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Twist-Valued Models for Three-Valued Paraconsistent Set Theory

Abstract. We propose in this paper a family of algebraic models of **ZFC** based on the three-valued paraconsistent logic **LPT0**, a linguistic variant of da Costa and D'Ottaviano's logic **J3**. The semantics is given by twist structures defined over complete Boolean agebras. The Boolean-valued models of **ZFC** are adapted to twist-valued models of an expansion of **ZFC** by adding a paraconsistent negation. This allows for inconsistent sets w satisfying 'not $(w = w)$', where 'not' stands for the paraconsistent negation. Finally, our framework is adapted to provide a class of twist-valued models generalizing Löwe and Tarafder's model based on logic $(\mathbb{PS}_3,*)$, showing that they are paraconsistent models of **ZFC**. The present approach offers more options for investigating independence results in paraconsistent set theory.

Keywords: Paraconsistent set theory; Boolean-valued models; axiomatic set theory; twist structures; logics of formal inconsistency; three-valued paraconsistent logics; Leibniz's Law

1. On models of set theory: Gödel shrinks, Cohen expands

The interest for — and the overall knowledge about — models for set theory changed dramatically after the famous invention (or discovery) of Paul Cohen's methods of forcing. Cohen was able to show that the notion of cardinal number is elastic and relative, in contrast with the methods of "inner models" that Gödel used. Gödel has shown that, by shrinking the totality of sets in a model, they would turn to be 'well-behaved'. As a consequence, the constructible sets could not be used to prove the relative consistency of the negation of the Axiom of Choice (AC) or of

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the Continuum Hypothesis (CH). Paul J. Cohen, on the contrary, had the idea of reverting the paradigm, and instead of cutting down the sets within models, found a way to expand a countable standard model into a standard model in which CH or AC can be false, doing this in a minimalist but controlled fashion. Cohen elements are 'bad-behaved', but finely guided so as to make 'logical space' for the independence of AC and CH.

As Dana Scott puts in the forward of Bell's book [1], "Cohen's achievement lies in being able to expand models (countable, standard models) by adding new sets in a very economical fashion: they more or less have only the properties they are forced to have by the axioms (or by the truths of the given model)." Cohen's methods, however, are not easy, being regarded by some researchers as somewhat lengthy and tedious — but were the only tool available until the Boolean-valued models of set theory put forward by Scott and Solovay (and independently by Vopěnka) in 1965 offered a more natural and rich alternative for describing forcing. This does not discredit the brilliant idea of Cohen, who did not have the machinery of Boolean-valued models available at his time.

What is a Boolean-valued model? The intuitive idea is to pick a suitable Boolean algebra $\mathcal{A}$, and define, by transfinite recursion, the set M of all $\mathcal{A}$-valued sets, generalizing the familiar $\{0, 1\}$ valued models. Then add to the language one constant symbol for each element of the model. After this, define a map $\varphi \mapsto [\![\varphi]\!]^{\mathcal{A}}$ from the sentences in the language of **ZF** to $\mathcal{A}$ which obey certain equations, so that it should assign 1 to all the axioms of **ZFC**.

The resulting structure $M_{\mathcal{A}}$ will not be a standard model of **ZFC**, because it will consist of "relaxed sets" somehow similar to fuzzy sets, and not sets properly. If we take an arbitrary sentence about sets (for instance, "Is Y is a member of X" ?) and ask whether it holds in $M_{\mathcal{A}}$, then the answer may be neither plain "yes" nor "no", but some element of the Boolean algebra $\mathcal{A}$ meaning the "degree" to which Y is a member of X. However, $M_{\mathcal{A}}$ will satisfy **ZFC**, and to turn $M_{\mathcal{A}}$ into an actual model of **ZFC** with certain desired properties it is sufficient to take a suitable quotient of M_B that eliminates the elements of fuzziness.

Boolean-valued models not only avoid tedious details of Cohen's original construction, but permit a great generalization by varying on any Boolean algebra.

2. Losing unnecessary weight: the role of alternative set theories

It is a well-known historical fact that the discovery of the paradoxes in set theory and in the foundations of mathematics was the fuse that fired the revolution in contemporary set theory around its efforts to attempt to rescue Cantor's naive theory from triviality. The usual culprit was the Principle of (unrestricted) Abstraction, also known as the Principle of Comprehension. Unrestricted abstraction allows sets to be defined by arbitrary conditions, and this freedom combined with the axiom of extensionality, leads to a contradiction, which by its turn leads to triviality in the sense that "everything goes", when the laws of the underlying logic obey the standard principles that comprise the so-called "classical" logic.

But there is a way out from this maze. Paraconsistent set theory is the theoretical move to maintain the freedom of defining sets, while stripping the theory of unnecessary principles so as to avoid triviality, a disastrous consequences of contradictions involving sets in **ZF**.

This philosophical maneuver is in frank opposition to traditional strategies, which deprive the freedom of set theory so appreciated by Cantor, by maintaining the underlying logic and weakening the Principle of Abstraction.

An analogy may be instructive. The basic goal of reverse mathematics is to study the relative logical strengths of theorems from ordinary non-set theoretic mathematics. To this end, one tries to find the minimal natural axiom system A that is capable of proving a theorem T.

In a perhaps vague, but illuminating analogy, paraconsistent logic tries to find the minimal natural principles that are capable of permitting us to reason in generic circumstances, even in the undesired circumstances of contradictions.

This does not mean that contradictions are necessarily real: [9] gives a formal system and a corresponding intended interpretation, according to which true contradictions are not tolerated. Contradictions are, instead, epistemically understood as conflicting evidence. There are indeed many cases of contradictions in reasoning, but the classical principle *Ex Contradictione Quodlibet*, or Principle of Explosion, is not even used in mathematics in general; it is not, therefore, a characteristic of good reasoning, and has to be abandoned.

Some people may be mislead by thinking that *Reductio ad Absurdum*, which is a useful and robust rule of inference, would be lost by abandoning the Principle of Explosion. This is not so: even if discarding such a principle, proofs by *Reductio ad Absurdum* get unaffected, as long as one can define a strong negation. This is achieved in many paraconsistent logics, in particular in all the logics of the family of the Logics of Formal Inconsistency (**LFI**s) [see 5, 6, 7]. Reasoning does not necessarily require the full power of *Ex Contradictione Quodlibet*, because contradictions reached in a *Reductio* proof are not really used to cause any deductive explosion; what is used is the manipulation of negation.

3. Expanding Cohen's expansion: twist-valued models

Boolean-valued models were adapted by Takeuti, Titani, Kozawa and Ozawa to lattice-valued models of set theory, with applications to quantum set theory and fuzzy set theory [see 19, 20, 21, 23, 24]. The guidelines of these constructions were taken by Löwe and Tarafder in [18] in order to obtain a three-valued model (in the form of a lattice-valued model) for a paraconsistent set theory based on **ZF**. They propose a class of algebras based on a certain kind of implication, called *reasonable implication algebras* (see Section 9) which satisfy several axioms of **ZF**. From this class, they found an especific three-valued model which satisfies all the axioms of **ZF**, and it can be expanded to an algebra $(\mathbb{PS}_3, *)$ with a paraconsistent negation $*$, obtaining so a paraconsistent model of **ZF**. As we discuss in Section 9, the matrix logic associated to $(\mathbb{PS}_3, *)$ with $\mathbf{0}$ as the only non-designated value, which will be also denoted in this paper by $(\mathbb{PS}_3, *)$ or $(\mathbb{PS}_3, \neg)$, is the same as the logic **MPT** introduced in [10], and coincides up to language with the logic **LPT0** adopted in the present paper, as well as with da Costa and D'Ottaviano's logic **J3**. Here, we will introduce the notion of twist-valued models for a paraconsistent set theory $\mathbf{ZF_{LPT0}}$ based on **QLPT0**, a first-order version of **LPT0**. Our models, defined for any complete Boolean algebra $\mathcal{A}$, constitute a generalization of the Boolean-valued models for set theory, at the same time generalizing Löwe and Tarafder's three-valued model. Indeed, in Section 9 the model of **ZF** based on $(\mathbb{PS}_3, *)$ will be generalized to twist-valued models over an arbitrary complete Boolean algebra, obtaining so a class of models of **ZFC**. The structure over $(\mathbb{PS}_3, *)$ will constitute a particular case, by considering the two-element complete Boolean algebra. As a consequence of this, it follows that Löwe and Tarafder's three-valued structure is, indeed, a model of **ZFC**.

Twist-structure semantics have been independently proposed by M. Fidel [14] and D. Vakarelov [25], in order to semantically characterize the well-known Nelson logic. A twist structure consists of operations defined on the cartesian product of the universe of a lattice, $L \times L$ so that the negative and positive algebraic characteristics can be treated separately. In terms of logic, a pair (a, b) in $L \times L$ is such that a represents a truth-value for a formula φ while b corresponds to a truth-value for the negation of φ. That is, a is a positive value for φ while b is a negative value for it, thus justifying the name 'twist structures' given for this kind of algebras. This strategy is especially useful for obtaining semantical characterizations for non-standard logics. As a limiting case, a Boolean algebra turns out to be a particular case of twist structures when there is no need to give separate attention to negative and positive algebraic characteristics, since the latter are uniquely obtained from the former by the dualizing Boolean complement $\sim$. In this case, every pair (a, b) is of the form $(a, \sim a)$, hence the second coordinate is redundant. Our proposal is based on models for **ZF** based on twist structures, thus the sentences of the language of **ZF** will be interpreted as pairs (a, b) in a suitable twist structure, such that the supremum $a \vee b$ is always 1, but the infimum $a \wedge b$ is not necessarily equal to 0. This corresponds to the validity of the third-excluded middle for the non-classical negation of the underlying logic, while the explosion law $\varphi \wedge \neg\varphi \to \psi$ is not valid in general in the underlying paraconsistent logic **LPT0**. A somewhat related approach was proposed by Libert in [17]: he proposes models for a naive set theory in which the truth-values are pairs of sets (A, B) of a universe U such that $A \cup B = U$ where A and B represent, respectively, the extension and the anti-extension of a set a. However, besides this similarity, our approach is quite different: we are interested in giving paraconsistent models for **ZFC** and not in new models for naive set theory.

It is important to notice that there exists in the literature several approaches to paraconsistent set theory, under different perspectives. In particular, we propose in [4] a paraconsistent set theory based on several **LFI**s, but that approach differs from the one in the present paper. First, in the previous paper the systems were presented axiomatically, by means of suitable modifications of **ZF**. Moreover, in that logics a consistency predicate $C(x)$ was considered, with the intuitive meaning that 'x is a consistent set'. On the other hand, in the present paper a model for standard **ZFC** will be presented instead of a Hilbert calculus for a modified version of **ZF**. We will return to this point in Section 10, in which the possibility of defining a consistency predicate $C(x)$ within $\mathbf{ZF_{LPT0}}$ will be discussed.

As mentioned above, twist structures over a Boolean algebra generalize Boolean algebras, and are by their turn generalized by the *swap structures* introduced in [5, Chapter 6] (a previous notion of swap structures was given in [3]). Swap structures are non-deterministic algebras defined over the three-fold Cartesian product $\mathcal{A} \times \mathcal{A} \times \mathcal{A}$ of a given Boolean algebra so that in a triple (a, b, c) the first component a represents the truth-value of a given formula φ while b and c represent, respectively, possible values for the paraconsistent negation $\neg\varphi$ of φ, and for the consistency $\circ\varphi$ of φ.

Swap structures are committed to semantics with a non deterministic character, while twist structures are used when the semantics are deterministic (or truth-functional). Definition 4.6 below shows how the definition of twist structures for the three-valued logic $\mathbf{LFI1}_\circ$ introduced in [11, Definition 9.2] can be adapted to **LPT0**.

As discussed in Section 9, the three-valued logic $(\mathbb{PS}_3, *)$ used in [22] already appears in [10] under the name **MPT**, and it is equivalent to **LPT0** and also to $\mathbf{LFI1}_\circ$. Variants of this logic have been independently proposed by different authors with different motivations in several occasions (for instance, as the well-known da Costa and D'Ottaviano's logic **J3**). The naturalness of this logic is reflected by the fact that the three-valued algebra of **LPT0** (see Definition 4.2 below) is equivalent, up to language, to the algebra underlying Łukasiewicz three-valued logic Ł3. The only difference is that in the former the set of distinguished (or designated) truth values is $\{\mathbf{1}, \frac{\mathbf{1}}{\mathbf{2}}\}$ instead of $\{\mathbf{1}\}$, and this is why **LPT0** is paraconsistent while Ł3 is paracomplete (taking into account that the negation is the same in both logics).

Twist-valued models work beautifully as enjoying many properties similar to Boolean-valued models (when restricted to pure **ZF**-languages). Such similarities lead to a natural proof that **ZFC** is valid w.r.t. twist-valued models, as our central Theorem 8.22 shows. This paper deals with the paraconsistent set theory $\mathbf{ZF_{LPT0}}$, defined by using as the underlying logic a first-order version of **LPT0**, called **QLPT0**, proposed in [12] under the form of $\mathbf{QLFI1}_\circ$ (that is, by replacing the strong negation $\sim$ by the consistency operator $\circ$).

The paraconsistent character of twist-valued models as regarding $\mathbf{ZF_{LPT0}}$ as rival of **ZFC** is emphasized. Despite having some limitative results, as much as Löwe and Tarafder's model, $\mathbf{ZF_{LPT0}}$ has a great potential as generator of models for paraconsistent set theory. A subtle, but critical advantage of our models is that the implication operator of **LPT0** is much more suitable for a paraconsistent set theory than the one of $\mathbb{PS}_3$. Indeed, our models allow for inconsistent sets, and this is of paramount importance, as we argue below. Moreover, as pointed out above, our models generalize the three-valued model based on $\mathbb{PS}_3$, since they can be defined for any complete Boolean algebra. In this way, we have several models at our disposal, and in principle this can be used to investigate independence results in paraconsistent set theory.

Albeit Boolean-valued models and their generalization in the form of twist-valued models are naturally devoted to study independence results, this paper does not tackle this big questions yet. The paper, instead, is dedicated to clarifying such models while establishing their basic properties.

4. The logic LPT0

In this section the logic **LPT0** will be briefly discussed, including its twist structures semantics. From now on, if Σ' is a propositional signature then, given a denumerable set $\mathcal{V} = \{p_1, p_2, \ldots\}$ of propositional variables, the propositional language generated by Σ' from $\mathcal{V}$ will be denoted by $\mathcal{L}_{\Sigma'}$. The paraconsistent logics considered in this paper belong to the class of logics known as *logics of formal inconsistency*, introduced in [7] [see also 5, 6].

Definition 4.1. Let $\boldsymbol{L} = \langle \Sigma', \vdash \rangle$ be a Tarskian, finitary and structural logic defined over a propositional signature Σ', which contains a negation $\neg$, and let $\circ$ be a (primitive or defined) unary connective. The logic $\boldsymbol{L}$ is said to be a *logic of formal inconsistency* (**LFI**) with respect to $\neg$ and $\circ$ if the following holds:

(i) $\varphi, \neg\varphi \nvdash \psi$ for some φ and ψ;

(ii) there are two formulas φ and ψ such that

  (a) $\circ\varphi, \varphi \nvdash \psi$;

  (b) $\circ\varphi, \neg\varphi \nvdash \psi$;

(iii) $\circ\varphi, \varphi, \neg\varphi \vdash \psi$ for all φ and ψ.

Recall the logic $\mathcal{A}_{\mathrm{PT0}}$ presented in[5] as a linguistic variant of the logic **MPT** introduced in [10].

Definition 4.2 (Modified propositional logic of pragmatic truth **MPT0**; 5, Definition 4.4.51). Let $\mathcal{M}_{\mathrm{PT0}} = \langle M, D\rangle$ be the three-valued logical matrix over $\Sigma = \{\wedge, \vee, \to, \sim, \neg\}$ with domain $M = \{\mathbf{1}, \frac{\mathbf{1}}{\mathbf{2}}, \mathbf{0}\}$ and set of designated values $D = \{\mathbf{1}, \frac{\mathbf{1}}{\mathbf{2}}\}$ such that the operators are defined as follows:

$\wedge$	$\mathbf{1}$	$\frac{\mathbf{1}}{\mathbf{2}}$	$\mathbf{0}$
$\mathbf{1}$	$\mathbf{1}$	$\frac{\mathbf{1}}{\mathbf{2}}$	$\mathbf{0}$
$\frac{\mathbf{1}}{\mathbf{2}}$	$\frac{\mathbf{1}}{\mathbf{2}}$	$\frac{\mathbf{1}}{\mathbf{2}}$	$\mathbf{0}$
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$

$\vee$	$\mathbf{1}$	$\frac{\mathbf{1}}{\mathbf{2}}$	$\mathbf{0}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
$\frac{\mathbf{1}}{\mathbf{2}}$	$\mathbf{1}$	$\frac{\mathbf{1}}{\mathbf{2}}$	$\frac{\mathbf{1}}{\mathbf{2}}$
$\mathbf{0}$	$\mathbf{1}$	$\frac{\mathbf{1}}{\mathbf{2}}$	$\mathbf{0}$

$\to$	$\mathbf{1}$	$\frac{\mathbf{1}}{\mathbf{2}}$	$\mathbf{0}$
$\mathbf{1}$	$\mathbf{1}$	$\frac{\mathbf{1}}{\mathbf{2}}$	$\mathbf{0}$
$\frac{\mathbf{1}}{\mathbf{2}}$	$\mathbf{1}$	$\frac{\mathbf{1}}{\mathbf{2}}$	$\mathbf{0}$
$\mathbf{0}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$

	$\sim$
$\mathbf{1}$	$\mathbf{0}$
$\frac{\mathbf{1}}{\mathbf{2}}$	$\mathbf{0}$
$\mathbf{0}$	$\mathbf{1}$

	$\neg$
$\mathbf{1}$	$\mathbf{0}$
$\frac{\mathbf{1}}{\mathbf{2}}$	$\frac{\mathbf{1}}{\mathbf{2}}$
$\mathbf{0}$	$\mathbf{1}$

The logic associated to the logical matrix $\mathcal{M}_{\mathrm{PT0}}$ is called **MPT0**. The three-valued algebra underlying $\mathcal{M}_{\mathrm{PT0}}$ will be called $\mathcal{A}_{\mathrm{PT0}}$.

