

## Subjective probability and quantum certainty

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### Abstract

In the Bayesian approach to quantum mechanics, probabilities—and thus quantum states—represent an agent's degrees of belief, rather than corresponding to objective properties of physical systems. In this paper we investigate the concept of *certainty* in quantum mechanics. Particularly, we show how the probability-1 predictions derived from pure quantum states highlight a fundamental difference between our Bayesian approach, on the one hand, and Copenhagen and similar interpretations on the other. We first review the main arguments for the general claim that probabilities always represent degrees of belief. We then argue that a quantum state prepared by some physical device always depends on an agent's prior beliefs, implying that the probability-1 predictions derived from that state also depend on the agent's prior beliefs. Quantum certainty is therefore always some agent's certainty. Conversely, if facts about an experimental setup could imply agent-independent certainty for a measurement outcome, as in many Copenhagen-like interpretations, that outcome would effectively correspond to a preexisting system property. The idea that measurement outcomes occurring with certainty correspond to preexisting system properties is, however, in conflict with locality. We emphasize this by giving a version of an argument of Stairs [(1983). Quantum logic, realism, and value-definiteness. *Philosophy of Science*, 50, 578], which applies the Kochen–Specker theorem to an entangled bipartite system.

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## 1. Introduction

At the heart of Bayesian probability theory (Bernardo & Smith, 1994; de Finetti, 1990) is a strict category distinction between propositions and probabilities. Propositions are either true or false; the truth value of a proposition is a fact. A probability is an agent's degree of belief about the truth of some proposition. A probability assignment is neither true nor false; probability assignments are not propositions.

This category distinction carries over to probabilities for events or, more particularly, for the outcomes of observations or measurements. This particular context, of measurements and their outcomes, is the most relevant for this paper. The proposition corresponding to an outcome of an observation is the statement that the outcome occurs. Ascertaining or eliciting<sup>1</sup> the outcome determines the truth value of the proposition for the agent. The outcome is thus a fact for the agent. He can use the fact to modify his probabilities, but the probabilities themselves are not facts.

Any actual usage of probability theory starts from an agent's prior probability assignment. Gathering data allows the agent to update his probability assignments by using Bayes's rule. The updated probabilities always depend on the agent's prior probabilities as well as on the data and thus can be different for agents in possession of the same data. It is in this sense that probability assignments can be called *subjective*, meaning they depend upon the agent. For lack of a better term, we adopt this usage in this paper.

Subjective probabilities are not arbitrary. They acquire an operational meaning in decision theory (Savage, 1972). The Dutch-book argument (de Finetti, 1990) shows that, to avoid sure loss, an agent's gambling commitments should obey the usual probability axioms. By maintaining a strict category distinction between facts and probabilities, Bayesian arguments make an explicit distinction between the objective and subjective parts of any application of probability theory. The *subjective* part of a statistical argument is the initial judgment that leads to prior probability assignments. The *objective* part is any given data and the application of the rules of probability theory to the (subjective) prior probabilities. This part is objective because neither the data nor the rules of probability theory depend upon the agent's beliefs.

Bayesian theory is conceptually straightforward. It provides simple and compelling accounts of the analysis of repeated trials in science (Savage, 1972), statistical mechanics and thermodynamics (Jaynes, 1957a, 1957b), and general statistical practice (Bernardo & Smith, 1994). Still, it might seem to have a limited purview. For instance, a subjectivist interpretation of probability is natural in a deterministic world, where the outcome of any observation can be predicted with certainty given sufficient initial information. Probabilities then simply reflect an agent's ignorance. But what of an indeterministic world?

In quantum mechanics the usual perception is that not all probabilities can be interpreted as subjective degrees of belief (Giere, 1973, 1979; Ismael, 2006; Loewer, 2001, 2004; Suppes, 1973). Particularly, the probabilities of the outcomes of a quantum measurement on a system in a pure quantum state are given by physical law and are therefore objective—i.e., not depending on any agent's belief. Or so goes the usual

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<sup>1</sup>We introduce “eliciting” here as an alternative to “ascertaining” because “ascertaining” has the connotation of determining a preexisting property. This, however, is in conflict with the central point of our paper in the quantum-mechanical case. “Eliciting” at least suggests subtly that the outcome might have no prior existence.

argument. We have shown in a series of previous publications (Brun, Caves, & Schack, 2001; Caves, Fuchs, & Schack, 2002a, 2002b, 2002c; Caves & Schack, 2005; Fuchs, 2002; Fuchs, Schack, & Scudo, 2004; Schack, 2003) that, despite this common perception, all probabilities in quantum mechanics can be interpreted as Bayesian degrees of belief and that the Bayesian approach leads to a simple and consistent picture, which resolves several of the conceptual difficulties of the interpretation of quantum mechanics. A consequence of the Bayesian approach is that all quantum states, even pure states, must be regarded as subjective.

This is not to say, however, that everything in the quantum formalism is subjective. For instance, the Born rule for calculating probabilities from quantum states is not subjective. Instead it is akin to the rules of probability theory itself, which, as pointed out above, make no reference to an agent's particular beliefs. We will return to this point and make more of it in the Conclusion.

In this paper, our main aim is to address more carefully than previously the problem of *certainty* in the Bayesian approach to quantum mechanics. We show that a consistent treatment of quantum probabilities requires that even if a measurement outcome has probability 1, implying certainty about the outcome,<sup>2</sup> that probability has to be interpreted as a Bayesian degree of belief. This is the case if the premeasurement state is an eigenstate of the measured observable. Even in this case we maintain, as we must if we are to hold that pure states are subjective,<sup>3</sup> that the measurement has no preassigned outcome. There is no element of reality that guarantees the particular measurement outcome.<sup>4</sup> Certainty is a function of the agent, not of the system.<sup>5</sup>

Along with this, we give a precise account of a fundamental difference between our Bayesian approach, on the one hand, and various Copenhagen-like interpretations of

<sup>2</sup>In this paper, we make no distinction between probability 1 and certainty. This is unproblematic because we only consider measurements with a finite or countably infinite number of outcomes.

<sup>3</sup>In a previous publication (Caves et al., 2002c), the authors were confused about the status of certainty and pure-state assignments in quantum mechanics and thus made statements about state preparation that we would now regard as misleading or even wrong. For much discussion surrounding this point and sidelights on the material in the present paper, see C. A. Fuchs, *Quantum States: What the Hell Are They?* (particularly, the essays on pp. 35–113), available at <http://netlib.bell-labs.com/who/cafuchs/PhaseTransition.pdf>.

<sup>4</sup>This distinction of ours can also be reworked in terms drawn from the philosophy of language, as has been done by Timpson (2006). It contrasts with what is sometimes called the “eigenstate–eigenvalue link,” which is almost universally adopted (in the strong form quoted below) in the quantum foundations literature, even by treatments otherwise sympathetic to a Bayesian view of quantum states. For instance, Brukner & Zeilinger (2001) give this description of the case of certainty:

Only in the exceptional case of the qubit in an eigenstate of the measurement apparatus the bit value observed reveals a property already carried by the qubit. Yet in general the value obtained by the measurement has an element of irreducible randomness and therefore cannot be assumed to reveal the bit value or even a hidden property of the system existing before the measurement is performed.

