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On Proof-Theoretic Approaches to the Paradoxes: Problems of Undergeneration and Overgeneration in the Prawitz-Tennant Analysis

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Abstract

When a doctor finds a patient, he diagnoses what the illness the patient has and prescribes it in accordance with his diagnosis. Likewise, when a logician faces a problematic argument (or proof), he characterizes the problem and solves it on the basis of his characterization. It is often believed that solutions to the paradoxes are closely tied with the characterization of the paradoxes. For instance, an informal characterization of a paradox proposed by Sainsbury (2009, p. 1) says that it is an unacceptable conclusion elicited from the acceptable premises via acceptable reasoning. A diagnosis of the paradoxes through Sainsbury's characterization can be that it is a trouble that acceptability leads to unacceptability. Thus, from the diagnosis with the characterization, three responses to the paradoxes can be proposed such that either the premises or the reasoning is not in fact acceptable, or else the conclusion is acceptable. We shall call the first response the premise-rejection, the second the reasoning-rejection, and the last the conclusion-acceptance. Of course, it is not to say that traditional characterizations of and solutions to the informal notion of a 'paradox' are in full conformity with Sainsbury's definition. However, his informal definition is the simplest way to understand 'paradox' and the solution to it. In this dissertation, we presume that the traditional understandings of 'paradox' are quite coherent with Sainsbury's definition.

It seems to be that a proof-theoretic solution to the paradoxes relies on how we characterize the informal notion of a 'paradox' in a proof-theoretic fashion. In this regard, it is possible that the proof-theoretic criterion for and the solution to the paradoxes differ from Sainsbury's definition. The present dissertation aims to investigate the proof-theoretic criterion for and the solution to the paradoxes from the perspectives on the Prawitz-Tennant analysis of the paradoxes.

First of all, we will mainly deal with the set-theoretic/semantic paradoxes which were primarily discussed in the late 19th to the early 20th century for the foundation of mathematics. In other words, we will center on paradoxes, often called self-referential paradoxes. This dissertation consists of five chapters. Chapter 1 will summarize the traditional approaches to the paradoxes by dividing the cases into the set-theoretic paradox and the semantic paradox. The traditional approaches consist of three types of responses: the

premise-rejection, the reasoning-rejection, and the conclusion-acceptance. Traditional approaches to the paradoxes have some aspects that a constructivist can hardly accept. Those approaches use a model-theoretic method which often applies constructively invalid inferences, such as *classical reductio*. Also, the proof-theoretic investigation of the paradoxes may offer the uniform solution to the set-theoretic and semantic paradoxes on the perspectives of constructivism.

In the last part of Chapter 1, Section 1.3, we will introduce the Prawitz-Tennant analysis of the paradoxes. While investigating Russell's paradox in natural deduction, Prawitz (1965, p. 95) first remarks, 'the set-theoretic paradoxes are ruled out by the requirement that the [derivations] shall be in normal.' His derivation formalizing Russell's paradox falls into a non-terminating reduction sequence and so is not reducible to a normal derivation. The requirement of a normal derivation may be a promising proof-theoretic solution to the paradoxes and it can be interpreted as below.

The Requirement of a (Full) Normal Derivation(*RND*): For any derivation $\mathfrak{D}$ in natural deduction, $\mathfrak{D}$ is acceptable only if $\mathfrak{D}$ is (in principle) convertible into a (full) normal derivation.

Neil Tennant (1982, 1995, 2016, 2017) regards the non-terminating reduction sequence as the primary feature of genuine paradoxes and proposes his criterion for paradoxicality (TCP).

Tennant's Criterion for Paradoxicality:(TCP) Let $\mathfrak{D}$ be any derivation of a given natural deduction system *S*. $\mathfrak{D}$ is a *T-paradox* iff

(i) $\mathfrak{D}$ is a (closed or open) derivation of $\bot$,

(ii) *id est* inferences (or rules) are used in $\mathfrak{D}$,

(iii) a reduction procedure of $\mathfrak{D}$ generates a non-terminating reduction sequence, such as a reduction loop.

When he first introduces his criterion, Tennant (1982, p. 268) wants to regard the criterion as the conjecture for genuine paradoxes that for any derivation $\mathfrak{D}$, $\mathfrak{D}$ formalizes a genuine paradox iff $\mathfrak{D}$ is a T-paradox. if his conjecture is true, any derivation of a genuine paradox is T-paradox. Also, since a T-paradox is unable to be reduced to a normal derivation, *RND* can block the T-paradox and becomes to be a proof-theoretic solution to the paradoxes.

In this dissertation, we shall investigate whether *TCP* can be a correct criterion for genuine paradoxes and whether *RND* can be a proof-theoretic solution to the paradoxes. Tennant's criterion has two types of counterexamples. The one is a case which raises the problem of overgeneration that *TCP* makes a paradoxical derivation non-paradoxical. The other is one which generates the problem of undergeneration that *TCP* renders a non-paradoxical derivation paradoxical. Chapter 2 deals with the problem of undergeneration and Chapter 3 concerns the problem of overgeneration. Chapter 2 discusses that Tenant's diagnosis of the counterexample which applies *CR*−rule and causes the undergeneration problem is not correct and presents a solution to the problem of undergeneration. Chapter 3 argues that Tennant's diagnosis of the counterexample raising the overgeneration problem is wrong and provides a solution to the problem. Finally, Chapter 4 addresses what should be explicated in order for *RND* to be a proof-theoretic solution to the paradoxes. The contents of Chapter 2–4 are summarized as follows:

Abstract of Chapter 2. In order to solve the problem of undergeneration raised by Rogerson-type counterexamples, Tennant (2015) seems to presume that the application of *Classical Reductio*, i.e. *CR*−rule, is the culprit of the trouble that it disguises the main feature of paradoxicality, such as a non-terminating reduction sequence. Tennant may not take the problem of undergeneration seriously. We will claim that the undergeneration problem is not solved by simply accusing *CR*−rule of the trouble. In order to show that the occurrence of a non-terminating reduction sequence is independent of the use of *CR*−rule. We suggest two examples of the Liar paradox. First, we suggest derivations of the Liar paradox and Curry's paradox which neither use *CR*−rule nor generate a non-terminating reduction sequence. In addition, we provide derivations of the Liar paradox in which the non-terminating reduction sequence is produced even though the *CR*−rule is used. After we diagnose the culprit of preventing a non-terminating reduction sequence, it will be discussed that the problem of undergeneration will be solved by adding the condition to *TCP* that only harmonious rules are to be used.

Abstract of Chapter 3. Tennant(2016) asserts that if all elimination rules are stated in generalized form, the problem of overgeneration can be solved. However, we claim that the mere choice of generalized elimination rules fails to solve the problem because there exist Ekman-type reductions which are stated in generalized form and produce a non-terminating reduction sequence. Thus, we claim that the real issue is which set of reductions is proper. In order to find a criterion for a proper reduction, we shall investigate Schroeder-Heister and Tranchini's Triviality test and argue that Triviality test does not block every Ekman-type reduction procedure since it works relative to a system. At last, we will propose an alternative way to evaluate a proper reduction, called Translation test.

Abstract of Chapter 4. In order for *RND* to be a proof-theoretic solution, there are three things to be explicated: (i) 'which paradoxes are genuine paradoxes?', (ii) 'why should we accept only a normalizable derivation?', and (iii) 'should we consider only $\perp$ as an unacceptable conclusion?' With regard to the first question (i), we will discuss that Tennant does not have a clear standard for genuine paradoxes. In addition, with respect to the second question (ii), if proof-theoretic validity implies normalizability, then *RND* can be the proof-theoretic solution. However, it will be noted that the relation should be extended to a general case. Moreover, it will additionally discussed that if *RND* could be a proof-theoretic solution, it would be a different type of solution rather than a reasoning-rejection solution which constrains a particular inference rule. Lastly, with the third question (iii), we shall consider a normal derivation of $\neg\varphi \wedge \varphi$ which seems to be a paradoxical derivation and argue that if any formula having the form $\neg\varphi \wedge \varphi$ is regarded as an unacceptable conclusion, since *RND* fails to block the normal derivation of $\neg\varphi \wedge \varphi$, it cannot be the proof-theoretic solution to the paradoxes. Hence, it should be explicated why $\perp$ should be the only unacceptable conclusion in proof theory.

More precisely, in chapter 2, we will introduce counterexamples proposed by Rogerson (2006) which raises the problem of undergeneration. Rogerson's derivation formalizes Curry's paradox that Tennant may regard it as a genuine one but it does not generate a non-terminating reduction sequence by using the rule for *Classical Reductio*, i.e. $CR-$rule. In other words, in spite of the fact that her derivation formalizes the genuine paradox, it is not a T-paradox and shows that TCP_E undergenerates. Section 2.1 introduces preliminary notations, rules, and the harmony relation between introduction and elimination rules.

Section 2.2 introduces Tennant's diagnosis to the problem of undergeneration occurred by the example of using the rule for *classical reductio*, $CR-$rule, and argues that his diagnosis is not correct. Perhaps he seems to assume that the $CR-$rule not only produce a *normal* derivation of $\perp$, but it also masks the key feature of a paradoxical derivation. He explains this phenomenon and expresses it as the 'classical rub.' Also, in the direction of avoiding the phenomenon, he presents the Methodological Conjecture that 'genuine paradoxes are never classical.' Even if his methodological conjecture is correct, it needs to discover the fact that which causes the problem of undergeneration. Tennant may believe that the $CR-$rule has the problem of causing a normal derivation of $\perp$ and concealing a non-terminating reduction sequence, i.e. a primary feature of the paradoxes. Section 2.3 provides derivations which cause the problem of undergeneration but do not use $CR-$rule. That is, $CR-$rule is not the culprit of the undergeneration problem. To find a solution to the problem, Section 2.4 diagnoses what preventing the occurrence of a non-terminating reduction sequence. With some observations, we propose a possible diagnosis that a non-terminating reduction sequence does not occur if a derivation in question includes (i) a major premise which has no reduction process to eliminate it or (ii) a formula having a principal constant which has no reduction procedure to get rid of it. Then, we suggest an additional condition to TCP that a derivation formalizing a genuine paradox only uses harmonious rules. If the suggested condition is acceptable, the condition can solve the problem of undergeneration.

Chapter 3 will cover the problem of overgeneration. In particular, Ekman's paradox presented by Schroeder-Heister and Tranchini (2017) will be introduced. Ekman's paradox is not to be considered a genuine paradox because it involves an inadequate reduction process, and so it causes the problem of overgeneration because it is a T-paradox with respect to Tennant's criterion. To begin with, we will see the response of Tennant (2016) to the Ekman's paradox. He argues that if all elimination rules are stated in generalized form, then the problem of overgeneration will be solved. However, in Section 3.2, we will argue that Tennant's response is inappropriate and that the problem of overgeneration will still occur, even if only generalized elimination rules are used. Furthermore, it will be discussed that Tennant's criterion needs to have an additional condition of which reduction procedure is proper. Section 3.3 introduces Triviality Test of Schroeder-Heister and Tranchini (2017) for appropriate reduction procedures. We shall argue that their Triviality test appears to be unsuitable for the evaluation of standard reduction procedures and it is inappropriate to test a reduction precess independently of a system. Then, Section 3.4 will present Translation test. According to Translation test, Ekmann-type reduction procedures are not proper because it is a detour-making process, and Translation test will have the advantage of being able to test the reduction procedure itself compared to Triviality test.

In Chapter 4, we will examine whether the requirement of a normal derivation(*RND*) can be a solution to the paradoxes. To this end, we will consider three questions of (i) which paradox is a genuine paradox and which formalization is legitimate for the genuine paradox, (ii) why the only normalizable derivations are acceptable, and (iii) why the only propositional constant $\perp$ for absurdity is an unacceptable conclusion. If *RND* is the solution to genuine paradoxes, it needs to be answered what genuine paradox is. Furthermore, even if *RND* could prevent paradoxical derivation, *RND* would not be justified to be a proof-theoretic solution to the paradoxes, unless we had reason to use only normal derivations. Also, if there is a derivation of a genuine paradox which is in normal form and leads to an unacceptable conclusion, *RND* fails to prevent the derivation. In this case, too, *RND* would not be a proof-theoretic solution to the paradoxes.

In Section 4.1, we shall introduce his argument on why Russell's paradox is not a genuine paradox, and argue that by following his argument, if Russell's paradox is not a genuine paradox, neither is the Liar paradox. Tennant has no standard for genuine paradoxes.

Our discussion comes into a question of which formalization is legitimate for the genuine paradox. *RND* only blocks non-normalizable derivations, such as T-paradoxes. If *RND* is regarded as a promising proof-theoretic solution to *genuine paradoxes*, it should be answered to the first question of which paradoxes are genuine paradoxes.

In Section 4.2, we will explore the second question of why it is desirable only to use normal derivations. One possibility is that proof-theoretic validity implies normalizability. In other words, if a paradoxical derivation is not normalizable, it can be ruled out by *RND* because it is not a proof-theoretically valid derivation, *RND* can be a solution to the paradoxes. Section 4.2 will establish that in a particular system, proof-theoretic validity implies normalizability. However, in order for *RND* to be a proof-theoretic solution, the result should be extended to a general case. Section 4.3 discusses that the requirement of a normal derivation is different from the reasoning-rejection solution commonly considered as a restriction of a particular inference rule. If a reasoning-rejection solution is regarded as a solution to constrains a certain inference rule, *RND* will not be the reasoning-rejection solution because it constrains every derivation in an intended system. Section 4.4 introduces a normal derivation of $\neg\varphi \wedge \varphi$ presented by Petrolo and Pistone (2018) and argues that *RND* cannot be a proof-theoretic solution if we accept a formula of the form $\neg\varphi \wedge \varphi$ as well as $\bot$ as an unacceptable conclusion. All in all, only when proof-theoretic validity *generally* implies normalizability and any formula having the form $\neg\varphi \wedge \varphi$ is not regarded as an unacceptable conclusion, *RND* can be a proof-theoretic solution to the paradoxes.

Contents

Chapter 1

Introduction: A Proof-Theoretic Criterion of and Solution to the Paradoxes

In the late 19th and early 20th centuries, a proof of the consistency of mathematics was the main theme of the foundation of mathematics. A contradiction raised by paradoxes was a significant issue to the foundations of logic and mathematics. Since the discovery of the paradoxes, involving fundamental notions and inferences which were considered to be acceptable, had an effect on the foundations, the paradoxes have acquired a significant role in contemporary logic.

There are multiple types of paradoxes, but in this dissertation, we will deal with paradoxes, often called, a 'self-referential paradox.' Self-referential paradoxes are related to a statement that refers to itself or its own referent. Following Frank Ramsey (1925), we divides the paradoxes into two classes: the set-theoretic and the semantic paradoxes. The set-theoretic paradoxes comprise Russell-Zermelo's paradox of the set-membership, Cantor's paradox of cardinality, Burali-Forti's paradox of ordinality etc. The semantic paradoxes are about the semantical concepts, such as the concepts of truth, denotation, predication, and so on. The semantic paradoxes comprise the liar, Grelling's Berry's etc. Most of our discussions in this dissertation will use Russell's and the Liar paradox as major examples, however, the discussions will not be limited to them.

The traditional responses to the set-theoretic paradoxes are to constrain the notion of 'set' by using the separation axiom instead of using the native comprehension axiom, or to constrain the concept of 'type' through the ramified type theory of Bertrand Russell (1908). In the case of the semantic paradoxes, such as the Liar paradox, following Alfred Tarski (1936a,b, 1944), it is proposed that the semantic concept '$\tilde{\text{i}}$s true' is not expressible in the object language. Moreover, those who believe that the concept of truth is expressible in the object language have developed Tarski's idea and suggested the gap theory which allows the sentence neither true nor false, or proposed the glut theory which claims that there exists a sentence both true and false.

Most classical logicians have adapted a way to follow or develop the traditional responses. On the other hand, it is unclear that intuitionists, sometimes more generally called constructivists, who claim that some classical inferences, like *the Law of Excluded Middle* and *the Double Negation Elimination*, can satisfy the traditional responses. There are at least two reasons why constructivists would be reluctant to accept the traditional approaches. First, the traditional approaches allow the use of non-constructive reasoning, such as *the Law of Excluded Middle* and *the Double Negation Elimination*, and so it is sometimes claimed that reasoning by *the Law of Excluded Middle* is an essential part of the semantic and set-theoretic paradoxes. For instance, Hartry Field (2008) considers that classical inferences are somehow the key reasoning that generates the paradoxes.