Observe that $x \to y = {\sim}x \vee y$ for every x, y. Recall that, by definition, the consequence relation $\vDash_{\mathbf{MPT0}}$ of $\mathcal{A}_{\mathrm{PT0}}$ is given as follows: for every $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_\Sigma$, $\Gamma \vDash_{\mathbf{MPT0}} \varphi$ iff, for every homomorphism $v \colon \mathcal{L}_\Sigma \to M$ of algebras over Σ, if $v[\Gamma] \subseteq D$ then $v(\varphi) \in D$.

From [5] a sound and complete Hilbert calculus for **MPT0**, called **LPT0**, can be defined. This calculus is an axiomatic extension of a Hilbert calculus for classical propositional logic **CPL** over the signature $\Sigma_c = \{\wedge, \vee, \to, \sim\}$. From now on, $\varphi \leftrightarrow \psi$ will be an abbreviation for the formula $(\varphi \to \psi) \wedge (\psi \to \varphi)$.

Definition 4.3 (The calculus **LPT0**; 5, Definition 4.4.52). The Hilbert calculus **LPT0** over Σ is defined as follows:[1]

Axiom schemas:

(Ax1) $\varphi \to (\psi \to \varphi)$
(Ax2) $(\varphi \to (\psi \to \gamma)) \to ((\varphi \to \psi) \to (\varphi \to \gamma))$
(Ax3) $\varphi \to (\psi \to (\varphi \wedge \psi))$
(Ax4) $(\varphi \wedge \psi) \to \varphi$

[1] To be rigorous, in [5, Theorem 4.4.56] an additional axiom schema is required: $\neg{\sim}\varphi \to \varphi$. However, it is easy to prove that this axiom is derivable from the others, by using MP.

(Ax5)	$(\varphi \wedge \psi) \to \psi$
(Ax6)	$\varphi \to (\varphi \vee \psi)$
(Ax7)	$\psi \to (\varphi \vee \psi)$
(Ax8)	$(\varphi \to \gamma) \to ((\psi \to \gamma) \to ((\varphi \vee \psi) \to \gamma))$
(Ax9)	$\varphi \vee (\varphi \to \psi)$
(TND)	$\varphi \vee \sim\varphi$
(exp)	$\varphi \to (\sim\varphi \to \psi)$
$(\mathrm{TND}_\neg)$	$\varphi \vee \neg\varphi$
(dneg)	$\neg\neg\varphi \leftrightarrow \varphi$
$(\mathrm{neg}_\vee)$	$\neg(\varphi \vee \psi) \leftrightarrow (\neg\varphi \wedge \neg\psi)$
$(\mathrm{neg}_\wedge)$	$\neg(\varphi \wedge \psi) \leftrightarrow (\neg\varphi \vee \neg\psi)$
$(\mathrm{neg}_\to)$	$\neg(\varphi \to \psi) \leftrightarrow (\varphi \wedge \neg\psi)$

Inference rule:

$$\text{(MP)} \quad \frac{\varphi \quad \varphi \to \psi}{\psi}$$

It is worth noting that axioms (Ax1)–(Ax9), (TND) and (exp), together with (MP), constitute an adequate Hilbert calculus for classical propositional logic **CPL** in the signature $\Sigma_c = \{\wedge, \vee, \to, \sim\}$. Moreover, (Ax1)–(Ax9) plus (MP) is an adequate Hilbert calculus for classical positive propositional logic $\mathbf{CPL}^+$ in the signature $\Sigma_{cp} = \{\wedge, \vee, \to\}$.

THEOREM 4.4 (5, Theorem 4.4.56). *The logic* **LPT0** *is sound and complete w.r.t. the matrix logic of* $\mathcal{A}_{\mathrm{PT0}}$: $\Gamma \vdash_{\mathbf{LPT0}} \varphi$ *iff* $\Gamma \vDash_{\mathbf{MPT0}} \varphi$, *for every* $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_\Sigma$.

The latter result can be extended to twist-structures semantics, as shown in [11]. Indeed, **LPT0** coincides (up to signature) with $\mathbf{LFI1}_\circ$, an **LFI** defined over the signature $\Sigma_\circ = \{\wedge, \vee, \to, \neg, \circ\}$ such that the consistency operator $\circ$ is defined as

	$\circ$
1	**1**
$\mathbf{\frac{1}{2}}$	**0**
0	**1**

In $\mathbf{LFI1}_\circ$ the strong negation $\sim$ is defined as $\sim\varphi := \varphi \to \bot_\varphi$ such that $\bot_\varphi := (\varphi \wedge \neg\varphi) \wedge \circ\varphi$. On the other hand, the consistency operator $\circ$ is defined in **LPT0** as $\circ\varphi := \sim(\varphi \wedge \neg\varphi)$. The twist-structures semantics for $\mathbf{LFI1}_\circ$ introduced in [11, Definition 9.2] can be adapted to **LPT0** as follows:

DEFINITION 4.5. Let $\mathcal{A} = \langle A, \wedge, \vee, \to, \sim, 0, 1\rangle$ be a Boolean algebra.[2] The *twist domain* generated by $\mathcal{A}$ is the set $T_{\mathcal{A}} = \{(z_1, z_2) \in A \times A \ : \ z_1 \vee z_2 = 1\}$.

DEFINITION 4.6. Let $\mathcal{A}$ be a Boolean algebra. The *twist structure for* **LPT0** *over* $\mathcal{A}$ is the algebra $\mathcal{T}_{\mathcal{A}} = \langle T_{\mathcal{A}}, \tilde{\wedge}, \tilde{\vee}, \tilde{\to}, \tilde{\sim}, \tilde{\neg}\rangle$ over Σ such that the operations are defined as follows, for every $(z_1, z_2), (w_1, w_2) \in T_{\mathcal{A}}$:

(i) $(z_1, z_2) \,\tilde{\wedge}\, (w_1, w_2) = (z_1 \wedge w_1, z_2 \vee w_2)$;
(ii) $(z_1, z_2) \,\tilde{\vee}\, (w_1, w_2) = (z_1 \vee w_1, z_2 \wedge w_2)$;
(iii) $(z_1, z_2) \,\tilde{\to}\, (w_1, w_2) = (z_1 \to w_1, z_1 \wedge w_2)$;
(iv) $\tilde{\neg}(z_1, z_2) = (z_2, z_1)$
(v) $\tilde{\sim}(z_1, z_2) = ({\sim} z_1, z_1)$.

By recalling that the consistency operator $\circ$ is defined in **LPT0** as $\circ\,\varphi := {\sim}(\varphi \wedge \neg\,\varphi)$, it follows that $\tilde{\circ}(z_1, z_2) = ({\sim}(z_1 \wedge z_2), z_1 \wedge z_2)$.[3]

DEFINITION 4.7. The logical matrix associated to the twist structure $\mathcal{T}_{\mathcal{A}}$ is $\mathcal{MT}_{\mathcal{A}} = \langle \mathcal{T}_{\mathcal{A}}, D_{\mathcal{A}}\rangle$ where $D_{\mathcal{A}} = \{(z_1, z_2) \in T_{\mathcal{A}} \ : \ z_1 = 1\} = \{(1, a) \ : \ a \in A\}$. The consequence relation associated to $\mathcal{MT}_{\mathcal{A}}$ will be denoted by $\models_{\mathcal{T}_{\mathcal{A}}}$. Let $\mathcal{M}_{\mathbf{LPT0}} = \{\mathcal{MT}_{\mathcal{A}} \ : \ \mathcal{A} \text{ is a Boolean algebra}\}$ be the class of twist models for **LPT0**. The *twist-consequence relation* for **LPT0** is the consequence relation $\models_{\mathcal{M}_{\mathbf{LPT0}}}$ associated to $\mathcal{M}_{\mathbf{LPT0}}$, namely: $\Gamma \models_{\mathcal{M}_{\mathbf{LPT0}}} \varphi$ iff $\Gamma \models_{\mathcal{T}_{\mathcal{A}}} \varphi$ for every Boolean algebra $\mathcal{A}$.

Remark 4.8. In [11, Theorem 9.6] it was shown that **LPT0** is sound and complete w.r.t. twist structures semantics, namely: $\Gamma \vdash_{\mathbf{LPT0}} \varphi$ iff $\Gamma \models_{\mathcal{M}_{\mathbf{LPT0}}} \varphi$, for every set of formulas $\Gamma \cup \{\varphi\}$. On the other hand, if $\mathbb{A}_2$ is the two-element Boolean algebra with domain $\{0, 1\}$ then $\mathcal{T}_{\mathbb{A}_2}$ consists of three elements: $(1, 0)$, $(1, 1)$ and $(0, 1)$. By identifying these elements with $\mathbf{1}$, $\mathbf{\frac{1}{2}}$ and $\mathbf{0}$, respectively, then $\mathcal{T}_{\mathbb{A}_2}$ coincides with the three-valued algebra $\mathcal{A}_{\mathrm{PT0}}$ underlying the matrix $\mathcal{M}_{\mathrm{PT0}}$ (recall Definition 4.2). Moreover, $\mathcal{MT}_{\mathbb{A}_2}$ coincides with $\mathcal{M}_{\mathrm{PT0}}$. Taking into consideration Theorem 4.4, this situation is analogous to the semantical characterization of **CPL** w.r.t. Boolean algebras: it is enough to consider the two-element Boolean algebra $\mathbb{A}_2$.

[2] In this paper the symbol $\sim$ will be used for denoting the strong negation of **LPT0** as well as for denoting the classical negation and its semantical interpretation (the Boolean complement in a Boolean algebra). The context will avoid possible confusions.

[3] This is why, in [11, Definition 9.2], clause (v) of Definition 4.6 was replaced by the clause defining $\tilde{\circ}$.

5. The logic QLPT0

A first-order version of **LPT0**, called **QLPT0**, was proposed in [12] under the equivalent (up to language) form of $\mathbf{QLFI1}_{\circ}$.[4] For convenience, we reproduce here the main features of **QLPT0**.

Definition 5.1. Let $\mathrm{Var} = \{v_1, v_2, \ldots\}$ be a denumerable set of individual variables. A first-order signature Θ for **QLPT0** is given as follows:

- a set $\mathcal{C}$ of individual constants;
- for each $n \geqslant 1$, a set $\mathcal{F}_n$ of function symbols of arity n,
- for each $n \geqslant 1$, a set $\mathcal{P}_n$ of predicate symbols of arity n such that $\mathcal{P}_n$ is nonempty for some $n \geqslant 1$.

The sets of terms and formulas generated by a signature Θ will be denoted by $\mathrm{Ter}(\Theta)$ and $\mathrm{For}(\Theta)$, respectively. The set of closed formulas (or sentences) and the set of closed terms (terms without variables) over Θ will be denoted by $\mathrm{Sen}(\Theta)$ and $\mathrm{CTer}(\Theta)$, respectively. The formula obtained from a given formula φ by substituting every free occurrence of a variable x by a term t will be denoted by $\varphi[x/t]$.

Definition 5.2. Let Θ be a first-order signature. The logic **QLPT0** is obtained from **LPT0** by adding the following axioms and rules:

Axiom Schemas:

(Ax∃) $\varphi[x/t] \to \exists x\varphi$, if t is a term free for x in φ
(Ax∀) $\forall x\varphi \to \varphi[x/t]$, if t is a term free for x in φ
(Ax¬∃) $\neg\exists x\varphi \leftrightarrow \forall x \neg\varphi$
(Ax¬∀) $\neg\forall x\varphi \leftrightarrow \exists x \neg\varphi$

Inference rules:

(∃-In) $\dfrac{\varphi \to \psi}{\exists x\varphi \to \psi}$, where x does not occur free in ψ

(∀-In) $\dfrac{\varphi \to \psi}{\varphi \to \forall x\psi}$, where x does not occur free in φ

The consequence relation of **QLPT0** will be denoted by $\vdash_{\mathbf{QLPT0}}$.

[4] That is, by taking $\circ$ instead of $\sim$ as a primitive connective.

6. Twist structures semantics for QLPT0

In [12] a semantics of first-order structures based on twist structures for **LFI1**$_\circ$ was proposed for **QLFI1**$_\circ$. That semantics will be briefly recalled here, adapted to **QLPT0**. From now on, only complete, non-trivial Bolean algebras will be considered (that is, such that $0 \neq 1$).

Definition 6.1. Let $\mathcal{A}$ be a complete Boolean algebra. Let $\mathcal{MT}_\mathcal{A}$ be the logical matrix associated to a twist structure $\mathcal{T}_\mathcal{A}$ for **LPT0**, and let Θ be a first-order signature (see Definition 5.1). A (first-order) *structure* over $\mathcal{MT}_\mathcal{A}$ and Θ (or a **QLPT0**-structure over Θ) is pair $\mathfrak{A} = \langle U, I_\mathfrak{A}\rangle$ such that U is a nonempty set (the domain or universe of the structure) and $I_\mathfrak{A}$ is an interpretation function which assigns:

- an element $I_\mathfrak{A}(c)$ of U to each individual constant $c \in \mathcal{C}$;
- a function $I_\mathfrak{A}(f)\colon U^n \to U$ to each function symbol f of arity n;
- a function $I_\mathfrak{A}(P)\colon U^n \to T_\mathcal{A}$ to each predicate symbol P of arity n.

Notation 6.2. From now on, we will write $c^\mathfrak{A}$, $f^\mathfrak{A}$ and $P^\mathfrak{A}$ instead of $I_\mathfrak{A}(c)$, $I_\mathfrak{A}(f)$ and $I_\mathfrak{A}(P)$ to denote the interpretation of an individual constant symbol c, a function symbol f and a predicate symbol P, respectively.

Definition 6.3. Given a structure $\mathfrak{A}$ over $\mathcal{MT}_\mathcal{A}$ and Θ, an *assignment* over $\mathfrak{A}$ is any function $\mu\colon \mathrm{Var} \to U$.

Definition 6.4. Given a structure $\mathfrak{A}$ over $\mathcal{MT}_\mathcal{A}$ and Θ, and given an assignment $\mu\colon \mathrm{Var} \to U$ we define recursively, for each term t, an element $[\![t]\!]^\mathfrak{A}_\mu$ in U as follows:

- $[\![c]\!]^\mathfrak{A}_\mu = c^\mathfrak{A}$ if c is an individual constant;
- $[\![x]\!]^\mathfrak{A}_\mu = \mu(x)$ if x is a variable;
- $[\![f(t_1,\ldots,t_n)]\!]^\mathfrak{A}_\mu = f^\mathfrak{A}([\![t_1]\!]^\mathfrak{A}_\mu,\ldots,[\![t_n]\!]^\mathfrak{A}_\mu)$ if f is a function symbol of arity n and $t_1,\ldots,t_n$ are terms.

Definition 6.5. Let $\mathfrak{A}$ be a structure over $\mathcal{MT}_\mathcal{A}$ and Θ. The *diagram language* of $\mathfrak{A}$ is the set of formulas $\mathrm{For}(\Theta_U)$, where Θ_U is the signature obtained from Θ by adding, for each element $a \in U$, a new individual constant $\bar{a}$.

Definition 6.6. The structure $\widehat{\mathfrak{A}} = \langle U, I_{\widehat{\mathfrak{A}}}\rangle$ over Θ_U is the structure $\mathfrak{A}$ over Θ extended by $I_{\widehat{\mathfrak{A}}}(\bar{a}) = a$ for every $a \in A$.

It is worth noting that $s^{\widehat{\mathfrak{A}}} = s^\mathfrak{A}$ whenever s is a symbol (individual constant, function symbol or predicate symbol) of Θ.

Notation 6.7. The set of sentences or closed formulas (that is, formulas without free variables) of the diagram language $\mathrm{For}(\Theta_U)$ is denoted by $Sen(\Theta_U)$, and the set of terms and of closed terms over Θ_U will be denoted by $\mathrm{Ter}(\Theta_U)$ and $\mathrm{CTer}(\Theta_U)$, respectively. If t is a closed term we can write $[\![t]\!]^{\mathfrak{A}}$ instead of $[\![t]\!]^{\mathfrak{A}}_{\mu}$, for any assignment μ, since it does not depend on μ.

Notation 6.8. From now on, if $z \in T_{\mathcal{A}}$ then $(z)_1$ and $(z)_2$ (or simply z_1 and z_2) will denote the first and second coordinate of z, respectively.

Definition 6.9 (**QLPT0** interpretation maps). Let $\mathcal{A}$ be a complete Boolean algebra, and let $\mathfrak{A}$ be a structure over $\mathcal{MT}_{\mathcal{A}}$ and Θ. An *interpretation map* for **QLPT0** over $\mathfrak{A}$ and $\mathcal{MT}_{\mathcal{A}}$ is a function $[\![\cdot]\!]^{\mathfrak{A}}\colon \mathrm{Sen}(\Theta_U) \to T_{\mathcal{A}}$ satisfying the following clauses (using Notation 6.8 in clauses (iv) and (v)):

(i) $[\![P(t_1,\ldots,t_n)]\!]^{\mathfrak{A}} = P^{\mathfrak{A}}([\![t_1]\!]^{\widehat{\mathfrak{A}}},\ldots,[\![t_n]\!]^{\widehat{\mathfrak{A}}})$, if $P(t_1,\ldots,t_n)$ is atomic;
(ii) $[\![\#\,\varphi]\!]^{\mathfrak{A}} = \tilde{\#}[\![\varphi]\!]^{\mathfrak{A}}$, for every $\# \in \{\neg, \sim\}$;
(iii) $[\![\varphi\,\#\,\psi]\!]^{\mathfrak{A}} = [\![\varphi]\!]^{\mathfrak{A}}\,\tilde{\#}\,[\![\psi]\!]^{\mathfrak{A}}$, for every $\# \in \{\wedge, \vee, \to\}$;
(iv) $[\![\forall x\varphi]\!]^{\mathfrak{A}} = (\bigwedge_{a\in U}([\![\varphi[x/\bar{a}]]\!]^{\mathfrak{A}})_1, \bigvee_{a\in U}([\![\varphi[x/\bar{a}]]\!]^{\mathfrak{A}})_2)$.
(v) $[\![\exists x\varphi]\!]^{\mathfrak{A}} = (\bigvee_{a\in U}([\![\varphi[x/\bar{a}]]\!]^{\mathfrak{A}})_1, \bigwedge_{a\in U}([\![\varphi[x/\bar{a}]]\!]^{\mathfrak{A}})_2)$.

