporary constructivism, for instance in intuitionistic type theory, see (Martin-Löf 1984).

justified or unjustified. An elementary assertion $\vdash \alpha$ can be considered justified, iff there exists an (intuitive) proof (conclusive evidence or verification) for the truth of the content α. In this framework, "the pragmatic notion of justification (or proof) presupposes the semantic notion of truth as a regulative concept, since, intuitively, a proof of a proposition amounts to a proof that its truth value is true" (Dalla Pozza and Garola 1995, p. 101).

Let π be a pragmatic justification function from assertive formulas to the justification values indicated as "J" (justified) and "U" (unjustified). We can then state that $(\vdash \alpha)$ = J when there exists a proof of the truth of α. Notice that, if α is atomic, then the notion of intuitive proof has to be intended as an empirical verification that justifies the assertion, since a logical proof of an atomic formula cannot be given. In detail, the following justification rules apply to (complex) assertive formulas in LP:

JR1 – Let α be a radical formula. Then:

(JR1.1) $\pi(\alpha) = J$ iff a proof exists that α is true.

(JR1.2) $\pi(\alpha) = U$ iff no proof exists that α is true.

JR2 – Let δ be an assertive formula. Then:

$\pi(\sim\delta) = J$ iff a proof exists that δ is unjustified, i.e. that $\pi(\delta) = U$.

JR3 – Let δ_1 and δ_2 be assertive formulas. Then:

(JR3.1) $\pi(\delta_1 \cap \delta_2) = J$ iff $\pi(\delta_1) = J$ and $\pi(\delta_2) = J$;

(JR3.2) $\pi(\delta_1 \cup \delta_2) = J$ iff $\pi(\delta_1) = J$ or $\pi(\delta_2) = J$;

(JR3.3) $\pi(\delta_1 \supset \delta_2) = J$ iff a proof exists that $\pi(\delta_2) = J$ whenever $\pi(\delta_1) = J$.

The justification function π is *partial*: it won't assign a justification value to all complex formulas.[10] This is in line with the intuitionistic flavour of the pragmatic connectives. Unlike truth-functions that can lead to truth or falsity, justification-conditions reflect the idea that propositional content can be proven, its negation can be proven or it may be the case that a proof of it is lacking altogether.

Pragmatic connectives operating on assertive formulas and truth-functional connectives operating on radical formulas are logically related by the following *Bridge Principles*:

(a) $\vdash (\neg \alpha) \supset (\sim \vdash (\alpha))$

(b) $((\vdash \alpha_1) \cap (\vdash \alpha_2)) \equiv (\vdash (\alpha_1 \wedge \alpha_2))$

(c) $((\vdash \alpha_1) \cup (\vdash \alpha_2)) \supset (\vdash (\alpha 1 \vee \alpha_2))$

(d) $(\vdash (\alpha_1 \rightarrow \alpha_2)) \supset (\vdash \alpha_1 \supset \vdash \alpha_2)$.

The principle (a) states that from the assertion of the negation of α it is possible to infer the *pragmatic negation* of α. Principle (b) shows that the conjunction of two assertions is equivalent to the assertion of a conjunction. As discussed before, this is

[10] In a strict sense, the function π depends on the semantic function used to evaluate radical formulas.

a fundamental property of normal logical systems for assertion. Formula (c) states that from the disjunction of two assertions it is possible to infer the assertion of the disjunction. Finally, principle (d) expresses the fact that from the assertion of a material implication $\alpha_1 \rightarrow \alpha_2$, it is possible to derive that the assertion of α_1 *pragmatically implies* the assertion of α_2.