> ... we ought to seriously consider restricting classical logic to deal with all these paradoxes. In particular, we should seriously consider restricting the law of excluded middle. ... I take excluded middle to be clearly suspect only for certain sentences that have a kind of "inherent circularity" ... (Field, 2008, p. 15).

Second, the traditional responses are basically based on the model-theoretic approaches which often allow non-constructive methods. So constructivists tend to prefer to use proof-theory to explain the concept of validity of arguments or their theory of meaning. Therefore, as an alternative to the model-theoretic approaches, a proof-theoretic solution to the paradoxes can be presented and it would be a promising way for constructivists.

It seems to be that a proof-theoretic solution to the paradoxes relies on how we characterize the informal notion of a 'paradox' in a proof-theoretic fashion. The present dissertation aims to investigate a proof-theoretic criterion for and solution to the paradoxes from the perspectives on the Prawitz-Tennant analysis of the paradoxes, centered on a view from Neil Tennant (1982, 2015, 2016, 2017). There are two types of counterexamples to the proof-theoretic criterion for paradoxicality: problems of under- and overgeneration. The undergeneration problem is raised by the case which shows that the criterion makes a paradoxical derivation non-paradoxical. The problem of overgeneration is the case which represents that the criterion includes a non-paradoxical derivation into the realm of paradoxical derivations. In this chapter, we will introduce Tennant's early criterion for paradoxicality introduced by Tennant (1982). Chapter 2 and 3 shall investigate whether Tennant's criterion can be a necessary and sufficient condition for genuine paradoxes. That is to say, Chapter 2 deals with the problem of undergeneration and Chapter 3 is about the problem of overgeneration. In Chapter 4, we will discuss that there are some difficulties for the requirement of a normal derivation to be a solution to the paradoxes.

Before we introduce Tennant's criterion for paradoxicality and its related solution, we will briefly explore traditional responses to the paradoxes and some critics to those responses. Section 1.1 introduces traditional approaches to the paradoxes and their problems. Preliminary notions and rules will be introduced in Section 1.2. Tennant's criterion for paradoxicality and the requirement of a normal derivation will be introduced in Section 1.3.

1.1 Traditional Responses to the Paradoxes and Dialetheism

When a doctor finds a patient, he diagnoses what the illness the patient has and prescribes it in accordance with his diagnosis. Likewise, when a logician faces a problematic argument (or proof), he characterizes the problem and solves it on the basis of his characterization. It is often believed that solutions to the paradoxes are closely tied with the characterization of the paradoxes. For instance, an informal characterization of a paradox proposed by Richard M. Sainsbury (2009) says, a paradox is generally conceived as 'an apparently unacceptable conclusion derived by apparently acceptable reasoning from apparently acceptable premises.' Since acceptable reasoning hardly draws an unacceptable conclusion from acceptable premises, there are three ways to solve the paradox. Either the premise or the reasoning is not actually acceptable, or else the conclusion is acceptable. The first response is the premise-rejection. The second is the reasoning-rejection, and the last is the conclusion-acceptance. The last response is often supported by dialetheism which is the view that there exists a true contradiction.

It would be difficult to consider that the traditional approaches to the paradoxes are properly distinguished in accordance with Sainsbury's informal characterization of a 'paradox.' For instance, it is uncertain whether the premise-rejection and the reasoning-rejection can be precisely distinguished. However, it might be the simplest way to understand the informal notion of a 'paradox,' and so we investigate the traditional responses while following Sainsbury's characterization.

Traditional responses to the (self-referential) paradoxes seem to embrace the premise-rejection and the reasoning-rejection. Traditional approaches to the paradoxes can be found in both the set-theoretic and the semantic paradoxes. For the case of the set-theoretic paradox, we will examine Russell's paradox. The Liar paradox will be used for the semantic paradox.

1.1.1 Traditional Approaches to Russell's and the Liar Paradox.

To begin with, Russell's paradox occurs in naive set theory by considering the set of all sets not members of themselves. Let us consider that Our language has constants and quantifiers, $\wedge$, $\rightarrow$, $\bot$, $\neg$, $\exists$, $\forall$, $\in$ for conjunction, implication, absurdity, negation, existential/universal quantifiers, and a two-place set-membership relation respectively. Additional expressions can be introduced into the language. Let x, y be any free variables and t be any term not free. φ, ψ, σ be any formulas. Then, the naive comprehension principle has the main role to derive a contradiction from the paradox. It states that there is a set y such that for any object x, x is an element of y iff the condition expressed by the formula φ holds for x.[1] We define '$\varphi \leftrightarrow \psi$' as $(\varphi \to \psi) \wedge (\psi \to \varphi)$. The naive comprehension principle can be written as follows:

The Naive Comprehension Principle: $\exists y \forall x (x \in y \leftrightarrow \varphi)$

Russell's paradox arises by taking φ to be the formula: x not in x, i.e. $x \notin x$. A contradiction is a formula having the form $\varphi \wedge \neg\varphi$. A contradictory conclusion is easily derived from the following three steps.

Premise $\exists y \forall x (x \in y \leftrightarrow x \notin x)$.

(1) $a \in a \leftrightarrow a \notin a$ where a is a parameter.

(2) $\exists y (y \in y \wedge y \notin y)$.

First, the naive comprehension principle allows to use the concept of the set of all sets not members of themselves. We have the premise that $\exists y \forall x (x \in y \leftrightarrow x \notin x)$. By the existential and universal instantiations, we have (1), and then have (2) by the applications of the *Law of Excluded Middle* which states that for any formula φ, either φ or $\neg\varphi$ is true, and the existential generation.

A traditional response to Russell's paradox is to restrict the naive comprehension principle and to use the concept of a set in a limited way. It restricts to use the set of all sets not members of themselves, and rejects the premise $\exists y \forall x (x \in y \leftrightarrow x \notin x)$. The response seems to be accepted by claiming that the concept of a set, i.e. a collection of arbitrary objects, is too vague to count as a mathematical concept and can be constrained. Since the response prevents to use the premise, it can be the premise-rejection.

The other response to Russell's paradox is to put a constraint on the use of the *Law of Excluded Middle*. It is often believed that classical inference, such as the *Law of Excluded*

[1] 'iff' is an abbreviation of 'if and only if.'

Middle has the main role to derive an unacceptable conclusion from the paradoxes. In particular, Priest (2006, pp. 28–29) proposes a similar argument of deriving a contradiction from Russell's paradox above, and considers that classical inferences are somehow the key reasoning that raises the paradoxes.

> The last step [from (1) to (2)] is an application of *reductio*, or the law of excluded middle. ... Reasoning by the law of excluded middle is a well entrenched part of orthodox set theoretic practice. And if one is tempted by this line, one can dismiss it quickly. Essentially the same replies can be made of it as to the corresponding suggestion with the semantic paradoxes.

If the law of excluded middle has the main role to generate the paradoxes, to reject the application of the law of excluded middle would be a response to Russell's paradox. Our examination of Russell's paradox has used the *Law of Excluded Middle* in order to derive (1) $\exists y(y \in y \wedge y \notin y)$ from (2) $\exists y(y \in y \leftrightarrow y \notin y)$. If the rejection of the *Excluded Middle* can block the derivation of a contradiction, it can be the reasoning-rejection solution. Therefore, we may consider that there are two traditional responses to Russell's paradox: the premise- and the reasoning-rejection.

Similarly, the Liar paradox has two responses from the traditional perspective. The Liar paradox is the most well-known paradox among the semantic paradoxes. Alfred Tarski (1936a,b, 1944) mainly deals with the Liar paradox when he gives a classical characterization of the formal concept of truth. Tarski (1936b, p. 401) and Tarski (1944, p. 345) use the word 'semantic' in a narrower sense such that it is a discipline dealing with the relation between expressions of a given language and their references, i.e. the objects or states of affairs. His notion of semantic may be suitable for the correspondence theory of truth which is the view that truth is correspondence to a fact (or broadly any view which embraces the idea that truth consists in a relation to reality). Tarski (1944, p. 344) thinks that the usage of the expression, 'is true,' is *adequate* when it satisfies the schematic relation that, for some formula φ and a name 'φ' for it, 'φ' is true iff φ. Let us use the left and right corner quotes, $\ulcorner\ \urcorner$. Let the function $\ulcorner - \urcorner$ be any injective mapping from formulas into expressions (or coded numerals). So to speak, $\ulcorner - \urcorner$ codes the expressions in a given language.[2] For instance, if φ is a given sentence, then $\ulcorner \varphi \urcorner$ refers to φ. If $\psi(x)$ is a formula with one free variable x then $\psi(\ulcorner \varphi \urcorner)$ is a sentence describing that a sentence φ denoted by $\ulcorner \varphi \urcorner$ is ψ. Then, Tarski's materially adequate notion of truth satisfies the following T-schema.

T-schema: for any formula φ,

$$\ulcorner \varphi \urcorner \text{ is true if and only if } \varphi.$$

The Liar paradox gives rise to a problem to T-schema as a *formally correct* definition of truth. We shall roughly say that $T(x)$ is a truth predicate for a given language $\mathfrak{L}$ if $T(\ulcorner \varphi \urcorner)$ is well-formed for any formula φ in $\mathfrak{L}$. We have a liar sentence Φ by defining a particular formula Φ as $\neg T(\ulcorner \Phi \urcorner)$. Tarski (1936a, pp. 157–159) and Tarski (1944, pp. 347–348) introduce the problem of T-schema while examing the Liar paradox. His materially adequate truth predicate $T(x)$ should satisfy T-schema, so we have the equivalence relation $T(\ulcorner \Phi \urcorner) \leftrightarrow \Phi$. By the meaning of the liar sentence Φ, either we have $\Phi \leftrightarrow \neg T(\ulcorner \Phi \urcorner)$. Then, our usual inferential practice derives the sentence $T(\ulcorner \Phi \urcorner) \leftrightarrow \neg T(\ulcorner \Phi \urcorner)$ which implies a contradictory conclusion.

Tarski (1944, pp. 348–349) diagnoses that a *semantically closed* language causes the Liar paradox and the rejection of the use of such language can solve the paradox.

> If we now analyze the assumptions which lead to the [Liar paradox], we notice the following.
>
> (I) We have implicitly assumed that the language in which the [paradox] is constructed contains, in addition to its expressions, also the names of these expressions, as well as semantic terms such as the term 'true' referring to sentences of this language; we have also assumed that all sentences which determine the adequate usage of this term can be asserted in the language. A language with these properties will be called '*semantically closed*.'

[2]For coding processes, in this dissertation, we follow Dirk van Dalen (2013, pp. 245-250).

> (II) We have assumed that in this language the ordinary laws of logic hold. ... the assumption (I) and (II) prove essential. Since every language which satisfies both of these assumptions is inconsistent, we must reject at least one of them.

Tarski's ordinary laws of logic were laws in classical logic. His notion of 'semantically closed' is not limited in classical inferences. We will summarize that a language $\mathfrak{L}$ is *semantically closed* if it is such that

(i) for any formula φ in $\mathfrak{L}$, $\mathfrak{L}$ has a term $\ulcorner\varphi\urcorner$ which refers to φ,

(ii) $\mathfrak{L}$ has a semantical concept in question, such as the term 'true,' and the term in $\mathfrak{L}$ satisfies its adequacy condition. For the example of the concept of truth, any instances of the concept satisfy T-schema,

Tarski prefers to use classical logic and focuses on formal languages for consistent scientific discourses. Since he did not want to revise classical logic, the rejection of classical inferences, i.e. the reasoning-rejection, was not his option. In order to solve the problem raised by the Liar paradox, he claims that the semantically closed language is not to be used in any consistent discourses. Tarski (1944, p. 349) said,

> It would be superfluous to stress here the consequences of rejecting [classical logic], that is, of changing our logic (supposing this were possible) even in its more elementary and fundamental parts. We thus consider only the possibility of rejecting the assumption [(i) and (ii)]. Accordingly, we decide *not to use any language which is semantically closed* in the sense given.

As he constructs a formal language for the consistent scientific discourses, he thinks that in a single formal language the paradox cannot be solved. He distinguishes the language between the object- and the meta-language. The former is the language which contains expressions as the subject-matter and the other is the language in which we deal with the subject-matter. It is assumed that any expressions in the object language can be translated into the meta-language, but the inverse cannot. From the paradox, he suggests a stricture against the semantically closed language that no language can have its own semantic concepts. Let $\mathfrak{L}_{\mathbb{O}}$ be an object-language and $\mathfrak{L}_{\mathbb{M}}$ be a meta-language of $\mathfrak{L}_{\mathbb{O}}$. $\mathfrak{L}_{\mathbb{O}}$ must not contain its truth-predicate and the truth-predicate must be found only in $\mathfrak{L}_{\mathbb{M}}$. Likewise, the truth-predicate for $\mathfrak{L}_{\mathbb{M}}$ is to be found only in the meta-language of $\mathfrak{L}_{\mathbb{M}}$. Since $\mathfrak{L}_{\mathbb{O}}$ does not have its semantic concepts, it fails to satisfy the condition (ii). He concludes that no language for the consistent scientific discourse can be semantically closed.

Tarski's stricture against the semantic closure may be the premise-rejection solution to the Liar paradox. We apply his stricture to T-schema and have the following hierarchical T-schema by adding the supplementary condition on the object- and the meta-languages.

Hierarchical T-schema : Let $\mathfrak{L}_{\mathbb{O}}$ be an object-language and $\mathfrak{L}_{\mathbb{M}}$ be a meta-language of $\mathfrak{L}_{\mathbb{O}}$. (i.e. any expressions in $\mathfrak{L}_{\mathbb{O}}$ is to be (translated) in $\mathfrak{L}_{\mathbb{M}}$ but not vice versa.[3]) For any sentence φ in $\mathfrak{L}_{\mathbb{O}}$ and its name $\ulcorner\varphi\urcorner$ in $\mathfrak{L}_{\mathbb{M}}$,

$$\ulcorner\varphi\urcorner \text{ is true if and only if } \varphi.$$

Tarski (1944, p. 350) thinks that the hierarchical T-schema and every instance of it should be formulated in the meta-language, $\mathfrak{L}_{\mathbb{M}}$. With respect to his stricture against the semantically closed language, the truth-predicate for the object-language $\mathfrak{L}_{\mathbb{O}}$ is to be found only in the meta-language $\mathfrak{L}_{\mathbb{M}}$. Let us consider the liar sentence Φ which is equivalent to $\neg T(\ulcorner\Phi\urcorner)$. Let us assume that $\mathfrak{L}_{\mathbb{O}}$ has the liar sentence Φ. Then, by the hierarchical T-schema, we have the relation

$$\ulcorner\Phi\urcorner \text{ is true if and only if } \Phi.$$

Now, in virtue of the meaning of Φ, we have the following

$$\ulcorner\Phi\urcorner \text{ is true if and only if } \ulcorner\Phi\urcorner \text{ is not true (i.e. } \neg T(\ulcorner\Phi\urcorner)).$$

The above relation says that the object-language $\mathfrak{L}_{\mathbb{O}}$ contains $\neg T(\ulcorner\Phi\urcorner)$. However, by Tarski's stricture, any language cannot have its truth-predicate, $T(x)$. Any sentences which are equivalent to $\neg T(\ulcorner\Phi\urcorner)$ are not to be formulated in $\mathfrak{L}_{\mathbb{O}}$. There is no liar sentence in $\mathfrak{L}_{\mathbb{O}}$.

[3]It we regard a formal language as a set of expressions, $\mathfrak{L}_{\mathbb{O}}$ is a proper subset of $\mathfrak{L}_{\mathbb{M}}$, so to speak, $\mathfrak{L}_{\mathbb{O}} \subset \mathfrak{L}_{\mathbb{M}}$.

Furthermore, because any language has no truth-predicate, no liar sentence exist in any language. Therefore, there is no Liar paradox. Tarski's stricture against the semantically closed language may be the premise-rejection solution to the Liar paradox in the sense that it prevents to use the liar sentence. Of course, Tarski's stricture may be regarded as the reasoning-rejection solution to the Liar paradox because his stricture does not allow the inference from $\neg T(\ulcorner \varphi \urcorner)$ to φ and vice versa in the object language. Hence, his view can be interpreted as the premise-rejection or the reasoning-rejection solution if his stricture is the solution to the paradoxes.

As we have seen in this subsection, traditional approaches to Russell's and the Liar paradoxes may be considered to be the premise-rejection or the reasoning-rejection or both. In the next subsection, we will investigate problems of traditional responses to the paradoxes and dialetheism.

