teorema
Vol. XXXI/1, 2012, pp. 147-150
[BIBLID 0210-1602 (2012) 31:1; pp. 147-150]

REVISTA DE LIBROS/BOOK REVIEWS

Defending the Axioms – On the Philosophical Foundations of Set Theory, by PENELOPE MADDY, OXFORD, OXFORD UNIVERSITY PRESS, 2011, pp. 160.

Defending the Axioms (*DA*) is a sort of part three in a trilogy. *Naturalism in Mathematics* (1997) and *Second Philosophy* (2007) were the two previous books of the series. All books are related to each other and they practice a variety of post-Quinean naturalism. This variety is an austere form of naturalism. The philosopher is the busy sailor of Neurath's boat but, unlike Quine's busy sailor, he is born native to the boat. *Naturalism in Mathematics* applied this variety of naturalism to mathematics. *Second Philosophy* introduced a character in the series – called the Second Philosopher – and described his thoughts and practices concerning science, logic and mathematics. Now, this idealized inquirer proceeds with his investigation focusing on pure mathematics. What are the proper methods of pure mathematics? How could we defend the set-theoretic axioms? What set theory is about? These are the main questions of *DA*.

Chapter 1, "The Problem", surfs some historical aspects of the relation between mathematics and the other sciences in order to see how we came to the present state of mathematics. The point is to show how mathematics became pure around the 19th century with the arising of mathematical concepts with no direct physical meaning. Three examples are supplied: the theory of groups of Galois; the non-Euclidean geometries of Gauss; the radical change in the view how applied mathematics is related to the physical world, at the end of the 19th century. Maddy details this third strand and shows how Galileo's *credo*, according to which the book of nature is written in mathematical language, starts being questioned in the 19th century. A long story, from Newton until the atomic theory of 20th century, is related. This third strand establishes an astonishing conclusion: "it now appears that even applied mathematics is pure" [p. 27].

What is "The problem" of chapter 1? Set theory, ZFC, is considered as *the* foundation of pure mathematics. However, ZFC is not sufficiently powerful. There are some questions – known as "independent questions" – that cannot be answered in light of ZFC, such as the continuum hypothesis, the measurability of Lebesgue and the Whitehead's free groups. A solution to these questions is to add new axioms to set theory. However, since science is no longer a methodological guide for pure mathematics, how to choose the axioms? Here is the (double) problem for the Second Philosopher. The methodological problem: what are the proper methods of set theory? The "philosophical" problem: why are these proper methods of set theory? The following chapters of *DA* try to answer these problems.

Chapter 2, "Proper Method", is concerned with the methodological problem. Maddy analyses four cases of the set theory practice: "Cantor's introduction of sets", "Dedekind's introduction of sets", "Zermelo's defense of his axiomatization" and "the case for determinacy". Based on these cases, it is concluded that the mathematical methods are rational and autonomous of the empirical methods. For example, sets were posited with mathematical goals in mind: to solve local mathematical problems; to give mathematical foundations; to promise rich and deep mathematical extensions. This chapter ends with the famous Benacerraf's challenge for Robust Realism: how human beings can attain abstract mathematical knowledge? Maddy rejects Robust Realism, since it implies a supplementation. It postulates an objective abstract mathematical reality that is behind sound mathematical reasoning. With this rejection, Benacerraf's challenge is bypassed.

The next two chapters are about two conceptions: "Thin Realism" (chapter 3) and "Arealism" (chapter 4). Thin Realism defends that sets exist and they are abstract entities described by a true theory – set theory. Arealism defends that there is no reason to suppose that sets exist and set theory is a true theory. After introducing these positions they are contrasted with similar positions, in order to clarify what these positions are and are not. Thin Realism is contrasted with Robust Realism; Arealism is contrasted with nominalism (the belief that abstract objects do not exist), fictionalism (mathematical objects are a fiction of our mathematical theories) and formalism (mathematics is like a game under the rule if-thenism). Finally, Thin Realism is compared with Arealism. The main goal of these two chapters is to show that both positions – Thin Realism and Arealism – are accurate descriptions of the pure mathematics practice.