3. Diagrammatic Notations for Assertion

3.1 *Assertion and Existential Graphs*

Historically, an important contribution to the logic of assertions was set up in Peirce's work on algebraic and graphical logics since 1880 (Pietarinen 2004, 2015). In his later works, he clearly endorsed the Geach Point, as follows from the statement that "[o]ne and the same proposition may be affirmed, denied, judged, doubted, inwardly inquired into, put as a question, wished, asked for, effectively commanded, taught, or merely expressed, and does not thereby become a different proposition" (R 517, c.1901).[11]

Both in Peirce's algebraic and graphical logics, writing a formula down on the sheet of paper is to make an assertion that the content of that formula or a graph is true in the universe of discourse that the sheet represents.[12] Writing down a complex formula, such as a conditional, is to assert that such conditional relation obtains between the antecedent and consequent propositions in the conditional structure. It is not to assert the antecedent or to assert the consequent of that conditional structure.

The key notion in Peirce's logic is thus the *sheet of assertion*. The sheet is the surface (topologically, an open-compact, unoriented manifold) on which instances of assertions are *scribed*. Important thing is that the sheet itself is also an assertion. It asserts that in the dialogue between the 'Graphist' (Peirce's term for the theoretical agent of our make-believe who proposes and defends assertions) and the 'Interpreter' (the agent who interprets and accepts those assertions), the discourse is understood to run over a commonly accepted *universe of discourse*, the domain which is taken to be common knowledge to both parties (R 517). The sheet is thus an assertion of the assumption of the *common ground* in communication (R 615, 1908). These facts, Peirce explains, "render it certain that when I assert something, that is, endeavour by my utterances to induce you to recognize it as true, you will (if not invariably, at least sometimes) know what it is to which I intend to bring you to assent" (R 339, 1907; Peirce 2019: 620).

This strikingly Gricean perspective has some important consequences. If scribing a proposition on the sheet is an assertion of it, and if the sheet itself is an assertion, then "the very first proposition that is scribed on the Sheet is scribed on a sheet already bearing a proposition" (R S-30, 1906). Expressing a proposition upon the sheet is indeed commonly and reciprocally understood to be asserted, and "whatever proposition is to be asserted shall be scribed on that sheet" (*ibid.*).

[11] This manuscript was ill-dated to 1904 in EP II and in the subsequent literature.
[12] Witness how Peirce had put this in R 650 ("Diversions of Definitions", 1910): "Any shape or combination of shapes that put on the Sheet of Assertion would be an assertion, I term a *graph*, and your act of putting it on any surface by writing or drawing or a mixture of the two I express by saying that you *scribe* that sign on that *area*" (Peirce 2019: 164-165).

Now the assertion of '*p*' by scribing it on the sheet is in fact also something more: it is to add or *juxtapose* the assertion of '*p*' with a proposition which is already scribed upon the same sheet and hence already asserted. This is to affirm the presence of a well-understood and mutually agreed-upon universe of discourse of which '*p*' is asserted. And this, then, means that whatever is laid upon the sheet as an assertion, is laid upon it *independently* of any other assertion that may have been scribed upon that same sheet.

Peirce's graphical method of logic thus leads to the assignment of a certain further signification to the notion of the sheet of assertion: it is not only a representation of a tautology (as a blank sheet in the propositional case, or the domain of discourse in the case of predicate logic, or the space of all proofs in intuitionistic logic, etc.). The sheet is at once also a representation of *logical conjunction*.

It follows, for instance, that the equivalence expressed in the Bridge Principle (b) above is generically assumed to hold in graphical logics even if the conjunction of an assertion cannot directly be expressed and only the equivalent assertion of conjunction is expressed.

In Peirce's graphical logic, the Dudman Point clearly holds, too, since assertion is notationally represented by the sheet of assertion. But the sheet is not a sign specific to assertions, so the Frege Point does not hold. And since assertions juxtaposed on the sheet are connected by logical conjunctions (and in Peirce's graphical logic of EGs this is effected by the device of the "cuts" and hence connected by other truth-functional logical connectives), the Reichenbach Point does not hold, either. The Elementary-restriction Point remains indeterminate, in so far as the basic and standard system of logical graphs is concerned (namely the Alpha part of EGs), as those agree with the Boolean algebra (Ma & Pietarinen 2018a) and have only the classical, truth-functional connectives in their repertoire. (But see Ma & Pietarinen 2018b,c for some non-classical modifications and extensions of the Alpha part.)