Remark 6.10. A partial order can be naturally introduced in $\mathcal{T}_{\mathcal{A}}$ as follows: $z \leq w$ iff $z_1 \leq w_1$ and $z_2 \geq w_2$. It is easy to see that, with this order, $\mathcal{T}_{\mathcal{A}}$ is a complete lattice (since $\mathcal{A}$ is a complete Boolean algebra), in which

$\bigwedge_{i\in I} z_i = (\bigwedge_{i\in I}(z_i)_1, \bigvee_{i\in I}(z_i)_2)$, and
$\bigvee_{i\in I} z_i = (\bigvee_{i\in I}(z_i)_1, \bigwedge_{i\in I}(z_i)_2)$.

Note that $\mathbf{1} := (1,0)$ and $\mathbf{0} := (0,1)$ are the top and bottom elements of $\mathcal{T}_{\mathcal{A}}$, respectively. These considerations justify the definition of the interpretation of the quantifiers given in Definition 6.9(iv) and (v).

Recall the notation stated in Definition 6.5. The interpretation map can be extended to arbitrary formulas as follows:

Definition 6.11. Let $\mathcal{A}$ be a complete Boolean algebra, and let $\mathfrak{A}$ be a structure over $\mathcal{MT}_{\mathcal{A}}$ and Θ. Given an assignment μ over $\mathfrak{A}$, the *extended interpretation map* $[\![\cdot]\!]^{\mathfrak{A}}_{\mu}\colon \mathrm{For}(\Theta_U) \to T_{\mathcal{A}}$ is given by $[\![\varphi]\!]^{\mathfrak{A}}_{\mu} = [\![\varphi[x_1/\overline{\mu(x_1)},\ldots,x_n/\overline{\mu(x_n)}]]\!]^{\mathfrak{A}}$, provided that the free variables of φ occur in $\{x_1,\ldots,x_n\}$.

Definition 6.12. Let $\mathcal{A}$ be a complete Boolean algebra, and let $\mathfrak{A}$ be a structure over $\mathcal{MT}_{\mathcal{A}}$ and Θ. Given a set of formulas $\Gamma \cup \{\varphi\} \subseteq \mathrm{For}(\Theta_U)$,

φ is said to be a *semantical consequence of Γ w.r.t.* $\langle\mathfrak{A}, \mathcal{MT}_{\mathcal{A}}\rangle$, denoted by $\Gamma \models_{\langle\mathfrak{A}, \mathcal{MT}_{\mathcal{A}}\rangle} \varphi$, if the following holds: if $[\![\gamma]\!]^{\mathfrak{A}}_{\mu} \in D$, for every formula $\gamma \in \Gamma$ and every assignment μ, then $[\![\varphi]\!]^{\mathfrak{A}}_{\mu} \in D$, for every assignment μ.

DEFINITION 6.13 (Semantical consequence relation in **QLPT0** w.r.t. twist structures). Let $\Gamma \cup \{\varphi\} \subseteq \mathrm{For}(\Theta)$ be a set of formulas. Then φ is said to be a *semantical consequence of Γ in* **QLPT0** *w.r.t. first-order twist structures*, denoted by $\Gamma \models_{\mathbf{QLPT0}} \varphi$, if $\Gamma \models_{\langle\mathfrak{A}, \mathcal{MT}_{\mathcal{A}}\rangle} \varphi$ for every pair $\langle\mathfrak{A}, \mathcal{MT}_{\mathcal{A}}\rangle$.

THEOREM 6.14 (Soundness and completeness of **QLPT0** w.r.t. first-order twist structures; 12). *For any $\Gamma \cup \{\varphi\} \subseteq \mathrm{For}(\Theta)$: $\Gamma \vdash_{\mathbf{QLPT0}} \varphi$ if and only if $\Gamma \models_{\mathbf{QLPT0}} \varphi$.*[5]

In Remark 4.8 was observed that $\mathcal{T}_{\mathbb{A}_2}$, the twist structure for **LPT0** defined over the two-element Boolean algebra $\mathbb{A}_2$, coincides (up to names) with the three-valued algebra $\mathcal{A}_{\mathrm{PT0}}$ underlying the matrix $\mathcal{M}_{\mathrm{PT0}}$ and, moreover, $\mathcal{MT}_{\mathbb{A}_2}$ coincides with the three-valued characteristic matrix $\mathcal{M}_{\mathrm{PT0}}$ of **LPT0**. In [12] it was proven that **QLPT0** can be characterized by first-order structures defined over $\mathcal{M}_{\mathrm{PT0}}$.[6]

THEOREM 6.15 (Soundness and completeness of **QLPT0** w.r.t. first-order structures over $\mathcal{M}_{\mathrm{PT0}}$; 12). *For any set $\Gamma \cup \{\varphi\} \subseteq \mathrm{For}(\Theta)$: $\Gamma \vdash_{\mathbf{QLPT0}} \varphi$ iff $\Gamma \models_{\langle\mathfrak{A}, \mathcal{M}_{\mathrm{PT0}}\rangle} \varphi$, for every structure $\mathfrak{A}$ over Θ and $\mathcal{M}_{\mathrm{PT0}}$.*

Remark 6.16. It is worth observing that Theorem 6.15 constitutes a variant of the soundness and completeness theorem of first-order **J3** w.r.t. first-order structures given in [13]. Indeed, both logics are the same (up to language), and the semantic structures are the same, up to presentation.

7. Twist-valued models for set theory

As mentioned before, a three-valued model for a paraconsistent set theory based on lattice-valued models for **ZF**, as a non-classical variant of

[5] As observed above, in [12] the logic **QLFI1**$_\circ$ was analyzed instead of **QLPT0**. However, both logics are equivalent, the only difference being the use of $\circ$ instead of $\sim$ as primitive connective. The adaptation of the soundness and completeness result for **QLFI1**$_\circ$ given in [12] to the logic **QLPT0** is straightforward.

[6] Once again, it is worth observing that the result obtained in [12] concerns the logic **QLFI1**$_\circ$ instead of **QLPT0**.

the well-known Scott-Solovay-Vopěnka Boolean-valued models for **ZF**, was proposed by Löwe and Tarafder in [18]. Specifically, they introduce a three-valued algebra called $\mathbb{PS}_3$ which can be expanded with a paraconsistent negation $\neg$ (which they denote by $*$) and then a model for **ZF** is constructed over the three-valued algebra $\mathbb{PS}_3$, as well as over its expansion $(\mathbb{PS}_3, \neg)$, along the same lines as the traditional Boolean-valued models. It is known that the matrix logic of $(\mathbb{PS}_3, \neg)$, introduced in [10] as **MPT**, coincides up to language with **LPT0**. We will return to this point in Section 9.

In this section, a twist-valued model for a paraconsistent set theory $\mathbf{ZF_{LPT0}}$ based on **QLPT0** will be defined, for any complete Boolean algebra $\mathcal{A}$. It will be shown that these models constitute a generalization of the Boolean-valued models for set theory, as well as of Löwe-Tarafder's three-valued model. Our constructions, as well as the proof of their formal properties, are entirely based on the exposition of Boolean-valued models given in the book [1], which constitutes a fundamental reference to this subject.

Consider the first order signature $\Theta_{\mathbf{ZF}}$ for set theory **ZF** which consists of two binary predicates $\in$ (for membership) and $\approx$ (for identity). The logic $\mathbf{ZF_{LPT0}}$ will be defined over the first-order language $\mathcal{L}$ generated by $\Theta_{\mathbf{ZF}}$ based on the signature of **QLPT0**, that is: the set of connectives is $\Sigma = \{\wedge, \vee, \rightarrow, \sim, \neg\}$, together with the quantifiers $\forall$ and $\exists$ and the set $\mathrm{Var} = \{v_1, v_2, \ldots\}$ of individual variables. As usual, $\mathrm{Fun}(f)$ means that f is a relation (ie, a set of ordered pairs) such that f is a function, while $\mathrm{dom}(f)$ and $\mathrm{im}(f)$ will denote the *domain* and *image* of a given function f.

DEFINITION 7.1. Let $\mathcal{A}$ be a complete Boolean algebra, and let α be an ordinal number. Define, by transfinite recursion on α, the following:

$\mathbf{V}_\alpha^{\mathcal{T}_\mathcal{A}} = \{x : \mathrm{Fun}(x), \mathrm{im}(x) \subseteq T_\mathcal{A} \text{ and } \mathrm{dom}(x) \subseteq \mathbf{V}_\xi^{\mathcal{T}_\mathcal{A}} \text{ for some } \xi < \alpha\}$,

$\mathbf{V}^{\mathcal{T}_\mathcal{A}} = \{x : x \in \mathbf{V}_\alpha^{\mathcal{T}_\mathcal{A}} \text{ for some } \alpha\}$.

The class $\mathbf{V}^{\mathcal{T}_\mathcal{A}}$ is called the *twist-valued model* over the complete Boolean algebra $\mathcal{A}$.

DEFINITION 7.2. Expand the language $\mathcal{L}$ by adding a constant $\bar{u}$ to each element u of $\mathbf{V}^{\mathcal{T}_\mathcal{A}}$, obtaining a language denoted by $\mathcal{L}(\mathcal{T}_\mathcal{A})$. The fragments of $\mathcal{L}$ and $\mathcal{L}(\mathcal{T}_\mathcal{A})$ without the connective $\neg$ will be denoted by $\mathcal{L}_p$ and $\mathcal{L}_p(\mathcal{T}_\mathcal{A})$, respectively. They will be called the *pure* **ZF**-*languages*. Observe that $\mathcal{L}(\mathcal{T}_\mathcal{A})$ and $\mathcal{L}_p(\mathcal{T}_\mathcal{A})$ are proper classes. Finally, a formula φ in $\mathcal{L}_p$ is called *restricted* if every occurrence of a quantifier in φ is of the form $\forall x(x \epsilon y \to \dots)$ or $\exists x(x \epsilon y \wedge \dots)$, or if it is proved to be equivalent in **ZFC** to a formula of this kind.

Notation 7.3. By simplicity, and as it is done with Boolean-valued models, we will identify the element u of $\mathbf{V}^{\mathcal{T}_{\mathcal{A}}}$ with its name $\bar{u}$ in $\mathcal{L}(\mathcal{T}_{\mathcal{A}})$, simply writting u. Moreover, if φ is a formula in which x is the unique variable (possibly) occurring free, we will write $\varphi(u)$ instead of $\varphi[x/u]$ or $\varphi[x/\bar{u}]$.

At this point, it will be convenient to recall the specific presentation of **ZFC** given in [1]. Since some axioms are indeed schemas which depend on the formulas of the language, we must differentiate between the formulations of **ZF** (or **ZFC**) expressed in $\mathcal{L}$ or in $\mathcal{L}_p$. In the first case, formulas with the paraconsistent negation $\neg$ are allowed in the axiom schemas, while this symbol is not allowed in the second case, obtaining so the standard theory of sets **ZF** (or **ZFC**). Let $\emptyset$ be the (definable) term representing the (classical) empty set $\{x : {\sim}(x \approx x)\}$. Then, the formula $(\emptyset \epsilon u)$ is equivalent in **ZF** to $\exists y(\forall x \,{\sim}(x \epsilon y) \wedge y \epsilon u)$ where x, y and u are three different variables. On the other hand, $(x \approx \emptyset)$ and ${\sim}(x \approx \emptyset)$ are equivalent to $\forall y \,{\sim}(y \epsilon x)$ and $\exists y(y \epsilon x)$, respectively.

DEFINITION 7.4 (**ZF** and **ZFC**; see 1). The *Zermelo-Fraenkel set theory* (**ZF**) is the theory of $\mathcal{L}$ consisting of the following axiom schemas:

- $\forall x \forall y \big(\forall z(z \epsilon x \leftrightarrow z \epsilon y) \to (x \approx y)\big)$ *Extensionality*
- $\forall u \exists v \forall x \big(x \epsilon v \leftrightarrow (x \epsilon u \wedge \varphi(x))\big)$
  where v is not free in the formula $\varphi(x)$ *Separation*
- $\forall u \big(\forall x(x \epsilon u \to \exists y \varphi(x,y)) \to \exists v \forall x(x \epsilon u \to \exists y(y \epsilon v \wedge \varphi(x,y)))\big)$
  where v is not free in the formula $\varphi(x,y)$ *Replacement*
- $\forall u \exists v \forall x \big(x \epsilon v \leftrightarrow \exists y(y \epsilon u \wedge x \epsilon y)\big)$ *Union*
- $\forall u \exists v \forall x \big(x \epsilon v \leftrightarrow \forall y(y \epsilon x \to y \epsilon u)\big)$ *Power set*
- $\exists u \big(\emptyset \epsilon u \wedge \forall x(x \epsilon u \to \exists y(y \epsilon u \wedge x \epsilon y))\big)$ *Infinity*
- $\forall x \big(\forall y(y \epsilon x \to \varphi(y)) \to \varphi(x)\big) \to \forall x \varphi(x)$
  where y is not free in the formula $\varphi(x)$ *Regularity*

The set theory **ZFC** is the theory of $\mathcal{L}$ obtained from **ZF** by adding the axiom of choice (AC), which is given by

- $\forall u \exists f \big(\mathrm{Fun}(f) \wedge (\mathrm{dom}(f) \approx u) \wedge \forall x(x \epsilon u \wedge {\sim}(x \approx \emptyset) \to f(x) \epsilon x)\big)$ *AC*

Remark 7.5 (Induction principles). Recall that, from the regularity axiom of **ZF**, the membership relation ϵ is well-founded, hence the sets $V_\alpha = \{x : x \subseteq V_\xi \text{ for some } \xi < \alpha\}$ are definable for every ordinal α.

Moreover, in **ZF** every set x belongs to some V_α, that is: $\forall x \exists \alpha (x \in V_\alpha)$. This induces a function rank(x) := least α such that $x \in V_{\alpha+1}$. Since rank$(x) <$ rank(y) is well-founded, it induces a *principle of induction on rank*:

> Let Ψ be a property over sets. Assume, for every set x, the following: if $\Psi(y)$ holds for every y such that rank$(y) <$ rank(x) then $\Psi(x)$ holds. Hence, $\Psi(x)$ holds for every x.

From this, the following *Induction Principle* (IP) holds in $\mathbf{V}^{\mathcal{T}_\mathcal{A}}$ (similar to the one for Boolean-valued models):

> Let Ψ be a property over individuals in $\mathbf{V}^{\mathcal{T}_\mathcal{A}}$. Assume, for every $x \in \mathbf{V}^{\mathcal{T}_\mathcal{A}}$, the following: if $\Psi(y)$ holds for every $y \in \mathrm{dom}(x)$ then $\Psi(x)$ holds. Hence, $\Psi(x)$ holds for every $x \in \mathbf{V}^{\mathcal{T}_\mathcal{A}}$.

Both induction principles are fundamental tools in order to prove properties in $\mathbf{V}^{\mathcal{T}_\mathcal{A}}$. Moreover, as it was done in [1], it will be convenient to consider the following relations: for $u, v \in \mathbf{V}^{\mathcal{T}_\mathcal{A}}$, $u \ll v$ iff $u \in \mathrm{dom}(v)$; and, for $(x, y), (u, v) \in \mathbf{V}^{\mathcal{T}_\mathcal{A}} \times \mathbf{V}^{\mathcal{T}_\mathcal{A}}$, $(x, y) \prec (u, v)$ iff either $x \ll u$ and $y = v$, or $x = u$ and $y \ll v$. Both relations are well-founded, which allows to define the denotation map $[\![\cdot]\!]^{\mathbf{V}^{\mathcal{T}_\mathcal{A}}}$ for atomic formulas by transfinite recursion on $\prec$, see Definition 7.6 below.

