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Some Remarks on Assertion and Proof

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Abstract

In our *introduction* we make some remarks on the main topics of this issue: assertion and proof. We briefly describe how each of the papers in the present publication has contributed from either different or complementary perspectives to the logical reflection on assertion and proof, while also specifying the relation between them.

1 Introductory remarks

It may sound like a philosophical *cliché*, but one could not stress enough the importance of the notions of assertion and proof for logic, philosophy of logic and philosophy of language.

Although these two notions have been undergoing development since the second half of the 19th century in a relatively independent way within research programs in logic and linguistics alike, the conceptual relationships between them are undeniable. In this short contribution to the Special Issue (**Assertion and Proof**) we will illustrate some of the (possible) links between proof and assertion.

With "assertion" we denote *prima facie* at least two rather different entities; the first is a kind of act, i.e. an illocutionary act, namely the act of asserting something;

the other entity is the outcome of the same act, that is, the asserted thing. We will see that such *prima facie* duality of assertion is reflected on its logical treatment.

Consider a proposition. One and the same proposition can occur both asserted and unasserted in different contexts. In particular, Geach [10, p. 449] pointed out that "[a] thought may have just the same content whether you assent to its truth or not; a proposition may occur in a discourse now asserted, now unasserted, and yet be recognizably the same proposition". A standard example is the justification of *modus ponens:* assuming (1) $\alpha \to \beta$ and (2) α, one infers (3) β. In (2), α is usually considered asserted, while in (1), α is not asserted, because only the whole conditional $\alpha \to \beta$ is asserted (on this, see Russell 1903). This means that the very same proposition may be used in both its asserted and unasserted forms[1].

Moreover, the same line of thought could be, somehow, extended: within any argumentative structure, the same propositions may be the subject of various illocutionary acts. For instance, one could assume that φ, then hypothesize that ψ, conjecture that θ, and so on.

In the light of previous considerations, one can also ask how to provide the expressive resources in order to describe the formal features of assertions. In a letter to Frege, Peano wrote that the various positions in which a formula can be placed determines whether it occurs asserted or unasserted in some truth-functional context. Using Peano's words, "the several positions that a proposition can have in a formula completely determine what is asserted of it" [12, p. 191]. Frege observed that this is because "the principal relation sign invariably carries assertoric force" [8, p. 148] without any specific sign for assertion being present in the notation. This means that in Peano's notation it is impossible to show whether a complex formula occurs asserted or unasserted.

In this regard, Frege introduced an *ad hoc* sign of assertion as a notational requirement of the logical language. He indicated it with "$\vdash$", which stands for the sign of assertion. (Nowadays, the sign "$\vdash$", has acquired the name "turnstile" and expresses the concept of derivability or provability.)

Once given the expressive resources in order to describe assertions, it becomes crucial to provide some criteria concerning the logical behaviour of assertions. From this point of view, it is normally assumed that assertions, intended as acts, cannot be iterated and cannot be connected by truth-functional connectives [15]. Reichenbach

[1] Bell [1] comments Russell's views on *modus ponens* in the following way: "Now this would imply that either all inferences having the *modus ponens* form (to take but one example) are invalid, or, at least, that all those with either a true antecedent or a true consequent in the conditional premiss are invalid. This is, of course, quite unacceptable" (pp. 87-88). On the analysis of Russell's views on *modus ponens* and Bell's untoward consequences, see the justification of *modus ponens* in pragmatic logic [3].

[15, p. 346] claims this on the basis of the fact that the term "assertion" is used in three different ways. Namely, "it denotes, first, the act of asserting; second, the result of this act, i.e., an expression of the form '⊢p'; third, a statement which is asserted, i.e. a statement 'p' occurring within an expression '⊢p'". Regarding the result of an assertion, Reichenbach claims that "since assertive expressions are not propositions, they cannot be combined by propositional operators" [15, §57, p. 337]. The assertion sign works, according to him, in its "pragmatic capacity", since it cannot be, for instance, negated with a propositional connective. And if this is so, then inferences can be understood as processes that allow us to derive justified asserted conclusions once the asserted premises are also justified. This means that there can be no nested or iterated occurrences of the assertion sign, because the truth-functional connectives only operate on propositions and never on assertions. Furthermore, inferences operate only on assertions and never on propositions.

The structure of an elementary assertion is thus composed by a unique assertion sign prefixed to the asserted content. This is, for instance, the case with Frege's [8] *Begriffsschrift*. A similar restriction is presented in Reichenbach's treatise, where the distinction between assertions and (propositional) content is developed.