We can summarise the upshot of these considerations and arguments—which Peirce presents in a much greater length in his writings that can be covered here—in terms of the following,

(vii) *The Peirce Point*: In any conceivable logical notation, you need to write down (scribe) what you assert.

According to the Peirce Point, there is nothing (no propositions in logic, abstract forms in topology, concepts in cognition, etc.) that could be expressed without writing or scribing them down on some*thing*. A representation of that thing in some media, manifold, space, imagination or cognitive schema is necessary. The form of such representation thus becomes the first and indispensable sign of all logical notations.

In the next section, we see how to modify such system of logical graphs to be consistent with the Dalla Pozza Point.

3.2 *Assertion and Assertive Graphs*

A new logical system of assertions inspired by EGs is Assertive Graphs (AGs) (Bellucci, Chiffi, Pietarinen 2018; Pietarinen and Chiffi 2018). AGs explicitly have assertions as elements of its language. Similarly to EGs, there is no specific sign for assertion, since propositional letters are in the language of AGs scribed on the sheet of assertion just as they are in the language of EGs. That sheet itself is both an assertion and a representation of the universe of discourse. Anything scribed on the sheet is asserted. The blank sheet expresses the collection of all justified assertions, analogously to the sheet of assertion in EGs in which it expresses tautology, the set of all truths.

Unlike EGs, the system of AGs has an intuitionistic flavour. It operates on asserted formulas in the sense of the aforementioned assertion-based variant of the BHK interpretation of intuitionistic constants. Because of this, all connective types are primitive signs of AGs, since unlike in the classical case of EGs, those connectives are not fully inter-definable. Moreover, there is no cut in AGs, which implies that there are less subgraphs to be nested to express complex formulas. Compared to EGs, this improves readability of complex formulas of AGs.

The main conventions and elements of the language of AGs are the following. Writing P on the sheet of assertion means that P is a justified assertion (Fig. I).

P

Figure I

Surrounding P with a thinly-lined box is a technical device that help us to focus on certain formula, much in the same way as we may do by parentheses. This means that the box has the role of grouping those asserted formulas together that are enclosed within the box (Fig. II).

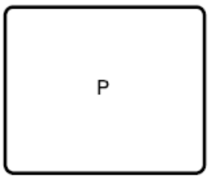

Figure II

Conjunction is scribed by juxtaposing the formula P and the formula Q (Fig. III).

P Q

Figure III

The graph in Figure III is equivalent to the following graph in which P and Q are boxed (Fig. IV):

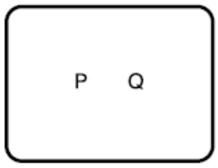

Figure IV

However, since -- and as we have already seen above -- in logical systems for assertions the assertion of a conjunction is equivalent to the conjunction of assertions, it follows that the graph of Figure IV is equivalent to the graph of Figure V:

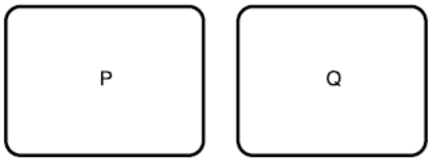

Figure V

A disjunction of assertions is represented by the graph of Figure VI, namely by adding a line (with a cross-mark to make it resemble the traditional '+' sign of disjunction) connecting two assertions such as the two boxed graphs in Figure VI:

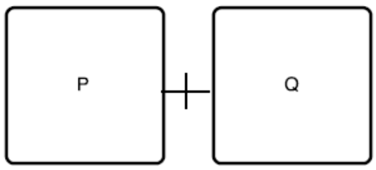

Figure VI

The implication of two assertions is expressed by the construction presented in Figure VII, and termed a *cornering*. Given the intuitionistic flavour of AGs, this conditional is not a material implication but a constructive conditional. From the assertion of P it is possible to conclude to the assertion of Q.