DEFINITION 7.6. A mapping $[\![\cdot]\!]^{\mathbf{V}^{\mathcal{T}_\mathcal{A}}}$ (or simply $[\![\cdot]\!]$), assigning to each closed formula in $\mathcal{L}(\mathcal{T}_\mathcal{A})$ a value in $T_\mathcal{A}$, can be defined by induction on the complexity in $\mathcal{L}(\mathcal{T}_\mathcal{A})$ as follows:

$$
\begin{aligned}
[\![u \in v]\!] &= \bigvee_{x \in \mathrm{dom}(v)} (v(x) \mathbin{\tilde{\wedge}} [\![x \approx u]\!]) \\
&= \Big(\bigvee_{x \in \mathrm{dom}(v)} ((v(x))_1 \wedge [\![x \approx u]\!]_1), \bigwedge_{x \in \mathrm{dom}(v)} ((v(x))_2 \vee [\![x \approx u]\!]_2) \Big)
\end{aligned}
$$

$$
\begin{aligned}
[\![u \approx v]\!] &= \bigwedge_{x \in \mathrm{dom}(u)} (u(x) \mathbin{\tilde{\to}} [\![x \in v]\!]) \mathbin{\tilde{\wedge}} \bigwedge_{x \in \mathrm{dom}(v)} (v(x) \mathbin{\tilde{\to}} [\![x \in u]\!]) \\
&= \Big(\bigwedge_{x \in \mathrm{dom}(u)} ((u(x))_1 \to [\![x \in v]\!]_1), \bigvee_{x \in \mathrm{dom}(u)} ((u(x))_1 \wedge [\![x \in v]\!]_2) \Big) \\
&\quad \tilde{\wedge} \Big(\bigwedge_{x \in \mathrm{dom}(v)} ((v(x))_1 \to [\![x \in u]\!]_1), \bigvee_{x \in \mathrm{dom}(v)} ((v(x))_1 \wedge [\![x \in u]\!]_2) \Big)
\end{aligned}
$$

$$[\![\phi \# \psi]\!] = [\![\phi]\!] \mathbin{\tilde{\#}} [\![\psi]\!] \quad \text{for } \# \in \{\wedge, \vee, \to\}$$

$$[\![\# \psi]\!] = \tilde{\#}[\![\psi]\!] \quad \text{for } \# \in \{\sim, \neg\}$$

$$\llbracket\forall x\varphi(x)\rrbracket = \bigwedge_{u\in\mathbf{V}^{\mathcal{T}_\mathcal{A}}}\llbracket\varphi(u)\rrbracket = \big(\bigwedge_{u\in\mathbf{V}^{\mathcal{T}_\mathcal{A}}}\llbracket\varphi(u)\rrbracket_1, \bigvee_{u\in\mathbf{V}^{\mathcal{T}_\mathcal{A}}}\llbracket\varphi(u)\rrbracket_2\big)$$

$$\llbracket\exists x\varphi(x)\rrbracket = \bigvee_{u\in\mathbf{V}^{\mathcal{T}_\mathcal{A}}}\llbracket\varphi(u)\rrbracket = \big(\bigvee_{u\in\mathbf{V}^{\mathcal{T}_\mathcal{A}}}\llbracket\varphi(u)\rrbracket_1, \bigwedge_{u\in\mathbf{V}^{\mathcal{T}_\mathcal{A}}}\llbracket\varphi(u)\rrbracket_2\big).$$

$\llbracket\varphi\rrbracket^{\mathbf{V}^{\mathcal{T}_\mathcal{A}}}$ is called the *twist truth-value* of the sentence $\varphi \in \mathcal{L}(\mathcal{T}_\mathcal{A})$ in the twist-valued model $\mathbf{V}^{\mathcal{T}_\mathcal{A}}$ over the complete Boolean algebra $\mathcal{A}$.

Remark 7.7. Observe that $\mathbf{V}^{\mathcal{T}_\mathcal{A}}$ can be seen as a structure for **QLPT0** over $\mathcal{MT}_\mathcal{A}$ and $\Theta_{\mathbf{ZF}}$ in a wide sense, given that its domain is a proper class. Under this identification, the twist truth-value $\llbracket\varphi\rrbracket^{\mathbf{V}^{\mathcal{T}_\mathcal{A}}}$ of the sentence φ in $\mathbf{V}^{\mathcal{T}_\mathcal{A}}$ is exactly the value assigned to φ by the interpretation map for **QLPT0** over $\mathbf{V}^{\mathcal{T}_\mathcal{A}}$ and $\mathcal{MT}_\mathcal{A}$ (recall Definition 6.9). In this case we assume that the mappings $(\cdot \in \cdot)^{\mathbf{V}^{\mathcal{T}_\mathcal{A}}}$ and $(\cdot \approx \cdot)^{\mathbf{V}^{\mathcal{T}_\mathcal{A}}}$ are as in Definition 7.6.

Recall the notion of semantical consequence relation in **QLPT0** (see Definitions 6.12 and 6.13). This motivates the following:

DEFINITION 7.8. A sentence φ in $\mathcal{L}(\mathcal{T}_\mathcal{A})$ is said to be *valid in* $\mathbf{V}^{\mathcal{T}_\mathcal{A}}$, which is denoted by $\mathbf{V}^{\mathcal{T}_\mathcal{A}} \models \varphi$, if $\llbracket\varphi\rrbracket^{\mathbf{V}^{\mathcal{T}_\mathcal{A}}} \in D_\mathcal{A}$.

It is worth noting that $\mathbf{V}^{\mathcal{T}_\mathcal{A}} \models \varphi$ iff $\llbracket\varphi\rrbracket_1^{\mathbf{V}^{\mathcal{T}_\mathcal{A}}} = 1$. The semantical notions introduced above can easily be generalized to formulas with free variables. Recall from Notation 7.3 that $\overline{u}$ is identified with u in $\mathbf{V}^{\mathcal{T}_\mathcal{A}}$. Then:

DEFINITION 7.9. Let φ be a formula in $\mathcal{L}$ whose free variables occur in $\{x_1,\ldots,x_n\}$. Given a twist-valued model $\mathbf{V}^{\mathcal{T}_\mathcal{A}}$ and an assignment $\mu : Var \to \mathbf{V}^{\mathcal{T}_\mathcal{A}}$, the *twist truth-value* of φ in $\mathbf{V}^{\mathcal{T}_\mathcal{A}}$ and μ is defined as follows: $\llbracket\varphi\rrbracket_\mu^{\mathbf{V}^{\mathcal{T}_\mathcal{A}}} := \llbracket\varphi[x_1/\mu(x_1),\ldots,x_n/\mu(x_n)]\rrbracket^{\mathbf{V}^{\mathcal{T}_\mathcal{A}}}$. The formula φ is *valid* in $\mathbf{V}^{\mathcal{T}_\mathcal{A}}$ if $\llbracket\varphi\rrbracket_\mu^{\mathbf{V}^{\mathcal{T}_\mathcal{A}}} \in D_\mathcal{A}$ for every μ.

DEFINITION 7.10. $\mathbf{ZF_{LPT0}}$ is the logic of the class of twist-valued models, seen as **QLPT0**-structures over the signature $\Theta_{\mathbf{ZF}}$. That is, $\mathbf{ZF_{LPT0}}$ is the set of formulas of $\mathcal{L}$ which are valid in every twist-valued model $\mathbf{V}^{\mathcal{T}_\mathcal{A}}$.

8. Boolean-valued models versus twist-valued models

In this section, the relationship between twist-valued models and Boolean-valued models will be briefly analized. It will be shown that these models enjoy similar properties than the Boolean-valued models (when restricted to pure **ZF**-languages). These similarities will be fundamental in order to prove that **ZFC** is valid w.r.t. twist-valued models (see Theorem 8.22 below).

The following basic results for twist-valued models are analogous to the corresponding ones for Boolean-valued models obtained in [1, Theorem 1.17]. All these results will be proven by using the Induction Principle (IP) (recall Remark 7.5). From now on we assume that the reader is familiar with the book [1].

First, it is interesting to notice that no element of $\mathbf{V}^{\mathcal{T}_\mathcal{A}}$ can be a ϵ-member of itself:

Proposition 8.1. *Let $\mathcal{A}$ be a complete Boolean algebra, and let $u \in \mathbf{V}^{\mathcal{T}_\mathcal{A}}$. Then $[\![u \,\epsilon\, u]\!]_1 = 0$. That is, $[\![u \,\epsilon\, u]\!] = \mathbf{0} = (0, 1)$ for every u in $\mathbf{V}^{\mathcal{T}_\mathcal{A}}$.*

Proof. Assume the inductive hypothesis $[\![y \,\epsilon\, y]\!]_1 = 0$ for every $y \in \mathrm{dom}(u)$. Note that

$$[\![u \,\epsilon\, u]\!]_1 = \bigvee_{y \in \mathrm{dom}(u)} ((u(y))_1 \wedge [\![y \approx u]\!]_1).$$

Let $y \in \mathrm{dom}(u)$. Then

$$\begin{aligned}(u(y))_1 \wedge [\![y \approx u]\!]_1 &\leq (u(y))_1 \wedge \bigwedge_{x \in \mathrm{dom}(u)} ((u(x))_1 \rightarrow [\![x \,\epsilon\, y]\!]_1)\\ &\leq (u(y))_1 \wedge ((u(y))_1 \rightarrow [\![y \,\epsilon\, y]\!]_1)\\ &\leq [\![y \,\epsilon\, y]\!]_1 = 0.\end{aligned}$$

Then $u(y)_1 \wedge [\![y \approx u]\!]_1 = 0$ for every $y \in \mathrm{dom}(u)$, hence $[\![u \,\epsilon\, u]\!]_1 = 0$. □

From the previous result, it follows that $\mathbf{ZF}_{\mathbf{LPT0}}$ does not allow the existence of 'paradoxical sets' such as Russell's set or the universal set, despite being a paraconsistent set theory as it will be shown below (see Corollaries 10.2 and 10.3).

Theorem 8.2. *Let $\mathcal{A}$ be a complete Boolean algebra. Then for all $u, v, w \in \mathbf{V}^{\mathcal{T}_\mathcal{A}}$ we have:*

(i) $[\![u \approx u]\!]_1 = 1$,
(ii) $u(x)_1 \leq [\![x \,\epsilon\, u]\!]_1$, *for every* $x \in \mathrm{dom}(u)$,
(iii) $[\![u \approx v]\!]_1 = [\![v \approx u]\!]_1$,
(iv) $[\![u \approx v]\!]_1 \wedge [\![v \approx w]\!]_1 \leq [\![u \approx w]\!]_1$,
(v) $[\![u \approx v]\!]_1 \wedge [\![u \,\epsilon\, w]\!]_1 \leq [\![v \,\epsilon\, w]\!]_1$,
(vi) $[\![v \approx w]\!]_1 \wedge [\![u \,\epsilon\, v]\!]_1 \leq [\![u \,\epsilon\, w]\!]_1$,

(vii) $[\![u \approx v]\!]_1 \wedge [\![\varphi(u)]\!]_1 \leq [\![\varphi(v)]\!]_1$, *for any formula* $\varphi(x)$ *in* $\mathcal{L}_p(\mathcal{T}_\mathcal{A})$ (Leibniz's Law).

Proof. The proof of items (i)-(vi) is analogous to the proof of the corresponding items found in [1, Theorem 1.17]. The proof of item (vii) is easily done by induction on the complexity of $\varphi(x)$ by observing that: the proof when φ is atomic uses items (i)–(vi) for the other cases. For complex formulas the result follows easily by induction hypothesis. □

Lemma 8.3. *Let* $\mathcal{A}$ *be a complete Boolean algebra. Then, for any formula* $\varphi(x)$ *in* $\mathcal{L}_p(\mathcal{T}_\mathcal{A})$ *and any* $u \in \mathbf{V}^{\mathcal{T}_\mathcal{A}}$: $[\![\exists y((u \approx y) \wedge \varphi(y))]\!]_1 = [\![\varphi(u)]\!]_1$.

Proof. It follows from Theorem 8.2 items (i), (iii) and (vii). Indeed,

$$\begin{aligned} [\![\exists y((u \approx y) \wedge \varphi(y))]\!]_1 &= \bigvee_{v \in \mathbf{V}^{\mathcal{T}_\mathcal{A}}} ([\![u \approx v]\!]_1 \wedge [\![\varphi(v)]\!]_1) \\ &\leq [\![\varphi(u)]\!]_1 = [\![u \approx u]\!]_1 \wedge [\![\varphi(u)]\!]_1 \\ &\leq [\![\exists y((u \approx y) \wedge \varphi(y))]\!]_1. \end{aligned}$$ □

Now, the so-called *bounded quantification properties* will be considered.

Notation 8.4. The following notation from [1] will be adopted from now on:

$$\forall x \,\epsilon\, u\; \varphi(x) := \forall x(x \,\epsilon\, u \to \varphi(x)),$$
$$\exists x \,\epsilon\, u\; \varphi(x) := \exists x(x \,\epsilon\, u \wedge \varphi(x)).$$

Definition 8.5. For any formula φ and every $u \in \mathbf{V}^{\mathcal{T}_\mathcal{A}}$, the *universal bounded quantification property* UBQ^u_φ and the *existential bounded quantification property* EBQ^u_φ are defined as follows:

(UBQ^u_φ) $\quad [\![\forall x \,\epsilon\, u\; \varphi(x)]\!]_1 = \bigwedge_{x \in \mathrm{dom}(u)}((u(x))_1 \to [\![\varphi(x)]\!]_1)$,

(EBQ^u_φ) $\quad [\![\exists x \,\epsilon\, u\; \varphi(x)]\!]_1 = \bigvee_{x \in \mathrm{dom}(u)}((u(x))_1 \wedge [\![\varphi(x)]\!]_1)$.

In Boolean-valued models, the Leibniz's Law is a sufficient condition to ensure the validity of the bounded quantification properties. Hence, by adapting the proof of [1, Corollary 1.18], and taking into account Theorem 8.2 and Lemma 8.3, it can be proven the following:

Theorem 8.6. *For any negation-free formula* φ *(i.e.,* $\varphi \in \mathcal{L}_p(\mathcal{T}_\mathcal{A})$*) and any* $u \in \mathbf{V}^{\mathcal{T}_\mathcal{A}}$*, the bounded quantification properties* UBQ^u_φ *and* EBQ^u_φ *hold in* $\mathbf{V}^{\mathcal{T}_\mathcal{A}}$.

Recall that a complete Boolean algebra $\mathcal{A}'$ is a complete subalgebra of the complete Boolean algebra $\mathcal{A}$ provided that $\mathcal{A}'$ is a subalgebra of $\mathcal{A}$ and $\bigvee_{\mathcal{A}'} X = \bigvee_{\mathcal{A}} X$ and $\bigwedge_{\mathcal{A}'} X = \bigwedge_{\mathcal{A}} X$ for every $X \subseteq A'$. Analogously, we say that a twist-structure $\mathcal{T}_{\mathcal{A}'}$ is a *complete subalgebra* of the twist-structure $\mathcal{T}_{\mathcal{A}}$ if $\mathcal{T}_{\mathcal{A}'}$ is a subalgebra of $\mathcal{T}_{\mathcal{A}}$ and $\bigvee_{\mathcal{T}_{\mathcal{A}'}} X = \bigvee_{\mathcal{T}_{\mathcal{A}}} X$ and $\bigwedge_{\mathcal{T}_{\mathcal{A}'}} X = \bigwedge_{\mathcal{T}_{\mathcal{A}}} X$ for every $X \subseteq T_{\mathcal{A}'}$, recalling Remark 6.10.

PROPOSITION 8.7. *If $\mathcal{A}'$ is a complete subalgebra of $\mathcal{A}$, then $\mathcal{T}_{\mathcal{A}'}$ is a complete subalgebra of $\mathcal{T}_{\mathcal{A}}$.*

PROOF. If follows from Definition 4.6 and Remark 6.10. □

THEOREM 8.8. *Let $\mathcal{A}'$ be a complete subalgebra of the complete Boolean algebra $\mathcal{A}$. Then*:

(i) $\mathbf{V}^{\mathcal{T}_{\mathcal{A}'}} \subseteq \mathbf{V}^{\mathcal{T}_{\mathcal{A}}}$.

(ii) *For all $u, v \in \mathbf{V}^{\mathcal{T}_{\mathcal{A}'}}$: $[\![u \,\epsilon\, w]\!]^{\mathbf{V}^{\mathcal{T}_{\mathcal{A}'}}} = [\![u \,\epsilon\, w]\!]^{\mathbf{V}^{\mathcal{T}_{\mathcal{A}}}}$ and $[\![u \approx w]\!]^{\mathbf{V}^{\mathcal{T}_{\mathcal{A}'}}} = [\![u \approx w]\!]^{\mathbf{V}^{\mathcal{T}_{\mathcal{A}}}}$.*

COROLLARY 8.9. *Suppose that $\mathcal{A}'$ is a complete subalgebra of $\mathcal{A}$. Then, for any restricted formula $\varphi(x_1, \ldots, x_n)$ in $\mathcal{L}_p$ (recall Definition 7.2) and for all $u_1, \ldots, u_n \in T_{\mathcal{A}'}$: $[\![\varphi(u_1, \ldots, u_n)]\!]^{\mathbf{V}^{\mathcal{T}_{\mathcal{A}'}}} = [\![\varphi(u_1, \ldots, u_n)]\!]^{\mathbf{V}^{\mathcal{T}_{\mathcal{A}}}}$.*

PROOF. The proof is analogous to that for [1, Corollary 1.21]. □

Remark 8.10. Recall from Remark 4.8 that $\mathcal{T}_{\mathbb{A}_2}$, the twist structure for **LPT0** defined over the two-element Boolean algebra $\mathbb{A}_2$, coincides (up to names) with the three-valued algebra $\mathcal{A}_{\mathrm{PT0}}$ underlying the matrix $\mathcal{M}_{\mathrm{PT0}}$, where $\mathbf{1}$, $\frac{\mathbf{1}}{\mathbf{2}}$ and $\mathbf{0}$ are identified with $(1,0)$, $(1,1)$ and $(0,1)$, respectively. Hence, the twist-valued structure $\mathbf{V}^{\mathcal{T}_{\mathbb{A}_2}}$ will be denoted by $\mathbf{V}^{\mathcal{A}_{\mathrm{PT0}}}$. Since $\mathbb{A}_2$ is a complete subalgebra (up to isomorphisms) of any complete Boolean algebra $\mathcal{A}$ then $\mathbf{V}^{\mathcal{A}_{\mathrm{PT0}}}$ is a complete subalgebra of $\mathbf{V}^{\mathcal{T}_{\mathcal{A}}}$, for any $\mathcal{T}_{\mathcal{A}}$. By Theorem 8.8, $[\![u \,\epsilon\, v]\!]^{\mathbf{V}^{\mathcal{A}_{\mathrm{PT0}}}} = [\![u \,\epsilon\, v]\!]^{\mathbf{V}^{\mathcal{T}_{\mathcal{A}}}}$ and $[\![u \approx v]\!]^{\mathbf{V}^{\mathcal{A}_{\mathrm{PT0}}}} = [\![u \approx v]\!]^{\mathbf{V}^{\mathcal{T}_{\mathcal{A}}}}$ for every $u, v \in \mathbf{V}^{\mathcal{A}_{\mathrm{PT0}}}$ and every $\mathcal{T}_{\mathcal{A}}$. As happens with the Boolean-valued model $\mathbf{V}^{\mathbb{A}_2}$, the twist-valued model $\mathbf{V}^{\mathcal{A}_{\mathrm{PT0}}}$ is, in some sense, isomorphic to the standard universe $\mathbf{V}$, as it will be shown in Theorem 8.14 below.