<sup>5</sup>More philosophical, quantum-independent precedents for this notion of ‘certainty’ can be found in the discussion given by White (1972). White makes a distinction between the certainty of agents and the certainty of things in themselves, and puts it this way:

The certainty of persons and the certainty of things are logically independent of each other. Somebody can be (or feel) certain of something which is not itself certain, while something can be certain without anybody's being (or feeling) certain of it. ‘He is certain that *p*’ neither implies nor is implied by the impersonal ‘It is certain that *p*’. The same thing cannot be both certain and not certain, though one person can be certain of it and another of its opposite ... People can become more or less certain of something which itself has not become any more or less certain. ... A gambler need not feel that *it* [our italics, for White is speaking of the thing in this case] is certain that red will turn up next in order to feel certain that it will.

quantum mechanics on the other.<sup>6</sup> There are, of course, many versions of the Copenhagen interpretation; here we focus on realist readings of it (Cohen & Stachel, 1979; Faye & Folse, 1994; Jammer, 1974; Murdoch, 1987)—as opposed to anti-realist readings (Faye, 1991; Plotnitsky, 1994) and other more subtle interpretations (Folse, 1985)—which are the most predominant in the physics community. To our knowledge, these all have in common that a system's quantum state is determined by a sufficiently detailed, agent-independent *classical description* of the preparation device, which is itself thought of as an agent-independent physical system (Peres, 1984; Stapp, 1972). This is the only salient feature of the interpretations we consider here. Although this feature is often associated with Copenhagen-like interpretations,<sup>7</sup> we prefer to use the neutral term “objective-preparations view” to refer to it throughout the remainder of this paper. The most important consequence of this feature is that, with it, quantum states must be regarded as objective.<sup>8</sup> Measurement outcomes that have probability 1 bring this difference into stark relief. In the objective-preparations view, a probability-1 outcome is *objectively certain*, guaranteed by facts about the preparation device, and thus corresponds to a preexisting property of the agent's external world.

The paper is organized as follows. In Section 2, we discuss the strict category distinction between facts and probabilities within the setting of applications of probability theory to classical systems. We argue, following de Finetti (1931), that in the last analysis probability

<sup>6</sup>There exist many alternative attempts to solve the conceptual problems of quantum theory. Most notable are the many-worlds (or Everett) interpretation, hidden-variable theories (e.g., Bohmian mechanics), and spontaneous-collapse theories (e.g., the Ghirardi–Rimini–Weber model). All of these are the subjects of ongoing debates. In this paper, we do not comment on any of these alternatives.

<sup>7</sup>We offer the following quotes as a small justification for associating the objective-preparations view with Copenhagen-like interpretations and, specifically, with Bohr. The first is drawn from a letter from Niels Bohr to Wolfgang Pauli, 2 March 1955 (provided to us by H. J. Folse):

In all unambiguous account it is indeed a primary demand that the separation between the observing subject and the objective content of communication is clearly defined and agreed upon. . . . Of course, one might say that the trend of modern physics is the attention to the observational problem and that just in this respect a way is bridged between physics and other fields of human knowledge and interest. But it appears that what we have really learned in physics is how to eliminate subjective elements in the account of experience, and it is rather this recognition which in turn offers guidance as regards objective description in other fields of science. To my mind, this situation is well described by the phrase ‘detached observer’. . . . Just as Einstein himself has shown how in relativity theory ‘the ideal of the detached observer’ can be retained by emphasizing that coincidences of events are common to all observers, we have in quantum physics attained the same goal by recognizing that we are always speaking of well defined observations obtained under specified experimental conditions. These conditions can be communicated to everyone who also can convince himself of the factual character of the observations by looking on the permanent marks on the photographic plates. In this respect, it makes no difference that in quantum physics the relationship between the experimental conditions and the observations are of a more general type than in classical physics.

The second comes from Bohr (1939):

In the system to which the quantum mechanical formalism is applied, it is of course possible to include any intermediate auxiliary agency employed in the measuring process [but] some ultimate measuring instruments must always be described entirely on classical lines, and consequently kept outside the system subject to quantum mechanical treatment.

<sup>8</sup>Even more extreme is this reading of Bohr, from Murdoch (1987, p. 107):

Bohr, then, held what I shall call the *objective-values theory* of measurement, according to which successful observation or measurement reveals the objective, pre-existing value of an observable. . . . It is important here to note that the objective-values theory should not be confused with what I shall call the *intrinsic-values theory* of properties . . . according to which all the observables of an object have, at any moment, definite values. The latter theory, which Bohr rejected, is logically independent of the former.

assignments are always subjective in the sense defined earlier. We briefly consider the concept of objective probability, or *objective chance*, and review the main argument showing that this concept is problematic within the classical setting.

Section 3 addresses the role of prior belief in quantum state preparation. In our interpretation, the quantum state of a system cannot be determined by facts alone. In addition to the facts an agent acquires about the preparation procedure, his quantum state assignment inevitably depends on his prior beliefs. This constitutes the central difference between the Bayesian approach to quantum mechanics and the objective-preparations view. We argue that, by positing that states are fully determined by facts alone, the objective-preparations view neglects to take into account that quantum mechanics applies to preparation devices, even when derived from the “ultimate measuring instruments” of Bohr (1939).

In Section 4 we turn to the question of certainty in quantum mechanics and emphasize that in the objective-preparations view, facts about an experimental setup imply objective, agent-independent certainty for the outcomes of appropriate measurements. The objective-preparations view thus implies a preexisting property of the agent’s external world guaranteeing the measurement outcome in question—a point we find untenable. In Section 5, we support the view that measurement outcomes occurring with certainty cannot correspond to preexisting properties by showing it to be in conflict with locality. For this purpose, we use a modification of an argument of Stairs (1983)—independent variations of the argument can be found in Heywood and Redhead (1983) and Brown and Svetlichny (1990)—which applies the Kochen–Specker noncolorability theorem to an entangled bipartite system.

Finally, in Section 6 we give a short general discussion on the meaning of certainty in a world without preexisting instruction sets for quantum measurement outcomes (Mermin, 1985). We emphasize that certainty is always an agent’s certainty; there is nothing in the physical world itself that makes a quantum-mechanical probability-1 prediction true before the act of finding a measurement outcome. We conclude with brief discussions of the status of the Born rule and directions for further work on the Bayesian approach.

## 2. Subjective probability versus objective chance

The starting point for our considerations is a category distinction. Probability theory has two main ingredients. Firstly, there are *events*, or *propositions*. Mathematically, the space of events forms a sigma algebra. Conceptually, what is important about events is that, at least in principle, an agent can unambiguously determine whether an event has occurred or not. Expressed in terms of propositions, the criterion is that the agent can determine unambiguously in what circumstances he would call a proposition true and in what others false (de Finetti, 1990). The occurrence or nonoccurrence of an event is a *fact* for the agent. Similarly, the truth or falsehood of a proposition is a fact. We shall say that facts are *objective*, because they are not functions of the agent’s beliefs.

Secondly, there are *probabilities*. Mathematically, probabilities are measures on the space of events. Probabilities are fundamentally different from propositions. Theorems of probability theory take an event space and a probability measure as their starting point. Any usage of probability theory starts from a *prior probability assignment*. The question of whether a prior probability assignment is true or false cannot be answered.

The subjectivist Bayesian approach to probability theory (Bernardo & Smith, 1994; de Finetti, 1990; Jeffrey, 2004; Kyburg & Smokler, 1980; Savage, 1972) takes this category distinction as its foundation. Probabilities are degrees of belief, not facts. Probabilities cannot be derived from facts alone. Two agents who agree on the facts can legitimately assign different prior probabilities. In this sense, probabilities are not objective, but *subjective*.

Even though we hold that *all* probabilities, classical and quantum, are Bayesian, the reader is encouraged to view this section as predominantly a discussion of applications of probability theory to classical systems and the subsequent sections as having to do mainly with quantum probabilities. In both settings, when we refer to facts, we are usually thinking about data, in the form of outcomes or results, gathered from observations or measurements.

It is often said that one can verify a probability, thereby making it a fact, by performing repeated trials on independent, identically distributed systems. What is missed in this statement is that the repeated trials involve a bigger event space, the space of all potential sequences of outcomes. To apply probability theory to repeated trials requires assigning probabilities to these potential sequences, and additional subjective judgments are required to do this. (For two very nice expositions of this point, see Appleby, 2005a, 2005b.) Moreover, the outcome frequencies observed in repeated trials are not probabilities. They are facts. Like any fact, an observed frequency can be used, through Bayes's rule, to update subjective probabilities, in this case the subjective probabilities for subsequent trials.

