LANDINI, GREGORY [1996]: 'The definability of the set of natural numbers in the 1925 *Principia Mathematica*', *Journal of Philosophical Logic* **25**, 597–615.
LOWE, VICTOR [1990]: *Alfred North Whitehead: The Man and His Work*, Vol. 2. Baltimore: The Johns Hopkins University Press.
MYHILL, JOHN [1974]: 'The undefinability of the set of natural numbers in the ramified *Principia*', in George Nakhnikian, ed., *Bertrand Russell's Philosophy*, pp. 19–27. London: Duckworth.
RUSSELL, BERTRAND [2001]: *The Selected Letters of Bertrand Russell*. Vol. 2. Nicholas Griffin, ed. London: Routledge.
URQUHART, ALASDAIR [2011]: 'Henry M. Sheffer and notational relativity', *History and Philosophy of Logic* **33**, 33–47.
WITTGENSTEIN, L. [1995]: Cambridge Letters. Brian McGuinness and G.H. von Wright, eds. Oxford: Blackwell.

doi:10.1093/phimat/nkt011
Advance Access publication April 12, 2013

ALBERT LAUTMAN. ***Mathematics, Ideas and the Physical Real***. Simon B. Duffy, trans. London and New York: Continuum, 2011. 978-1-4411-2344-2 (pbk); 978-1-44114656-4 (hbk); 978-1-44114433-1 (pdf e-bk); 978-1-44114654-0 (epub e-bk). Pp. xlii + 310.

Reviewed by PIERRE CASSOU-NOGUÈS*

The volume is a translation from the French of the latest edition of the work of Albert Lautman. Except for the proceedings of the lecture that Lautman gave together with Cavaillès at the French Philosophical Society in 1938, it comprises all the texts published by Lautman, as well as a biographical sketch from Jacques Lautman (Albert Lautman's son), a substantial introduction by F. Zalamea, the editor of the volume in French, and a short foreword by the mathematician J. Dieudonné to a preceding edition. S. Duffy has added a note explicating his translation of Lautman's key concepts and a bibliography, which mentions Lautman's sources and their translations in English together with some of the secondary literature on his work.

I will first give a few biographical elements concerning Lautman as they explain several aspects of his work. Albert Lautman was born in 1908. His father, a Jewish emigrant from Vienna, had become a medical doctor and fought in the First World War where he was seriously wounded. In high school, young Albert became a close friend of Jacques Herbrand, the future logician. Though his main subject was philosophy, he acquired a solid mathematical education and, after the death of Herbrand,

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was closely linked to Chevalley and Ehresmann, two of the founding members of Bourbaki. On the philosophical side, he studied at the École Normale Superieure in the same class as Sartre and Merleau-Ponty, which may explain his later interest in Heidegger. However, his main philosophical influence came from the circle of Brunschvicg, whose meetings Lautman attended together with Cavaillès and, occasionally, Bachelard. In 1929, he spent a year in Berlin, and acquired perfect German which enabled him to pass for a German when he later escaped from a prison camp in Silesia during the war. In the 1930s, he taught philosophy, while writing the two theses which he published in 1938. Nevertheless, like his friend Cavaillès, he took every kind of military preparation and was a strong supporter of the war, in which he fought as a captain in the artillery. As already mentioned, he was made prisoner, escaped, and when he came back to France was fired from his position as professor in philosophy because he was a Jew. He entered the resistance, was arrested in 1944, and was executed August 1st.

This context should be kept in mind, particularly with regards to the enigmatic but important reference to Heidegger in 1939. Previous to 1937, Lautman had published several papers which mainly discussed existing positions concerning the foundations of mathematics and the status of logic, Russell's and Whitehead's *Principia Mathematica*, Hilbert's program, and logical positivism. However, the bulk of his work is made up of the two theses of 1938. These are not particularly concerned with the problem of foundations, and in them Lautman puts in place his unusual platonism. The theses are complemented by a short note from 1937 summarizing their conclusion, the more important paper of 1939 which aims in particular at reformulating the platonism of the theses through the philosophy of Heidegger, and an article published posthumously in 1946 which extends this platonistic perspective to physics. One must also mention the interesting lecture of 1938 at the French Philosophical Society and the discussion that followed between Cavaillès, Lautman, and several mathematicians such as Fréchet and Cartan. It is reprinted in Cavaillès's collected works (*Œuvres complètes de philosophie des sciences*. Paris: Hermann, 1994) but is not included in Lautman's volume, nor has it been translated into English.