DEFINITION 8.11. For each $x \in \mathbf{V}$ define, by transfinite recursion on the well-founded relation $y \in x$, the following: $\hat{x} := \{\langle \hat{y}, \mathbf{1}\rangle : y \in x\}$.

It is clear that $\hat{x} \in \mathbf{V}^{\mathcal{A}_{\mathrm{PT0}}}$ and so $\hat{x} \in \mathbf{V}^{\mathcal{T}_{\mathcal{A}}}$ for any $\mathcal{T}_{\mathcal{A}}$. Hence, if $\varphi(v_1, \ldots, v_n)$ is a restricted formula in $\mathcal{L}_p$ and $x_1, \ldots, x_n \in \mathbf{V}$ then $[\![\varphi(\widehat{x_1}, \ldots, \widehat{x_n})]\!]^{\mathbf{V}^{\mathcal{A}_{\mathrm{PT0}}}} = [\![\varphi(\widehat{x_1}, \ldots, \widehat{x_n})]\!]^{\mathbf{V}^{\mathcal{T}_{\mathcal{A}}}}$ for any $\mathcal{T}_{\mathcal{A}}$, by Corollary 8.9.

LEMMA 8.12. *Let $\varphi(v_1,\dots,v_n)$ be a formula in $\mathcal{L}_p$ and $x_1,\dots,x_n \in \mathbf{V}$. Then, $[\![\varphi(\widehat{x_1},\dots,\widehat{x_n})]\!]^{\mathbf{V}^{\mathcal{A}_{\mathrm{PT0}}}} \in \{\mathbf{0},\mathbf{1}\}$.*

PROOF. The result is proven by induction on the complexity of φ. By Definition 7.6, and taking into account that $\{\mathbf{0},\mathbf{1}\}$ is a subalgebra of $\mathcal{A}_{\mathrm{PT0}}$ which is also closed under arbitrary infima and suprema, it is enough to prove that the denotation in $\mathbf{V}^{\mathcal{A}_{\mathrm{PT0}}}$ of any atomic formula of the form $\hat{x} \approx \hat{y}$ or $\hat{x} \in \hat{y}$ takes a value in $\{\mathbf{0},\mathbf{1}\}$. Observe that $u \in \mathrm{dom}(\hat{x})$ iff $u = \hat{y}$ for some $y \in x$. Thus, given $\hat{x}$ and $\hat{y}$, assume the inductive hypothesis $[\![\hat{z} \,\epsilon\, \hat{y}]\!]^{\mathbf{V}^{\mathcal{A}_{\mathrm{PT0}}}} \in \{\mathbf{0},\mathbf{1}\}$ for every $\hat{z} \in \mathrm{dom}(\hat{x})$, and $[\![\hat{z} \,\epsilon\, \hat{x}]\!]^{\mathbf{V}^{\mathcal{A}_{\mathrm{PT0}}}} \in \{\mathbf{0},\mathbf{1}\}$ for every $\hat{z} \in \mathrm{dom}(\hat{y})$. Then:

$$[\![\hat{x} \approx \hat{y}]\!]^{\mathbf{V}^{\mathcal{A}_{\mathrm{PT0}}}} = \bigwedge_{\hat{z}\in\mathrm{dom}(\hat{x})} (\mathbf{1} \,\tilde{\to}\, [\![\hat{z} \,\epsilon\, \hat{y}]\!]^{\mathbf{V}^{\mathcal{A}_{\mathrm{PT0}}}}) \;\;\tilde{\wedge} \bigwedge_{\hat{z}\in\mathrm{dom}(\hat{y})} (\mathbf{1} \,\tilde{\to}\, [\![\hat{z} \,\epsilon\, \hat{x}]\!]^{\mathbf{V}^{\mathcal{A}_{\mathrm{PT0}}}}).$$

Using the inductive hypothesis and the fact that $\{\mathbf{0},\mathbf{1}\}$ is a subalgebra of $\mathcal{A}_{\mathrm{PT0}}$ closed under infima, it follows that $[\![\hat{x} \approx \hat{y}]\!]^{\mathbf{V}^{\mathcal{A}_{\mathrm{PT0}}}} \in \{\mathbf{0},\mathbf{1}\}$. Analogously it can be proven that $[\![\hat{x} \,\epsilon\, \hat{y}]\!]^{\mathbf{V}^{\mathcal{A}_{\mathrm{PT0}}}} \in \{\mathbf{0},\mathbf{1}\}$. ☐

COROLLARY 8.13. *Let $\varphi(v_1,\dots,v_n)$ be a restricted formula in $\mathcal{L}_p$, and let $x_1,\dots,x_n \in \mathbf{V}$. Then, $[\![\varphi(\widehat{x_1},\dots,\widehat{x_n})]\!]^{\mathbf{V}^{\mathcal{T}_{\mathcal{A}}}} \in \{\mathbf{0},\mathbf{1}\}$ for every $\mathcal{A}$.*

PROOF. It follows by Lemma 8.12 and by Corollary 8.9. ☐

THEOREM 8.14. (i) *For any $x \in \mathbf{V}$ and $u \in \mathbf{V}^{\mathcal{T}_{\mathcal{A}}}$: $[\![u \,\epsilon\, \hat{x}]\!] = \bigvee_{y\in x} [\![u \approx \hat{y}]\!]$.*

(ii) *For all $x, y \in \mathbf{V}$:*
*$(x \,\epsilon\, y)$ holds in **ZFC** iff $\mathbf{V}^{\mathcal{T}_{\mathcal{A}}} \models (\hat{x} \,\epsilon\, \hat{y})$ for every $\mathcal{A}$;*
*$(x \approx y)$ holds in **ZFC** iff $\mathbf{V}^{\mathcal{T}_{\mathcal{A}}} \models (\hat{x} \approx \hat{y})$ for every $\mathcal{A}$.*

(iii) *The function $x \mapsto \hat{x}$ is one-to-one from $\mathbf{V}$ to $\mathbf{V}^{\mathcal{A}_{\mathrm{PT0}}}$.*

(iv) *For any $u \in \mathbf{V}^{\mathcal{A}_{\mathrm{PT0}}}$ there is a (unique) $x \in \mathbf{V}$ such that $\mathbf{V}^{\mathcal{T}_{\mathcal{A}}} \models (u \approx \hat{x})$ for all $\mathcal{A}$.*

(v) *For every formula $\varphi(v_1,\dots,v_n)$ in $\mathcal{L}_p$ and all $x_1,\dots,x_n \in \mathbf{V}$:*
*$\varphi(x_1,\dots,x_n)$ holds in **ZFC** iff $\mathbf{V}^{\mathcal{A}_{\mathrm{PT0}}} \models \varphi(\widehat{x_1},\dots,\widehat{x_n})$.*

In addition, if φ is restricted (recall Definition 7.2), then for all x_1, …, $x_n \in \mathbf{V}$ we have:
*$\varphi(x_1,\dots,x_n)$ holds in **ZFC** iff $\mathbf{V}^{\mathcal{T}_{\mathcal{A}}} \models \varphi(\widehat{x_1},\dots,\widehat{x_n})$, for every $\mathcal{A}$.*

PROOF. It follows by an easy adaptation of the proof of from [1]. The only points to be considered are the following:

(i) Note that $\mathbf{1} \,\tilde{\wedge}\, a = a$ for every $a \in T_{\mathcal{A}}$. Then, the adaptation of the proof of this item is immediate.

(ii) Both assertions are simultaneously proven by induction on $\mathrm{rank}(y)$ (see Remark 7.5), where the induction hypothesis is: for any z with $\mathrm{rank}(z) < \mathrm{rank}(y)$.

$$\text{(IH1)} \qquad x \in z \text{ iff } [\![\hat{x} \,\epsilon\, \hat{z}]\!]_1^{\mathbf{V}^{\mathcal{T}_\mathcal{A}}} = 1 \ \text{ for every } x \text{ and } \mathcal{A};$$

$$\text{(IH2)} \qquad x = z \text{ iff } [\![\hat{x} \approx \hat{z}]\!]_1^{\mathbf{V}^{\mathcal{T}_\mathcal{A}}} = 1 \ \text{ for every } x \text{ and } \mathcal{A}; \text{ and}$$

$$\text{(IH3)} \qquad z \in x \text{ iff } [\![\hat{z} \,\epsilon\, \hat{x}]\!]_1^{\mathbf{V}^{\mathcal{T}_\mathcal{A}}} = 1 \ \text{ for every } x \text{ and } \mathcal{A}.$$

Thus, fix $x \in \mathbf{V}$ and let $\mathcal{A}$ be a complete Boolean algebra. Then: $x \in y$ iff $x = u$ for some $u \in y$ iff $[\![\hat{x} \approx \hat{u}]\!]_1^{\mathbf{V}^{\mathcal{T}_\mathcal{A}}} = 1$ for some $u \in y$, by (IH2) iff $\bigvee_{u\in y}[\![\hat{x} \approx \hat{u}]\!]_1^{\mathbf{V}^{\mathcal{T}_\mathcal{A}}} = 1$ iff $[\![\hat{x} \,\epsilon\, \hat{y}]\!]_1^{\mathbf{V}^{\mathcal{T}_\mathcal{A}}} = 1$ (by item (i)). This proves the preservation of (IH1) for y. For (IH2), observe that $1 \to a = a$ for every $a \in A$. Hence, $[\![\hat{x} \approx \hat{y}]\!]_1^{\mathbf{V}^{\mathcal{T}_\mathcal{A}}} = \bigwedge_{u\in x}[\![\hat{u} \,\epsilon\, \hat{y}]\!]_1^{\mathbf{V}^{\mathcal{T}_\mathcal{A}}} \wedge \bigwedge_{v\in y}[\![\hat{v} \,\epsilon\, \hat{x}]\!]_1^{\mathbf{V}^{\mathcal{T}_\mathcal{A}}} = 1$ iff $\bigwedge_{u\in x}[\![\hat{u} \,\epsilon\, \hat{y}]\!]_1^{\mathbf{V}^{\mathcal{T}_\mathcal{A}}} = \bigwedge_{v\in y}[\![\hat{v} \,\epsilon\, \hat{x}]\!]_1^{\mathbf{V}^{\mathcal{T}_\mathcal{A}}} = 1$ iff $[\![\hat{u} \,\epsilon\, \hat{y}]\!]_1^{\mathbf{V}^{\mathcal{T}_\mathcal{A}}} = 1$ for every $u \in x$ and $[\![\hat{v} \,\epsilon\, \hat{x}]\!]_1^{\mathbf{V}^{\mathcal{T}_\mathcal{A}}} = 1$ for every $v \in y$ iff $u \in y$ for every $u \in x$ (by the preservation of (IH1) for y just proved above) and $v \in x$ for every $v \in y$ (by (IH3)) iff $x \subseteq y$ and $y \subseteq x$ iff $x = y$, by extensionality. This proves the preservation of (IH2) for y. Finally: $y \in x$ iff $y = u$ for some $u \in x$ iff $[\![\hat{y} \approx \hat{u}]\!]_1^{\mathbf{V}^{\mathcal{T}_\mathcal{A}}} = 1$ for some $u \in x$ (by the preservation of (IH2) for y) iff $\bigvee_{u\in x}[\![\hat{y} \approx \hat{u}]\!]_1^{\mathbf{V}^{\mathcal{T}_\mathcal{A}}} = 1$ iff $[\![\hat{y} \,\epsilon\, \hat{x}]\!]_1^{\mathbf{V}^{\mathcal{T}_\mathcal{A}}} = 1$ (by item (i)). This proves the preservation of (IH3) for y.

(iii) It follows from (ii).

(iv) We first observe that, given $u \in \mathbf{V}^{\mathcal{A}_{\mathrm{PT0}}}$, the uniqueness of $x \in \mathbf{V}$ satisfying the required property is an immediate consequence of item (ii) and the properties of $\approx$. The existence of such x (given u) will be proved by induction on the well-founded relation $z \in \mathrm{dom}(u)$, i.e., $z \ll u$. Thus, the induction hypothesis is as follows: for any $z \in \mathrm{dom}(u)$

$$\text{(IH)} \qquad \text{there exists } y \in \mathbf{V} \text{ such that } \ [\![z \approx \hat{y}]\!]_1^{\mathbf{V}^{\mathcal{T}_\mathcal{A}}} = 1 \ \text{ for any } \mathcal{A}.$$

It will be shown that there exists $v \in \mathbf{V}$ such that $[\![u \approx \hat{v}]\!]_1^{\mathbf{V}^{\mathcal{T}_\mathcal{A}}} = 1$, for any $\mathcal{A}$. Since $1 \to a = a$ for every $a \in A$,

$$[\![u \approx \hat{v}]\!]_1^{\mathbf{V}^{\mathcal{T}_\mathcal{A}}} = \bigwedge_{z\in\mathrm{dom}(u)} ((u(z))_1 \to [\![z \,\epsilon\, \hat{v}]\!]_1^{\mathbf{V}^{\mathcal{T}_\mathcal{A}}}) \wedge \bigwedge_{y\in v} [\![\hat{y} \,\epsilon\, u]\!]_1^{\mathbf{V}^{\mathcal{T}_\mathcal{A}}}.$$

Thus, $[\![u \approx \hat{v}]\!]_1^{\mathbf{V}^{\mathcal{T}_\mathcal{A}}} = 1$ iff the following two conditions hold:

(C1) $(u(z))_1 \leq [\![z \,\epsilon\, \hat{v}]\!]_1^{\mathbf{V}^{\mathcal{T}_\mathcal{A}}}$, for any $z \in \mathrm{dom}(u)$;

(C2) $1 = [\![\hat{y} \,\epsilon\, u]\!]_1^{\mathbf{V}^{\mathcal{T}_\mathcal{A}}} = \bigvee_{z\in\mathrm{dom}(u)}((u(z))_1 \wedge [\![z \approx \hat{y}]\!]_1^{\mathbf{V}^{\mathcal{T}_\mathcal{A}}})$, for any $y \in v$.

In order to fulfill (C2), consider the set $v = \{y \in \mathbf{V} :$ for some $z \in \mathrm{dom}(u)$, $((u(z))_1 = 1$ and $([\![z \approx \hat{y}]\!]^{\mathbf{V}^{\mathcal{T}_{\mathcal{A}}}})_1 = 1$, for any $\mathcal{A}\}$. By item (ii) and the replacement axiom, it can be shown that $v \in \mathbf{V}$. Clearly, $[\![\hat{y} \,\epsilon\, u]\!]_1^{\mathbf{V}^{\mathcal{T}_{\mathcal{A}}}} = 1$ for every $y \in v$ and every $\mathcal{A}$. Hence v satisfies condition (C2). Finally, by (IH) it can be shown that v also satisfies (C1).

(v) It is proved by induction on the complexity φ. The atomic cases follow from item (ii). The cases $\varphi = {\sim}\psi$ and $\varphi = (\psi \,\#\, \gamma)$ for $\# \in \{\wedge, \vee, \to\}$ are straightforward. Finally, suppose that $\varphi = \exists x\psi$ (the case $\varphi = \forall x\psi$ follows from this and by $\sim$, taking into account that $\forall x\psi$ is equivalent to ${\sim}\exists x {\sim}\psi$). If $\varphi(x_1, \ldots, x_n)$ holds in **ZFC** then $\psi(x, x_1, \ldots, x_n)$ holds in **ZFC**, for some $x \in \mathbf{V}$. By induction hypothesis, $[\![\psi(\widehat{x}, \widehat{x_1}, \ldots, \widehat{x_n})]\!]_1^{\mathbf{V}^{\mathcal{A}_{\mathrm{PT0}}}} = 1$. This means that $\bigvee_{u \in \mathbf{V}^{\mathcal{A}_{\mathrm{PT0}}}} [\![\psi(u, \widehat{x_1}, \ldots, \widehat{x_n})]\!]_1^{\mathbf{V}^{\mathcal{A}_{\mathrm{PT0}}}} = 1$. That is, $[\![\varphi(\widehat{x_1}, \ldots, \widehat{x_n})]\!]_1^{\mathbf{V}^{\mathcal{A}_{\mathrm{PT0}}}} = 1$. Conversely, suppose that $[\![\varphi(\widehat{x_1}, \ldots, \widehat{x_n})]\!]_1^{\mathbf{V}^{\mathcal{A}_{\mathrm{PT0}}}} = 1$. Hence, $\bigvee_{u \in \mathbf{V}^{\mathcal{A}_{\mathrm{PT0}}}} [\![\psi(u, \widehat{x_1}, \ldots, \widehat{x_n})]\!]_1^{\mathbf{V}^{\mathcal{A}_{\mathrm{PT0}}}} = 1$. Recalling that $M = \{\mathbf{1}, \frac{\mathbf{1}}{\mathbf{2}}, \mathbf{0}\}$ is the domain of $\mathcal{A}_{\mathrm{PT0}}$, observe that, if $\emptyset \neq X \subseteq M$ is such that $\bigvee_{\mathcal{A}_{\mathrm{PT0}}} X = \mathbf{1}$, then $\mathbf{1} \in X$. From this, $[\![\psi(u, \widehat{x_1}, \ldots, \widehat{x_n})]\!]_1^{\mathbf{V}^{\mathcal{A}_{\mathrm{PT0}}}} = 1$ for some $u \in \mathbf{V}^{\mathcal{A}_{\mathrm{PT0}}}$. By item (iv), there exists $x \in \mathbf{V}$ such that $[\![u \approx \hat{x}]\!]_1^{\mathbf{V}^{\mathcal{A}_{\mathrm{PT0}}}} = 1$. Then,

$$1 = [\![\psi(u, \widehat{x_1}, \ldots, \widehat{x_n}]\!]_1^{\mathbf{V}^{\mathcal{A}_{\mathrm{PT0}}}} \wedge [\![u \approx \hat{x}]\!]_1^{\mathbf{V}^{\mathcal{A}_{\mathrm{PT0}}}} \leq [\![\psi(\hat{x}, \widehat{x_1}, \ldots, \widehat{x_n}]\!]_1^{\mathbf{V}^{\mathcal{A}_{\mathrm{PT0}}}}.$$

by Theorem 8.2(vii). Hence, $[\![\psi(\hat{x}, \widehat{x_1}, \ldots, \widehat{x_n}]\!]_1^{\mathbf{V}^{\mathcal{A}_{\mathrm{PT0}}}} = 1$ and so, by induction hypothesis, $\psi(x, x_1, \ldots, x_n)$ holds in **ZFC**, for some $x \in \mathbf{V}$. That is, $\varphi(x_1, \ldots, x_n)$ holds in **ZFC**. $\square$

Now it will be shown that the *Maximum Principle* of Boolean-valued models [see 1, Lemma 1.27] is also valid in twist-valued models. The adaptation to our framework of the proof of this result found in [1] is straightforward.