Consider tossing a coin. The probabilities for Heads and Tails cannot be derived from physical properties of the coin or its environment. To say, for example, that the coin is "fair" is ultimately a subjective judgment, equivalent to assigning a symmetric prior probability to the possible outcomes. A symmetry argument applied to facts of the mass distribution of the coin does not determine probabilities, because the outcome of a toss also depends on the initial conditions. These, too, are facts, but a symmetry argument applied to the initial conditions must be phrased in terms of probabilities for the initial conditions, i.e., in terms of judgments, not facts. In coming to the judgment that a coin is a fair coin, an agent is well advised, of course, to take into account all known facts about the physical constitution of the coin, perhaps even developing a detailed model of the mass distribution of the coin and the coin-tossing mechanism. In that case, however, the agent must still make probabilistic judgments at an earlier stage of the model, say, regarding the initial conditions for the tossing mechanism or the state of the surrounding gas. Probability assignments are not arbitrary, but they always have an irreducibly subjective component.

To summarize, even probabilities that follow from symmetry arguments are subjective, because the symmetry argument is applied to the probabilities, not to facts. The assumed symmetry is an agent's judgment about the events in question, and the resulting probabilities are the expression of that judgment. This paper takes one important further step: our central claim is that even probabilities that appear to be given by physical law, as in quantum theory, are subjective.

The Bayesian approach is immediately applicable to physics experiments because it accounts effortlessly for repeated trials (Caves et al., 2002b). For an excellent general discussion of the use of subjective probability in science, see Savage (1972, chapter 4). An example of an area where subjective probabilities have had notable success is classical statistical mechanics (Jaynes, 1957a, 1957b), where the Bayesian approach draws a strict category distinction between a system's microstate, which is a fact, and the subjective

probabilities assigned to microstates, which are a reflection of an agent's ignorance of the microstate. Even here, however, the success is often belittled because of a failure to appreciate the category distinction (North, 2003; Shalizi & Moore, 2003). Suppose an agent assigns an epistemic uniform probability distribution to an ice cube's microstates. The ice cube melts. Does the ice cube melt, it is asked, because of the agent's probability assignment (North, 2003)? How can an agent's epistemic state have anything whatsoever to do with the ice's melting? The answer to these questions is simple: an agent's epistemic state is part of the reason for his prediction that the ice cube melts, not part of the reason for the melting. The ice will either melt or not melt; it is indifferent to the agent's ignorance. "This ice cube will melt" is a proposition whose truth value is a fact about the world. "Ice melts" is an abbreviated version of the subjective judgment that the probability is close to 1 that a typical ice cube will melt. If an agent were able to determine the ice cube's initial microstate, this would have no effect on whether the ice melts, but it would mean that the agent could extract more energy from the process than somebody else who is ignorant of the microstate.

Despite the successes of the Bayesian approach to probability in physics, there appears to be a strong desire, among a sizeable number of physicists, for an objective probability concept. We now review the main argument against the validity of such a concept within the classical setting.

In physics, probabilities appear side by side with physical parameters such as length and are used in a superficially similar way. Both length and probability appear in mathematical expressions used for predicting measurement outcomes. This has led to attempts, for instance by Braithwaite (1968), to treat probability statements as physical parameters residing in the category of facts. A probability statement such as "the probability of this atom decaying in the next 5 min is  $p = 0.3$ " would thus be a proposition, analogous to, e.g., a geometrical statement such as "the length of this ruler is  $l = 0.3$  m". This geometrical analogy (Feller, 1968) is inherently flawed, however (de Finetti, 1931). Whereas the truth of a statement concerning the length of a ruler can be unambiguously decided (at least in the approximate form  $0.29 \text{ m} \leq l \leq 0.31 \text{ m}$ ), the truth of statements concerning probabilities cannot be decided, not even approximately, and not even in principle. The usual method of "verifying" probabilities, through the outcomes of repeated trials, yields outcome frequencies, which belong to the category of events and propositions and are not probabilities. Probability theory allows one to assign a probability (e.g.,  $p = 0.99$ ) to the proposition "the outcome frequency is in the interval  $0.29 \leq f \leq 0.31$ ", and the truth of this proposition can be unambiguously decided, but this is a proposition about an outcome frequency, not about probability.

In order to bridge the category distinction between probability and physical parameters, a new principle or axiom is needed. In Braithwaite's theory, for instance, the new principle is introduced in the form of the acceptance and rejection rules of orthodox statistics (Lehmann, 1986). These rules must be postulated, however, because they cannot be systematically derived from probability theory (Braithwaite, 1968). Even though most of non-Bayesian statistics is based on these rules, they are essentially *ad hoc*, and the way they are used in statistics is highly problematical (Berger & Sellke, 1987; Jeffreys, 1961).

Instead of discussing the conceptual difficulties of orthodox statistics, we focus on a simpler and more direct postulate designed to bridge the category distinction between probability and physical parameters, namely, Lewis's principal principle (Lewis, 1986a, 1986b). The principal principle (PP) distinguishes between *chance* and Bayesian

probability. Chance is supposed to be objective. The numerical value of chance is a fact. Chance therefore belongs to the same category as a physical parameter. If  $E$  is an event, and  $0 \leq q \leq 1$ , the statement “the chance of  $E$  is  $q$ ” is a proposition. Denote this proposition by  $C$ . The PP links chance and probability by requiring that an agent’s conditional probability of  $E$ , given  $C$ , must be  $q$ , irrespective of any observed data. More precisely, if  $D$  refers to some other compatible event, e.g., frequency data, then the Principal Principle states that the Bayesian probability must satisfy

$$\Pr(E|C\&D) = q. \quad (1)$$

Within the context of experimental situations with large sample sizes, where Bayesian updating leads to similar posteriors for exchangeable priors, the geometric analogy, combined with the PP to connect chance with probability, would appear to work quite well. This gives rise to the idea that the PP accounts for the concept of objective chance in physics. However, from a Bayesian perspective, the introduction of chance is completely unmotivated (de Finetti, 1931; Jeffrey, 1997, 2004). More urgently, in those cases where the idea is not already fraught with obvious difficulties, it serves no role that Bayesian probability itself cannot handle (Jeffrey, 2004).

To illustrate one such difficulty, return to the coin-tossing example discussed above, and assume that there is an objective chance  $q$  that a coin-tossing event will produce Heads. As we have seen in the discussion above, the chance cannot be deduced from physical properties of the coin alone, because the probability of Heads also depends on initial conditions and perhaps other factors. An advocate of objective chance is forced to say that the chance is a property of the entire “chance situation,” including the initial conditions and any other relevant factors. Yet a sufficiently precise specification of these factors would determine the outcome, leaving no chance at all. The circumstances of successive tosses must be different to give rise to chance, but if chance aspires to objectivity, the circumstances must also be the same. Different, but the same—there is no way out of this conundrum as long as objective and chance are forced to co-exist in a single phrase. Subjective probabilities easily dispense with this conundrum by maintaining the category distinction. The differences between successive trials are differences in the objective facts of the initial conditions; the sameness is an agent’s judgment that he cannot discern the differences in initial conditions and thus assigns the same probability to every trial.

But what of probabilities in quantum mechanics? Given the last paragraph, one might well think—and many have thought—there is something different going on in the quantum case. For, in repeating a preparation of a pure state  $|\psi\rangle$ , are not all the conditions of preparation the same by definition? Any subsequent probabilities for measurement outcomes will then be determined by applying the Born rule to  $|\psi\rangle$ . They are not subjective probabilities that come about by an inability to take all circumstances into account. Thus quantum states (and hence quantum “chances”) are objective after all, and the PP is just the kind of thing needed to connect these quantum chances to an agent’s subjective probabilities—or so a very beguiling account might run.