Of course, the logic of assertions must engage with the possibility of connecting elementary assertions in order to construct more complex ones. At play here is in fact the intertwining between the logical form of asserted contents and the logic of assertions. It is plausible to claim that asserting a conjunction $(\varphi \wedge \psi)$ is equivalent to juxtaposing two elementary assertions, respectively of φ and ψ.

Things are not always so easy. Asserting a disjunction $(\varphi \vee \psi)$ does not seem to be equivalent to asserting φ or asserting ψ. And the same holds for implication. The logic of assertions is, thus, more complex and, somehow, different from the logic dealing with the asserted content.

This discrepancy becomes more evident if we look at Dalla Pozza's system [5]. Within it, complex assertions may be logically combined by an application of intuitionistic-like connectives. This means that complex asserted formulas may be expressed through connectives that explicate intuitionistic meanings of logical constants, without a truth-functional behaviour. Moreover, intuitionistic connectives can indicate the (pragmatic) justification-conditions for (acts of) assertions.

On the basis of this short overview on some fundamental features of the logic of assertion, we are able to cast some light on the link between the concept of assertion and the concept of proof.

As we said, when assertions are intended as acts, they are neither true nor false. Therefore, it is quite natural to employ a non-truth-functional kind of semantics in order to construe the formula of a logic for assertion. A natural candidate is the concept of *proof*: an assertion is justified (or unjustified) depending on the existence (or not) of a proof of the asserted content. The reference to the concept of proof emphasizes the constructive feature of the logic of assertions.

By way of example, it is interesting to notice the justification clause pertaining to the assertion according to Dalla Pozza and Garola's [5] approach: in that system, the implication between two assertions ($\vdash \varphi \supset \vdash \psi$) is justified if and only if we have a proof which transforms any proof of φ into a proof of ψ at our disposal. It is, thus, clear why the outer logic of assertion can be different from the inner logic of asserted contents. We can, for instance, be justified in asserting every instance of excluded middle ($\varphi \vee \neg\varphi$) without being justified either to assert φ or to assert $\neg\varphi$. It is sufficient to assume that φ describes a proposition for which we do not have conclusive proofs.

We said that an assertion is justified if and only if we have a proof at disposal. But what is a *proof*? Here below we propose some introductory remarks. Let us start by observing that this is a central notion of the *proof-theorethic approach to logic*. According to this, a consequence is identified with *deducibility*: an argument $\langle \Gamma \therefore \alpha \rangle$ is valid if and only if there exists a *proof* or a derivation of α from Γ, each of whose steps is intuitively sound. The account is formal, insofar as logical consequence is identified with derivability in a system of rules of a certain form. *Proof-theoretic semantics* is standardly taken as an alternative to *truth-condition semantics*. In a nutshell, *proof-theoretic semantics* is based on the assumption that the meanings of the logical constants are assigned in terms of *proof* and of their inferential role rather than in terms of *truth*.

Proof-theoretic accounts of consequence are sometimes quickly dismissed. Field, for example, writes that "proof-theoretic definitions proceed in terms of some definite proof procedure", and observes that "it seems pretty arbitrary which proof procedure one picks" and "it isn't very satisfying to rest one's definitions of fundamental metalogical concepts on such highly arbitrary choices" [7, p. 2]. Etchemendy similarly observes that "the intuitive notion of consequence cannot be captured by any *single* deductive system" [6, p. 2], since the notion of consequence is neither tied to any particular language, nor to any particular deductive system.

In order to understand what a proof is one can first specify a notion of *formal proof* or of *informal proof*. The formalisation of the idea of *proof* as a *given set of sequences of symbols* underlies the meta-mathematical research pioneered by Hilbert and Bernays and subsequently developed by Gödel, Gentzen and others. Boolos [2] famously explored the intensional representation of formal proof through systems of modal logic in which ($\Box$) is interpreted as "it is provable that" in a formal sense. A characteristic axiom (called the "Gödel-Löb axiom") of the notion of formal provability is $\Box(\Box\varphi \rightarrow \varphi) \rightarrow \Box\varphi$. If this axiom is added to the modal system K4, we obtain the modal logic G. Such a system was formalized by Solovay [17]. It is complete with respect to transitive and conversely well-founded frames. In this system the reflection principle $\Box\varphi \rightarrow \varphi$ does not hold.