Lautman's work may be studied from two perspectives. First, Lautman is a key figure in the history of philosophy in France. Indeed, like Cavaillès, Lautman makes the transition from the idealism that prevailed before the Second World War under Brunschvicg's influence to the philosophy developed in the 1960s and 1970s. Though his theses are undoubtedly influenced by Brunschvicg, his paper of 1939 represents one of the first discussions of Heidegger in French. Together with Cavaillès, Sartre, Merleau-Ponty, and Levinas, Lautman illustrates a phenomenological turn of philosophy in the French language. Later, Lautman was also an important reference for Deleuze in *Difference and Repetition*, where his platonistic ideas constituted a model for Deleuze's 'virtual'. I will not delve more into these matters here but will concentrate on Lautman's importance for the philosophy of mathematics. From this second perspective, Lautman's work delineates an unusual platonistic position. If Lautman adheres to the existence of ideas independent of the human mind and the mathematical theories that it creates, his platonism has two original characteristics. Here is a rather long quotation from the note of 1937 in which Lautman defines the main aspects of his platonism:

> [It goes with] the reality inherent to mathematics as [with] all reality in which the mind encounters an objectivity that is imposed on it. [. . .] The reality of mathematics is not made in the act of the intellect that creates or understands, but it is in this act that it appears to us and it cannot be fully characterized independently of the mathematics that is its indispensable support. In other words, we think that the proper movement of a mathematical theory lays out the schema of connections that support certain abstract ideas that are dominating with respect to mathematics. (p. 28)

The first point I want to stress here is that it is in the 'proper movement' or, as Lautman writes elsewhere, in the 'living movement' of mathematical theories that ideas express themselves. It is not in the axioms. The reference to an ideal reality is not intended to give a foundation to the theory, nor justify the introduction of new axioms. It is rather a way to describe mathematical practice or the recent history of mathematical theories. Lautman delineates a platonism of mathematical practice. The question that it answers is not: how can such an axiom be justified, or such a mode of reasoning characterized by such schemata? It is rather: what goes on in the development of such a theory? And the answer, for Lautman, is that one sees in the way in which the theory develops the expression of certain ideas that are transcendent to the theory itself and could be found in various forms in various theories. What the mathematician does is in fact to give a new form to conflicting ideas that existed before the birth of the theory and can already be found in other parts of mathematics. The *locus* of Lautman's ideas is not in the axioms of the theory but in its development, its immediate history, or in mathematical practice.

This orientation, by itself, gives to Lautman's platonism an unusual aspect. It certainly is related to Brunschvicg's influence. Brunschvicg once wrote that history was the philosopher's laboratory. It is in the field of history (and in fact history of science) that the philosopher checks her, or his, conceptions. Following Brunschvicg, philosophers such as Bachelard and Cavaillès turn to the history of science. The case of Lautman is slightly more complicated. Lautman studied recent developments in mathematics (around non-commutative algebra, Riemann's zeta function, the work of

Hopf and of Alexandroff in topology, Galois theory, Chevalley's class field theory, singularities in analytical functions). He did not intend to describe the development of a theory in the long term but neither was he really interested in the way in which a theory, or the whole of mathematics, might be formally presented. His aim was rather to analyze recent results and the way mathematicians proceed. Indeed, his center of interest, what he called the 'living movement' of mathematical theories, is the practice of the mathematician or the immediate history of mathematical theories. Another consequence of this orientation is that, contrary to most philosophers of his time, Lautman studied 'real' mathematics, the mathematics that mathematicians do, and logic is only taken into account as one mathematical theory among others.

The second point that I want to stress concerns the nature of this 'objectivity that is imposed upon' the mind and the (at first glance obscure) expression of 'schema of connections' in the above quotation. Indeed, the ideas, which make the objectivity, or the reality, underlying the theory, are not the objects of the theory, nor the concepts that are defined in the theory. The ideas beneath set theory would not be a certain domain of sets nor the concept of set itself. The reality of analysis lies not in the objective existence of real numbers or functions. Lautman's ideas are constituted by a couple of abstract and opposite notions such as continuous and discontinuous, local and global, essence and existence. What Lautman puts in the spotlight is the interplay of these opposite notions in mathematical theories. In these couples, each term calls for the other and needs to be conciliated with the other. And mathematical theories offer various 'schema[ta] of connections', various ways to put these notions in harmony. The Continuous or the Discrete, which is more fundamental? Number theory seems to be concerned with the Discrete but it uses methods from analysis, or even probability theory, to establish theorems concerning prime numbers, so it makes room for the Continuous. Cantor's theory would obviously give another kind of connection between the same two ideas, the continuous and the discrete.

Local or global? Which point of view is the more fruitful and how does one go from one to the other? In differential geometry, one may have to describe the manifold by its local properties or, on the contrary, try and find the local aspects of a manifold from a given general equation. Essence or existence, which matters most? The ontological proof aims at establishing the existence of God from its essence. Mathematicians sometimes proceed in the same direction, starting from certain sets of properties and showing there exists an object satisfying these properties in the domain. However, they may also establish the existence of a certain object satisfying a certain property (for example a well-ordering of the continuum) without being able to tell its other properties. Here the existence seems to precede the essence.