The objective-preparations view supports the seeming need for a PP-style account by positing that classical facts about a preparation device determine the prepared quantum state and its associated measurement probabilities. The subjective Bayesian interpretation of quantum probabilities contends, in contrast, that facts alone never determine a quantum state. What the objective-preparations view leaves out is the *essential* quantum nature of the preparation device, which means that the prepared quantum state always depends on



prior beliefs in the guise of a quantum operation that describes the preparation device. We turn now to a discussion of these issues in the next two sections.

### 3. Prior beliefs in quantum state preparation

In the subjectivist interpretation of quantum-mechanical probabilities advocated in this paper, the strict category distinction between (objective) facts and (subjective) probabilities holds for all probabilities, including probabilities for the outcomes (facts) of quantum measurements. Since probabilities are an agent's subjective degrees of belief about the possible outcomes of a trial and quantum states are catalogues of probabilities for measurement outcomes, it follows that quantum states summarize an agent's degrees of belief about the potential outcomes of quantum measurements. This approach underlines the central role of the agent, or observer, in the very formulation of quantum mechanics. In this sense our interpretation is close to Copenhagen-like interpretations, even when these interpretations incorporate the objective-preparations view, but the Bayesian approach differs markedly from the objective-preparations view in the way facts and quantum states are related.

In the objective-preparations view, the facts about a classical preparation procedure determine the quantum state (Peres, 1984; Stapp, 1972). According to the objective-preparations view, one can give, in unambiguous terms, a description of an experimental device that prepares a given quantum state; thus a quantum state is completely determined by the preparation procedure. In the objective-preparations view, there is no room for prior beliefs in quantum state preparation; quantum states and the probabilities derived from them are determined by objective facts about the preparation device.

In our interpretation, the quantum state of a system is not determined by classical facts alone. In addition to the facts, an agent's quantum state assignment depends on his prior beliefs. We now show why this must be so.

Classically, Bayes's rule,

$$\Pr(h|d) = \frac{\Pr(d|h)\Pr(h)}{\Pr(d)}, \quad (2)$$

is used to update probabilities for hypotheses  $h$  after acquiring facts in the form of data  $d$ . The posterior probability,  $\Pr(h|d)$ , depends on the observed data  $d$  and on prior beliefs through the prior probabilities  $\Pr(h)$  and the conditional probabilities  $\Pr(d|h)$ .<sup>9</sup>

In quantum mechanics, the most general updating rule has the form

$$\rho \mapsto \rho_d = \frac{\mathbf{A}_d(\rho)}{p_d}. \quad (3)$$

Here  $d$  is an observed measurement outcome (a fact);  $\rho_d$  is the post-measurement (posterior) state;  $\rho$  is the premeasurement (prior) state; and  $\mathbf{A}_d$  is a completely positive

<sup>9</sup>Bayesian updating is consistent, as it should be, with logical deduction of facts from other facts, as when the observed data  $d$  logically imply a particular hypothesis  $h_0$ , i.e., when  $\Pr(d|h) = 0$  for  $h \neq h_0$ , thus making  $\Pr(h_0|d) = 1$ . Since the authors disagree on the implications of this consistency, it is fortunate that it is irrelevant to the point of this paper. That point concerns the status of quantum measurement outcomes and their probabilities, and quantum measurement outcomes are not related by logical implication. Thus we do not discuss further this consistency, or its implications or lack thereof.

linear map, called a *quantum operation*, corresponding to outcome  $d$  and given by

$$\mathbf{A}_d(\rho) = \sum_j A_{dj} \rho A_{dj}^\dagger. \tag{4}$$

The linear operators  $A_{dj}$  define POVM elements

$$E_d = \sum_j A_{dj}^\dagger A_{dj}, \tag{5}$$

which obey the normalization condition

$$\sum_d E_d = 1, \tag{6}$$

and

$$p_d(\rho) = \text{tr } \rho E_d = \text{tr } \sum_j A_{dj} \rho A_{dj}^\dagger \tag{7}$$

is the probability for outcome  $d$ . Similar to the classical case, the posterior state depends on the measurement outcome, the prior state, and the completely positive map  $\mathbf{A}_d$ , which is analogous to the conditional probabilities of Bayesian updating (Fuchs, 2002; Leifer, 2006a, 2006b).

We now argue that the posterior state always depends on prior beliefs, even in the case of *quantum state preparation*, which is the special case of quantum updating in which the posterior state is independent of the prior state. To be precise, in the state-preparation case, there is a state  $\sigma$  such that for outcome  $d$ ,

$$\mathbf{A}_d(\rho) = \sum_j A_{dj} \rho A_{dj}^\dagger = p_d(\rho) \sigma \tag{8}$$

for all states  $\rho$  for which  $p_d(\rho) \neq 0$ . If the preparation depends on obtaining a particular measurement outcome  $d$ , i.e., if  $p_d(\rho) < 1$  for some  $\rho$ , the preparation operation is called *stochastic*; if  $p_d(\rho) = 1$  for all  $\rho$ , the preparation device is *deterministic*. Notice that if the posterior state  $\sigma$  is a pure state  $|\psi\rangle$ , it corresponds to certainty for the outcome of a yes-no measurement of the observable  $O = |\psi\rangle\langle\psi|$ .

It is tempting to conclude that objective facts, consisting of the measurement outcome  $d$  and a classical description of the preparation device, determine the prepared quantum state  $\sigma$ . This would violate the category distinction by allowing facts to fully determine probabilities derived from  $\sigma$ . What this can only mean for a thoroughgoing Bayesian interpretation of quantum probabilities is that the posterior quantum state  $\sigma$  *must* depend on prior beliefs through the quantum operation (Fuchs, 2002; Fuchs & Schack, 2004; Fuchs et al., 2004). We now consider this crucial difference in more detail.

The quantum operation depends, at least partly, on an agent’s beliefs about the device that executes the state-preparation procedure. Any attempt to give a complete specification of the preparation device in terms of classical facts (i.e., observations or measurements of the device and its method of operating) and thus to derive the quantum operation from classical facts alone comes up against the device’s quantum-mechanical nature.

Classical facts cannot suffice to specify a preparation device completely because a complete description must ascribe to the device an initial quantum state, which inevitably represents prior beliefs of the agent who is attempting to describe the device. It is quite

possible—indeed, likely—that other subjective judgments are involved in an agent’s description of the preparation device, but to find a prior belief that is unavoidably part of the description, it is sufficient to recall the usual justification for the mapping given in Eq. (3) (Nielsen & Chuang, 2000). The mapping can always be modeled as coming about from an unitary interaction between the system and an apparatus, followed by an observation on the apparatus alone. To say what the unitary operation actually does, however, one must specify initial quantum states for all systems concerned. But which quantum state for the apparatus? That, the subjective Bayesian would say, is subjective. The objective-preparations assumption that a preparation device can be given a complete classical description neglects that any such device is quantum mechanical and thus cannot be specified completely in terms of classical facts. An example of the dependence of the system’s output state on the input state of the apparatus is given in Fig. 1. Mermin (2006) analyzes the same preparation apparatus, but reaches quite different conclusions.