DEFINITION 8.15. Let $\mathcal{A}$ be a complete Boolean algebra. Given sets $E = \{a_i : i \in I\} \subseteq |\mathcal{A}|$ and $F = \{u_i : i \in I\} \subseteq \mathbf{V}^{\mathcal{T}_{\mathcal{A}}}$, the *twist mixture* of F with respect to E is the element $u = \sum_{i \in I} a_i \odot u_i$ of $\mathbf{V}^{\mathcal{T}_{\mathcal{A}}}$ defined as follows:[7]

- $\mathrm{dom}(u) = \bigcup_{i \in I} \mathrm{dom}(u_i)$, and
- $u(z) = \big(\bigvee_{i \in I}(a_i \wedge [\![z \,\epsilon\, u_i]\!]_1), {\sim}\bigvee_{i \in I}(a_i \wedge [\![z \,\epsilon\, u_i]\!]_1)\big)$, for any $z \in \mathrm{dom}(u)$.

[7] It is worth observing that the definition of the second coordinate of $u(z)$ will be irrelevant.

LEMMA 8.16 (Mixing Lemma). *Let* $\{a_i : i \in I\} \subseteq |\mathcal{A}|$ *and* $\{u_i : i \in I\} \subseteq \mathbf{V}^{\mathcal{T}_\mathcal{A}}$, *and let* $u = \sum_{i \in I} a_i \odot u_i$. *Suppose that, for every* $i, j \in I$, $a_i \wedge a_j \leq [\![u_i \approx u_j]\!]_1$. *Then* $a_i \leq [\![u \approx u_i]\!]_1$ *for every* $i \in I$.

PROOF. It can be proved by a straightforward adaptation of the proof of [1, Lemma 1.25], taking into account Theorem 8.2 items (ii), (iii) and (vi). □

The next fundamental result shows that the set of pure **ZF**-sentences validated by each twist-valued structure $\mathbf{V}^{\mathcal{T}_\mathcal{A}}$ is a Henkin theory:

LEMMA 8.17 (The Maximum Principle). *Let* $\mathcal{A}$ *be a complete Boolean algebra. Then, for every formula* $\varphi(x)$ *in* $\mathcal{L}_p(\mathcal{T}_\mathcal{A})$, *there is* $u \in \mathbf{V}^{\mathcal{T}_\mathcal{A}}$ *such that*

$$[\![\exists x \varphi(x)]\!]_1 = [\![\varphi(u)]\!]_1.$$

In particular, if $\mathbf{V}^{\mathcal{T}_\mathcal{A}} \models \exists x \varphi(x)$ *then* $\mathbf{V}^{\mathcal{T}_\mathcal{A}} \models \varphi(u)$ *for some* $u \in \mathbf{V}^{\mathcal{T}_\mathcal{A}}$.

PROOF. The proof is obtained by a straightforward adaptation of the proof of [1, Lemma 1.27]. The collection $X = \{[\![\varphi(u)]\!] : u \in \mathbf{V}^{\mathcal{T}_\mathcal{A}}\}$ is a set, since $T_\mathcal{A}$ is a set. By the axiom of choice, there is an ordinal α and a set $\{u_\xi : \xi < \alpha\} \subseteq \mathbf{V}^{\mathcal{T}_\mathcal{A}}$ such that $X = \{[\![\varphi(u_\xi)]\!] : \xi < \alpha\}$, hence $[\![\exists x \varphi(x)]\!]_1 = \bigvee_{\xi < \alpha} [\![\varphi(u_\xi)]\!]_1$. For each $\xi < \alpha$ let $a_\xi = [\![\varphi(u_\xi)]\!]_1 \wedge {\sim} \bigvee_{\eta < \xi} [\![\varphi(u_\eta)]\!]_1$, and let $u = \sum_{\xi < \alpha} a_\xi \odot u_\xi$. By the Mixing Lemma 8.16 and by Theorem 8.2 items (ii) and (vii) it follows that $[\![\exists x \varphi(x)]\!]_1 = [\![\varphi(u)]\!]_1$. □

COROLLARY 8.18. *Let* $\varphi(x)$ *be a formula in* $\mathcal{L}_p(\mathcal{T}_\mathcal{A})$ *such that* $\mathbf{V}^{\mathcal{T}_\mathcal{A}} \models \exists x \varphi(x)$. *Then:*

(i) *For any* $v \in \mathbf{V}^{\mathcal{T}_\mathcal{A}}$ *there exists* $u \in \mathbf{V}^{\mathcal{T}_\mathcal{A}}$ *such that* $[\![\varphi(u)]\!]_1 = 1$ *and* $[\![\varphi(v)]\!]_1 = [\![u \approx v]\!]_1$.

(ii) *Let* $\psi(x)$ *be a formula in* $\mathcal{L}_p(\mathcal{T}_\mathcal{A})$ *such that* $\mathbf{V}^{\mathcal{T}_\mathcal{A}} \models \varphi(u)$ *implies that* $\mathbf{V}^{\mathcal{T}_\mathcal{A}} \models \psi(u)$, *for any* $u \in \mathbf{V}^{\mathcal{T}_\mathcal{A}}$. *Then* $\mathbf{V}^{\mathcal{T}_\mathcal{A}} \models \forall x(\varphi(x) \to \psi(x))$.

PROOF. It is an easy adaptation of the proof of [1, Corollary 1.28], taking into account Lemma 8.17 and Theorem 8.2 items (ii) and (vii). □

The notion of *core* for a Boolean-valued set [see 1] can be easily adapted to twist-valued sets:

DEFINITION 8.19. Let $u \in \mathbf{V}^{\mathcal{T}_\mathcal{A}}$. A *core* for u is a set $v \subseteq \mathbf{V}^{\mathcal{T}_\mathcal{A}}$ such that: (i) $[\![x \,\epsilon\, u]\!]_1 = 1$ for every $x \in v$; and (ii) for every $y \in \mathbf{V}^{\mathcal{T}_\mathcal{A}}$ such that $[\![y \,\epsilon\, u]\!]_1 = 1$, there is a unique $x \in v$ such that $[\![x \approx y]\!]_1 = 1$.

LEMMA 8.20. *Any $u \in \mathbf{V}^{\mathcal{T}_\mathcal{A}}$ has a core.*

PROOF. It is an easy adaptation of the proof of [1, Lemma 1.31]. $\square$

Let $\varnothing$ be the empty element of $\mathbf{V}^{\mathcal{T}_\mathcal{A}}$. As happens with Boolean-valued models, if $u \in \mathbf{V}^{\mathcal{T}_\mathcal{A}}$ is such that $\mathbf{V}^{\mathcal{T}_\mathcal{A}} \models {\sim}(u \approx \varnothing)$ then, by the Maximum Principle, any core of u is nonempty. From Corollary 8.18 we obtain:

COROLLARY 8.21. *Let $u \in \mathbf{V}^{\mathcal{T}_\mathcal{A}}$ such that $\mathbf{V}^{\mathcal{T}_\mathcal{A}} \models {\sim}(u \approx \varnothing)$, and let v be a core for u. Then, for any $x \in \mathbf{V}^{\mathcal{T}_\mathcal{A}}$ there exists $y \in v$ such that $[\![x \approx y]\!]_1 = [\![x \,\epsilon\, u]\!]_1$.*

From the results obtained above, one of the main results of the paper can be established:

THEOREM 8.22. *All the axioms (hence all the theorems) of* **ZFC**, *when restricted to pure* **ZF**-*languages $\mathcal{L}_p(\mathcal{T}_\mathcal{A})$ (recall Definition 7.2), are valid in $\mathbf{V}^{\mathcal{T}_\mathcal{A}}$, for every $\mathcal{A}$.*

PROOF. It is a relatively easy (but arduous) adaptation of the proof of [1, Theorem 1.33], taking into account the auxiliary results obtained within this section, which are similar to the ones required in [1]. Indeed, Theorem 8.2 is used in the proof of validity of several axioms of **ZF**, while Theorem 8.14(v) is useful in order to establish the validity of the infinity axiom. Finally, the notion of core, together with their properties shown above, is used in order to state the validity of Zorn's lemma. Since the latter is set-theoretically equivalent to the axiom of choice, this guarantees the validity of all the axioms of **ZFC** restricted to $\mathcal{L}_p(\mathcal{T}_\mathcal{A})$. $\square$

9. Twist-valued models for $(\mathbb{PS}_3, \neg)$

In this section the three-valued model for set theory introduced by Löwe and Tarafder in [18] will be extended to a class of twist-valued models.

As observed in Section 7, the three-valued logic of $(\mathbb{PS}_3, \neg)$ (denoted as $(\mathbb{PS}_3, *)$ in [18]) was already considered in [10] under the name **MPT**. Indeed, this logic has been independently proposed by different authors at several times, and with different motivations.[8] For instance, the same logic was proposed in 1970 by da Costa and D'Ottaviano's as **J3**. It was reintroduced in 2000 by Carnielli, Marcos and de Amo as **LFI1**, and by

[8] As mentioned in Section 3, $\mathbf{LFI1}_\circ$ is another presentation of this logic.

Batens and De Clerq as the propositional fragment of the first-order logic **CLuNs**, in 2014. As observed by Batens, this logic was firstly proposed by Karl Schütte in 1960 under the name Φ_v [see 5 for details and specific references]. Each of the three-valued algebras above is equivalent, up to language, to the three-valued algebra of Łukasiewicz three-valued logic Ł$_3$. Hence, these logics are equivalent to Ł$_3$ with $\{\mathbf{1}, \frac{\mathbf{1}}{\mathbf{2}}\}$ as designated values. Moreover, as it was shown by Blok and Pigozzi in [2], the class of algebraic models of **J3** (and so the class of twist structures for **LPT0**) coincides with the algebraic models of Łukasiewicz's three-valued logic Ł$_3$. More remarks about these three-valued equivalent logics can be found in [5, Chapters 4 and 7].

As shown in [10, p. 407], the implication $\Rightarrow$ given by

$\Rightarrow$	$\mathbf{1}$	$\frac{\mathbf{1}}{\mathbf{2}}$	$\mathbf{0}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{0}$
$\frac{\mathbf{1}}{\mathbf{2}}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{0}$
$\mathbf{0}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$

(which is the same implication $\Rightarrow$ of $\mathbb{PS}_3$ and the primitive implication of **MPT**) can be defined in the language of **LFI1** (hence in the language of **LPT0**) as follows: $\varphi \Rightarrow \psi := \neg\sim(\varphi \to \psi)$. From this, it is easy to adapt Definition 4.6 of twist-structures for **LPT0** to $(\mathbb{PS}_3, \neg)$ (see Definition 9.1 below). Hence, the logic of $(\mathbb{PS}_3, \neg)$ will be considered as defined over the signature $\Sigma_\Rightarrow = \{\wedge, \vee, \Rightarrow, \neg\}$. As observed in [10, pp. 395 and 407], the strong negation $\sim$ can be defined as $\sim\varphi := \varphi \Rightarrow \neg(\varphi \Rightarrow \varphi)$, while $\varphi \to \psi := \sim\varphi \vee \psi$.

Definition 9.1. Let $\mathcal{A}$ be a complete Boolean algebra, and let $T_\mathcal{A}$ as in Definition 4.5. The *twist structure for* $(\mathbb{PS}_3, \neg)$ *over* $\mathcal{A}$ is the algebra $\mathcal{T}_{\mathcal{A}^*} = \langle T_\mathcal{A}, \tilde{\wedge}, \tilde{\vee}, \tilde{\Rightarrow}, \tilde{\neg}\rangle$ over $\Sigma_\Rightarrow$ such that the operations $\tilde{\wedge}$, $\tilde{\vee}$ and $\tilde{\neg}$ are defined as in Definition 4.6, and $\tilde{\Rightarrow}$ is defined as follows, for every $(z_1, z_2), (w_1, w_2) \in T_\mathcal{A}$:

$$(z_1, z_2) \tilde{\Rightarrow} (w_1, w_2) = (z_1 \to w_1, z_1 \wedge \sim w_1).$$

By considering (as mentioned above) $\sim$ and $\to$ as derived connectives in $\mathcal{T}_{\mathcal{A}^*}$, it is clear that $\tilde{\sim}(z_1, z_2) = (\sim z_1, z_1)$ and $(z_1, z_2) \tilde{\to} (w_1, w_2) = (z_1 \to w_1, z_1 \wedge w_2)$. Hence, the original operations of Definition 4.6 can be recovered in $\mathcal{T}_{\mathcal{A}^*}$.

As it will be discussed below, we will adopt a technique different to the one used in [18] in order to show the satisfaction of **ZFC** in the twist-valued models based on $\mathcal{T}_{\mathcal{A}^*}$. However, it is interesting to observe that a nice property of $(\mathbb{PS}_3, \neg)$ is preserved by any $\mathcal{T}_{\mathcal{A}^*}$. Indeed, in [18] the following notion of *reasonable implication algebras* was proposed in order to provide suitable lattice-valued model for **ZF**:

DEFINITION 9.2. An algebra $\mathcal{A} = \langle A, \wedge, \vee, \Rightarrow, \mathbf{0}, \mathbf{1}\rangle$ is an *reasonable implication algebra* if the reduct $\langle A, \wedge, \vee, \mathbf{0}, \mathbf{1}\rangle$ is a complete lattice with bottom $\mathbf{0}$ and top $\mathbf{1}$, and $\Rightarrow$ is a binary operator satisfying the following, for all $z, w, u \in A$:

(P1) $z \wedge w \leq u$ implies that $z \leq (w \Rightarrow u)$;
(P2) $z \leq w$ implies that $(u \Rightarrow z) \leq (u \Rightarrow w)$;
(P3) $z \leq w$ implies that $(w \Rightarrow u) \leq (z \Rightarrow u)$.

PROPOSITION 9.3. *For every complete Boolean algebra $\mathcal{A}$, the twist structure $\mathcal{T}_{\mathcal{A}^*}$ for $(\mathbb{PS}_3, \neg)$ is a reasonable implication algebra such that $\mathbf{0} = (0, 1)$ and $\mathbf{1} = (1, 0)$.*[9]

PROOF. Let $(z_1, z_2), (w_1, w_2), (u_1, u_2) \in T_{\mathcal{A}}$.

(P1): Assume that $(z_1, z_2) \mathbin{\tilde{\wedge}} (w_1, w_2) \leq (u_1, u_2)$. That is, $(z_1 \wedge w_1, z_2 \vee w_2) \leq (u_1, u_2)$. Then $z_1 \wedge w_1 \leq u_1$ and $z_2 \vee w_2 \geq u_2$. From $z_1 \wedge w_1 \leq u_1$ it follows that $z_1 \leq w_1 \to u_1$. Besides, since $z_1 \vee z_2 = 1$ then $\sim z_2 \leq z_1 \leq w_1 \to u_1$. Hence $z_2 \geq \sim(w_1 \to u_1) = w_1 \wedge \sim u_1$. From this, $(z_1, z_2) \leq (w_1 \to u_1, w_1 \wedge \sim u_1) = (w_1, w_2) \mathbin{\tilde{\Rightarrow}} (u_1, u_2)$.

(P2): Assume that $(z_1, z_2) \leq (w_1, w_2)$. Then $z_1 \leq w_1$, hence $u_1 \to z_1 \leq u_1 \to w_1$ and so $u_1 \wedge \sim z_1 = \sim(u_1 \to z_1) \geq \sim(u_1 \to w_1) = u_1 \wedge \sim w_1$. This means that $(u_1, u_2) \mathbin{\tilde{\Rightarrow}} (z_1, z_2) \leq (u_1, u_2) \mathbin{\tilde{\Rightarrow}} (w_1, w_2)$.

(P3): It is proved analogously, but now taking into account that $z_1 \leq w_1$ implies that $w_1 \to u_1 \leq z_1 \to u_1$. □

Now, the three-valued model of set theory presented in [18] will be generalized to twist-valued models over any complete Boolean algebra. The structure $\mathbf{V}^{\mathcal{T}_{\mathcal{A}^*}}$ is defined as the structure $\mathbf{V}^{\mathcal{T}_{\mathcal{A}}}$ given in Definition 7.1. This does not come as a surprise, given that the domain of $\mathcal{T}_{\mathcal{A}}$ and $\mathcal{T}_{\mathcal{A}^*}$ is the same, the set $T_{\mathcal{A}}$. However, $\mathbf{V}^{\mathcal{T}_{\mathcal{A}}}$ and $\mathbf{V}^{\mathcal{T}_{\mathcal{A}^*}}$ are different as first-order structures, namely, the way in which the formulas are interpreted. The only difference, besides using different implications in the underlying logics, will be in the form in which the predicates ϵ and $\approx$ are interpreted. Thus, the twist truth-value $[\![\varphi]\!]^{\mathbf{V}^{\mathcal{T}_{\mathcal{A}^*}}}$ of a sentence φ in $\mathbf{V}^{\mathcal{T}_{\mathcal{A}^*}}$ will

[9] To be rigorous, the $\neg$-less reduct of $\mathcal{T}_{\mathcal{A}^*}$ expanded with $\mathbf{0}$ and $\mathbf{1}$ is a reasonable implication algebra.

be defined according to the recursive clauses in Definition 7.6, with the following difference: any occurrence of the operator $\tilde{\to}$ must be replaced by the operator $\tilde{\Rightarrow}$. Note that the clause interpreting $\sim\varphi$ is now derived from the others, taking into account the observation after Definition 9.1.