1.1.2 Problems of Traditional Responses and Dialetheism

In this subsection, after we introduce problems of traditional responses to the liar and Russell's paradox, we will briefly introduce dialetheism which says that a true contradiction exists. Dialetheism may be an alternative solution to the paradoxes, however we shall argue that not every logician should accept it as the primary response to the paradoxes.

With respect to the Liar paradox, Tarski's stricture, while regarding it as the response to the Liar paradox, has been subject to some criticisms. The main criticism against Tarski's solution is made by Saul Kripke (1975, pp. 690–698) that whether or not a sentence is paradoxical is dependent not only on formal properties which are intrinsic to the syntax and semantics of the sentence, but also on empirical facts. In addition, it is not easy to place non-paradoxical claims within Tarski's syntactically fixed set of levels.

(a) Everything Kim Jong-un says in his language $\mathfrak{L}_K$ is true.

(b) Everything Donald Trump says in his language $\mathfrak{L}_T$ is true.

For example, we consider that Donald Trump claims (a) in $\mathfrak{L}_T$ and Kim Jong-un claims (b) in $\mathfrak{L}_K$. Tarski (1944, p. 350) has noticed that the distinction between the object- and the meta-languages is only a relative one. With respect to Trump's claim (a), his language $\mathfrak{L}_T$ has to be higher than Kim's language $\mathfrak{L}_K$ in virtue of Tarski's stricture. That is, $\mathfrak{L}_K$ is an object-language and $\mathfrak{L}_T$ is a meta-language of $\mathfrak{L}_K$ because the semantical concept, 'true,' is to be found only on the meta-language $\mathfrak{L}_T$. On the other hand, with respect to Kim's claim, $\mathfrak{L}_K$ is a meta-language of $\mathfrak{L}_T$ since his claim (b) contains the semantical concept, 'true,' which is to be found only on $\mathfrak{L}_K$. Thus, $\mathfrak{L}_K$ and $\mathfrak{L}_T$ are both object- and meta-languages. It is not easy to apply Tarski's hierarchical distinction to the Kim-Trump case.

A possible defense to this criticism is that the meanings of 'true' in (a) and (b) are different. (a) has a true-predicate, 'is true_T,' which is expressible in $\mathfrak{L}_T$ but not in $\mathfrak{L}_K$. (b) has a truth-predicate, 'is true_K,' expressible in $\mathfrak{L}_K$ but not in $\mathfrak{L}_T$.

(a') Everything Kim Jong-un says in $\mathfrak{L}_K$ is true_T.

(b') Everything Donald Trump says in $\mathfrak{L}_T$ is true_K.

Each has different truth-predicate, so we can distinguish between object- and meta-languages. However, Kim and Trump may not share the same meaning of the semantical notion of 'true.' They cannot communicate with each other. Consequently, Tarski's *hierarchical* T-schema fails to explain a general notion of truth.

Tarski rejects the view that the semantical notion of 'true' is expressible in the object language and pursues the consistency of the object language. On the other hand, Graham Priest (2006, pp.17–18) wants to use a truth predicate in the object language. Considers the sentence 'All the sentences on page 11 of the dissertation entitled *On Proof-Theoretic Approaches to the Paradoxes* are true.' Priest (2006, pp. 18–20) thinks that this sentence is a perfectly good English sentence but not a sentence of the hierarchy, and so the hierarchy is not English. He claims that Kripke's criticism against Tarski's stricture explicates that any languages satisfying Tarskian hierarchical stricture are expressively weaker than English and hence it discusses that it is hard to apply the hierarchical concept of truth to our use of 'true' in natural language, especially English.

> This illustrates a general criticism of the mooted solution to the semantic paradoxes made by Kripke (1975) ... Any semantico-syntactic constraint which

> succeeds in ruling out paradoxes will therefore also rule out perfectly ordinary, non-paradoxical assertions too. In other words, all languages (or hierarchies thereof) which satisfy these constraints will be expressively weaker than English.

If these arguments are right, traditional responses to the paradoxes, such as Tarski's stricture, are unable to be a general solution to the paradoxes.

As Priest accepts T-schema in natural language, the Liar paradox generates a contradiction. His view, called 'dialetheism', is that some contradictions can be true, and the set-theoretic and the semantic paradoxes show that there are true contradictions. As we have said, if a paradox is grasped as an apparently unacceptable conclusion derived by apparently acceptable reasoning from apparently acceptable premises, dialetheism can be a plausible response to the paradox because it claims that the seeming unacceptable conclusion is actually acceptable. It is the conclusion-acceptance response to the paradoxes. It seems to be that Priest's dialetheism is an alternative way to solve the paradoxes if the premise-rejection and the reasoning-rejection are not a general solution to the paradoxes. However, dialetheism is not a general solution to the paradoxes either. Especially, an intuitionist does not need to be a dialetheist.

To discuss why the intuitionist would not enjoy dialetheism as the solution to the paradoxes, we give some additional terminologies. In accordance with standard practice, for a given formal system S with its language $\mathfrak{L}$ containing $\bot$ and $\neg$, we write '$S \vdash \varphi$' to mean that S derives φ and '$S \nvdash \varphi$' means that S does not derive φ. We say that S is *complete* if for each formula φ in $\mathfrak{L}$, either $S \vdash \varphi$ or $S \vdash \neg\varphi$; otherwise incomplete. S is *consistent* if $S \nvdash \bot$; otherwise inconsistent. S is *trivial* if for any sentence φ in $\mathfrak{L}$ $S \vdash \varphi$; otherwise non-trivial. A dialetheic system derives a true contradiction, so it is inconsistent. Since it rejects *ex contradictione quodlibet* which means that a contradiction implies everything, the dialetheic system is inconsistent but non-trivial.

Graham Priest (2006, Ch.1 – 3 and 7) argues that true contradictions are derivable from the semantic paradoxes, the set-theoretic paradoxes, and Gödel's incompleteness theorem. In addition, Priest (2006, p. 66) offers a prospect of an intuitionistic dialetheism.

> It would be equally possible to have an "intuitionistic dialetheism", which took a constructive stance on negation (so that a proof of the impossibility of a proof of [φ] was required for the truth of [$\neg\varphi$]) and the other logical constants. (We noted ... that the proofs of many logical paradoxes do not require the law of excluded middle or other intuitionistically invalid principles.)

Intuitionistic (relevant) logic is one of the primary candidate logics for intuitionism. If an intuitionistic dialetheism is plausible, the intuitionist can accept dialetheism as the solution to the paradoxes. Unlike the prospect of Priest, an intuitionistic relevant logician, Tennant (1994, p.110), says that nice (or correct) logic is adequate for uncovering all inconsistencies and any intuitionistic consequences of any consistent set of axioms. Also, Tennant (2004) has argued that there is no true contradiction in intuitionistic (relevant) logic. Priest (2006) replies to Tennant in a footnote 6 at page 286.

> In the final section of [Tennant (2004)], Tennant also critici[z]es my account of the paradoxes of self reference by giving his own. But he does not address the arguments of the 1st edn that would appear to apply to his account. For example, he says that the liar sentence is 'radically truth-valueless' ... but he does not address the extended version of the paradox: this sentence is false or radically truth-valueless. ... Nor does he address the paradoxes that do not use the [law of excluded middle], such as Berry's. Similarly, he claims that the "Gödel Paradox" shows that the notion of navie proof cannot be formali[z]ed. He does not address the consideration ... as to why this is false or irrelevant.

Tennant did not answer, but the reason why the intuitionist would not accept dialetheism is enough.

For the issue of Berry's paradox, Priest (1983) and Priest (2006, pp. 25–27) have attempted to show that *the Law of Excluded Middle* is unnecessary for the derivation of a contradiction from Berry's paradox. First, as Ross Brady (1984) explains how Priest implicitly assumes the excluded middle, it is arguable whether the excluded middle is necessary to derive a contradiction from Berry's paradox. Although Priest (2006, pp. 25–27) suggests a different argument from that of Priest (1983) to derive a contradiction from Berry's paradox, his proof uses the double negation elimination which is provably equivalent to the excluded middle. Regardless of the issued of whether classical inferences are essential for paradoxical arguments, his argument for intuitionistic dialetheism is overly loose.

With regard to Gödel's incompleteness theorem, Priest (2006, Ch. 3) maintains that any correct formalization of our naive proof procedure is inconsistent and it is tantamount to establish that a true contradiction exists. On the other hand, Tennant (2004) asserts that Gödel's theorem merely shows that one cannot have the complete characterization of our naive proof procedure. Tennant's interpretation may be a consistent counterpart whereas Priest's view may presume that our inconsistent linguistic practice leads to an inconsistency of our naive proof procedure. They may have a different conception of our naive proof procedure. The tension between them is based on their different intuitions of the naive proof procedure, so it seems to be hard to find any ways to ease the tension. As Seungrak Choi (2017) notes, however, a contradiction is derivable from Gödel sentence only in the complete system. The intuitionist does not have to suppose the completeness of the system unless the principle of bivalence is assumed.[4] Likewise, for a given natural system S that the prooflessness is expressible, Choi (2018) shows that $\perp$ is derivable from the strengthened liar sentence in S only when S is complete. If the completeness assumption of logical systems (or theories) is not necessary for intuitionism, the intuitionist need not follow the solution of Priest. In this respect, dialetheism will be excluded from the intuitionist's solution to the paradox.

In this section, we have examined three types of responses to the paradoxes and discussed that all three responses have room for criticism. Since these responses are based on the model-theoretic approaches, some constructivists (or intuitionists) are reluctant to accept it. Interestingly, a well-known proof of the consistency of mathematics was suggested in proof-theory but the proof-theoretic criterion of and solution to the paradoxes have not yet been well investigated. After we will have some requisite notions and rules of proof-theory in Section 1.2, Section 1.3 shall introduce Tennant's criterion for paradoxicality and

[4]The principle of bivalence states that every sentence φ is determinately true or false, independently of our method to know the truth-value of φ.

the requirement of a normal derivation as a plausible solution to the paradoxes. Proof-theoretic approaches can apply to both the set-theoretic and semantic paradoxes. In other words, the proof-theoretic approaches opened up the possibility of the uniform solution to the paradoxes. .

1.2 Preliminaries

As we have discussed in Section 1.1, the naive comprehension axiom has the main role to derive a contradiction from Russell's paradox. Since the notion of a 'set' in the principle allows the set of all sets which are not members of themselves, it is often considered that the application of the notion of a 'set' causes a contradiction from Russell's paradoxes. Hence, the restriction of the naive comprehension principle may be read as the constraint on the application of the notion of a 'set.'

When Gerhard Gentzen (1936) suggests his second proof of the consistency of arithmetic, he seems to think that an error of Russell's paradox is not our use of the notion of a 'set' but in the logical inferences involved in. In pursuing his consistency proof, he first set the analysis of purely logical deduction which was intended to be extended to arithmetic and analysis. Gentzen (1935, Sec. 2) introduces pairs of introduction and elimination rules for natural deduction system as the natural method of reasoning in mathematics. He invented another logical calculus that he called 'sequent calculus,' and his two most important results, a proof of Hauptsatz (cut-elimination theorem) and a consistency proof of arithmetic, were established in sequent calculus. Although two results were proved in sequent calculus, they both were clearly inspired by insights that he got by reflecting on his natural deduction system. The same result can be established by normalization theorem in natural deduction system, and a consistency proof can be suggested as a corollary of the theorem by Prawitz (1965, 1971, 2015).

Prawitz's normalization theorem is deeply related to the proof-theoretic criterion of and solution to the paradoxes that we will concern. In this section, we will introduce primary notions, rules and results in Prawitz's natural deduction system.[5]

Our language has constants and quantifiers, $\wedge$, $\vee$, $\rightarrow$, $\bot$, $\neg$, $\exists$, $\forall$ for conjunction, disjunction, implication, absurdity, negation, existential universal quantifiers respectively. Additional expressions can be introduced into the language. Let x, y be any free variables and t be any term not free. φ, ψ, σ be any formulas. Let $\mathfrak{D}$ be a derivation of a natural deduction system, used in the same manner as 'deduction' in Prawitz (1965). Following Prawitz, we shall use the following conventions: if a derivation $\mathfrak{D}$ ends with a formula φ, we shall write $\begin{matrix}\mathfrak{D}\\ \varphi\end{matrix}$ and φ is called, an 'end-formula.' If it depends on a formula ψ, we shall write $\begin{matrix}\psi\\ \mathfrak{D}\\ \varphi\end{matrix}$.

Natural deduction rules have introduction and elimination rules. Also, natural deduction has two forms of elimination rules: standard and generalized forms. Generalized elimination rules will be introduced in Chapter 2. In this chapter, we shall have natural deduction rules stated in the standard form. Now, we have rules in the natural deduction style proposed by Prawitz (1965).

$$\dfrac{\begin{matrix}\mathfrak{D}_1\\ \varphi_1\end{matrix} \quad \begin{matrix}\mathfrak{D}_2\\ \varphi_2\end{matrix}}{\varphi_1 \wedge \varphi_2}\wedge I \qquad \dfrac{\varphi_1 \wedge \varphi_2}{\varphi_i}\wedge E_{(i=1,2)} \qquad \dfrac{\begin{matrix}[\varphi]^1\\ \mathfrak{D}_1\\ \psi\end{matrix}}{\varphi \rightarrow \psi}\rightarrow I_{,1} \qquad \dfrac{\varphi \rightarrow \psi \quad \begin{matrix}\mathfrak{D}_2\\ \varphi\end{matrix}}{\psi}\rightarrow E$$

$$\dfrac{\begin{matrix}\mathfrak{D}_1\\ \varphi_i\end{matrix}}{\varphi_1 \vee \varphi_2}\vee I_{(i=1,2)} \qquad \dfrac{\varphi_1 \vee \varphi_2 \quad \begin{matrix}[\varphi_1]^1\\ \mathfrak{D}_2\\ \psi\end{matrix} \quad \begin{matrix}[\varphi_2]^2\\ \mathfrak{D}_3\\ \psi\end{matrix}}{\psi}\vee E_{,1,2} \qquad \dfrac{\begin{matrix}[\varphi]^1\\ \mathfrak{D}_1\\ \bot\end{matrix}}{\neg\varphi}\neg I_{,1} \qquad \dfrac{\neg\varphi \quad \begin{matrix}\mathfrak{D}_2\\ \varphi\end{matrix}}{\bot}\neg E$$

$$\dfrac{\begin{matrix}\mathfrak{D}_1\\ \varphi[y/x]\end{matrix}}{\forall x\varphi(x)}\forall I \qquad \dfrac{\forall x\varphi(x)}{\varphi[t/x]}\forall E \qquad \dfrac{\begin{matrix}\mathfrak{D}_1\\ \varphi(t)\end{matrix}}{\exists x\varphi[x/t]}\exists I \qquad \dfrac{\exists x\varphi(x) \quad \begin{matrix}[\varphi[y/x]]^1\\ \mathfrak{D}_2\\ \psi\end{matrix}}{\psi}\exists E_{,1}$$

[5] In this dissertation, we only consider a natural deduction system suggested by Prawitz (1965).

$\varphi[x/y]$ means the substitution of x for y in φ. We call the formulas directly above the line in each rule, 'premise,' and the formula directly below the line, 'conclusion.' *Assumptions* which can be discharged are in the square brackets, e.g. $[\varphi]$. The *open assumptions* of a derivation are the assumptions on which the end-formula depends. A derivation is called *closed* if it contains no open assumptions, otherwise it is called *open*. A *major premise* of the elimination rule for a constant is the premise containing the constant in the elimination rule and all other premises are *minor premises*. The *maximum formula* is the conclusion of an application of an introduction rule and is at the same time the major premise of an elimination rule. We follow Prawitz (1965) for the standard variable restriction of $\forall I-$ and $\exists E-$rules: the eigenvariable y must not free in the conclusion of each rule, nor in any assumption that the conclusion depends on, except for the discharged assumption $[\varphi[y/x]]$ in $\exists-$rule. Let S and S' be any natural deduction systems. S' is an *extension* of S if S' is S itself or results from S by adding further rules. We call a natural deduction system containing the rules given above a (first order) *minimal* natural deduction system. An *intuitionistic* system is an extension of the minimal system plus $EFQ-$rule. We have a *classical* system by adding $CR-$rule to the intuitionistic system.

$$\frac{\bot}{\varphi}\,EFQ \qquad \frac{\begin{array}{c}[\neg\varphi]^1\\ \mathfrak{D}\\ \bot\end{array}}{\varphi}\,CR_{,1}$$

Moreover, we have definitions of 'immediate subderivation' and 'subformula.'