Lautman's terminology varies and he sometimes calls an 'idea' one of the notions of the couple and sometimes the whole couple (usually with a capital I, 'Idea'). Anyhow, his aim is to describe precisely the multiplicity of ways in which such opposite ideas connect, conciliate, interplay in mathematical theories. In this way, he gives a synthetic presentation of the mathematics of his time. He then comes to this unusual platonism where, first, ideas are located not in the axioms but in the practice and, second, consist not in a presumed domain of a theory, nor in a concept to be defined in the theory, but in what Lautman calls 'dialectic opposites'. It is to be noted that, although Lautman insists that it is mathematics only which express these ideas in a definite way (in the quotation above, these ideas 'cannot be fully characterized independently of the mathematics that is [their] indispensable support'), they may also appear in other contexts. Lautman's work has been used for example with reference to music.

A brief comparison with Gödel's platonism may also shed light on the status of Lautman's ideas. In the famous postscript to 'What is Cantor's Continuum Problem', Gödel notes :

> It should be noted that mathematical intuition need not be conceived of as a faculty giving an *immediate* knowledge of the objects concerned. Rather it seems that, as in the case of physical experience, we *form* our ideas also of those objects on the basis of something else, which *is* immediately given. Only this something else here is *not*, or not primarily, the sensations.[1]

Gödel is certainly concerned with foundations and what he intends to give reality to is a domain of sets, or a concept of set. However, Lautman's platonism illustrates the difference that Gödel puts emphasis on between the ideal reality and mathematical theories. These could, in Lautman's platonism, be formed on the basis on an intuition of ideas which are not the objects of our theories. In fact, Gödel had read Lautman and kept in his papers a note with the quotation in French given above.[2]

Now Lautman's main work was published in 1938 when he was 30. He died at 36. There is no doubt that his work was unfinished and, until the end, he struggled with unresolved difficulties. What status should be given to these ideas? What relationship do they have to mathematical theories and to the mathematicians' thinking? It is in that context, as a way to answer such questions, that Lautman referred to Heidegger in 1939. Crudely, Lautman seems to be hesitating between two perspectives. In the first perspective, the mathematical development would be based on an

[1] Kurt Gödel, *Collected Works*, Vol. II, p. 268. S. Feferman, *et al.*, eds. Oxford: Clarendon Press.

[2] Gödel's papers, Box 10a, f42 item 050144.

intuition, as in the above quotation from Gödel. The mathematician would have an intuition of these conflicting ideas; he would see a new way to bring together the opposite notions, or to make room for one opposite in a theory dominated by the other, and from there he would develop a mathematical work properly speaking. In the second perspective, the mathematician would rather aim at solving open problems in the field; he would not look straightforwardly so to speak at the ideal reality, but keep his eyes set on the mathematical field. However, he would be forced by the mathematical reality to use these conflicting ideas that philosophers may then reveal for themselves. These two perspectives, crudely delineated, would give rise to different models of the history of mathematics. It seems that Lautman cannot really decide which one to promote. That accounts for the multiplicity of references (from Plotinus to Heidegger) that Lautman discusses in relation to the status of his ideas.

I have already mentioned the key position of Lautman in the history of philosophy in France. With regards to philosophy of mathematics itself, Lautman's platonism is important for two reasons. First, from a historical point of view, it illustrates (together with Bernays's and, later on, Gödel's) the various platonisms that flourished in the aftermath of Hilbert's program. Second, because it is not primarily concerned with foundations but with the 'living movement' of mathematical theories, it seems particularly akin to the contemporary turn of the philosophy of mathematics towards questions of practice.

doi:10.1093/phimat/nkt019
Advance Access publication May 30, 2013

Gordon Belot. ***Geometric Possibility***. Oxford and New York: Oxford University Press, 2011. ISBN 978-0-19-959532-7. Pp. x + 219.

Reviewed by Chris Smeenk*†

How are we to understand the truth conditions for claims about space-time geometry, *e.g.*, that a cyclist's front tire is trailing the rear tire of another cyclist by 10 cm, or that both cyclists are accelerating as they go downhill? A substantivalist regards the truth or falsity of such claims as underwritten by geometrical relations among the regions of space-time occupied by the

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† I am grateful to Erik Curiel for discussions that led to a co-authored contribution to a book symposium and for comments on an earlier version and to Gordon Belot for his willingness to participate in the book symposium, his careful replies to various questions and objections, and comments on an earlier version of this review.