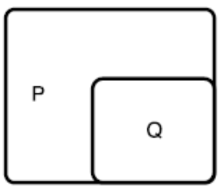

Figure VII

The last primitive notational element of AGs is the sign of absurdum, indicated by "•", and called "the blot". The graph of Figure VIII indicates an assertion that is always false and never justified. It is a pseudo-assertion.

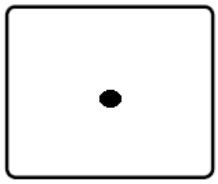

Figure VIII

In EGs cuts are used to express also (classical) negation, besides other functions such as grouping and order of interpretation. In AGs there is no primitive sign expressing negation. Instead, negation is expressed as the fact that a formula implies the *absurdum*. Therefore, the assertion of 'not P' is expressed as depicted in Figure IX:

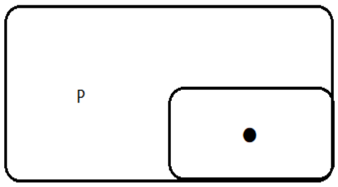

Figure IX

The definition of the deducibility relation and the set of rules of transformations have been presented and discussed in more detail in Bellucci, Chiffi, and Pietarinen (2018) and in Pietarinen and Chiffi (2020). They result in a true deep-inference of the proof rules in the system of AGs (Ma & Pietarinen 2019). Here, we add a few remarks on the notational aspects of the system.

As noted, AGs is a system with a precise intended interpretation. It is not serving only as a calculus but as a system that conveys intended interpretations to formulas on the basis of the notion of assertion. This implies that the syntactical (diagrammatical) features of AGs are guided by some relatively strong semantic/pragmatic intuitions to do with pre-formal aspects of reasoning. According to this view, inference takes place among acts of assertions. (This is quite along the fashion of Frege's works). The notion of inference is a crucial ingredient of the logical system, supplanting the semantic notion of logical consequence. Unlike in Dalla Pozza's system of pragmatic logic, however, the justification value of an assertion in AGs is not explicit, since everything that sits on the sheet of assertion has to be considered as justified. The only exception is the assertion of the *absurdum*, i.e. the pseudo-assertion of the blot, which is devoid of any such justification.[13]

[13] See Bellucci & Pietarinen (2017) on how denials contrast with (justified) assertions and how they are notationally played out in the graphical systems of EGs.

Finally, it is possible to notationally differentiate the assertion of the conjunction of two propositions (Figure IV) from that the conjunction of two assertions (Figure V). This validates the Dalla Pozza Point according to which logical connectives are to have an intuitionistic basis. Given an intuitionistic basis of the logical constants of the theory of AGs, the assertion of a disjunction of propositions cannot be directly expressed, for example.[14]

4. Comparison from the Point of View of Philosophy of Notation

Let us revisit the seven points delineated in the introduction, summarised here:

(i) *The Geach Point*: One and the same proposition can occur both asserted and unasserted in different contexts.

(ii) *The Dudman Point*: The distinction between asserted and unasserted propositions should be notationally distinguished, *whatever form* the notational expression of this difference happens to take.

(iii) *The Frege Point*: Assertion must be represented by a specific logical sign.

(iv) *The Reichenbach Point*: Assertions cannot be iterated and they cannot be connected by truth-functional connectives.

(v) *The Elementary-restriction Point*: No connective (of any sort) is ever applied to different assertions.

(vi) *The Dalla Pozza Point*: Complex assertions may be logically combined by an application of intuitionistic-like connectives.

And finally, let us derive a notational generalisation of these points from the previous considerations:

(vii) *The Peirce Point*: In any conceivable logical notation, you need to write down (or scribe) what you assert.