Definition 1.2.1. Let $\mathfrak{D}$ and $\mathfrak{D}'$ be any derivation. A derivation $\mathfrak{D}'$ is an *immediate subderivation* of $\mathfrak{D}$ iff $\mathfrak{D}'$ is an initial part of $\mathfrak{D}$ ending with a premise of the last inference step in $\mathfrak{D}$.

Definition 1.2.2. (Subformulas) The notion of *subformula* is defined inductively by (1) φ is a subformula of φ, (2) if $\psi \circ \sigma$ is a subformula of φ then so are ψ, σ where $\circ$ is $\vee$ or $\wedge$ or $\rightarrow$, (3) if $\forall x\psi$ or $\exists x\psi$ is a subformula of φ, then so is $\psi[x/t]$.

When Gentzen (1935) introduces a natural deduction system, he explains the roles of introduction and elimination rules as below:

> The introductions represent, as it were, the 'definitions' of the symbols concerned, and the eliminations are no more, in the final analysis, than the consequences of these definitions. This fact may be expressed as follows: In eliminating a symbol, we may use the formula with whose terminal symbol we are dealing only 'in the sense afforded it by the introduction of that symbol.' (Gentzen, 1935, p. 80)

To realize his idea, he thinks that there needs to be a certain requirement.

> By making these ideas more precise it should be possible to display the $E-$inferences as unique functions of their corresponding $I-$inferences, on the basis of certain requirements.(Gentzen, 1935, p. 81)

Gentzen's idea is often interpreted as the meaning of an operator (or a constant) is exhaustively determined by its introduction rule and determines its elimination rule. Prawitz (1965) borrowed the idea and has developed it in natural deduction system. Prawitz (1965, p. 32) first suggests sufficient conditions to derive $\varphi_1 \wedge \varphi_2$, $\varphi_1 \vee \varphi_2$, $\varphi \to \psi$, $\exists x\varphi(x)$, and $\forall x\varphi(x)$.

The Sufficient Conditions for $\wedge$, $\vee$, $\to$, $\exists$ and $\forall$: For any formula φ_1 and φ_2, *sufficient conditions* to derive

(1) $\varphi_1 \wedge \varphi_2$ is a pair $(\mathfrak{D}_1, \mathfrak{D}_2)$ of derivations such that $\begin{matrix}\mathfrak{D}_1\\ \varphi_1\end{matrix}$ and $\begin{matrix}\mathfrak{D}_2\\ \varphi_2\end{matrix}$,

(2) $\varphi_1 \vee \varphi_2$ is a pair $(\mathfrak{D}_1, \mathfrak{D}_2)$ of derivations such that $\begin{matrix}\mathfrak{D}_1\\ \varphi_1\end{matrix}$ or $\begin{matrix}\mathfrak{D}_2\\ \varphi_2\end{matrix}$,

(3) $\varphi \to \psi$ is a derivation $\mathfrak{D}_1$ such that $\begin{matrix}\varphi\\ \mathfrak{D}\\ \psi\end{matrix}$,

(4) $\exists x\varphi(x)$ is a derivation $\mathfrak{D}_1$ such that $\begin{matrix}\mathfrak{D}_1\\ \varphi(t)\end{matrix}$,

$$\mathfrak{D}_1$$

(5) $\forall x\varphi(x)$ is a derivation $\mathfrak{D}_1$ such that $\varphi[y/x]$where y is not free in $\forall x\varphi(x)$ nor any assumption that $\forall x\varphi(x)$ depends on.

Each sufficient condition for an operator $\circ$ is an immediate subderivation of an introduction rule for $\circ$. Prawitz (1965, p. 32) describes that 'The I-rule for [$\circ$] thus gives a sufficient condition for [deriving] formulas that have [$\circ$] as principal sign, which is stated in terms of subformulas of these formulas.'

To realize Gentzen's idea that an elimination rule is determined by the meaning of the conclusion of an introduction rule, there must be a certain requirement that fixes the elimination rule as the inverse of the corresponding introduction rule. For such requirement, Prawitz (1965, p. 33) suggests his inversion principle which states that whatever follows from a formula must follow from the direct ground for deriving that formula:

> Let α be an application of an elimination rule that has φ as consequence, Then, [derivations] that satisfy the sufficient condition ... for deriving the major premis[es] of α, when combined with [derivations] of the minor premis[es] of α (if any), already 'contain' a [derivation] of [φ]; the [derivation] of [φ] is thus obtainable directly from the given [derivations] without the addition of α.

We summarize his principle as follows:

The Inversion Principle: Let $\mathfrak{D}_i$ be any immediate subderivation of an introduction rule for deriving the major premise of an elimination rule, $\mathfrak{D}_j$ be any derivation of minor premises of the elimination rule, and φ be any conclusion of the elimination rule. $\mathfrak{D}_i$ together with $\mathfrak{D}_j$ already derives φ without the application of the elimination rule. (i.e. any consequences of the major premise is derivable by $\mathfrak{D}_i$ together with $\mathfrak{D}_j$.)

The inversion principle reflects Gentzen's idea and says that nothing is gained by an application of an elimination rule when its major premise has been derived by means of an introduction rule. In order to show that a pair of introduction and elimination rules of each operator satisfy the inversion principle, Prawitz (1965, pp. 35–38) proposes reduction procedures for $\wedge$, $\vee$, $\rightarrow$, $\neg$, $\forall$, and $\exists$. For any derivation $\mathfrak{D}_1$ and $\mathfrak{D}_2$ ending with the same formula. Let $\mathfrak{D}_1 \rhd \mathfrak{D}_2$ mean that $\mathfrak{D}_1$ reduces to $\mathfrak{D}_2$ by applying a single reduction step to an immediate subderivation $\mathfrak{D}'$ of $\mathfrak{D}_1$. Then, the standard reduction procedures for $\wedge$, $\neg$, $\rightarrow$, $\vee$, $\forall$, and $\exists$ are as follows:

1. The standard reduction procedure for $\wedge$.

$$\dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1 & \mathfrak{D}_2\\ \varphi_1 & \varphi_2\end{matrix}}{\varphi_1 \wedge \varphi_2}\wedge I}{\varphi_i}\wedge E_{(i=1,2)} \qquad \rhd_{\wedge} \qquad \begin{matrix}\mathfrak{D}_i\\ \varphi_i\end{matrix}$$

2. The standard reduction procedure for $\rightarrow$.

$$\dfrac{\dfrac{\begin{matrix}[\varphi]^1\\ \mathfrak{D}_1\\ \psi\end{matrix}}{\varphi \rightarrow \psi}\rightarrow I_{,1} \quad \begin{matrix}\mathfrak{D}_2\\ \varphi\end{matrix}}{\psi}\rightarrow E \qquad \rhd_{\rightarrow} \qquad \begin{matrix}\mathfrak{D}_2\\ \varphi\\ \mathfrak{D}_1\\ \psi\end{matrix}$$

3. The standard reduction procedure for $\vee$

$$\dfrac{\dfrac{\begin{matrix}\mathfrak{D}_i\\ \varphi_i\end{matrix}}{\varphi_1 \vee \varphi_2}\vee I_{i=1,2} \quad \begin{matrix}[\varphi_1]^1\\ \mathfrak{D}_3\\ \psi\end{matrix} \quad \begin{matrix}[\varphi_2]^2\\ \mathfrak{D}_4\\ \psi\end{matrix}}{\psi}\vee E_{,1,2} \qquad \rhd_{\vee} \qquad \begin{matrix}\mathfrak{D}_1\\ \varphi_1\\ \mathfrak{D}_3\\ \psi\end{matrix} \quad or \quad \begin{matrix}\mathfrak{D}_2\\ \varphi_2\\ \mathfrak{D}_4\\ \psi\end{matrix}$$

4. The standard reduction procedure for $\neg$

$$\dfrac{\dfrac{\begin{array}{c}[\varphi]^1\\ \mathfrak{D}_1\\ \bot\end{array}}{\neg\varphi}\neg I_{,1} \qquad \begin{array}{c}\mathfrak{D}_2\\ \varphi\end{array}}{\bot}\neg E \qquad \rhd_{\neg} \qquad \begin{array}{c}\mathfrak{D}_2\\ \varphi\\ \mathfrak{D}_1\\ \bot\end{array}$$

5. The standard reduction procedure for $\forall$

$$\dfrac{\dfrac{\begin{array}{c}\mathfrak{D}_1\\ \varphi(y)\end{array}}{\forall x\varphi[x/y]}\forall I}{\varphi[t/x]}\forall E \qquad \rhd_{\forall} \qquad \begin{array}{c}\mathfrak{D}_1\\ \varphi[t/y]\end{array}$$

6. The standard reduction procedure for $\exists$

$$\dfrac{\dfrac{\begin{array}{c}\mathfrak{D}_1\\ \varphi(t)\end{array}}{\exists x\varphi[x/t]}\exists I \qquad \begin{array}{c}[\varphi[y/x]]^1\\ \mathfrak{D}_2\\ \psi\end{array}}{\psi}\exists E_{,1} \qquad \rhd_{\exists} \qquad \begin{array}{c}\mathfrak{D}_1\\ \varphi[y/t]\\ \mathfrak{D}_2\\ \psi\end{array}$$

These standard reduction procedures are also known as the process of reducing the degree of a maximum formula. A reduction process which reduces the degree of a maximum formula in accordance with the inversion principle will be called a *standard reduction* procedure. The notions of 'degree' and 'length' of a derivation are defined as below.

Definition 1.2.3. The *degree* $d(\varphi)$ of a formula φ is defined by $d(\bot) = 0$, $d(\alpha) = 0$ for an atomic formula α, $d(\varphi \circ \psi) = d(\varphi) + d(\psi) + 1$ for binary operators $\circ$, $d(\circ\varphi) = d(\varphi) + 1$ for unary operators $\circ$. The *degree* $d(\mathfrak{D})$ of a derivation $\mathfrak{D}$ is defined as the highest degree of a maximum formula in $\mathfrak{D}$; $d(\mathfrak{D}) = 0$ if there is no such occurrences. The *length* of a derivation $\mathfrak{D}$ is the number of formula occurrences.

When the derivation has no maximum formula, we say that it is in *normal form*. Let $\mathbb{R}$ be a set of reduction procedures. Every reduction procedure in $\mathbb{R}$ is to be closed under substitution of derivations for open assumptions, and the notions of 'normal' and 'normalizable' are defined in the following ways.

Definition 1.2.4. A derivation $\mathfrak{D}$ *immediately reduce* to $\mathfrak{D}'$ ($\mathfrak{D} \rhd \mathfrak{D}'$) iff $\mathfrak{D}'$ is obtained from $\mathfrak{D}$ by replacing a subderivation of $\mathfrak{D}$ through a single-step reduction procedure for it. A sequence $< \mathfrak{D}_1, ..., \mathfrak{D}_i, \mathfrak{D}_{i+1}, ... >$ of derivations is a *reduction sequence* relative to $\mathbb{R}$ iff $\mathfrak{D}_i \rhd \mathfrak{D}_{i+1}$ relative to $\mathbb{R}$ where $1 \leqslant i$ for any natural number i.[6] A derivation $\mathfrak{D}_1$ is *reducible* to $\mathfrak{D}_i$ ($\mathfrak{D}_1 \succ \mathfrak{D}_i$) relative to $\mathbb{R}$ iff there is a sequence $< \mathfrak{D}_1, \mathfrak{D}_2, ..., \mathfrak{D}_i >$ relative to $\mathbb{R}$ where for each $j < i$, $\mathfrak{D}_j \rhd \mathfrak{D}_{j+1}$; $\mathfrak{D}_1$ is *irreducible* relative to $\mathbb{R}$ iff there is no derivation $\mathfrak{D}'$ to which $\mathfrak{D}_1 \rhd \mathfrak{D}'$ relative to $\mathbb{R}$ except $\mathfrak{D}_1$ itself.

Definition 1.2.5. A derivation $\mathfrak{D}$ is *normal* (or in *normal form*) relative to $\mathbb{R}$ iff $\mathfrak{D}$ is irreducible relative to $\mathbb{R}$, i.e. $\mathfrak{D}$ has no maximum formula. A reduction sequence *terminates* iff it has a finite number of derivations and its last derivation is in normal form. A derivation $\mathfrak{D}$ is *normalizable* relative to $\mathbb{R}$ iff there is a terminating reduction sequence relative to $\mathbb{R}$ starting from $\mathfrak{D}$. $\mathfrak{D}$ is *strongly normalizable* relative to $\mathbb{R}$ iff every reduction sequence relative to $\mathbb{R}$ that starts from $\mathfrak{D}$ terminates.[7]

Definition 1.2.6. A reduction procedure $\begin{array}{c}\varphi_1, ..., \varphi_n\\ \mathfrak{D}\\ \psi\end{array} \rhd \begin{array}{c}\varphi_1, ..., \varphi_n\\ \mathfrak{D}'\\ \psi\end{array}$ in $\mathbb{R}$ is *closure under substitution* iff, for any derivation $\begin{array}{ccc}\mathfrak{D}_1 & & \mathfrak{D}_n\\ \varphi_1 & , ..., & \varphi_n\end{array}$, a reduction procedure $\begin{array}{c}\mathfrak{D}_1 \quad \mathfrak{D}_n\\ \varphi_1, ..., \varphi_n\\ \mathfrak{D}\\ \psi\end{array} \rhd \begin{array}{c}\mathfrak{D}_1 \quad \mathfrak{D}_n\\ \varphi_1, ..., \varphi_n\\ \mathfrak{D}'\\ \psi\end{array}$ is in $\mathbb{R}$ as well.

Prawitz (1965, 1971) has proved (weak) normalization theorem for the first order intuitionistic logic which says that each derivation in a system for the first order intuitionistic

[6] For any term x and y, let $x \leqslant y$ mean that x is less than or equal to y.

[7] When all elimination rules are stated in generalized form, the set $\mathbb{R}$ of reduction procedures has additional reduction procedures, called permutation conversion. Then, an irreducible derivation is the same as the derivation that all major premises are assumptions or not derived by any rules in a given system. In that case, we shall use 'full normal form' and 'full normalizable' rather than 'normal form' and 'normalizable.' Generalized elimination rules and related terminologies will be introduced in Chapter 2. In addition, if there is no misunderstanding, for our convenience sake, we drop the 'relative to $\mathbb{R}$ in the suggested notions.

logic reduces to be in normal form. The corollary of the result is that every (closed) derivation in the system can be reduced to one using an introduction rule in the last step. Since no introduction rule derive an absurdity ($\perp$) as its conclusion, normalization theorem implies that there is no derivation of $\perp$ in the system, i.e. the consistency of the system. In addition, he has shown the normalization theorem for the weak first order classical predicate logic which only contains rules for $\perp$, $\rightarrow$, $\wedge$, $\forall$. Prawitz (2015) extends the result to the weak classical first order arithmetic and proposes the consistency proof of the weak classical arithmetic.

An interesting point is that the normalization theorem seems to show that any normal derivation does not derive $\perp$. So to speak, any absurdities derived by the paradoxes may be eliminated by the requirement of a normal derivation that every derivation must be in normal form. While investigating Russell's paradox, Prawitz (1965, p. 95) first remarks, 'the set-theoretic paradoxes are ruled out by the requirement that the [derivations] shall be in normal.' His derivation formalizing Russell's paradox falls into a non-terminating reduction sequence and so is not reducible to a normal derivation. The requirement of a normal derivation may be a proof-theoretic solution to the paradoxes.

Neil Tennant (1982, 2017) has a similar perspective of the genuine paradoxes. He has proposed the proof-theoretic conjecture for genuine paradoxes and believed that the conjecture provides a proof-theoretic criterion for paradoxicality.

> The original proof-theoretic thesis stands:
>
> > Genuine paradoxes are those whose associated *proofs of absurdity*, when formalized as natural deductions, cannot be converted into normal form.
>
> This conjecture provides a proof-theoretic criterion for the identification of genuine paradoxes ... (Tennant, 2017, p. 288)

Tennant (1982) proposed a proof-theoretic criterion for paradoxicality that a genuine paradox is a derivation of an unacceptable conclusion which employs a certain form of *id est* inferences and generates an infinite reduction sequence. In the next section, we will see

Tennant's criterion for paradoxicality and the requirement of a normal derivation as a plausible solution to the paradoxes.