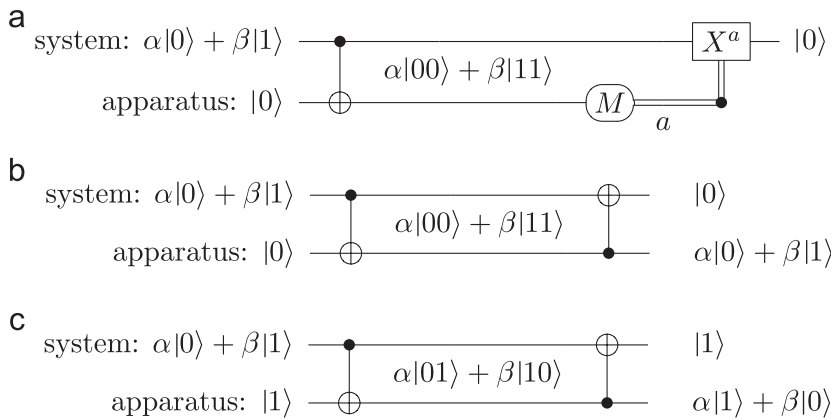


Fig. 1. (a) Quantum-circuit diagram for a device that prepares the system qubit in the state  $|0\rangle$ : The controlled-NOT gate puts the system qubit and the apparatus qubit in the entangled state  $\alpha|00\rangle + \beta|11\rangle$ ; a measurement of the apparatus qubit yields result  $a = 0$  or  $1$ ; the system state is flipped if  $a = 1$ , thus always preparing the system qubit in state  $|0\rangle$ , regardless of the system’s initial state. The single lines in the circuit diagram carry quantum states, which are subjective in the Bayesian view, and the double lines carry the outcome  $a$  of the measurement, which is an objective fact that is used to conditionally flip the system qubit. This quantum circuit describes passing a spin- $\frac{1}{2}$  particle (photon) through a Stern–Gerlach magnetic field (polarizing beam splitter), which sends spin-up (horizontal polarization) and spin-down (vertical polarization) along different paths; measuring which path; and then flipping the spin (rotating the polarization from vertical to horizontal) if the particle is determined to be moving along the spin-down (vertical-polarization) path. This is a deterministic preparation device. Any deterministic preparation operation can be realized in this way: entangle system and apparatus, measure apparatus, and then change the system state conditional on the measurement outcome. (b) In the circuit of (a), the flip based on the measurement result  $a$  can be moved in front of the measurement, becoming a controlled-NOT gate. The measurement can then be omitted, making the preparation device into a unitary interaction between system and apparatus. Any deterministic preparation operation can be realized by such a purely unitary interaction. (c) If the initial apparatus state in (b) is changed to  $|1\rangle$ , the system qubit is prepared in the state  $|1\rangle$ . Indeed, (b) and (c) show that the system qubit is prepared in a state identical to the initial apparatus state, whatever that state is. This highlights our conclusion that the operation of a preparation device always depends on prior beliefs about the device, in particular, its initial quantum state. The objective-preparations view, by positing that the operation of a preparation device can be specified completely in terms of facts, is forced to conclude that the input state to the apparatus is an objective fact. Thus if one adopts the objective-preparations view, one is forced to regard quantum states as objective, a conclusion we reach by a different route in Section 4.

There is an analogy between the coin-tossing example discussed above and the quantum-mechanical analysis of a preparation device. By examining the coin and the tossing mechanism, a scientist cannot derive the probabilities for Heads and Tails. These always depend on some prior judgment. Similarly, by examining a preparation device, a scientist cannot derive the output quantum state and its associated probabilities for measurement outcomes. These also always depend on some prior judgment. The analogy cannot be pushed too far, however, even though in both cases, facts never determine probabilities. In the classical setting, a complete specification of the coin's physical properties, the tossing mechanism, and the initial conditions leads to certainty for the outcome. For a quantum preparation device, a complete specification of the device in terms of facts is simply not allowed by the quantum formalism.

In practice, of course, experimenters depend on their experience and on manufacturers' specifications to inform the prior judgment. They also use repeated trials to *test* the entire setup (D'Ariano, Maccone, & Lo Presti, 2004). To analyze the test results, however, requires the use of the quantum-mechanical formalism, which inevitably involves a prior judgment as input. An example of such a prior judgment is the assumption of exchangeability for repeated trials; for thorough discussions of exchangeability in quantum tomography of states and operations, the reader is urged to consult our previous papers on these subjects (Caves et al., 2002b; Fuchs & Schack, 2004; Fuchs et al., 2004).

An important consequence of our argument is that a quantum operation assigned to a preparation device belongs to the same category in our category distinction as quantum states. This must be so because such a quantum operation determines its output state irrespective of the input. The quantum operations assigned to preparation devices are therefore subjective. The subjectivity of these quantum operations is akin to the subjectivity of conditional probabilities in probability theory.

It follows from all this that two agents who agree on all the facts relevant to a quantum experiment can disagree on the state assignments. In general, two agents starting from the same facts, but different priors, arrive at different (posterior) state assignments. For sufficiently divergent priors, the two agents might even legitimately assign different pure states (Caves et al., 2002a), as in the example of Fig. 1.

#### 4. Certainty and objective properties

The previous section can be summarized as follows. A crucially important difference between the objective-preparations view and the Bayesian approach lies in the way quantum state preparation is understood. In the Bayesian view, a prepared quantum state is not determined by facts alone, but always depends on prior beliefs, in the form of a prior assignment of a quantum operation to a preparation device. Facts, in the form of measurement outcomes, are used to update the prior state (or, as in quantum process tomography, the prior quantum operation), but they never determine a quantum state. By contrast, according to the objective-preparations view, the state of a system is determined by the preparation procedure, which can be completely specified in unambiguous, classical terms. The objective-preparations view holds that a quantum state is determined by the facts about the experimental setup. This means that, according to the objective-preparations view, quantum states are *objective*.

These considerations have an important implication for the concept of certainty in the objective-preparations view. Let  $|\psi\rangle$  be a state prepared by a preparation device, and

consider the observable  $O = |\psi\rangle\langle\psi|$ , which has eigenvalues 0 and 1. If the state is  $|\psi\rangle$ , a measurement of  $O$  will give the outcome 1 with certainty. In the objective-preparations view, this certainty is implied by the facts about the experimental setup, independently of any observer's information or beliefs. Effectively, in the objective-preparations view, it is a fact, guaranteed by the facts about the experimental setup, that the measurement outcome will be 1. The measurement outcome is thus *objectively certain*. Whatever it is that guarantees the outcome, that guarantor is effectively an objective property. It might be a property of the system alone, or it might be a property of the entire experimental setup, including the system, the preparation device, and the measurement apparatus. In any case, the guarantor is a property of the world external to the agent.

The above constitutes a major problem for any interpretation that incorporates the objective-preparations view while also maintaining that quantum states are not part of physical reality, but are *epistemic* (Spekkens, 2007), i.e., representing information or knowledge (Brukner & Zeilinger, 2001; Mermin, 2002). It is simply inconsistent to claim that a quantum state is not part of physical reality if there are facts that guarantee that the measurement of  $O$  defined above has the outcome 1. This is a point made by Roger Penrose in his book *The Emperor's New Mind* (Penrose, 1989, p. 340):

It is an implication of the tenets of the theory that for *any state whatever*—say the state  $|\chi\rangle$ —there is a yes/no measurement that can *in principle* be performed for which the answer is **YES** if the measured state is (proportional to)  $|\chi\rangle$  and **NO** if it is orthogonal to  $|\chi\rangle$ . [...] This seems to have the strong implication that state-vectors must be *objectively real*. Whatever the state of a physical system happens to be—and let us call that state  $|\chi\rangle$ —there is a measurement that can in principle be performed for which  $|\chi\rangle$  is the *only* state (up to proportionality) for which the measurement yields the result **YES**, with *certainty*. [...]

A more concise version (Busch, 2002) of the same argument is that the existence of this yes/no measurement “is sufficient to warrant the objective reality of a pure quantum state.” The objective reality of the quantum state follows here from the notion that there is something in the world, independent of any agent, that guarantees the outcome YES. This notion in turn is implied by the objective-preparations interpretation. The objective-preparations view is, therefore, inconsistent with the idea that a quantum state does not represent a property of the external world.

How does our Bayesian approach escape the same conclusion? In the Bayesian view, the quantum state of the system is not fully determined by facts about the preparation device, since prior beliefs about the preparation device inevitably enter into the assignment of the quantum state it prepares. The statement that the measurement outcome is 1 with certainty is thus not a proposition that is true or false of the system, but an agent's belief—and another agent might make a different prediction. Certainty resides in the agent, not in the physical world. In the Bayesian approach there is no property of the system, or of the system plus the preparation device and measurement apparatus, that guarantees that the outcome will be 1.