This scribing of propositions as assertions of them is implemented in the theory of logical graphs in terms of the sheet of assertion.[15]

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[14] Nonetheless, a classical variant of AGs has been proposed (Pietarinen & Chiffi 2019), in order to deal with classical principles. This consists in extending AGs with the rule called "elimination of coincident corners" (ECC); a rule that amounts to the rule of elimination of double negation. However, this strategy imposes a global re-interpretation of assertive formulas of AGs in a classical sense. If so, then all formulas have to be interpreted as a single assertion of different propositions, along the lines of classical systems of Frege, Reichenbach and Peirce. Only in Dalla Pozza's pragmatic logic are all the propositional and assertive connectives fully notationally disambiguated while logically related, as attested by the Bridge Principles (a)-(d).

[15] The notion of the sheet of assertion is not an exclusive property of graphical notations. Peirce used the notion of the sheet in his general algebra of logic, before the advent of logical graphs in 1896. The transitions between algebraic and graphical points of view were without much difference for Peirce. Sometimes he employs terminology in the logic of the algebra of the copula that may be more familiar from his theory of logical graphs (such as "scriptibility", "sheet of assertion"). For example, in the context of the *Minute Logic* (R 430, manuscript page 70, 1902), the writing down of a proposition "on some duly validated *sheet of assertions*" makes the proposition so uttered an assertion that "becomes a binding act". This "we will pretend" to be so "[f]or the sake of fixing our ideas", he further states (*ibid.*). The supposition that one takes there to be the "sheet" upon which an utterance or writing down of a proposition makes it an act of assertion is common in Peirce's algebra of logic just as it is in his graphical method (see Pietarinen 2019). Likewise is the application of the term "to scribe" or "scriptibility" (R 501): any algebraic or graphical constituent that has a signification by virtue of the fact that it has been asserted as having that signification, is said to be *scriptible* whenever "it is applicable to V, the *veritas*, in some understood sense" (*ibid.*).

Our analysis supports the following interrelations. Frege accepts

- (i), (ii), (iii), (iv) and (v), but not (vi): Frege does not accept that complex assertions may be logically combined by an application of intuitionistic-like connectives.

Dalla Pozza, instead, accepts:

- (i), (ii), (iii), (iv) and (vi), but not (v). He does not accept that any connective (of any sort) is ever applied to different assertions.

Peirce with his Existential Graphs accepts:

- (i) and (ii), but not (iii), (iv) or (vi). Peirce does not accept that an assertion must be represented by a specific logical sign, or that assertions cannot be iterated and cannot be connected by truth-functional connectives. He does not explicitly consider that complex assertions may be logically combined by an application of intuitionistic-like connectives.

Finally, in the framework of Assertive Graphs we have that:

- (i), (ii), (iii, with qualifications), (iv), and (vi) hold, while (v) does not, that is, that no one connective (of any sort) is ever applied to different assertions.

According to this analysis, the question of finding good notations for assertions has not so much to do on adding new signs to the vocabulary to capture assertions than it has on finding notationally economic means of expressing (acts of) assertion, as acts of writing down propositions as formulas and graphs. In this way, the sheet on which we scribe assertions is at the same time both (i) a representation that an assertion has been made, (ii) a proposition (expressing 'all truths', 'all assertions', 'all transformations', 'all proofs' etc., depending on the intended interpretations of underlying logical systems), and (iii) a sign of a logical connective (typically, that of a logical conjunction).

5. Conclusion

The pragmatic notion of assertion plays an important role in different logical systems, either explicitly or implicitly. Frege's *Begriffsschrift*, Peirce's Existential Graphs and Heyting's explication of intuitionistic constants are examples of such pragmatically motivated approaches. In this paper, the key pragmatic and notational aspects were considered, taking into account both (i) systems that explicitly express logical assertions introducing a specific sign of assertion, in particular (a) Frege's logical system and (b) Dalla Pozza's pragmatic logic, and (ii) those systems where no specific or *ad hoc* sign for assertion is introduced, in particular (c) Peirce's Existential Graphs and (d) its recently developed variant of Assertive Graphs. It was found that the question of finding the best notation for assertions has not so much to do on adding new signs to the vocabulary than finding notationally economic means of expressing assertion as acts of scribing or writing down logical formulas. The way this works in the fashion of intuitionistic logic is implemented in the theory of Assertive Graphs.

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