1.3 Tennant's Criterion for Paradoxicality (TCP) and the Requirement of a Normal Derivation (RND)

Prawitz (1965) firstly investigates that a derivation of an absurdity from the set-theoretic paradox falls into a non-terminating reduction sequence. Though he did not explicitly mention that the non-terminating reduction sequence is the distinguishing feature of the paradoxes, it is often said that he would do so. For instance, Schroeder-Heister and Tranchini (2017, p. 568) said, 'Prawitz proposed [the non-terminating reduction sequence] to be the distinguished feature of Russell's paradox.' Further to Prawitz, Tennant (1982) suggests the proof-theoretic criterion for paradoxicality and it has been developed by Tennant (1995, 2016, 2017).

Let us take the natural deduction system S_N for the naive set theory in the same manner of Prawitz (1965, Appendix B) which only contains the rules for $\wedge$, $\rightarrow$, $\neg$, and the following additional rules:

$$\dfrac{\begin{array}{c}\mathfrak{D}\\ \varphi[t/x]\end{array}}{t \in \{x|\varphi(x)\}} \in I \qquad \dfrac{t \in \{x|\varphi(x)\}}{\varphi[t/x]} \in E$$

$\in -$rules have the following standard reduction process.

$$\dfrac{\dfrac{\begin{array}{c}\mathfrak{D}\\ \varphi[t/x]\end{array}}{t \in \{x|\varphi(x)\}} \in I}{\varphi[t/x]} \in E \quad \rhd_{\in} \quad \begin{array}{c}\mathfrak{D}\\ \varphi[t/x]\end{array}$$

Let us define a parameter a as $\{x|\neg x \in x\}$. Then, an application of $\in I-$rule to $\neg a \in a$ derives $a \in a$ and an application of $\in E-$rule to $a \in a$ derives $\neg a \in a$. Prawitz (1965, p. 95) investigates that a derivation of $\perp$ from Russell's paradox cannot be transformed into a normal derivation because it raises an infinite reduction sequence.

Proposition 1.3.1. *Let us define a parameter a as $\{x|\neg x \in x\}$. Then, there is a closed derivation of $\bot$ in S_N which generates a non-terminating reduction sequence and so is not normalizable.*

Proof. Two claims justify the result.
Claim 1. there exists a closed derivation $\mathfrak{D}_3$ of $\bot$.

First, there is an open derivation $\mathfrak{D}_1$ of $\bot$ from $[a \in a]$.

$$\dfrac{\dfrac{\dfrac{[a \in a]^1}{a \in \{x|\neg x \in x\}}\,def}{\neg a \in a}\in E \qquad [a \in a]^1}{\bot}\,\neg E$$

With the derivation $\mathfrak{D}_1$, we have a closed derivation $\mathfrak{D}_2$ of $a \in a$.

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}[a \in a]^1\\ \mathfrak{D}_1\\ \bot\end{array}}{\neg a \in a}\,\neg I_{,1}}{a \in \{x|\neg x \in x\}}\in I}{a \in a}\,def$$

Then, we have a closed derivation $\mathfrak{D}_3$ of $\bot$.

$$\dfrac{\dfrac{\begin{array}{c}[a \in a]^1\\ \mathfrak{D}_1\\ \bot\end{array}}{\neg a \in a}\,\neg I_{,1} \qquad \begin{array}{c}\mathfrak{D}_2\\ a \in a\end{array}}{\bot}\,\neg E$$

Claim 2. $\mathfrak{D}_3$ initiates a non-terminating reduction sequence and is not normalizable.

$\mathfrak{D}_3$ has a maximum formula $\neg a \in a$ in the last $\neg E-$rule and, by applying $\rhd_{\neg}-$reduction, it reduces to the derivation $\mathfrak{D}_4$ below.

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}[a \in a]^1 \\ \mathfrak{D}_1 \\ \bot\end{matrix}}{\neg a \in a}\,\neg I_{,1}}{a \in \{x|\neg x \in x\}}\in I}{\neg a \in a}\in E \qquad \begin{matrix}\mathfrak{D}_2 \\ a \in a\end{matrix}}{\bot}\,\neg E$$

$\mathfrak{D}_4$ has a maximum formula $a \in \{x|\neg x \in x\}$, i.e. $a \in a$ by definition, in $\in E-$rule either. The application of $\in -$reduction provides the same derivation with $\mathfrak{D}_3$ which we started. Therefore, the reduction procedures of $\mathfrak{D}_3$ generates a non-terminating reduction sequence and $\mathfrak{D}_3$ is not a normalizable derivation. □

The reduction process of $\mathfrak{D}_3$ ends up oscillating infinitely between $\neg-$ and $\in -$reductions. The reduction process cannot eliminate every maximum formula because it always yields maximum formulas, such as $a \in \{x|\neg x \in x\}$ ans $\neg a \in a$. Tennant (1982) describes the reduction process as falling into a looping reduction sequence.

The derivation $\mathfrak{D}_3$ allows both inferences from $a \in a$ to $\neg a \in a$ and $\neg a \in a$ to $a \in a$ which are what Tennant (1982, p. 271) calls *id est* inferences. The id est inferences may be any inferences having a formula interdeducible with its own negation (or its predication). The paradoxical inferences seem to have the circularity of the id est inferences. He may assume that the circularity of the paradoxical inference and the non-terminating reduction sequence are the same phenomenon. He regards the non-terminating reduction sequence as the distinguishing feature of all paradoxes.

> It is clear that paradoxicality hinges partly on the nature of the inferences from $[a \in a]$ to $[\neg a \in a]$ and from $[\neg a \in a]$ to $[a \in a]$. ... But not every paradox need display this feature so clearly. It is .. of considerable interest to enquire after

> techniques for discerning ... whether something at root similar to this circularity of inference at work in all paradoxes. I wish to maintain that it is indeed their distinguishing feature. I propose precisely the test of non-terminating reduction sequence. (Tennant, 1982, p. 271)

If a given derivation of a paradox falls into a reduction loop, it is unable to be reduced to a normal derivation. It is often said that only a normal derivation represents a (real) proof of the true statement. Provided that there exists a legitimate requirement that every derivation representing a proof must be (in principle) reducible to a normal derivation, the requirement may block the derivation of a paradox which falls into a non-terminating reduction sequence and so is not normalizable. In Chapter 4, we shall interpret a plausible requirement from the Prawitz-Tennant analysis of the paradoxes as below.

The Requirement of a (Full) Normal Derivation(*RND*): For any derivation $\mathfrak{D}$ in natural deduction, $\mathfrak{D}$ is acceptable only if $\mathfrak{D}$ is (in principle) convertible into a normal derivation.

If *RND* is an essential one for the acceptable reasoning, the derivation $\mathfrak{D}_3$ of Proposition 1.3.1 can be rejected because it cannot be in normal form. *RND* is likely to be regarded as a similar solution to the reasoning-rejection. Whereas the reasoning-rejection is to reject a specific rule, such as the law of excluded middle, to block the derivation of an absurdity, *RND* seems to be a stronger constraint on the whole structure of the system in question. It is too hasty to consider *RND* to be a reasoning-rejection. The related issues will be discussed in Section 4.2.

Regarding the non-terminating reduction sequence as the main feature of the paradoxes, Tennant (1982) proposes the criterion for genuine paradoxes. He admits the lesson of Kripke (1975) that some paradoxes are relative to the empirical facts, and he put forward to the criterion with respect to a given model which can contain the empirical facts. Let *M* be any model and $\theta(M)$ be a set of sentences relative to *M*. Tennant (1982, p. 283) said,

> A set of sentences is paradoxical relative to *M* iff there is some proof of [$\perp$]

from $\theta(M)$, involving those sentences in *id est* inferences, that has a looping reduction sequence.

His first stipulation of the criterion for paradoxicality is not purely described in proof-theoretic fashion because he uses the notion of 'model.' Since Tennant (1982) admits the lesson of Saul Kripke (1975) that some paradoxes are relative to the empirical facts, he proposes the criterion for paradoxicality with respect to a given model of the empirical facts. Instead of using the notion of 'model' and 'set of sentences,' we will only use a 'derivation.' Tennant (1982) appears to think that some id est inferences are legitimated by empirical facts and some paradoxical derivations have open assumptions. We will consider that both open and closed derivations of $\perp$ can be paradoxical. Let us summarize the early version of Tennant's criterion for paradoxicality, TCP_E, as follows:

The Early Version of Tennant's Criterion for Paradoxicality(TCP_E): Let $\mathfrak{D}$ be any derivation of a given natural deduction system S. $\mathfrak{D}$ is a *T-paradox* if and only if

(i) $\mathfrak{D}$ is a (closed or open) derivation of $\perp$ [8],

(ii) *id est* inferences (or rules) are used in $\mathfrak{D}$,

(iii) a reduction procedure of $\mathfrak{D}$ generates a non-terminating reduction sequence, such as a reduction loop.[9]

While he confines his attention to intuitionistic proofs, Tennant (1982, p. 285) conjectures that every derivation which formulates a genuine paradox is a T-paradox, by saying,

> I undertook at the beginning to confine my attention as much as possible to intuitionistic proofs. ... It appears to me an open question whether every para-

[8] $\perp$ is not the only unacceptable conclusion. We can use a propositional variable p as an unacceptable conclusion while formulating Curry's paradox. For the examination of other cases, the reader can consult Tennant (1982)

[9] The condition (iii) can include a spiraling reduction sequence. Tennant (1995) extends his criterion for paradoxicality by embracing the spiraling reduction sequence generated by non-self-referential paradoxes, such as Yablo's paradox. As Tennant (1995, p. 207) thinks that a looping reduction sequence is the main feature of the self-referential paradoxes, we only focus on the self-referential paradoxes. A spiraling reduction sequence will not be dealt with in this paper.

doxical set of sentences (relative to a model) can be shown to be paradoxical by means of an *intuitionistic* proof within a looping reduction sequence.

He appears to think that TCP_E applies to every derivation of a genuine paradox.[10] When he introduces his criterion, Tennant (1982, p. 268) wants to regard the criterion as the conjecture for genuine paradoxes that for any derivation $\mathfrak{D}$, $\mathfrak{D}$ formalizes a genuine paradox iff $\mathfrak{D}$ is a T-paradox. For our convenience sake, we consider Tennant's criterion as the proof-theoretic criterion for genuine paradoxes. The problems of TCP_E will be discussed in the next chapter. TCP_E might have a counterexample if there exists a derivation which satisfies TCP_E but does not formalize a genuine paradox.

TCP_E has faced some counterexamples by Susan Rogerson (2006) and Schroeder-Heister and Tranchini (2017, 2018). Although Tennant's criterion for paradoxicality may not be a necessary and sufficient condition for genuine paradoxes, the revision of the criterion would provide an interesting proof-theoretic criterion for paradoxicality. This dissertation will suggest plausible counterexamples to Tennant's criterion for paradoxicality and propose additional conditions to revise the criterion to be a necessary and sufficient condition for genuine paradoxes. Hence, the present dissertation deals with two topics: (i) the proof-theoretic criterion for paradoxicality and (ii) the proof-theoretic solution to the paradoxes. In the next chapter, we will introduce counterexamples to TCP_E which raise the problem of undergeneration. Chapter 3 will investigate the overgeneration problem of Tennant's criterion and the revised version of the criterion. Chapter 4 shall discuss whether the requirement of a normal derivation can be a solution to the paradoxes.

[10] In Tennant (1982), he did not use the expression, 'genuine paradox.' However, when he suggests the later version of his criterion for paradoxicality, Tennant (2016) calls the derivation of the Liar paradox satisfying *TCP* a genuinely paradoxical derivation. Moreover, in Tennant (2017, p. 288), he said, 'Genuine paradoxes are those whose associated proofs of absurdity, when formalized as natural deductions, cannot be converted into normal form.' In this way, it is natural to think that his conjecture is about genuine paradox.

Chapter 2

Classical Reductio and A Problem of Undergeneration

There are two types of counterexamples to Tennant's criterion for genuine paradoxes: the problems of overgeneration and undergeneration. When the criterion overgenerates, it includes a non-paradoxical derivation into the realm of genuine paradoxes. The undergeneration problem gives rise to the opposite phenomenon. The criterion excludes a derivation formalizing a genuine paradox from the scope of genuine ones.

Under the assumption that Curry's paradox is a genuine paradox, Rogerson (2006, p. 174) first put forward to a derivation formalizing Curry's paradox which employs classical *reductio ad absurdum* and the derivation does not generate a non-terminating reduction sequence. Curry sentence uses a propositional variable p for any formula, but we shall instead apply $\bot$ and define a parameter a as a set $\{x|x \in x \to \bot\}$. She said,

> So, according to Tennant's theories and claims, Curry's paradox is a genuine paradox as its proof can't be normalized ... and it is characterized by the sentence $[\{x|x \in x \to \bot\} \in \{x|x \in x \to \bot\}]$ and $[\{x|x \in x \to \bot\} \in \{x|x \in x \to \bot\}]$ abbreviated to $a \in a$ and $a \in a \to \bot$ when we let a abbreviate $[\{x|x \in x \to \bot\}]$.
>
> However, I do not think Tennant has the answer. Something else has to

be going on. The Curry sentence can be used to trivialize [the principle of naive comprehension $t \in \{x|\varphi(x) \leftrightarrow \varphi(t)\}$] in a few different ways, at least one of which does not appear to generate a non-terminating reduction sequence. (Rogerson, 2006, p. 172)

In order to introduce her derivation of Curry's paradox, we borrow Prawitz's system S_N for the naive set theory introduced in Section 1.3 of Chapter 1. We use the following instances of $\in -$rules and have a natural deduction system S_{NC} by adding the rule for *classical reductio* to S_N.

$$\frac{a \in a \to \bot}{a \in \{x|x \in x \to \bot\}} \in I \qquad \frac{a \in \{x|x \in x \to \bot\}}{a \in a \to \bot} \in E \qquad \dfrac{\begin{matrix}[\neg\varphi]^1 \\ \mathfrak{D} \\ \bot\end{matrix}}{\varphi}\, CR_{,1}$$

Let $\mathbb{R}$ be a set of reduction procedures. A set $\mathbb{R}'$ is an *extension* of $\mathbb{R}$ ($\mathbb{R}' \supseteq \mathbb{R}$) if $\mathbb{R}'$ results from $\mathbb{R}$ by adding reduction procedures which are closed under substitution in $\mathbb{R}'$. Let S and S' be any natural deduction system. S' is an *extension* of S iff S' is S itself or results from S by adding further rules. Then, S_{NC} is an extension of S_N. Also, we have a set $\mathbb{R}_N$ of reduction procedures for $\wedge$, $\to$, and $\neg$. S_{NC} has a set $\mathbb{R}_{NC}$ by adding auxiliary reductions for $CR-$rule.[1] While using $CR-$rule, auxiliary reduction procedures for the conclusion of $CR-$rule are added in the set of reductions. As Prawitz (1965, p. 34) does, the notion of a *maximum formula* is redefined as a major premise which is at the same time the conclusion of $I-$ or $CR-$rules. A similar version of Rogerson's example is stated as below.

Proposition 2.0.1. *Let us define a parameter a as* $\{x|x \in x \to \bot\}$. *Then, there is a closed derivation of* $\bot$ *in* S_{NC} *with respect to* $\mathbb{R}_{NC}$ *which neither does generate a non-terminating reduction sequence nor is in normal form.*

Proof. Two claims show the result.

Claim 1. There is a closed derivation Σ_2 of $\bot$ in S_{NC}.

[1]Auxiliary reduction procedures for $CR-$rule will be introduced in Section 2.2.

First, there is an open derivation Σ_1 of $\bot$ from $[\neg a \in a]$.

$$\dfrac{[\neg a \in a]^1 \qquad \dfrac{\dfrac{\dfrac{\dfrac{[\neg a \in a]^1 \quad [a \in a]^2}{\bot}\,\neg E}{a \in a \to \bot}\to I_{,2}}{a \in \{x|x \in x \to \bot\}}\in I}{a \in a}\,def}{\bot}\,\neg E$$

With the derivation Σ_1, we have a closed derivation Σ_2 as follows.

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}[\neg a \in a]^1\\ \Sigma_1\\ \bot\end{array}}{a \in a}\,CR_{,1}}{a \in \{x|x \in x \to \bot\}}\,def}{a \in a \to \bot}\in E \qquad \dfrac{\begin{array}{c}[\neg a \in a]^3\\ \Sigma_1\\ \bot\end{array}}{a \in a}\,CR_{,3}}{\bot}\to E$$

Claim 2. Σ_2 neither does generate a non-terminating reduction sequence nor is in normal form.