## 5. Certainty and locality

In the preceding section, we established that the objective-preparations view of quantum state preparation implies that there are physical properties of the agent's external world

guaranteeing measurement outcomes occur with certainty. In this section, we show that such properties must necessarily be nonlocal. We show that the assumption of locality rules out the existence of a preassigned outcome even in a measurement where the premeasurement state is an eigenstate of the observable. This buttresses our previous argument that probability-1 predictions are not preordained.

Consider a measurement of the observable

$$O = \sum_{k=1}^d \lambda_k |\phi_k\rangle \langle \phi_k|, \tag{9}$$

where the states  $|\phi_k\rangle$  form an orthonormal basis. If the system state,  $|\phi\rangle$ , before the measurement is an eigenstate of  $O$ , say  $|\phi\rangle = |\phi_j\rangle$  for some  $j \in \{1, \dots, d\}$ , the measurement outcome will be  $\lambda_j$  with probability 1. In other words, the measurement outcome is *certain*. It is tempting to say, in this situation, that  $\lambda_j$  is a property that was attached to the system already before the measurement. All the measurement would do in this case would be to reveal this preexisting property of the system. This property of the system would guarantee that the result of the measurement will be  $\lambda_j$ .

We now give a version of an argument by [Stairs \(1983\)](#) showing that the idea that a measurement of a system in an eigenstate of an observable reveals a preexisting property of the system, or indeed a preexisting property of the world external to the agent, conflicts with locality, i.e., with the assumption that a system property at a point  $x$  in space-time cannot depend on events outside the light cone centered at  $x$ .

For this we consider, for the sake of concreteness, the set  $S$  of 33 states in three dimensions introduced by Peres in his version of the proof of the Kochen–Specker theorem ([Peres, 1993](#)). The 33 states in  $S$  can be completed to form 40 orthonormal bases,  $\{|\psi_1^k\rangle, |\psi_2^k\rangle, |\psi_3^k\rangle\}$ ,  $k = 1, \dots, 40$ , consisting of a total of 57 distinct states ([Larsson, 2002](#)). These bases are, of course, not disjoint. For this set of states, one proves the Kochen–Specker theorem by showing that there is no map,

$$f : S \rightarrow \{0, 1\}, \tag{10}$$

such that, for each basis  $\{|\psi_1^k\rangle, |\psi_2^k\rangle, |\psi_3^k\rangle\}$ , exactly one vector is mapped to 1 and the other two are mapped to 0; i.e., there are no integers  $j_k \in \{1, 2, 3\}$ , for  $k = 1, \dots, 40$ , such that a function on the 57 states can be defined consistently by the conditions

$$f(|\psi_j^k\rangle) = \begin{cases} 1 & \text{if } j = j_k, \\ 0 & \text{if } j \neq j_k. \end{cases} \tag{11}$$

We show now that the assumption of preexisting properties for eigenstates, combined with the assumption of locality, implies the existence of such an impossible map and is therefore ruled out by locality.

Take two particles, at spatially separated locations  $A$  and  $B$ , in the maximally entangled state

$$|\Psi\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle), \tag{12}$$

where the states  $|0\rangle, |1\rangle, |2\rangle$  form an orthonormal basis. For any single-system state

$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle + c_2|2\rangle, \tag{13}$$

define the complex conjugate state

$$|\tilde{\psi}\rangle = c_0^*|0\rangle + c_1^*|1\rangle + c_2^*|2\rangle. \quad (14)$$

For some  $k \in \{1, \dots, 40\}$ , let a von Neumann measurement in the basis  $\{|\tilde{\psi}_1^k\rangle, |\tilde{\psi}_2^k\rangle, |\tilde{\psi}_3^k\rangle\}$  be carried out on the particle at  $A$ . Denote the outcome by  $j_k \in \{1, 2, 3\}$ . The resulting state of the particle at  $B$  is  $|\psi_{j_k}^k\rangle$ . It follows that a measurement of the observable

$$\mathcal{O}_{j_k}^k = |\psi_{j_k}^k\rangle\langle\psi_{j_k}^k| \quad (15)$$

on the particle at  $B$  gives the outcome 1 with certainty and that a measurement of

$$\mathcal{O}_j^k = |\psi_j^k\rangle\langle\psi_j^k| \quad \text{for } j \neq j_k, \quad (16)$$

gives the outcome 0 with certainty. According to our assumption that such measurements correspond to preexisting properties, there must be a property of the world that guarantees the outcome 1 for the measurement of  $\mathcal{O}_{j_k}^k$  and the outcome 0 for the two orthogonal measurements.

Locality demands that this property be independent of the measurement at  $A$ . Since the conclusions of the last paragraph hold for any choice of measurement,  $k \in \{1, \dots, 40\}$ , the assumption of locality requires that the world have physical properties that guarantee, for each state  $|\psi_j^k\rangle \in S$ , a unique outcome  $\in \{0, 1\}$  for a measurement of  $|\psi_j^k\rangle\langle\psi_j^k|$ , and these properties define a map  $f : S \rightarrow \{0, 1\}$  satisfying the impossible conditions (11). It follows that the assumptions of locality and preexisting properties for eigenstates are mutually contradictory.

Of course, one could take the position that the quantum state of system  $B$ , after the measurement on  $A$ , is an objective property, but not an objective property of  $B$  alone (Grangier, 2005).<sup>10</sup> It would instead be an objective property of the two systems, the device that prepares them, and, more generally, of the entire history of  $B$  and anything it has interacted with. In our view, this inevitably involves nonlocal influences, or it leads one down the path of a many-worlds (or Everett) interpretation.

## 6. Discussion: quantum certainty

The arguments in the preceding two sections imply that *there are no preassigned values to quantum measurement outcomes, even outcomes that are certain*. In other words, there is nothing intrinsic to a quantum system, i.e., no objectively real property of the system, that guarantees a particular outcome of a quantum measurement. This means that we must abandon explanations in terms of preexisting properties.

The Bayesian approach sketched in the previous sections takes this conclusion fully on board by denying objective status to any state assignment, including pure-state

<sup>10</sup>N. D. Mermin (private communications, 2003 and 2006) characterizes as “dangerously misleading” the idea that the post-measurement quantum state  $|\psi\rangle$  of  $B$  is an objective property of system  $B$  alone. He “reject(s) the notion that objective properties (must) reside in objects or have physical locations.” Yet the quantum state of  $B$ , if objective, is a property of the world, external to the agent, and since it changes as the world changes, it is hard to see how it can have only the disembodied objectivity Mermin is describing. If the objectivity of  $|\psi\rangle$  *does* reside somewhere, say, in the entire experimental setup, including the device that prepares  $A$  and  $B$  and the measurement on  $A$  and its outcome, then consider a measurement of an observable of  $B$  for which  $|\psi\rangle$  is an eigenstate with eigenvalue  $\lambda$ . If  $\lambda$  is an objective property, how can it fail to reside in  $B$  (under the assumption of locality), thus making  $|\psi\rangle$  a property of  $B$  after all?

assignments. In the Bayesian approach, there is never one unique correct quantum state assignment to a system. Two scientists obeying all the rules of quantum mechanics can always in principle assign different pure states to the same system, without either of them being wrong. The statement that an outcome is certain to occur is always a statement relative to a scientist's (necessarily subjective) state of belief. "It is certain" is a state of belief, not a fact (see footnote 3).

A common objection to this goes as follows. Imagine a scientist who performs a sequence of  $Z$  measurements on a qubit. Quantum mechanics, plus his experience and prior judgment and perhaps the outcomes of a long sequence of previous measurements, make him certain that the outcomes will all be "up". Now he performs the measurements, and he always gets the result "up". Should not the agent be surprised that he keeps getting the outcome "up"? Does not this mean that it is a fact, rather than a mere belief, that the outcomes of his experiment will be "up"? Does not this repeated outcome demand an explanation independent of the agent's belief?