Let me highlight and criticize some aspects of these chapters. Why thin realism is not Robust Realism? Robust Realism appeals to an external epistemology that certifies the reliability of set-theoretic methods, i. e., mathematical propositions are true or false in virtue of a world of *abstracta*; Thin Realism denies that external epistemology, since the reliability of the set-theoretic methods is a plain fact. For example, for the robust realist the continuum hypothesis, CH, has a determinate truth value that depends of an objective abstract reality. Since our actual axioms do not give a complete description of that reality, we do not know whether CH is true or false. The Thin Realist takes *CH or not-CH* as a theorem about the universe of sets.

However, it seems to me that there is here a tension on Maddy's thought. On the one hand, Thin Realism must only describe the pure mathematics practice, but, on the other hand, *CH or not-CH* is not assumed as a theorem by pure mathematicians. CH is a hypothesis, *simpliciter*. The same goes for other undecidable propositions, such as Goldbach's conjecture.

Arealists and Thin Realists have the same position about the role of mathematics in science. Both consider that mathematics is applied in science because it is developed by purely mathematical goals. The truth (or not) of mathematics is irrelevant for the account of how mathematics works in applications. Why set-theoretic methods are reliable? Why these methods track the existence of sets? Why set theory is a body of truths? The Thin Realist considers that the interconnections between mathematics and empirical science give a good reason for the reliability of set-theoretic methods: pure mathematics arose from applied mathematics; one of the aims of pure mathematical practice is to give tools for empirical science; and pure mathematical theories continue to find scientific applications. On the contrary, the Arealist considers that set theory is not a body of truths, simply because there is no evidence to support the existence of sets. Consequently, set-theoretic methods are unreliable: they do not track anything. Thus, the Arealist considers that set-theoretic methods are rational, autonomous and unreliable, but only with a hard philosophical stomach we can swallow that a method could be rational and thoroughly unreliable at the same time.

How the Thin Realist deals with the sceptical challenges? Could the set-theoretic methods were completely wrong? After all, could sets to be an illusion of the evil demon and sets do not exist at all? According to Maddy, the Thin Realist adopts a "thin epistemology" – sets are known by set-theoretic methods – that avoids the sceptical challenges. Unlike the empirical knowledge, in pure mathematics there is no "great gulf" between sets and set-theoretic methods for sceptical attacks, such as perceptual beliefs. However, it seems that there is "some gulf" in *DA* for sceptical attacks. According to *DA*, the interconnections between the mathematical and the empirical sciences are the evidence for the reliability of set-theoretic methods. Thus, some of the sceptical challenges for the empirical sciences can be raised to set-theoretic methods, too.

According to ZFC, there is no set of all sets. Of course, there is a class that contains all sets – a proper class – that it is not a set. But ZFC is a theory about sets and it is not a theory about classes. One way to deal with this problem is to try intuitionist flights and to defend that the concept *set* is indefinitely extensible. However, *prima facie*, Thin Realism or Arealism could not accommodate intuitionist thoughts about mathematics. Intuitionism supposes a sort of extra abstract reality that it is constructed by humans, but Thin Realism and Arealism simply deny any extra mathematical reality for mathematical evidence.

If Thin Realism and Arealism are both consistent with the practice of pure mathematics, what morals can we draw from this? The last chapter of *DA*, “Morals”, tries to answer this question. First moral: questions of ontology and truth are red herrings in the mathematical objectivity dispute; mathematical fruitfulness is *the* constraint that underpins the objectivity of mathematics. Second moral: the Fregean version of Robust Realism resembles to Thin Realism and Arealism. Third (heretical) moral: mathematical intrinsic justifications (self-evident, intuitiveness and obviousness) are secondary to the extrinsic justifications (effectiveness, fruitfulness and deepness). This last moral is supported by additional examples from set-theoretic practice.

Penelope Maddy is the philosopher who has better understood contemporary pure mathematics practice. Philosophers of mathematics, as well as set theorists, should read this book. *

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* ACKNOWLEDGEMENT: this work was supported by grant SFRH/BPD/46847/2008 and the project PTDC/FIL-FCI/109991/2009, “Hilbert's Legacy in the Philosophy of Mathematics”, from the Fundação para a Ciência e Tecnologia.

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