Since there is no reduction procedure in $\mathbb{R}_{NC}$ applicable to Σ_1 and Σ_2, Σ_2 does not initiate a non-terminating reduction sequence. Also, $a \in \{x|x \in x \to \bot\}$ in $\in E-$rule is a major premise and a conclusion of $CR-$rule, and so is a maximum formula. Therefore, Σ_2 is not in normal form. □

With respect to Tennant's criterion, TCP_E, Proposition 2.0.1 says that Σ_2 is not a T-paradox because Σ_2 does not yield a non-terminating reduction sequence. When she deals with a similar derivation, Rogerson (2006, p. 174) concludes,

> No standard reduction steps given by [Prawitz (1965)] straightforwardly apply in this case as the use of the [∈] operator insulates the formulae from the normalization process. It seems plausible to conclude that this [derivation] does

not reduce to a normal form and does not generate a non-terminating reduction sequence in the sense of [Tennant (1982) or Tennant (1995)]. Thus, Tennant's criterion for paradoxicality does not apply here. (Rogerson, 2006, p. 174)

We call any derivation of a genuine paradox which employs $CR-$rule and does not yield a non-terminating reduction sequence, a Rogerson-type counterexample. The Rogerson-type counterexample establishes that if Curry's paradox is a genuine one, there is a derivation of a genuine paradox which is not a T-paradox. If she is right, there is a derivation of a genuine paradox which raises the problem of undergeneration.[2]

Rogerson seems to think that the non-terminating reduction sequence is not the primary feature of paradoxical derivations. However, as Tennant (2016, p. 2) focusses on, the non-terminating reduction sequence may be regarded as a proof-theoretic feature of the vicious circularity in the self-referential paradoxes.

> Tennant (1982) proposed a proof-theoretic criterion, or test, for paradoxicality – that of *non-terminating reduction sequences* initiated by the 'proofs of $\bot$' associated with the paradoxes in question (p. 271). In that paper, the subsequent focus was on *looping* reduction sequences. These are the proof-theorist's explication of the *vicious circularity* involved in paradoxes. (Tennant, 2016, p. 2)

It will be a retrogression in the proof-theoretic investigation of the paradoxes that the non-terminating reduction sequence is simply considered not to be related to any paradoxical features.

Rather, while we presume that the non-terminating reduction sequence is the main feature of the self-referential paradoxes, in this chapter, we deal with the problem of undergeneration. Introduction rules in Gentzen-Prawitz's natural deduction system have two forms of elimination rules: standard and generalized elimination rules. The counterexample which shows that TCP_E undergenerates can use the generalized elimination rules. As

[2] Rogerson (2006) only considers standard reduction processes suggested by Prawitz (1965). However, Schroeder-Heister and Tranchini (2017, pp. 572–573) borrow the reduction proposed by Gunnar Stålmarck (1991, pp. 131–132) and claim that her example can be further reduced. It will be seen in Proposition 2.4.1 that the reduced derivation generates a non-terminating reduction sequence.

a preliminary matter, Section 2.1 introduces generalized elimination rules with their reduction procedures and the harmony relation between introduction and elimination rules.

For our discussion about $CR-$rule and the problem of undergeneration, it should be discussed which paradoxes are genuine paradoxes. However, for our convenience, we presume in Section 2.2 that Curry and the Liar paradox are genuine paradoxes. Even though he did not mention Rogerson's counterexample, Tennant (2015, pp. 588–589) deals with a Rogerson-type counterexample of the Liar paradox which uses $CR-$rule and has no non-terminating reduction sequence. To solve the problem of undergeneration, he seems to presume that the application of $CR-$rule has a defect that the rule conceals the main feature of the paradoxes and proposes the methodological conjecture that genuine paradoxes are never strictly classical. However, the undergeneration problem is not solved by simply accusing $CR-$rule of disguising the feature because there are cases which do not use $CR-$rule but show that TCP_E undergenerates. Tennant seems not to consider seriously the problem of undergeneration. Section 2.3 proposes some counterexamples to TCP_E which represent that TCP_E undegenerates, however those counterexamples do not employ $CR-$rule. We will see that the occurrence of a non-terminating reduction sequence relies on our choice of reduction procedures. Section 2.4 deals with the question of what makes a non-terminating reduction sequence stops. With some observations, we propose a possible diagnosis that a non-terminating reduction sequence does not occur if a derivation in question includes (i) a major premise which has no reduction process to eliminate it or (ii) a formula having a principal constant which has no reduction procedure to get rid of it. Then, we suggest an additional condition to TCP_E that a derivation formalizing a genuine paradox only uses harmonious rules. If the suggested condition is acceptable, the condition can solve the problem of undergeneration.

2.1 Preliminaries: Generalized Elimination Rules and Harmony Relation.

We will see in the next chapter that the later version of Tennant's criterion for paradoxicality (TCP_L) accepts an additional condition that all elimination rules must be stated in the generalized form.[3] In addition, the early version of Tennant's criterion TCP_E does not restrict the form of elimination rules. There would be a counterexample to TCP_E which uses the generalized elimination rules. After we will introduce the generalized form of elimination rules with related terminologies in Section 2.1.1, Section 2.1.2 explains intrinsic and GE-harmony requirement for a desirable pair of introduction and elimination rules.

2.1.1 Generalized Elimination Rules

An introduction rule in natural deduction has two forms of elimination rules: standard and generalized forms. Based on the rules suggested in Section 1.2, $\wedge E-$, $\rightarrow E-$, $\neg E-$, and $\forall E-$rules have the form of standard elimination rules. $\vee-$ and $\exists-$rules have the form of generalized elimination rules.

Generalized elimination rules were first introduced by Schroeder-Heister (1984a). His purpose was to obtain a *general* schema for introduction and elimination rules for logical constants in propositional logic. His work was extended to quantifiers in Schroeder-Heister (1984b). Tennant (1992, 2002) borrows the generalized elimination rules and suggests the proof of what he calls 'ultimate normal form' for generalized intuitionistic relevant natural deductions which are isomorphic to cut- and weakening-free sequent calculus. For the proof, Tennant (1992, p. 47 and p. 50) proposes the requirement that all major premises of elimination rules *stand proud*, which means that every major premise does not stand as a conclusion of any rule. We say that a derivation is in *full normal form* iff all major premises are not derived by any rule, i.e. they are assumptions or axioms. Then, the requirement of

[3] Tennant prefers to say 'parallelized' and 'serial' rather than 'generalized' and 'standard.' In the present dissertation, however, we shall use the later names.

a full normal derivation is the same as his.[4] The main idea of his was that the requirement prevents the reorder of derivations which can make an additional reduction possible.

Roughly put, Tennant's proof of ultimate normal form has three steps. At first, Tennant (2002) shows that any derivation in an intuitionistic relevant system with generalized elimination rules are in normal form. Then, the proof establishes that every normal derivation is converted into full normal form, and thence into ultimate normal form which satisfies isomorphism between natural deduction and sequent calculus. In a similar perspective, Negri and Von Plato (2001) introduce the generalized forms of elimination rules and an isomorphic interpretation procedure between natural deduction and sequent calculus. When we use generalized elimination rules, we shall use the notion of 'full normal form' rather than 'normal form.' We adopt the isomorphic interpretation algorithm and the forms of generalized elimination rules for $\wedge$, $\vee$, $\rightarrow$, $\neg$, $\forall$, $\exists$ and their reduction procedures from Negri and Von Plato (2001).

$$\dfrac{\begin{matrix}\mathfrak{D}_1 & \mathfrak{D}_2\\ \varphi_1 & \varphi_2\end{matrix}}{\varphi_1 \wedge \varphi_2}\wedge I \qquad \dfrac{\varphi_1 \wedge \varphi_2 \quad \begin{matrix}[\varphi_1]^1, [\varphi_2]^1\\ \mathfrak{D}_3\\ \psi\end{matrix}}{\psi}\wedge E_{,1} \qquad \dfrac{\begin{matrix}[\varphi]^1\\ \mathfrak{D}_1\\ \psi\end{matrix}}{\varphi \rightarrow \psi}\rightarrow I_{,1} \qquad \dfrac{\varphi \rightarrow \psi \quad \begin{matrix}\mathfrak{D}_2\\ \varphi\end{matrix} \quad \begin{matrix}[\psi]^1\\ \mathfrak{D}_3\\ \chi\end{matrix}}{\chi}\rightarrow E_{,1}$$

$$\dfrac{\begin{matrix}\mathfrak{D}_1\\ \varphi_i\end{matrix}}{\varphi_1 \vee \varphi_2}\vee I_{(i=1,2)} \qquad \dfrac{\varphi_1 \vee \varphi_2 \quad \begin{matrix}[\varphi_1]^1\\ \mathfrak{D}_2\\ \psi\end{matrix} \quad \begin{matrix}[\varphi_2]^2\\ \mathfrak{D}_3\\ \psi\end{matrix}}{\psi}\vee E_{,1,2} \qquad \dfrac{\begin{matrix}[\varphi]^1\\ \mathfrak{D}_1\\ \bot\end{matrix}}{\neg\varphi}\neg I_{,1} \qquad \dfrac{\neg\varphi \quad \begin{matrix}\mathfrak{D}_2\\ \varphi\end{matrix} \quad \begin{matrix}[\bot]^1\\ \mathfrak{D}_3\\ \psi\end{matrix}}{\psi}\neg E_{,1}$$

$$\dfrac{\begin{matrix}\mathfrak{D}_1\\ \varphi[y/x]\end{matrix}}{\forall x\varphi(x)}\forall I \qquad \dfrac{\forall x\varphi(x) \quad \begin{matrix}[\varphi[t/x]]^1\\ \mathfrak{D}_2\\ \psi\end{matrix}}{\psi}\forall E_{,1} \qquad \dfrac{\begin{matrix}\mathfrak{D}_1\\ \varphi(t)\end{matrix}}{\exists x\varphi[x/t]}\exists I \qquad \dfrac{\exists x\varphi(x) \quad \begin{matrix}[\varphi[y/x]]^1\\ \mathfrak{D}_2\\ \psi\end{matrix}}{\psi}\exists E_{,1}$$

[4]Even though the requirement was first suggested by Tennant (1992), it looks as if Negri and Von Plato (2001) have introduced an explicit notion of full normal form. For our convenience, we shall use their notion of 'full normal form' in the sense that every major premise is an assumption, rather than Tennant's notion of 'stand proud.'

The standard reduction procedures for $\wedge$, $\rightarrow$, $\vee$, $\neg$, $\forall$, and $\exists$ are as below[5]:

1. The standard reduction procedure for $\wedge$.

$$\dfrac{\dfrac{\begin{array}{cc}\mathfrak{D}_1 & \mathfrak{D}_2\\ \varphi_1 & \varphi_2\end{array}}{\varphi_1 \wedge \varphi_2}\wedge I \qquad \begin{array}{c}[\varphi_1]^1, [\varphi_2]^1\\ \mathfrak{D}_3\\ \psi\end{array}}{\psi}\wedge E_{,1} \qquad \rhd_{\wedge} \qquad \begin{array}{c}\mathfrak{D}_1 \quad \mathfrak{D}_2\\ \varphi_1 \quad \varphi_2\\ \mathfrak{D}_3\\ \psi\end{array}$$

2. The standard reduction procedure for $\rightarrow$.

$$\dfrac{\dfrac{\begin{array}{c}[\varphi]^1\\ \mathfrak{D}_1\\ \psi\end{array}}{\varphi \rightarrow \psi}\rightarrow I_{,1} \qquad \begin{array}{c}\mathfrak{D}_2\\ \varphi\end{array} \qquad \begin{array}{c}[\psi]^2\\ \mathfrak{D}_3\\ \chi\end{array}}{\chi}\rightarrow E_{,2} \qquad \rhd_{\rightarrow} \qquad \begin{array}{c}\mathfrak{D}_2\\ \varphi\\ \mathfrak{D}_1\\ \psi\\ \mathfrak{D}_3\\ \chi\end{array}$$

3. The standard reduction procedure for $\vee$

$$\dfrac{\dfrac{\begin{array}{c}\mathfrak{D}_i\\ \varphi_i\end{array}}{\varphi_1 \vee \varphi_2}\vee I_{i=1,2} \qquad \begin{array}{c}[\varphi_1]^1\\ \mathfrak{D}_3\\ \psi\end{array} \qquad \begin{array}{c}[\varphi_2]^2\\ \mathfrak{D}_4\\ \psi\end{array}}{\psi}\vee E_{,1,2} \qquad \rhd_{\vee} \qquad \begin{array}{c}\mathfrak{D}_1\\ \varphi_1\\ \mathfrak{D}_3\\ \psi\end{array} \quad or \quad \begin{array}{c}\mathfrak{D}_2\\ \varphi_2\\ \mathfrak{D}_4\\ \psi\end{array}$$

4. The standard reduction procedure for $\neg$

[5] The special elimination rules for $\forall$ and $\exists$ have additional variable restrictions suggested by Prawitz (1965, pp. 37-38). For the reduction procedures for generalized elimination rules for $\forall$ and $\exists$, we apply the same restrictions. The all free variables in $\mathfrak{D}_1$ of both standard reduction procedures for $\forall$ and $\exists$ are different from both y and t. Since there can be no free occurrence of x in φ, $\varphi[x/y][t/x]$ is the same as $\varphi[t/y]$. The assumptions in $\mathfrak{D}_2$ of both stadard reduction procedures for $\forall$ and $\exists$ that ψ depends do not contain any occurrence of y, and y does not occur in ψ. For the process for $\forall$, the assumptions in $\mathfrak{D}_1$ on which $\forall x\varphi[x/y]$ depends do not contain any occurrence of y.

$$\dfrac{\dfrac{\begin{matrix}[\varphi]^1\\ \mathfrak{D}_1\\ \bot\end{matrix}}{\neg\varphi}\neg I_{,1} \quad \begin{matrix}\mathfrak{D}_2\\ \varphi\end{matrix} \quad \begin{matrix}[\bot]^2\\ \mathfrak{D}_3\\ \psi\end{matrix}}{\psi}\neg E_{,2} \qquad \rhd_\neg \qquad \begin{matrix}\mathfrak{D}_2\\ \varphi\\ \mathfrak{D}_1\\ \bot\\ \mathfrak{D}_3\\ \psi\end{matrix}$$

5. The standard reduction procedure for $\forall$

$$\dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1\\ \varphi(y)\end{matrix}}{\forall x\varphi[x/y]}\forall I \quad \begin{matrix}[\varphi[x/y][t/x]]^1\\ \mathfrak{D}_1\\ \psi\end{matrix}}{\psi}\forall E_{,1} \qquad \rhd_\forall \qquad \begin{matrix}\mathfrak{D}_1\\ \varphi[t/y]\\ \mathfrak{D}_2\\ \psi\end{matrix}$$

6. The standard reduction procedure for $\exists$

$$\dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1\\ \varphi(t)\end{matrix}}{\exists x\varphi[x/t]}\exists I \quad \begin{matrix}[\varphi[y/x]]^1\\ \mathfrak{D}_2\\ \psi\end{matrix}}{\psi}\exists E_{,1} \qquad \rhd_\exists \qquad \begin{matrix}\mathfrak{D}_1\\ \varphi(t)\\ \mathfrak{D}_2[y/t]\\ \psi\end{matrix}$$

When all elimination rules are formulated in generalized form, all normal derivations are not in full normal form. For instance, there is a derivation which has no maximum formula and so is in normal form, but is not in full normal form. We consider the following two derivations.

$$\dfrac{\dfrac{[\varphi\wedge(\psi\wedge\sigma)]}{\varphi\wedge\psi}\wedge E}{\sigma}\wedge E \qquad\qquad \dfrac{\dfrac{[\varphi\wedge(\psi\wedge\sigma)] \quad [\psi\wedge\sigma]^1}{\psi\wedge\sigma}\wedge E_{,1} \quad [\sigma]^2}{\sigma}\wedge E_{,2}$$

The elimination rules for $\wedge$ in the left side derivation are stated in standard form. On the other hand, the right side derivation consists of generalized elimination rules for $\wedge$. The right side derivation is in normal form but is not in full normal form. To reduce the degree of the major premise in the last $\wedge E-$rule, we need to apply *permutation conversion* to the derivation and have the following derivation.[6]

$$\dfrac{[\varphi \wedge (\psi \wedge \sigma)] \qquad \dfrac{[\varphi \wedge \psi]^1 \quad [\psi]^2}{\sigma} \wedge E_{,2}}{\sigma} \wedge E_{,1}$$

Hence, when we use generalized elimination rules, we will use the notion of 'full normal form' rather than 'normal form.' Moreover, permutation conversion is the essential processes to eliminate the major premise which is derived by an elimination rule. It will be an additional auxiliary reduction procedure.