The answer to the first question is easy: Surprised? To the contrary, he would bet his life on it. Since the agent was certain that he would get the outcome "up" every time, he is not going to be surprised when that happens. Given his prior belief, only observing "down" would surprise him, since he was certain this would not happen, though nature might choose to surprise him anyway.

The answer to the second question is similarly straightforward. According to our assumption, the agent has put together all his experience, prior beliefs, previous measurement outcomes, his knowledge of physics and in particular quantum theory, all to predict a run of "up" outcomes. Why would he want any further explanation? What could be added to his belief of certainty? He has consulted the world *in every way he can* to reach this belief; the world offers no further stamp of approval for his belief beyond all the factors that he has already considered.

The third question brings us closer to a deeper motivation for this challenge. The question could be rephrased as follows: Isn't not asking for a further explanation a betrayal of the very purpose of science, namely, never to give up the quest for an explanation (Garrett, 1993)? Shouldn't a naturally curious scientist never give up looking for explanations? The answer to these questions is that truth can be found at different levels. At one level, a scientist who accepts the Bell/EPR arguments should indeed stop looking for an explanation in terms of hidden variables or preassigned values. The Bell/EPR arguments show that there simply is no local and realistic explanation for the correlations predicted by quantum mechanics. Giving up the quest for such an explanation is unavoidable if one stays within the framework of quantum theory. On a different level, it appears that the absence of a mechanistic explanation is just the message that quantum mechanics is trying to send us. Accepting the Bell/EPR analysis at face value means accepting what might be the most important lesson about the world, or what we believe about the world, coming from quantum theory, namely, that there are no instruction sets behind quantum measurement outcomes. Go beyond quantum mechanics if you wish to formulate an explanation in terms of instruction sets, but accept the lesson of no instruction sets if you wish to interpret quantum mechanics.

It might still be argued that an agent could not be certain about the outcome "Yes" without an objectively real state of affairs guaranteeing this outcome, i.e., without the existence of an underlying instruction set. This argument, it seems to us, is based on a prejudice. What would the existence of an instruction set add to the agent's beliefs about



the outcome? Why would he be more confident about the outcome “up” if he knew that the particle carried an instruction set? The existence of instruction sets might make the agent feel better if he is bound by a classical world view, but from the perspective of quantum mechanics, would not contribute to his certainty about the outcome.

Let us end with a couple of points for future research. We have emphasized that one of the arguments often repeated to justify that quantum-mechanical probabilities are objective, rather than subjective, is that they are “determined by physical law.” But, what can this mean? Inevitably, what is being invoked is an idea that quantum states  $|\psi\rangle$  have an existence independent of the probabilities they give rise to through the Born rule,

$$p(d) = \langle \psi | E_d | \psi \rangle. \quad (17)$$

From the Bayesian perspective, however, these expressions are not independent at all, and what we have argued in this paper is that quantum states are every bit as subjective as any Bayesian probability. What then is the role of the Born rule? Can it be dispensed with completely?

It seems no, it cannot be dispensed with, even from the Bayesian perspective. But its significance is different than in other developments of quantum foundations: the Born rule is not a rule for *setting* probabilities, but rather a rule for *transforming* or *relating* them.

For instance, take a complete set of  $D + 1$  observables  $O^k$ ,  $k = 1, \dots, D + 1$ , for a Hilbert space of dimension  $D$  (Wootters, 1986). Subjectively setting probabilities for the  $D$  outcomes of each such measurement uniquely determines a quantum state  $|\psi\rangle$  (via inverting the Born rule). Thus, as concerns probabilities for the outcomes of any other quantum measurements, there can be no more freedom. All further probabilities are obtained through linear transformations of the originals. In this way, the role of the Born rule can be seen as having something of the flavor of Dutch-book coherence, but with an empirical content added on top of bare, law-of-thought probability theory: an agent interacting with the quantum world would be wise to adjust his probabilities for the outcomes of various measurements to those of quantum form if he wants to avoid devastating consequences. The role of physical law—i.e., the assumption that the world is made of quantum mechanical stuff—is codified in how measurement probabilities are related, not how they are set.<sup>11</sup>

This brings up a final consideration. What we have aimed for here is to show that the subjective Bayesian view of quantum probabilities is completely consistent, even in the case of certainty. One of our strong motivations for doing this is our belief that taking this approach to quantum mechanics alleviates many of the conceptual difficulties that have been with it since the beginning. But even so, this is no reason to stop digging deeper into the foundations of quantum mechanics. For all the things the Bayesian program seems to answer of quantum mechanics, there is still much more to question. For instance, from our point of view, the existence of Bell inequality violations is not particularly mysterious, but this conceptual point does not get us much closer to a technical understanding of the exact violations quantum mechanics does provide: What, from a Bayesian point of view, would justify that correlations be constrained by the Tsirelson bound (S. J. van Enk, private communications, 2000–2006)? Indeed, why is the structure of quantum probabilities

<sup>11</sup>These ideas mesh to some extent with Pitowsky’s development (Pitowsky, 2003). Pitowsky, however, suggests that quantum mechanics entails a modification of probability theory, whereas we think the Born rule is an empirical addition to probability, not a modification.

(Bayesian though they be) just the way it is? Why does that structure find its most convenient expression through the Hilbert-space formalism? Most importantly, let us pose a question we never lose sight of: given that the Bayesian approach promises a clear distinction between the subjective and objective, what features of the quantum formalism beyond the ones discussed here actually correspond to objective properties? All of these questions have no immediate answer. Yet finding answers to them will surely lead to a better understanding of quantum phenomena. As we see it, subjective probability is the firmest foundation for a careful approach to that quest.

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## References

- Appleby, D. M. (2005a). Facts, values and quanta. *Foundations of Physics*, 35, 627.
- Appleby, D. M. (2005b). Probabilities are single-case, or nothing. *Optics Spectrum*, 99, 447.
- Berger, J. O., & Sellke, T. (1987). Testing a point null hypothesis: The irreconcilability of P-values and evidence. *Journal of the American Statistical Association*, 82, 112.
- Bernardo, J. M., & Smith, A. F. M. (1994). *Bayesian theory*. Chichester: Wiley.
- Bohr, N. (1939). The causality problem in modern physics. In *New theories in physics* (pp. 11–45). Paris: International Institute of Intellectual Cooperation.
- Braithwaite, R. B. (1968). *Scientific explanation*. Cambridge: Cambridge University Press.
- Brown, H. R., & Svetlichny, G. (1990). Nonlocality and Gleason's lemma. Part I. Deterministic theories. *Foundations of Physics*, 20, 1379.
- Brukner, C., & Zeilinger, A. (2001). Conceptual inadequacy of the Shannon information in quantum measurements. *Physical Review A*, 63, 022113.
- Brun, T. A., Caves, C. M., & Schack, R. (2001). Quantum Bayes rule. *Physical Review A*, 63, 042309.
- Busch, P. (2002). Classical versus quantum ontology. *Studies in History and Philosophy of Modern Physics*, 33B, 517.
- Caves, C. M., Fuchs, C. A., & Schack, R. (2002a). Conditions for compatibility of quantum-state assignments. *Physical Review A*, 66, 062111.
- Caves, C. M., Fuchs, C. A., & Schack, R. (2002b). Unknown quantum states: The quantum de Finetti representation. *Journal of Mathematical Physics*, 43, 4537.
- Caves, C. M., Fuchs, C. A., & Schack, R. (2002c). Quantum probabilities as Bayesian probabilities. *Physical Review A*, 65, 022305.
- Caves, C. M., & Schack, R. (2005). Properties of the frequency operator do not imply the quantum probability postulate. *Annals of Physics*, 315, 123 [corrigendum: *Annals of Physics*, 321, 504].
- Cohen, R. S., & Stachel, J. J. (Eds.). (1979). *Selected papers of Léon Rosenfeld*. Dordrecht: D. Reidel.
- D'Ariano, G. M., Maccone, L., & Lo Presti, P. (2004). Quantum calibration of measuring apparatuses. *Physical Review Letters*, 93, 250407.
- de Finetti, B. (1931). Probabilismo. *Logos*, 14, 163 translation: Probabilism. *Erkenntnis*, 31, 169–223.
- de Finetti, B. (1990). *Theory of probability*. New York: Wiley.
- Faye, J. (1991). *Niels Bohr: His heritage and legacy. An antirealist view of quantum mechanics*. Dordrecht: Kluwer.
- Faye, J., & Folse, H. J. (Eds.). (1994). *Niels Bohr and contemporary philosophy*. Dordrecht: Kluwer.
- Feller, W. (1968). *An introduction to probability theory and its applications* (Vol. I, 3rd ed.). New York: Wiley.
- Folse, H. J. (1985). *The philosophy of Niels Bohr: The framework of complementarity*. Amsterdam: North-Holland.