In Theorem 9.4 below it is stated that every twist-valued structure $\mathbf{V}^{\mathcal{T}_{\mathcal{A}^*}}$ is a model of **ZFC**. This constitutes a generalization of [18, Corollary 11]. Indeed, instead of taking just a three-valued model (generated by the two-element Boolean algebra), we obtain a class of models, one for each complete Boolean algebra. Moreover, we also prove that these generalized models (including, of course, the original Löwe-Tarafder model) satisfy, in addition, the axiom of choice.

The proof of validity of **ZF** given in [18, Corollary 11] is strongly based on the particularities of the three-valued algebra $(\mathbb{PS}_3, \neg)$.[10] This forces us to adapt, to this setting, the proof for twist-valued models over $\mathcal{T}_{\mathcal{A}}$ given in the previous sections (which, by its turn, is adapted from the proof for Boolean-valued sets). Such adaptations from $\mathcal{T}_{\mathcal{A}}$ to $\mathcal{T}_{\mathcal{A}^*}$ are immediate, and all the results and definitions proposed in the previous sections work fine for $\mathcal{T}_{\mathcal{A}^*}$. Hence, we obtain the second main result of the paper:

Theorem 9.4. *All the axioms* (*hence all the theorems*) *of* **ZFC**, *when restricted to pure* **ZF**-*languages* $\mathcal{L}_p(\mathcal{T}_{\mathcal{A}})$, *are valid in* $\mathbf{V}^{\mathcal{T}_{\mathcal{A}^*}}$, *for any* $\mathcal{A}$.

Remark 9.5. Observe that, in [18, Corollary 11], it was proved that $\mathbb{PS}_3$ is a model of **ZF**, not of **ZFC**. Thus, Theorem 9.4 improves the above mentioned result in two ways: it is generalized to arbitrary Boolean algebras and, in addition, it proves that the axiom of choice AC is also satisfied by all that models, including the original three-valued structure $\mathbb{PS}_3$.

10. $\mathbf{ZF}_{\mathbf{LPT0}}$ as a paraconsistent set theory

After proving that the two classes of twist-valued models proposed here are models of **ZFC**, in this section the paraconsistent character of both classes of models will be investigated. It will be shown, at the end of Subsection 10.1, that twist-valued models over $\mathcal{T}_{\mathcal{A}}$ (that is, over the logic

[10] For instance, the fact that expressions like $[\![u \approx v]\!] \Rightarrow [\![u \in w]\!]$ can only take either the value $\mathbf{0}$ or $\mathbf{1}$ is used several times in [18]. Observe that, in $\mathcal{T}_{\mathcal{A}^*}$, the value of $z \tilde{\Rightarrow} w$ is always of the form $(a, \sim a)$ for some $a \in |\mathcal{A}|$. Hence $[\![u \approx v]\!]^{\mathbf{V}^{\mathcal{T}_{\mathcal{A}^*}}}$ is always of the form $(a, \sim a)$ for some $a \in |\mathcal{A}|$.

LPT0) are "more paraconsistent" that the ones over $\mathcal{T}_{\mathcal{A}^*}$ (that is, defined over $(\mathbb{PS}_3, \neg)$).

Recall from Theorem 8.2(i) that $[\![u \approx u]\!] \in D_{\mathcal{A}}$ for every u in every twist-valued model $\mathbf{V}^{\mathcal{T}_{\mathcal{A}}}$. The interesting fact of $\mathbf{ZF_{LPT0}}$ is that it allows "inconsistent" sets, that is, elements of $\mathbf{V}^{\mathcal{T}_{\mathcal{A}}}$ such that the value of $(u \not\approx u)$ is also designated. Observe that $\mathbf{1} = (1, 0)$, $\frac{\mathbf{1}}{\mathbf{2}} = (1, 1)$ and $\mathbf{0} = (0, 1)$ are defined in every $\mathcal{T}_{\mathcal{A}}$. Since $z \in D_{\mathcal{A}}$ iff $z = (1, a)$ for some $a \in A$, it follows that $z \in D_{\mathcal{A}}$ iff $\frac{\mathbf{1}}{\mathbf{2}} \leq z$ (recalling the partial order for $\mathcal{T}_{\mathcal{A}}$ considered in Remark 6.10).

PROPOSITION 10.1. *There exists* $u \in \mathbf{V}^{\mathcal{T}_{\mathcal{A}}}$ *such that* $[\![u \approx u]\!] = \frac{\mathbf{1}}{\mathbf{2}}$.

PROOF. Let w be any element of $\mathbf{V}^{\mathcal{T}_{\mathcal{A}}}$, and let $u = \{\langle w, \frac{\mathbf{1}}{\mathbf{2}}\rangle\}$. Since $[\![w \approx w]\!] \in D_{\mathcal{A}}$ then $[\![w \,\epsilon\, u]\!] = u(w) \mathbin{\tilde{\wedge}} [\![w \approx w]\!] = \frac{\mathbf{1}}{\mathbf{2}} \mathbin{\tilde{\wedge}} [\![w \approx w]\!] = \frac{\mathbf{1}}{\mathbf{2}}$. From this, $[\![u \approx u]\!] = u(w) \mathbin{\tilde{\to}} [\![w \,\epsilon\, u]\!] = \frac{\mathbf{1}}{\mathbf{2}} \mathbin{\tilde{\to}} \frac{\mathbf{1}}{\mathbf{2}} = \frac{\mathbf{1}}{\mathbf{2}}$. □

From the last result it can be proven that $\mathbf{ZF_{LPT0}}$ is strongly paraconsistent, in the sense that there is a contradiction which is valid in the logic:

COROLLARY 10.2. *Let* $\sigma = \forall x(x \approx x)$. *Then* $\mathbf{V}^{\mathcal{T}_{\mathcal{A}}} \models \sigma \wedge \neg\, \sigma$.

PROOF. Let $\mathbf{V}^{\mathcal{T}_{\mathcal{A}}}$ be a twist-valued model for $\mathbf{ZF_{LPT0}}$. As observed above, $\frac{\mathbf{1}}{\mathbf{2}} \leq z$ for every $z \in D_{\mathcal{A}}$. By Theorem 8.2(i), $[\![v \approx v]\!] \in D_{\mathcal{A}}$ for every v in $\mathbf{V}^{\mathcal{T}_{\mathcal{A}}}$ and so $\frac{\mathbf{1}}{\mathbf{2}} \leq [\![v \approx v]\!]$ for every v, that is, $\frac{\mathbf{1}}{\mathbf{2}} \leq [\![\forall x(x \approx x)]\!]$, by Definition 7.6. On the other hand, $[\![\forall x(x \approx x)]\!] \leq [\![u \approx u]\!] = \frac{\mathbf{1}}{\mathbf{2}}$ for u as in Proposition 10.1. This shows that $[\![\sigma]\!] = [\![\forall x(x \approx x)]\!] = \frac{\mathbf{1}}{\mathbf{2}}$ and so $[\![\neg\, \sigma]\!] = \tilde{\neg}\, [\![\sigma]\!] = \frac{\mathbf{1}}{\mathbf{2}}$. Hence $[\![\sigma \wedge \neg\sigma]\!] = [\![\sigma]\!] \mathbin{\tilde{\wedge}} [\![\neg\, \sigma]\!] = \frac{\mathbf{1}}{\mathbf{2}}$, a designated value. □

COROLLARY 10.3. *There are inconsistent sets in* $\mathbf{ZF_{LPT0}}$ *for any* $\mathcal{T}_{\mathcal{A}}$: $\mathbf{V}^{\mathcal{T}_{\mathcal{A}}} \models \exists x((x \approx x) \wedge \neg(x \approx x))$.

PROOF. An easy consequence of Corollary 10.2 and the validity of axiom (Ax¬∃). □

Since the extensionality axiom of **ZF** is satisfied by every twist-valued model $\mathbf{V}^{\mathcal{T}_{\mathcal{A}}}$ for $\mathbf{ZF_{LPT0}}$, and by virtue of Corollary 8.18(ii), $[\![u \approx v]\!] \in D_{\mathcal{A}}$ iff u and v have the same elements, that is: for every w in $\mathbf{V}^{\mathcal{T}_{\mathcal{A}}}$, $[\![w \,\epsilon\, u]\!] \in D_{\mathcal{A}}$ iff $[\![w \,\epsilon\, v]\!] \in D_{\mathcal{A}}$. However, nothing guarantees that u and v will have the same 'non-elements', namely: it could be possible that $[\![\neg(w \,\epsilon\, u)]\!] \in D_{\mathcal{A}}$ but $[\![\neg(w \,\epsilon\, v)]\!] \notin D_{\mathcal{A}}$, for some w in $\mathbf{V}^{\mathcal{T}_{\mathcal{A}}}$, even when $[\![u \approx v]\!] \in D_{\mathcal{A}}$. Given such w, consider the property $\varphi(x) := \neg(w \,\epsilon\, x)$, meaning that "$w$ is a non-element of x". Then, this situation shows that $\mathbf{V}^{\mathcal{T_A}} \not\models ((u \approx v) \wedge \varphi(u)) \to \varphi(v)$, which constitutes a violation of the Leibniz's Law for the equality predicate $\approx$ in $\mathbf{ZF_{LPT0}}$.

Theorem 10.4. *The formula $\varphi(x) := \neg(w \,\epsilon\, x)$ is such that the Leibniz's Law fails for it in every $\mathbf{V}^{\mathcal{T_A}}$, namely: $\mathbf{V}^{\mathcal{T_A}} \not\models \forall x \forall y((x \approx y) \wedge \varphi(x) \to \varphi(y))$.*

Proof. Let $\mathbf{V}^{\mathcal{T_A}}$ be a twist-valued model for $\mathbf{ZF_{LPT0}}$, and let $\varnothing$ be the empty element of $\mathbf{V}^{\mathcal{T_A}}$. Observe that $w = \{\langle \varnothing, \mathbf{1}\rangle\}$, $u = \{\langle w, \frac{\mathbf{1}}{\mathbf{2}}\rangle\}$ and $v = \{\langle w, \mathbf{1}\rangle\}$ belong to every model $\mathbf{V}^{\mathcal{T_A}}$. Now, $[\![\varnothing \,\epsilon\, w]\!] = w(\varnothing) \,\tilde{\wedge}\, [\![\varnothing \approx \varnothing]\!] = \mathbf{1} \,\tilde{\wedge}\, \mathbf{1} = \mathbf{1}$. From this, $[\![w \approx w]\!] = w(\varnothing) \,\tilde{\to}\, [\![\varnothing \,\epsilon\, w]\!] = \mathbf{1} \,\tilde{\to}\, \mathbf{1} = \mathbf{1}$ and so $[\![w \,\epsilon\, u]\!] = u(w) \,\tilde{\wedge}\, [\![w \approx w]\!] = \frac{\mathbf{1}}{\mathbf{2}} \,\tilde{\wedge}\, \mathbf{1} = \frac{\mathbf{1}}{\mathbf{2}}$. On the other hand, $[\![w \,\epsilon\, v]\!] = v(w) \,\tilde{\wedge}\, [\![w \approx w]\!] = \mathbf{1} \,\tilde{\wedge}\, \mathbf{1} = \mathbf{1}$. This implies that $[\![u \approx v]\!] = (u(w) \,\tilde{\to}\, [\![w \,\epsilon\, v]\!]) \,\tilde{\wedge}\, (v(w) \,\tilde{\to}\, [\![w \,\epsilon\, u]\!]) = (\frac{\mathbf{1}}{\mathbf{2}} \,\tilde{\to}\, \mathbf{1}) \,\tilde{\wedge}\, (\mathbf{1} \,\tilde{\to}\, \frac{\mathbf{1}}{\mathbf{2}}) = \frac{\mathbf{1}}{\mathbf{2}}$.

But $[\![\varphi(u)]\!] = [\![\neg(w \,\epsilon\, u)]\!] = \tilde{\neg}\, [\![w \,\epsilon\, u]\!] = \tilde{\neg}\, \frac{\mathbf{1}}{\mathbf{2}} = \frac{\mathbf{1}}{\mathbf{2}}$ and $[\![\varphi(v)]\!] = [\![\neg(w \,\epsilon\, v)]\!] = \tilde{\neg}\, [\![w \,\epsilon\, v]\!] = \tilde{\neg}\, \mathbf{1} = \mathbf{0}$. Thus, $[\![((u \approx v) \wedge \varphi(u)) \to \varphi(v)]\!] = (\frac{\mathbf{1}}{\mathbf{2}} \,\tilde{\wedge}\, \frac{\mathbf{1}}{\mathbf{2}}) \,\tilde{\to}\, \mathbf{0} = \mathbf{0}$, which implies that $\mathbf{V}^{\mathcal{T_A}} \not\models \forall x \forall y((x \approx y) \wedge \varphi(x) \to \varphi(y))$. $\square$

It is important to observe that the failure of the Leibniz's Law in $\mathbf{V}^{\mathcal{T_A}}$ shown in Theorem 10.4 does not contradict Theorem 8.2(vii): indeed, what Theorem 8.2(vii) states is the validity of the Leibniz's Law in $\mathbf{V}^{\mathcal{T_A}}$ for every formula $\varphi(x)$ in the pure $\mathbf{ZF}$-language $\mathcal{L}_p(\mathcal{T_A})$. On the other hand, the formula $\varphi(x)$ found in Theorem 10.4 which violates the Leibniz's Law in $\mathbf{V}^{\mathcal{T_A}}$ contains an occurrence of the paraconsistent negation $\neg$, that is, it does not belong to $\mathcal{L}_p(\mathcal{T_A})$. In that example, two sets which are equal have different 'non-elements', where 'non' refers to the paraconsistent negation $\neg$.

Besides the failure of the Leibniz's Law for the full language, $\mathbf{ZF_{LPT0}}$ does not validate the bounded quantification properties (recall Definition 8.5). Indeed, as shown in Theorem 8.6, these important properties hold in the pure $\mathbf{ZF}$-language. However, for formulas containing the paraconsistent negation, that result does not holds in general:

Proposition 10.5. *There is $u \in \mathbf{V}^{\mathcal{T_A}}$ and formulas $\varphi(x)$ and $\psi(x)$ such that the bounded quantification properties UBQ^u_ψ and EBQ^u_φ fail in $\mathbf{V}^{\mathcal{T_A}}$.*

Proof. It is enough to prove the failure of EBQ^u_φ given that the failure of UBQ^u_ψ is obtained from it by using $\psi(x) := {\sim}\varphi(x)$ and the duality between infimum and supremum through the Boolean complement $\sim$.

Thus, let $\mathbf{V}^{\mathcal{T_A}}$ and let $w = \{\langle\varnothing, \mathbf{1}\rangle\}$, $v = \{\langle w, \frac{\mathbf{1}}{\mathbf{2}}\rangle\}$, $y = \{\langle w, \mathbf{1}\rangle\}$ and $u = \{\langle y, \mathbf{1}\rangle\}$. Let $\varphi(x) := \neg(w \,\epsilon\, x)$. As in the proof of Theorem 10.4 it can be proven that $[\![v \approx y]\!] = [\![\varphi(v)]\!] = \frac{\mathbf{1}}{\mathbf{2}}$ and $[\![\varphi(y)]\!] = \mathbf{0}$. Hence $\bigvee_{x\in\mathrm{dom}(u)}((u(x))_1 \wedge [\![\varphi(x)]\!]_1) = (u(y))_1 \wedge [\![\varphi(y)]\!]_1 = 0$ while $[\![\exists x \,\epsilon\, u\, \varphi(x)]\!]_1 = [\![\exists x(x \,\epsilon\, u \wedge \varphi(x))]\!]_1 = \bigvee_{v'\in\mathbf{V}^{\mathcal{T_A}}} \bigvee_{x\in\mathrm{dom}(u)}((u(x))_1 \wedge [\![v' \approx x]\!]_1 \wedge [\![\varphi(v')]\!]_1) = \bigvee_{v'\in\mathbf{V}^{\mathcal{T_A}}}((u(y))_1 \wedge [\![v' \approx y]\!]_1 \wedge [\![\varphi(v')]\!]_1) \geq (u(y))_1 \wedge [\![v \approx y]\!]_1 \wedge [\![\varphi(v)]\!]_1 = 1$. This means that $[\![\exists x \,\epsilon\, u\, \varphi(x)]\!]_1 = 1 \neq 0 = \bigvee_{x\in\mathrm{dom}(u)}((u(x))_1 \wedge [\![\varphi(x)]\!]_1)$. □

It is worth noting that the limitations of $\mathbf{ZF_{LPT0}}$ pointed out above (namely, the Leibniz's Law and the bounded quantification properties for formulas containing the paraconsistent negation) are also present in Löwe-Tarafder's model [18].