General elimination rules will be used in Section 2.3 and in Chapter 3. It is often encouraged to use harmonious introduction and elimination rules. The next subsection will introduce intrinsic and GE-harmony relations between introduction and elimination rules.

2.1.2 An Intrinsic and a GE-Harmony Relation Between Introduction and Elimination rules.

Gentzen first proposed introduction and elimination rules for natural deduction. Prawitz (1965, 1971) has proved the normalization theorem for (weak) classical and intuitionistic natural deduction systems that every derivation is reducible to a normal derivation. The normalization theorem is proved based on Gentzen's idea that the meaning of an principal operator (or a principal constant) is exhaustively determined by introduction rules and determines corresponding elimination rules. Prawitz's inversion principle reflects the idea.

The Inversion Principle: Let $\mathfrak{D}_\mathrm{i}$ be any immediate subderivation of an introduction rule for deriving the major premise of an elimination rule, $\mathfrak{D}_\mathrm{j}$ be any derivation of minor premises of the elimination rule, and φ be any conclusion of the elimination rule. $\mathfrak{D}_\mathrm{i}$ together with $\mathfrak{D}_\mathrm{j}$ already derives φ without the application of the elimination rule. (i.e. any consequences of the major premise is derivable by $\mathfrak{D}_\mathrm{i}$ together with $\mathfrak{D}_\mathrm{j}$.)

[6]The *permutation conversion* was found by Gentzen (2008) and Prawitz (1965) for $\vee-$ and $\exists E-$rules. It is an essential process to reduce the degree of the major premise derived by an elimination rule. Particular instances of permutation conversions are introduced in Appendix 2.B.

The inversion principle says that nothing is gained by deriving a formula from a major premise of an elimination rule when the major premise is a conclusion of an introduction rule. When any pair of introduction and elimination rules satisfies the inversion principle, they have a method to eliminate a major premise of the elimination rules given by the introduction rules. Based on his theory of meaning, Michael Dummett (1991, p. 250) treats 'the eliminability of [major premises] as a criterion for intrinsic harmony.' Dummett's intrinsic harmony requirement can be explained via the inversion principle.

Definition 2.1.1. (Intrinsic Harmony) Let $\circ$ be an operator (or a constant). Introduction and elimination rules for $\circ$ are *intrinsically harmonious* iff every pair consisting of an $\circ I-$ and $\circ E-$rules satisfies the inversion principle.

Not all pairs of introduction and elimination rules satisfy intrinsic harmony. Intrinsic harmony requirement should demand that *every* pair of introduction and elimination rules satisfy the inversion principle. Let us consider a well-known example of $tonk-$rules introduced by Arthur Prior (1960) with some variation.

$$\dfrac{\begin{matrix}\mathfrak{D}_i\\ \varphi_i\end{matrix}}{\varphi_1\ tonk\ \varphi_2}\,tonkI_i(i=1,2) \qquad \dfrac{\varphi_1\ tonk\ \varphi_2}{\varphi_i}\,tonkE_i(i=1,2)$$

$tonkI-$rules have the same form of $\vee I-$rules, but $tonkE-$rules have the standard form of $\wedge E-$rules. $tonk-$rules derive any formulas and are problematic. For instance,

$$\dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1\\ \varphi_1\end{matrix}}{\varphi_1\ tonk\ \varphi_2}\,tonkI_1}{\varphi_2}\,tonkE_2$$

From the derivation $\mathfrak{D}_1$ of φ_1, $tonkI_1-$ and $tonkE_2-$rules can have any consequences. If the intrinsic harmony requirement only demands a pair of introduction and elimination rules satisfying the inversion principle, $tonk-$rules have a reduction process between $tonkI_1-$ and $tonkE_1-$rules and are not problematic.

$$\dfrac{\dfrac{\begin{array}{c}\mathfrak{D}_1\\ \varphi_1\end{array}}{\varphi_1\ tonk\ \varphi_2}\,tonkI_1}{\varphi_1}\,tonkE_1 \quad \rhd \quad \begin{array}{c}\mathfrak{D}_1\\ \varphi_1\end{array}$$

However, no reduction procedure exists between $tonkI_1-$ and $tonkE_2-$rules (also between $tonkI_2-$ and $tonkE_1-$rules). Therefore, in order to block $tonk-$rules, we will request *any* pair of introduction and elimination rules satisfies the inversion principle.

Dummett (1991, p. 287) considers that intrinsic harmony may be too weak requirement. Roughly, intrinsic harmony prevents elimination rules to be stronger than the corresponding introduction rules whereas it does not restrict elimination rules which is weaker than the corresponding introduction rules. For example, we consider the following form of rules for $\blacktriangle$.

$$\dfrac{\begin{array}{cc}\mathfrak{D}_1 & \mathfrak{D}_2\\ \varphi & \psi\end{array}}{\varphi\blacktriangle\psi}\,\blacktriangle I \qquad \dfrac{\varphi\blacktriangle\psi}{\varphi}\,\blacktriangle E$$

$\blacktriangle-$rules are intrinsically harmonious, but yet $\blacktriangle E-$rule does not fully infer any consequences of the meaning of $\varphi\blacktriangle\psi$ conferred by $\blacktriangle I-$rule. So to speak, elimination rules should be neither stronger nor weaker than the corresponding introduction rules. For such demand, he proposes the *stability* requirement.

> A little reflection shows that [intrinsic] harmony is an excessively modest demand. ... The fact that the consequences we conventionally draw from [a formula] are in harmony with these acknowledged grounds shows only that we draw no consequences its meaning does not entitle us to draw. It does not show that we fully exploit that meaning, that we are accustomed to draw all those consequences we should be entitled to draw. ... Such a balance is surely desirable ... . The demand that such a condition be met goes beyond the requirement of harmony: we may call it 'stability' (Dummett, 1991, p. 287)

Even though Dummett (1991, Ch. 13) dedicates one chapter of the *Logical Basis of Metaphysics* to stability, it is far from clear how the rigorous account of stability is to be filled in.

There are two accounts of harmony to realize the idea of stability: Tennant's account of harmony as *deductive equilibrium* and generalized elimination account of harmony. Florian Steinberger (2009) suggests a counterexample to Tennant's account that it sanctions as harmonious obviously invalid rules for existential quantifier, i.e. an existential elimination rules which lack the usual variable restriction on the parameter. Tennant (2010) responses to the counterexample by the requirement of a proof of admissibility that the employment of structural rules, such as *Cut*, has to be legitimated by a proof of their admissibility.[7] Steinberger (2011) rebuts again that once the requirement of admissibility is introduced, Tennant's account of harmony as deductive equilibrium no longer has any role to play in the realization of stability. Moreover, he said,

> It follows that harmony should be understood as a relational property of pairs of inference rules (and by extension of the logical constants governed by them) and not as a property of deductive systems. Consequently, the admissibility of CUT, a property of deductive system, is not a candidate for formalizing the intuitive notion of harmony. (Steinberger, 2011, p. 278)

Tennant did not yet respond to Steinberger's objection. If Steinberger is right, Tennant's account of harmony as deductive equilibrium has a serious defect to realize the idea of stability.

The other candidate account is the generalized elimination harmony (GE-harmony for short). Unlike intrinsic harmony, GE-harmony delivers a method for generating GE-harmonious rules. Recently, the application of generalized elimination rules and the notion of GE-harmony has been developed by Dyckhoff and Francez (2012) and Negri and Von Plato (2001). Especially, Negri and Von Plato (2001, p. 6) suggest the generalized version of

[7] Let S be a system of rules. A rule with the premises $\varphi_1, ..., \varphi_n$ and the conclusion ψ is *admissible* in S if, whenever the premises $\varphi_1, ..., \varphi_n$ are derivable in S, the corresponding conclusion ψ is derivable in S.

inversion principle that whatever follows from the direct grounds for deriving a formula must follow from that formula. We call it the *Generalized Inversion Principle*:

The Generalized Inversion Principle: Let $\mathfrak{D}_i$ be any immediate subderivation of an introduction rule for deriving a major premise of an elimination rule, $\mathfrak{D}_j$ be any derivation of minor premises of the elimination rule. Whatever is derivable by $\mathfrak{D}_i$ together with $\mathfrak{D}_j$ is a consequence of the major premise.

The principle is standardly taken to be formally represented by Dyckhoff and Francez (2012). They have proposed the general form of introduction and generalized elimination rules which satisfy the generalized inversion principle. The general forms of introduction and generalized elimination rules with their standard reduction procedures are called the *GE-schema*. All rules and their standard reductions introduced in Section 2.1.1 are instances of the GE-schema. GE-harmony relation is defined in terms of the GE-schema such that a pair of introduction and elimination rules is *GE-harmonious* iff an elimination rule has been induced from the introduction rule by means of the GE-schema.

The intrinsic harmony and the GE-harmony have been considered to be the main contemporary accounts of harmony. It is encouraged to use intrinsically harmonious rules (or GE-harmonious rules if generalized elimination rules are employed). It is still an open question of whether we should only use harmonious rules or not. Although it is desirable to use harmonious rules, it seems to be a too strong requirement that only harmonious rules are acceptable.

For our purpose of investigating the problem of undergeneration, the next three sections will argue that applications of axioms having a principal operator (or a principal constant) that no I-rule introduces or of rules without having its corresponding harmonious rules can block the occurrence of a non-terminating reduction sequence.

2.2 The Methodological Conjecture and the Problem of Undergeneration

Tennant (2016, Sec. 4) claims that the Liar paradox is a genuine paradox by suggesting the same result with Proposition 2.A.1 in Appendix 2.A. On the other hand, it is easily seen that there exists a Rogerson-type counterexample formalizing the Liar paradox which represents that TCP_E undergenerates, i.e. TCP_E excludes a derivation of the Liar paradox from the realm of genuine paradoxes. Tennant (2015, pp. 588–589) recognizes that a derivation of the Liar paradox employing classical inferences, such as $CR-$rule, gives rise to problems that it appears to be a normal derivation of $\perp$ and it does not initiate a non-terminating reduction sequence. For the answer to the problems, he proposes the methodological conjecture that genuine paradoxes are never strictly classical. When he introduces the methodological conjecture, he claims, 'The use of classical *reductio* has masked the real defect that lies at the heart of paradoxical reasoning'(Tennant, 2015, p. 589). He thinks that the application of $CR-$rule disguises the main feature of genuine paradoxes, i.e. the non-terminating reduction sequence. In this section, we introduce his formulation of the Liar paradox and his methodological conjecture as the answer against the Rogerson-type counterexample. Since the conjecture explicates that classical inferences do not need to be used in derivations of genuine paradoxes and he thinks that the applications of classical inferences cause to stop generating the main feature of genuine paradoxes, we understand in this section that his answer to the problem of undergeneration is that classical inferences, such as $CR-$rule must not be used in derivations of genuine paradoxes.

When he suggests his example of the Liar paradox, Tennant (2015, pp. 588–589) uses the rules for the unary truth-predicate, $T(x)$ which states that x is true.

$$\frac{\varphi}{T(\ulcorner \varphi \urcorner)}\,TI \qquad \frac{T(\ulcorner \varphi \urcorner)}{\varphi}\,TE$$

In addition, he employs similar rules for the reflexity of identity and the substitutivity of identity introduced in Per Martin-Löf (1971, p. 190). Tennant (2007, p. 1061) accepts the following rules for identity.

$$\frac{\varphi(t)}{t=t}{=}I \qquad \frac{t=u \quad \psi(t,t)}{\psi(t,u)}{=}E$$

where t and u are any terms and $\varphi(t)$ is an atomic formula. On Martin-Löf's account of $=E-$rule, given the major premise $t=u$, ψ defines any reflexive (binary) relation and $=E-$rule binds a term t in the conclusion $\psi(t,u)$ such that t bears any (binary) relation ψ to u. By using either $\rightarrow-$rules or $\wedge-$rules, the rule of substitutivity of identity is readily derivable from $=E-$rule. (Cf. Martin-Löf (1971, p. 190) and Tennant (2007, p. 1062)) So we shall take reflexivity and substitutivity, respectively, as the introduction and elimination rules for identity:

$$\frac{\varphi(t)}{t=t}{=}I \qquad \frac{t=u \quad \sigma(t)}{\sigma(u)}{=}E$$

Let S_E be a natural deduction system containing $\neg-$, $T-$, and $=-$rules. S_E has a set $\mathbb{R}_E$ of reduction procedures for $\neg$ and the following standard reduction and the auxiliary reduction processes.

1. The standard reduction procedure for $T(x)$

$$\dfrac{\dfrac{\begin{matrix}\mathfrak{D}\\ \varphi\end{matrix}}{T(\ulcorner\varphi\urcorner)}TI}{\varphi}TE \qquad \rhd_{T(x)} \qquad \begin{matrix}\mathfrak{D}\\ \varphi\end{matrix}$$

2. The auxiliary reduction procedure for the substitutivity of identity.

$$\dfrac{t_2=t_1 \qquad \dfrac{\begin{matrix}\mathfrak{D}\\ t_1=t_2 \quad \varphi(t_1)\end{matrix}}{\varphi(t_2)}{=}E}{\varphi(t_1)}{=}E \qquad \trianglerighteq_{Sub} \qquad \begin{matrix}\mathfrak{D}\\ \varphi(t_1)\end{matrix}$$

The role of the standard reduction procedures is to eliminate the degree of a maximum formula and satisfies the inversion principle whereas the auxiliary reduction procedures are often proposed independently of the inversion principle. For example, $\trianglerighteq_{Sub}-$reduction above is just lowering the length of a derivation and its role does not seem to pursue the inversion principle. Also, the following reductions are different from the standard reductions.

3. The auxiliary reduction procedure for $CR-$rule with regard to $\neg-$rules.

$$\dfrac{\begin{array}{c}[\neg\neg\varphi]^1\\ \mathfrak{D}\\ \bot\end{array}}{\neg\varphi}\,CR_{,1} \qquad \trianglerighteq_{CR(\neg)} \qquad \dfrac{\dfrac{\begin{array}{c}\dfrac{\dfrac{[\neg\bot]^3 \quad \dfrac{[\neg\varphi]^1 \quad [\varphi]^2}{\bot}\,\neg E}{\bot}\,\neg E}{\neg\neg\varphi}\,\neg I_{,1}\\ \mathfrak{D}\\ \bot\end{array}}{\bot}\,CR_{,3}}{\neg\varphi}\,\neg I_{,2}$$

4. The auxiliary reduction procedure for $CR-$rule with regard to $T-$rules.

$$\dfrac{\begin{array}{c}[\neg T(\ulcorner\varphi\urcorner)]^1\\ \mathfrak{D}\\ \bot\end{array}}{T(\ulcorner\varphi\urcorner)}\,CR_{,1} \qquad \trianglerighteq_{CR(T(x))} \qquad \dfrac{\dfrac{\begin{array}{c}\dfrac{\dfrac{[\neg\varphi]^2 \quad \dfrac{[T(\ulcorner\varphi\urcorner)]^1}{\varphi}\,TE}{\bot}\,\neg E}{\neg T(\ulcorner\varphi\urcorner)}\,\neg I_{,1}\\ \mathfrak{D}\\ \bot\end{array}}{\varphi}\,CR_{,2}}{T(\ulcorner\varphi\urcorner)}\,TI$$

$\trianglerighteq_{CR(\neg)}-$ and $\trianglerighteq_{CR(T(x))}-$reductions neither do eliminate a maximum formula nor lower the length of a derivation, but they lower the degree of the conclusion of $CR-$rule.

Tennant (2015, pp. 585–588) proposes a derivation of the Liar paradox which does not employ $CR-$rule and does generate a looping reduction sequence. We will use a natural deduction system S_E which has $\neg-$, $T-$, $=-$rules. A set $\mathbb{R}_E$ of reductions contains $\rhd_{\neg}-$, $\rhd_{T(x)}-$, and $\trianglerighteq_{Sub}$. Then, we have the result in S_E with respect to $\mathbb{R}_E$.