Things are different in case of *informal proofs* (intended, for instance, as good mathematical arguments) which are usually assumed to justify truth, thus accepting the reflection principle.

The notion of *informal* or *naïve proof* received some attention when Gödel, in the *Gibbs lecture* [11], asserted his famous *dichotomy* concerning the nature of the human mind. Then, Priest characterised a naïve proof as a process of deductive argumentation by which one establishes certain mathematical claims to be true. So, supposing there is a mathematical assertion whose truth or falsity is to be established, one can look for a proof or a refutation to justify it or not. The informal deductive arguments from basic statements are, according to Priest, "naïve proofs" [14, 40]. It is interesting to observe that Priest [13], in his "The logic of paradox," developed a controversial argument, grounded in the notion of naïve proof, showing some possible connections between Gödel's first incompleteness theorem and the presence of *dialetheias* (viz., sentences that are both true and false) in the standard model of arithmetic. This last point has been criticized especially regarding the notion of naïve proof itself.

2 The contents of the issue

Each of the papers in this special issue contributes from different and complementary perspectives to the logical reflection on assertion and proof as well as on the relations between these two concepts.

Barés Gómez and Fontaine in *Defeasibility and non-monotonicity in dialogues* show how to introduce the notions of defeasibility and non-monotonicity in dialogical logic, and discuss them in a framework of adaptive dialogical logics.

Bellucci, Chiffi and Pietarinen in *Beta assertive graphs: Proofs of assertions with quantification* introduce and investigate quantification in the diagrammatic system of assertive graphs.

Carrara and Strollo in DLEAC *and the rejection paradox* develop a Dialetheic Logic with exclusive assumptions and conclusions, both understood as speech acts. A new paradox – the *rejectability paradox* – is (first informally, then formally) introduced. Its derivation is possible in an extension of DLEAC contanining the *rejectability* predicate.

Chiffi in *Asserting boo! and horray! Pragmatic logic for assertion and moral attitudes* proposes a pragmatic logic for expressivist moral attitudes in order to deal with the logical problems of expressivism such as the Frege-Geach problem, the negation problem, etc. The second part of the paper makes some analytic comparisons with other classical logical systems for expressivist sentences.

D'Agostino, Larese and Modgil in *Towards depth-bounded natural deduction for classical first-order logic* present a new proof-theory for classical first-order logic that allows for a natural characterization of a notion of inferential depth. Unlike natural deduction, in this framework the rules fixing the meaning of the logical operators are symmetrical with respect to assent and dissent and do not involve the discharge of formulas.

De Florio in *Reflections on logics for assertion and denial* discusses and refines the justification conditions for assertion and denial in an extension of Dalla Pozza's pragmatic logic.

Fait and Primiero in *HTLC: hyperintensional typed lambda calculus* introduce a new logical system termed "HTLC". The system extends the typed lambda-calculus with hyperintensions and rules to govern them. This allows us to reason with expressions for extensional, intensional and hyperintensional entities.

Francez in *Bilateralism based on corrective denial* presents a new variant of bilateralism based on a strong notion of denial, called "corrective denial". In this framework, a ground for denial is an incompatible atomic alternative to the denied formula.

Jespersen in *Two tales of the turnstile* criticizes, from a hyperintensional perspective, the view held by act-theoretic 'internalists' who invert the Frege-Geach point by making force integral to content.

Kürbis in *Normalisation for bilateral classical logic with some philosophical remarks* presents two bilateral connectives, comparable to Prior's *tonk*, for which, unlike the case of *tonk*, there are reduction steps for the removal of maximal formulas, arising from introducing and eliminating formulas with those connectives as main operators.

Lemanski in *Extended syllogistics in calculus CL* addresses the question regarding to what extent a syllogistic representation in *CL* (Lange's *Cubus Logicus*) diagrams can be seen as a form of extended syllogistics. The author shows that the ontology of CL enables numerically exact assertions and inferences.

Morato in *Assertions of counterfactuals and epistemic irresponsibility* discusses the so-called "reverse Sobel sequences", problematic for the variably strict semantics for counterfactuals. Morato shows, in particular, some limitations of the "principle of epistemic irresponsibility", which is assumed to ground the pragmatic view on this type of counterfactual sequences.

Finally, Schang in *A general semantics for logics of affirmation and negation* proposes some semantic considerations on the notions of affirmation and negation that may help us understand the possible translations among different logical systems.

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