10.1. Considering a consistency predicate for sets

In Corollary 10.3 it was shown that there are inconsistent sets in $\mathbf{ZF_{LPT0}}$. The notion of 'inconsistent' set considered above was defined in semantical terms, namely: a set u is inconsistent in a twist-valued model when the value of $(u \not\approx u)$ is designated or, equivalently, when the value of $(u \approx u)$ is $\frac{\mathbf{1}}{\mathbf{2}}$. The notion of consistent and inconsistent sets can be formalized in the language, in the same way as consistent and inconsistent sentences are expressed by means of the consistency and inconsistency connectives in **LFI**s [see, e.g., 5]. As mentioned in Section 3, in [4] was presented a family of paraconsistent set theories based on diverse **LFI**s, such that the original **ZF** axioms were slightly modified in order to deal with a unary predicate $C(x)$ representing that 'the set x is consistent'. The consistency connective $\circ$ is primitive in **mbC**, but it is definable as $\circ\varphi := {\sim}(\varphi \wedge \neg\varphi)$ in any axiomatic extension of **mbC** which proves the schema (ciw): $\circ\varphi \vee (\varphi \wedge \neg\varphi)$ such as **LPT0**. In the same way, the consistency predicate $C(x)$ can be expressed, in extensions of **ZFmbC**, in terms of a formula of **ZFmbC** without using the predicate C, and the same happens with the inconsistency predicate $\neg C(x)$. For instance, **ZFmCi** is based on **mCi**, an extension of **mbC** in which $\neg\circ\varphi$ is equivalent to $\varphi \wedge \neg\varphi$. Thus, $\neg C(x)$ was defined to be equivalent to $(x \approx x) \wedge \neg(x \approx x)$ in **ZFmCi**. From this, $\neg C(x)$ is equivalent to $\neg\circ(x \approx x)$ in **ZFmCi**. Given that **LPT0** is an extension of **mCi**, if a consistency predicate for sets were added to the language of $\mathbf{ZF_{LPT0}}$ then it seems reasonable to require the equivalence between $\neg C(x)$ and

$(x \approx x) \wedge \neg(x \approx x)$ in $\mathbf{ZF_{LPT0}}$.[11] Hence, $\neg C(x)$ would be equivalent to $\neg \circ(x \approx x)$ in $\mathbf{ZF_{LPT0}}$. But $\circ C(x)$ is derivable **ZFmCi**, so it would be valid in $\mathbf{ZF_{LPT0}}$.[12] From this $C(x) \leftrightarrow \circ(x \approx x)$ would be also derivable in **QLPT0** and so it would be valid in $\mathbf{ZF_{LPT0}}$ expanded with a suitable predicate C denoting 'consistency for sets'. This motivates the following:

DEFINITION 10.6. Define in $\mathbf{ZF_{LPT0}}$ the *consistency predicate for sets*, $C(x)$, as follows: $C(x) := \sim\neg(x \approx x)$.

According to the previous discussion, $C(x)$ should be equivalent to $\circ(x \approx x)$ in $\mathbf{ZF_{LPT0}}$. But $\circ\varphi$ is equivalent to $\sim(\varphi \wedge \neg\varphi)$ in **LPT0**, and $(x \approx x)$ is valid in $\mathbf{ZF_{LPT0}}$, hence $C(x)$ should be equivalent to $\sim\neg(x \approx x)$ in $\mathbf{ZF_{LPT0}}$, which justifies Definition 10.6. If $\neg C(x)$ denotes that x is inconsistent then, clearly, $C(u)$ is designated in a twist-valued models iff the value of $(u \approx u)$ is different to $\mathbf{\frac{1}{2}}$, while $\neg C(u)$ is designated iff the value of $(u \approx u)$ is $\mathbf{\frac{1}{2}}$.

PROPOSITION 10.7. *The consistency predicate $C(x)$ is non-trivial: there exist $v, w \in \mathbf{V}^{\mathcal{T}_\mathcal{A}}$ such that $[\![C(v)]\!] = \mathbf{1}$ and $[\![C(w)]\!] = \mathbf{0}$. Moreover, $[\![C(u)]\!] \neq \mathbf{\frac{1}{2}}$ for every u in $\mathbf{V}^{\mathcal{T}_\mathcal{A}}$. Hence, the sentences '$x$ is consistent' and 'x is inconsistent' are consistent, that is: $\mathbf{V}^{\mathcal{T}_\mathcal{A}} \models \forall x \circ C(x)$ and $\mathbf{V}^{\mathcal{T}_\mathcal{A}} \models \forall x \circ\neg C(x)$. Moreover, both formulas are provable in* **QLPT0**.

PROOF. Let $\mathbf{V}^{\mathcal{T}_\mathcal{A}}$ be a twist-valued model for $\mathbf{ZF_{LPT0}}$, and consider $v = \{\langle\varnothing, \mathbf{1}\rangle\}$ and $w = \{\langle\varnothing, \mathbf{\frac{1}{2}}\rangle\}$ in $\mathbf{V}^{\mathcal{T}_\mathcal{A}}$. It is easy to see that $[\![C(v)]\!] = \mathbf{1}$ and $[\![C(w)]\!] = \mathbf{0}$. On the other hand, for every u in $\mathbf{V}^{\mathcal{T}_\mathcal{A}}$ it is the case that $[\![C(u)]\!] = \tilde{\sim} z$ for $z = [\![\neg(u \approx u)]\!]$. Hence $[\![C(u)]\!] = (\sim z_1, z_1) \neq \mathbf{\frac{1}{2}}$, for every u. By recalling that $\circ\varphi := \sim(\varphi \wedge \neg\varphi)$, it follows that $\mathbf{V}^{\mathcal{T}_\mathcal{A}} \models \circ\varphi$ iff $[\![\varphi]\!] \neq \mathbf{\frac{1}{2}}$, hence $\circ C(x)$ and $\circ\neg C(x)$ are valid in $\mathbf{ZF_{LPT0}}$. Finally, it is easy to see that the formulas $\circ\sim\varphi$ and $\circ\neg\sim\varphi$ are valid in $\mathcal{A}_{\mathrm{PT0}}$, for any formula φ (recall Definition 4.2), hence they are provable in **LPT0**, by Theorem 4.4. From this, the formulas $\circ C(x)$ and $\circ\neg C(x)$ are provable in **QLPT0**. □

Remarks 10.8. (1) Despite $\neg C(x) := \neg\sim\neg(x \approx x)$ being equivalent to $(x \approx x) \wedge \neg(x \approx x)$, the former formula is consistent in $\mathbf{ZF_{LPT0}}$ and in **QLPT0** (as stated in Proposition 10.7), while the latter is not: if $[\![u \approx u]\!] = \mathbf{\frac{1}{2}}$ then $[\![(u \approx u) \wedge \neg(u \approx u)]\!] = \mathbf{\frac{1}{2}}$. Analogously, by Theorem 4.4

[11] In this sense, Corollary 10.3 states the existence of inconsistent sets in $\mathbf{ZF_{LPT0}}$.

[12] Indeed, the proof in **ZFmCi** of $\circ C(x)$ given in [4, Proposition 3.10] can be done in **QLPT0**, assuming the axioms for C taken from **ZFmCi**.

it follows that $\circ((x \approx x) \wedge \neg(x \approx x))$ is not a theorem of **QLPT0**, despite $\circ\neg C(x)$ being a theorem.[13]

(2) It is worth noting that the failure of the Leibniz's Law in $\mathbf{ZF_{LPT0}}$ for a formula containing the paraconsistent negation, pointed out in the proof of Theorem 10.4, involves a set $u = \{\langle w, \frac{1}{2}\rangle\}$ (where $w = \{\langle \varnothing, \mathbf{1}\rangle\}$ and $\varnothing$ is the empty element of $\mathbf{V}^{\mathcal{T_A}}$) which is inconsistent. Indeed, it is easy to see that $[\![u \approx u]\!] = \frac{1}{2}$, hence $[\![C(u)]\!] = \mathbf{0}$. This suggests a possible way to deal with the Leibniz's Law in the full language of $\mathbf{ZF_{LPT0}}$, that is, allowing occurrences of the paraconsistent negation in the formula under consideration. It would be enough assuming that the involved sets are consistent, namely: $(x \approx y) \wedge C(x) \wedge C(y) \wedge \varphi(x) \rightarrow \varphi(y)$. Thus, it is an interesting question how to define a consistency predicate satisfying the latter property. We will return to this point in Subsection 10.2.

Finally, we can show now that twist-valued models over $\mathcal{T_A}$ (that is, over the logic **LPT0**) are "more paraconsistent" than the ones over $\mathcal{T_{A^*}}$ (that is, defined over $(\mathbb{PS}_3, \neg)$). Indeed, as we have seen, $\mathbf{ZF_{LPT0}}$ allow us to define in every twist-valued model $\mathbf{V}^{\mathcal{T_A}}$ an "inconsistent set", namely u, such that $(u \approx u) \wedge \neg(u \approx u)$ holds. In fact, any $u = \{\langle w, \frac{1}{2}\rangle\}$ is such that $[\![u \approx u]\!] = \frac{1}{2} \mathrel{\tilde{\rightarrow}} \frac{1}{2} = \frac{1}{2}$. The difference, of course, rests on the nature of the implication operator considered in each case: in $(\mathbb{PS}_3, \neg)$ the value of $(u \approx u)$ is always $\mathbf{1}$, since $\frac{1}{2} \Rightarrow \frac{1}{2} = \mathbf{1}$. Hence, $\neg(u \approx u)$ always gets the value $\mathbf{0}$. The same holds in any model over reasonable implicative algebras considered by Löwe and Tarafder [see 18, Proposition 1]. Because of this, within these models every set u is consistent; that is, the value of the formula $C(u)$ as in Definition 10.6 is designated.

10.2. Discussion: $\mathbf{ZF_{LPT0}}$ and the failure of the Leibniz's Law

At first sight, having a (paraconsistent) set theory as $\mathbf{ZF_{LPT0}}$ in which the Leibniz's Law is not satisfied for every formula $\varphi(x)$ that represents a property could seem to be a bit disappointing. After all, **ZF** is defined as a first-order theory with equality, which pressuposes the validity of the Leibniz's Law.

The Leibniz's Law states that the equality predicate preserves logical equivalence, namely: $(a \approx b) \rightarrow (\varphi(a) \leftrightarrow \varphi(b))$ for every formula $\varphi(x)$

[13] This is a consequence of the fact that **QLPT0** and $\mathbf{ZF_{LPT0}}$ are not self-extensional: in general, the paraconsistent negation does not preserve logical equivalence.

(clearly this can be generalized to formulas with $n \geqslant 1$ free variables, assuming $\bigwedge_{i=1}^{n}(a_i \approx b_i)$). In first-order theories based on classical logic, such as **ZF**, it is enough to require that this property holds for every atomic formula, and so the general case is proven by induction on the complexity of φ. Of course this proof cannot be reproduced in **QLPT0**, since, as mentioned above, $\neg$ is not congruential: $\varphi(a) \leftrightarrow \varphi(b)$ does not imply that $\neg\varphi(a) \leftrightarrow \neg\varphi(b)$ in general (and this is the key step in the proof by induction). The solution is to require the validity of the Leibniz's Law for every φ from the beginning, adjusting accordingly the class of interpretations for **QLPT0** expanded with equality [see 12]. However, the situation for $\mathbf{ZF_{LPT0}}$ is quite different: because of the extensionality axiom, the definition of the interpretation of the equality predicate depends strongly on the interpretation of the membership predicate. In fact, the interpretation of both predicates is simultaneously defined by transfinite recursion, according to Definition 7.6.

The validity of the Leibniz's Law, in the case of Boolean-set models for **ZFC**, is proven as a theorem. The simultaneous definition of the equality and membership predicates is designed to fit exactly the requirements of the extensionality axiom: two individuals (sets) are identical provided that they have the same elements. From this, it is proven by induction of the complexity of $\varphi(x)$ that $[\![u \approx v]\!] \wedge [\![\varphi(u)]\!] \leq [\![\varphi(v)]\!]$ in every Boolean-valued model. As we have seen in Theorem 8.2(vii), the same holds in twist-valued models w.r.t. the first coordinate, namely: $[\![u \approx v]\!]_1 \wedge [\![\varphi(u)]\!]_1 \leq [\![\varphi(v)]\!]_1$. But then, it is required that this property just holds for 'classical' formulas, that is, formulas φ without occurrences of the paraconsistent negation $\neg$. The explanation for this fact is simple, from the technical point of view: assuming that the property above holds for φ then, when considering $\neg\varphi$, the value of $[\![\neg\varphi(u)]\!]_1$ is $[\![\varphi(u)]\!]_2$, and we don't have enough information about the relationship between $[\![\varphi(u)]\!]_2$, and $[\![\varphi(v)]\!]_2$. The example given in the proof of Theorem 10.4 shows that it is impossible to satisfy the Leibniz's Law in $\mathbf{ZF_{LPT0}}$ for formulas containing the paraconsistent negation, hence this is an unsolvable problem with the current definitions.

As mentioned right before Theorem 8.6, the Leibniz's Law is a sufficient condition to ensure the validity of the bounded quantification properties in Boolean-valued models. These properties are crucial in order to prove the validity of the axioms of **ZF** w.r.t. Boolean-valued semantics. Since the Leibniz's Law is not valid — in general — in $\mathbf{ZF_{LPT0}}$ for formulas containing the paraconsistent negation $\neg$, it should be expected that some instances of schema axioms of **ZF** (such as separation, replacement or regularity) involving formulas containing occurrences of $\neg$ could fail to hold in $\mathbf{ZF}_{\mathbf{LPT0}}$. The same argument applies *mutatis mutandis* to the three-valued model of **ZF** based on $(\mathbb{PS}_3, \neg)$ presented in [18].

The definition in $\mathbf{ZF}_{\mathbf{LPT0}}$ of a suitable 'consistency' predicate $C(x)$ for sets, as discussed in Subsection 10.1, would open the possibility to develop a paraconsistent set theory extending **ZFC** which, in addition could satisfy the Leibniz's Law, bounded quantification and all the schema axioms of **ZFC** allowing formulas with paraconsistent negation. This would be possible by requiring that the sets involved in the axioms be consistent (that is, by assuming $C(u)$ for every set u occurring in the instance of the axiom being applied), as hinted in Remark 10.8 for the Leibniz's Law. This way of 'locally' recovering **ZFC** in the full language by suitable assumptions of $C(x)$ is analogous to the 'local' recovering of classical logic w.r.t. the paraconsistent negation inside **LFI**s by assuming the consistency $\circ\varphi$ of certain sentences φ inside a derivation. This aproach to paraconsistent set theory, along the same lines as the one presented in [4] (in which, for instance, the regularity axiom only applies to consistent sets), deserves future research.

The failure of the Leibniz's Law is not necessarily a predicament. Although it is commonly accepted that any relation of identity must comply with the Leibniz's Law, some special cases of identity are extensively discussed since the question was posed by John Locke in his *Essay Concerning Human Understanding* of 1689 [see 15 for details)]. This question has unfoldings in theoretical computer science and foundations of Artificial Intelligence. Some computer ontology theorists adopt the thesis that it is possible for two individuals to be identical in one circumstance but different in another. This theoretical possibility is relevant for developing higher foundational computer ontologies [see, e.g., 16].

Within the present approach to axiomatic set theory, paraconsistent situations such as the existence of 'inconsistent' sets u satisfying $\neg(u \approx u)$, or the existence of a set being simultaneously an element and a non-element of another set seems to be irreconcilable with the fullfillment of the Leibniz's Law for formulas behind the 'classical' language. Because of this, the predicate $\approx$ in $\mathbf{ZF}_{\mathbf{LPT0}}$ should be considered as representing 'indiscernibility by pure **ZF**-properties', exactly as happens with Boolean-valued models for **ZF**. In this manner $(u \approx v)$ implies that, besides having the same elements, u and v have, for instance, the same 'non^c-elements', where 'non^c' stands for the classical negation $\sim$. That is, $\forall w(\sim(w \in u) \leftrightarrow \sim(w \in v))$ is a consequence of $(u \approx v)$. On the other hand, as it was shown in Theorem 10.4, $(u \approx v)$ *does not imply* (in general) that u and v have the same 'non-elements', where 'non' stands for the paraconsistent negation $\neg$: $\forall w(\neg(w \in u) \leftrightarrow \neg(w \in v))$ *is not* a consequence of $(u \approx v)$.

Instead of being regarded as discouraging, the fact that $(u \approx v)$ does not necessarily imply that u and v have the same 'non-elements' can be seen as an auspicious property, because it can be a way to circumvent undesirable consequences of 'non-elements', as it happens with the well-known Hempel's Ravens Paradox: evidence, differently from proof, for instance, has its own idiosyncratic properties. This point, however, will be left for further discussion.

11. Concluding remarks

In this paper, we introduce a generalization of Boolean-valued models of set theory to a class of algebras represented as twist-structures, defining a class of models for **ZFC** that we called twist-valued models. This class of models is based on a three-valued paraconsistent logic called **LPT**, which was extensively studied in the literature of paraconsistent logics under different names and signatures as, for example, the well-known da Costa and D'Ottaviano's logic **J3** and the logic **LFI1** [cf. 8]. As it was shown by Blok and Pigozzi in [2], the class of algebraic models of **J3** (hence, the class of twist structures for **LPT0**) coincides with the algebraic models of Łukasiewicz three-valued logic $Ł_3$.

With small changes, in Section 9 the twist-valued models for **LPT0** were adapted in order to obtain twist-valued models for $(\mathbb{PS}_3, \neg)$, the three-valued paraconsistent logic studied by Löwe and Tarafder in [18] as a basis for paraconsistent set theory. Thus, their three-valued algebraic model of **ZF** was extended to a class of twist-valued models of **ZF**, each of them defined over a complete Boolean algebra. In addition, it was proved that these models (including the three-valued model over $(\mathbb{PS}_3, \neg)$) satisfy, in addition, the axiom of choice. Moreover, it was shown that the implication operator $\to$ of **LPT0** is, in a sense, more suitable for a paraconsistent set theory than the one $\Rightarrow$ of $\mathbb{PS}_3$: it allows inconsistent sets (i.e., $[\![(w \approx w)]\!] = \mathbf{\frac{1}{2}}$ for some w, see Proposition 10.1). It is worth noting that $\to$ *does not* characterize a 'reasonable implication algebra' (recall Definition 9.2): indeed, $\mathbf{1} \wedge \frac{\mathbf{1}}{\mathbf{2}} \leq \frac{\mathbf{1}}{\mathbf{2}}$ but $\mathbf{1} \not\leq \frac{\mathbf{1}}{\mathbf{2}} \to \frac{\mathbf{1}}{\mathbf{2}} = \frac{\mathbf{1}}{\mathbf{2}}$. This shows that reasonable implication algebras are just one way to define a paraconsistent set theory.

Despite having the same limitative results than Löwe-Tarafder's model (that is, the debatable failure of Leibniz's Law and the bounded quantification property for formulas containing the paraconsistent negation, recall Section 10) we believe that $\mathbf{ZF_{LPT0}}$ has a great potential as a paraconsistent set theory. In particular, the formal properties that a consistency predicate $C(x)$ could have and the axiomatization of $\mathbf{ZF_{LPT0}}$, are topics that deserve to be further investigated, especially towards the problem of the validity of independence results in paraconsistent set theory.

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