- Fuchs, C. A. (2002). Quantum mechanics as quantum information (and only a little more). e-print quant-ph/0205039 (see also C. A. Fuchs, Quantum mechanics as quantum information, mostly. *Journal of Modern Optics*, 50, 987, for an abridged version).
- Fuchs, C. A., & Schack, R. (2004). Unknown quantum states and operations, a Bayesian view. In M. G. A. Paris, & J. Řeháček (Eds.), *Quantum estimation theory* (p. 151). Berlin: Springer.
- Fuchs, C. A., Schack, R., & Scudo, P. F. (2004). A de Finetti representation theorem for quantum process tomography. *Physical Review A*, 69, 062305.
- Garrett, A. J. M. (1993). Making sense of quantum mechanics: Why you should believe in hidden variables. In A. Mohammed-Djafari, & G. Demoment (Eds.), *Maximum entropy and Bayesian methods (Paris, France, 1992)* (pp. 79–83). Dordrecht: Kluwer.
- Giere, R. N. (1973). Objective single-case probabilities and the foundations of statistics. In P. Suppes, L. Henkin, A. Jojo, & Gr. C. Moisil (Eds.), *Logic, methodology and philosophy of science IV* (pp. 467–483). Amsterdam: North-Holland.
- Giere, R. N. (1979). Propensity and necessity. *Synthese*, 40, 439–451.
- Grangier, P. (2005). Contextual objectivity and the quantum formalism. *International Journal of Quantum Information*, 3, 17–22.
- Heywood, P., & Redhead, M. L. G. (1983). Nonlocality and the Kochen-Specker paradox. *Foundations of Physics*, 13, 481.
- Ismael, J. (2006). Chance reconsidered. Unpublished, preliminary draft.
- Jammer, M. (1974). *The philosophy of quantum mechanics*. New York: Wiley and various references therein.
- Jaynes, E. T. (1957a). Information theory and statistical mechanics. *Physical Review*, 106, 620.
- Jaynes, E. T. (1957b). Information theory and statistical mechanics. II. *Physical Review*, 108, 171.
- Jeffrey, R. (1997). Unknown probabilities. *Erkenntnis*, 45, 327.
- Jeffrey, R. (2004). *Subjective probability. The real thing*. Cambridge: Cambridge University Press.
- Jeffreys, H. (1961). *Theory of probability* (3rd ed.). Oxford: Clarendon Press.
- Kyburg, H. E., Jr., & Smokler, H. E. (Eds.). (1980). *Studies in subjective probability* (2nd ed.). Huntington: Robert E. Krieger Publishing.
- Larsson, J.-Å. (2002). A Kochen–Specker inequality. *Europhysics Letters*, 58, 799.
- Lehmann, E. L. (1986). *Testing statistical hypotheses* (2nd ed.). New York: Wiley.
- Leifer, M. S. (2006a). Quantum dynamics as an analog of conditional probability. *Physical Review A*, 74, 042310.
- Leifer, M. S. (2006b). Conditional density operators and the subjectivity of quantum operations. quant-ph/0611233.
- Lewis, D. (1986a). A subjectivist's guide to objective chance. In *Studies in inductive logic and probability* (Vol. II, p. 83). Oxford: Oxford University Press.
- Lewis, D. (1986b). Postscripts to 'A subjectivist's guide to objective chance. In *Philosophical papers* (Vol. II, p. 114). Oxford: Oxford University Press.
- Loewer, B. (2001). Determinism and chance. *Studies in History and Philosophy of Modern Physics*, 32, 609.
- Loewer, B. (2004). David Lewis' Humean theory of objective chance. *Philosophy of Science*, 75, 1115.
- Mermin, N. D. (1985). Is the moon there when nobody looks? Reality and the quantum theory. *Physics Today*, April 1985, 38.
- Mermin, N. D. (2002). Whose knowledge? In R. Bertlmann, & A. Zeilinger (Eds.), *Quantum (Un)speakables: Essays in commemoration of John S. Bell* (p. 271). Berlin: Springer.
- Mermin, N. D. (2006). In praise of measurement. *Quantum Information Processes*, 5, 239–260.
- Murdoch, D. (1987). *Niels Bohr's philosophy of physics*. Cambridge: Cambridge University Press.
- Nielsen, M. A., & Chuang, I. L. (2000). *Quantum computation and quantum information*. Cambridge: Cambridge University Press.
- North, J. (2003). Understanding probabilities in statistical mechanics. Unpublished manuscript, available at ([aardvark.ucsd.edu/grad\\_conference/north.doc](http://aardvark.ucsd.edu/grad_conference/north.doc)).
- Penrose, R. (1989). *The emperor's new mind*. New York: Oxford University Press.
- Peres, A. (1984). What is a state vector? *American Journal of Physics*, 52, 644.
- Peres, A. (1993). *Quantum theory: concepts and methods*. Dordrecht: Kluwer Academic Publishers.
- Pitowsky, I. (2003). Betting on the outcomes of measurements: A Bayesian theory of quantum probability. *Studies in History and Philosophy of Modern Physics*, 34, 395.
- Plotnitsky, A. (1994). *Complementarity: Anti-epistemology after Bohr and Derrida*. Durham, NC: Duke University Press.
- Savage, L. J. (1972). *The foundations of statistics* (2nd ed.). New York: Dover.

- Schack, R. (2003). Quantum theory from four of Hardy's axioms. *Foundations of Physics*, 33, 1461.
- Shalizi, C. R., & Moore, C. (2003). What is a macrostate? Subjective observations and objective dynamics. e-print cond-mat/0303625.
- Spekkens, R. W. (2007). Evidence for the epistemic view of quantum states: A toy theory. *Physical Review A*, 75, 032110.
- Stairs, A. (1983). Quantum logic, realism, and value-definiteness. *Philosophy of Science*, 50, 578.
- Stapp, H. P. (1972). The Copenhagen interpretation. *American Journal of Physics*, 40, 1098.
- Suppes, P. (1973). New foundations for objective probability: Axioms for propensities. In P. Suppes, L. Henkin, A. Jojo, & Gr. C. Moisil (Eds.), *Logic, methodology and philosophy of science IV* (pp. 515–529). Amsterdam: North-Holland.
- Timpson, C. G. (2006). Philosophical aspects of quantum information theory. e-print quant-ph/0611187. In D. Rickles (Ed.), *The Ashgate companion to the new philosophy of physics* (particularly Sections 3.3.1 and 5.2.3). Ashgate Publishing, to appear.
- White, A. (1972). Certainty. *Proceedings of the Aristotelian Society*, 46, 1 (See also Wittgenstein, L. *On Certainty*. Malden, MA: Blackwell).
- Wootters, W. K. (1986). Quantum mechanics without probability amplitudes. *Foundations of Physics*, 16, 391.