Proposition 2.2.1. *Suppose that, for some formula Φ, $\ulcorner\Phi\urcorner = \ulcorner\neg T(\ulcorner\Phi\urcorner)\urcorner$ is an axiom of S_E. S_E relative to $\mathbb{R}_E$ has a closed derivation of $\bot$ which generates a non-terminating reduction sequence, so is not normalizable.*

Proof. The result consists of two claims

Claim 1. there is a closed derivation Δ_3 of $\bot$ in S_E

We begin with an open derivation Δ_1 of $\bot$ from $[T(\ulcorner\Phi\urcorner)]$.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\ulcorner\Phi\urcorner = \ulcorner\neg T(\ulcorner\Phi\urcorner)\urcorner}\,Ax_1 \qquad [T(\ulcorner\Phi\urcorner)]^1}{T(\ulcorner\neg T(\ulcorner\Phi\urcorner)\urcorner)}\,=E}{\neg T(\ulcorner\Phi\urcorner)}\,TE \qquad [T(\ulcorner\Phi\urcorner)]^1}{\bot}\,\neg E$$

With the open derivation Δ_1, we have the closed derivation Δ_2 of $T(\ulcorner\phi\urcorner)$.

$$\dfrac{\dfrac{}{\ulcorner\neg T(\ulcorner\Phi\urcorner)\urcorner = \ulcorner\Phi\urcorner}\,Ax_2 \qquad \dfrac{\dfrac{\begin{matrix}[T(\ulcorner\Phi\urcorner)]^1\\ \Delta_1\\ \bot\end{matrix}}{\neg T(\ulcorner\Phi\urcorner)}\,\neg I_{,1}}{T(\ulcorner\neg T(\ulcorner\Phi\urcorner)\urcorner)}\,TI}{T(\ulcorner\Phi\urcorner)}\,=E$$

Then, we have the closed derivation Δ_3 of $\bot$.

$$\dfrac{\dfrac{\begin{matrix}[T(\ulcorner\Phi\urcorner)]^1\\ \Delta_1\\ \bot\end{matrix}}{\neg T(\ulcorner\Phi\urcorner)}\,\neg I_{,1} \qquad \begin{matrix}\Delta_2\\ T(\ulcorner\Phi\urcorner)\end{matrix}}{\bot}\,\neg E$$

Claim 2. Δ_3 generates a non-terminating reduction sequence, and so is not normalizable.

Δ_3 is not in normal form since it has a maximum formula $\neg T(\ulcorner\Phi\urcorner)$. By applying $\rhd_{\neg}$ to Δ_3, we have the following derivation Δ_4.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\ulcorner\Phi\urcorner = \ulcorner\neg T(\ulcorner\Phi\urcorner)\urcorner}Ax_1 \quad \dfrac{\dfrac{}{\ulcorner\neg T(\ulcorner\Phi\urcorner)\urcorner = \ulcorner\Phi\urcorner}Ax_2 \quad \dfrac{\dfrac{\begin{matrix}[T(\ulcorner\Phi\urcorner)]^1\\ \Delta_1\\ \bot\end{matrix}}{\neg T(\ulcorner\Phi\urcorner)}\neg I_{,1}}{T(\ulcorner\neg T(\ulcorner\Phi\urcorner)\urcorner)}TI}{T(\ulcorner\Phi\urcorner)}{=}E}{T(\ulcorner\neg T(\ulcorner\Phi\urcorner)\urcorner)}{=}E}{\neg T(\ulcorner\Phi\urcorner)}TE \quad \begin{matrix}\Delta_2\\ T(\ulcorner\Phi\urcorner)\end{matrix}}{\bot}\neg E$$

The application of $\trianglerighteq_{Sub}$ and $\rhd_{T(x)}$ yield the same derivation with Δ_3. Hence, the reduction procedure generates non-terminating reduction sequences and so it is not normalizable. $\square$

According to TCP_E, Δ_3 is a T-paradox. If the Liar paradox is a genuine paradox and the non-terminating reduction sequence is the main feature of the genuine paradox, Δ_3 supports the view that TCP_E is an appropriate proof-theoretic criterion for genuine paradoxes. However, by employing $CR-$rule, we have a Rogerson-type counterexample to TCP_E.

We have a system S_{CE} by adding $CR-$rule to S_E, i.e. S_{CE} is an extension of S_E. S_{CE} has a set $\mathbb{R}_{CE}$ of reduction procedures which is an extension of $\mathbb{R}_E$ by adding $\trianglerighteq_{CR(\neg)}$ and $\trianglerighteq_{CR(T(x))}$. Then, S_{CE} has Tennant's derivation of the Liar paradox using $CR-$rule. As Tennant (2015, p. 588) does, we suppose that, for some formula Φ, $\ulcorner\Phi\urcorner = \ulcorner\neg T(\ulcorner\Phi\urcorner)\urcorner$ is an axiom of S_{CE} and have the result. Since his result in Tennant (2015, pp. 588–589) can be further reduced by the application of $\trianglerighteq_{CR(T(x))}-$reduction, we provide a reduced derivation of $\bot$.

Proposition 2.2.2. *Suppose that, for some formula* Φ, $\ulcorner\Phi\urcorner = \ulcorner\neg T(\ulcorner\Phi\urcorner)\urcorner$ *is an axiom of* S_{CE}*. Then, there is a closed derivation of* $\bot$ *in* S_{CE}*.*

Proof. First, we have an open derivation Δ_5 of $\bot$ from $[\neg\Phi]$.

$$
\begin{prooftree}
\AxiomC{$[\neg\Phi]^1$}
\AxiomC{}
\RightLabel{Ax_2}
\UnaryInfC{$\ulcorner\neg T(\ulcorner\Phi\urcorner)\urcorner = \ulcorner\Phi\urcorner$}
\AxiomC{$[\neg\Phi]^1$}
\AxiomC{$[T(\ulcorner\Phi\urcorner)]^2$}
\RightLabel{TE}
\UnaryInfC{Φ}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\neg I_{,2}$}
\UnaryInfC{$\neg T(\ulcorner\Phi\urcorner)$}
\RightLabel{TI}
\UnaryInfC{$T(\ulcorner\neg T(\ulcorner\Phi\urcorner)\urcorner)$}
\RightLabel{$=E$}
\BinaryInfC{$T(\ulcorner\Phi\urcorner)$}
\RightLabel{TE}
\UnaryInfC{Φ}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\end{prooftree}
$$

With the derivation Δ_5, there is a closed derivation Δ_6 of $\bot$.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{Ax_1}
\UnaryInfC{$\ulcorner\Phi\urcorner = \ulcorner\neg T(\ulcorner\Phi\urcorner)\urcorner$}
\AxiomC{$[\neg\Phi]^1$}
\noLine
\UnaryInfC{Δ_5}
\noLine
\UnaryInfC{$\bot$}
\RightLabel{$CR_{,1}$}
\UnaryInfC{Φ}
\RightLabel{TI}
\UnaryInfC{$T(\ulcorner\Phi\urcorner)$}
\RightLabel{$=E$}
\BinaryInfC{$T(\ulcorner\neg T(\ulcorner\Phi\urcorner)\urcorner)$}
\RightLabel{TE}
\UnaryInfC{$\neg T(\ulcorner\Phi\urcorner)$}
\AxiomC{$[\neg\Phi]^3$}
\noLine
\UnaryInfC{Δ_5}
\noLine
\UnaryInfC{$\bot$}
\RightLabel{$CR_{,3}$}
\UnaryInfC{Φ}
\RightLabel{TI}
\UnaryInfC{$T(\ulcorner\Phi\urcorner)$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\end{prooftree}
$$

□

There is no reduction process in $\mathbb{R}_{CE}$ that we can apply to Δ_6. Δ_6 does not generate a non-terminating reduction sequence. If the Liar paradox is a genuine paradox, Δ_6 is a counterexample to TCP_E which raises the problem of undergeneration. It is the derivation of the genuine paradox but does not satisfy TCP_E. Hence, TCP_E fails to be a necessary condition to be the test of the genuine paradoxes. Tennant considers that the classical inference makes a trouble in the derivation of the Liar paradox.

In his derivation of the Liar paradox employing $CR-$rule which does not apply $\trianglerighteq_{CR(T(x))}$-reduction, Tennant (2015, p. 289) describes a similar phenomenon as the *classical rub*.

> Now here's the classical rub: this proof *appears to be in normal form*. The

> use of classical *reductio* has masked the real defect that lies at the heart of paradoxical reasoning (according to my account) – the abnormality that makes itself evident only when one hews to a constructivist line ... .

He thinks that a classical inference causes the trouble that a constructive reasoning does not make. Of course, there is a significant issue of whether the constructivist can accept the *classical reductio* and many constructivists do not agree that the *reductio* is a constructive reasoning. However, here the trouble that Tennant points out is not the issue on whether the *classical reductio* is constructive or not. His issue is that $CR-$rule makes a trouble when we investigate the proof-theoretic structure of the paradoxes. He considers that at least two things are troublesome. First, the derivation of the Liar paradox using $CR-$rule, such as Δ_6 does not produce the heart of paradoxical reasoning, i.e. the non-terminating reduction sequence. Second, it appears to be a normal derivation.[8] He proposes the methodological conjecture as the answer to the trouble, instead of arguing that the trouble only happens when using classical inferences.

> Paradoxes are never strictly classical. The kind of conceptual trouble that a paradox reveals will afflict the intuitionist just as seriously as it does the classicist. Therefore, attempted solutions to the paradoxes, if they are to be genuine solutions, must be available to the intuitionist. Nothing about an attempted solution to a paradox should imply that the trouble it reveals lies with strictly clssical moves of reasoning. (Tennant, 2015, p. 589)

It is true that we do not need $CR-$rule to formulate many of (self-referential) paradoxes in natural deduction. His methodological conjecture seems to be true. However, independently of his methodological conjecture, there is room for discussion on whether only classical inferences cause the trouble that he mentions. Furthermore, he does not seriously

[8]With respect to the derivation Δ_6 of Proposition 2.2.2, Δ_6 appears to be in normal form but whether it is in normal form is dependent on how we deal with the axiom $\ulcorner\Phi\urcorner = \ulcorner\neg T(\ulcorner\Phi\urcorner)\urcorner$. If the axiom were derived by the application of $=I-$rule, $\ulcorner\Phi\urcorner = \ulcorner\neg T(\ulcorner\Phi\urcorner)\urcorner$ in $=E-$rule would be a maximum formula. Then, Δ_6 would be a non-normal derivation. If we deal with the axiom in the same manner of assumptions, $\ulcorner\Phi\urcorner = \ulcorner\neg T(\ulcorner\Phi\urcorner)\urcorner$ would not be a maximum formula, and so it would be in normal form. For Tennant said that his derivation *appears* to be in normal form, we regard Δ_6 as a seeming normal derivation.

consider the problem of undegeneration. His expected answer against the undergeneration problem is the rejection of the use of $CR-$rule. However, we will see in the next section that there are cases generating the problem of undergeneration without the application of $CR-$rule.

2.3 The Undergeneration Problem without $CR-$Rule.

At the beginning of this chapter, the non-normal derivation Σ_2 of Curry's paradox using $CR-$rule is proposed and it fails to generate a looping reduction sequence. Also, in the last section, we have the seeming normal derivation Δ_6 of the Liar paradox without generating an infinite reduction sequence. It is the Rogerson-type counterexample to TCP_E because it raises the problem of undergeneration. As we have seen in the last section, Tennant thinks that the derivation has two problems: (i) the derivation appears to be in normal form, i.e. a normal derivation of $\perp$ exists, (ii) the derivation does not generate a non-terminating reduction sequence in spite of the fact that it formalizes a genuine paradox, such as the Liar paradox. He believes that these problems are only caused by the classical inference. From the observation that it is not necessary to use classical inferences to formalizes genuine paradoxes in natural deduction, he put forward to the methodological conjecture that genuine paradoxes are never strictly classical. Although he did not seriously consider the problem of undergeneration, from his view, we can think that his solution to the undergeneration problem is not to use any classical inferences. However, it is too hasty to think so.

When we take the problem of undergeneration more seriously, we can see that the problem is not solely caused by the application of classical inferences. We shall argue in this section that the problem of undergeneration is not solved by simply accusing classical inferences, especially $CR-$rule, of two troubles above, due to the fact that there are counterexamples to TCP_E which does not use any classical inference but does raise the undergeneration problem.

At first, Tennant ignores the fact that a non-terminating reduction sequence can be occurred by an auxiliary reduction procedure. In the next chapter, we will see Ekman's paradox which shows that TCP_E overgenerates in the sense that TCP_E makes a non-paradoxical derivation paradoxical. Ekman's paradox uses a special auxiliary reduction procedure called Ekman reduction procedure which has the main role to generate a non-terminating reduction sequence.

$$\dfrac{[\psi \to \varphi] \qquad \dfrac{[\varphi \to \psi] \quad \overset{\mathfrak{D}}{\varphi}}{\psi} \to E}{\varphi} \to E \qquad \trianglerighteq_E \qquad \overset{\mathfrak{D}}{\varphi}$$

Ekman reduction $\trianglerighteq_E$ is an auxiliary reduction procedure and it has a similar form with the auxiliary reduction procedure for the substitutivity of identity.

$$\dfrac{t_2 = t_1 \qquad \dfrac{t_1 = t_2 \quad \overset{\mathfrak{D}}{\varphi(t_1)}}{\varphi(t_2)} = E}{\varphi(t_1)} = E \qquad \trianglerighteq_{Sub} \qquad \overset{\mathfrak{D}}{\varphi(t_1)}$$

Similar to Ekman's paradox, the derivation of the Liar paradox in Tennant (2015, pp. 585–588), i.e. Proposition 2.2.1, should use $\trianglerighteq_{Sub}-$reduction. If $\trianglerighteq_{Sub}-$reduction is not available, Δ_4 in Proposition 2.2.1 does not reduce to the same derivation with Δ_3 and so Δ_3 does not generate a looping reduction sequence. As for the view of Tennant (2015, p. 589), the derivation Δ_6 in Proposition 2.2.2 of the Liar paradox which uses $CR-$rule appears to be in normal form. Likewise, Δ_4 appears to be a closed normal derivation of $\bot$ and it does not initiate the non-terminating reduction sequence. These are the same phenomena of what he calls 'the classical rub.' However, Δ_4 without the application of $\trianglerighteq_{Sub}-$reduction does not use any classical inferences but raises the problem of undergeneration. Therefore, with the assumption that the Liar paradox is a genuine one, if $\trianglerighteq_{Sub}-$reduction is not proper, Δ_4 can be the counterexample to TCP_E which shows that TCP_E undegenerates. That is, the rejection of the application of classical inferences or $CR-$rule is unable to be a solution to the problem of undergeneration.

Of course, Tennant may disallow the view that $\trianglerighteq_{Sub}-$reduction is not proper. To claim that $\trianglerighteq_{Sub}-$reduction is proper, he should explain which reduction process is proper and which process is not, but he has never explained about it.[9] Moreover, even without taking issue with $\trianglerighteq_{Sub}-$reduction procedure, a counterexample to TCP_E can be presented that causes the problem of undergeneration.

Tennant (2017, pp. 109–110) has preferred to use generalized elimination rules for four reasons: the uniform presentation, the efficiency of proof search, making shorter formal proofs, and affording a solution to the problem of overgeneration. The fourth reason is related to our topic.

> Fourth, ... using Elimination rules in [generalized] form affords a solution to certain problems that would otherwise arise for the proof-theoretic criterion of paradoxicality ... . (Tennant, 2017, p. 110)

When Tennant (2016) and Tennant (2017, Ch. 11) deal with the problem of overgeneration occurred by Ekman's paradox, instead of accusing Ekman reduction $\trianglerighteq_E$ of the problem, he provides a solution that all elimination rules are stated in generalized form.[10] Though he believes that the choice of the form of generalized elimination rules can solve the overgeneration problem, it also provides a case that causes the undergeneration problem.

We now consider the second case that raises the problem of undergeneration without classical inferences. When we use generalized form of elimination rules with permutation conversions, there is a closed full normal derivation of $\bot$ which formalizes the Liar paradox. For the example, we have the generalized elimination rules for the truth-predicate $T(x)$ and its standard reduction process.

$$\dfrac{\begin{array}{c}\mathfrak{D}_1\\ \varphi\end{array}}{T(\ulcorner\varphi\urcorner)}\,TI \qquad \dfrac{T(\ulcorner\varphi\urcorner) \quad \begin{array}{c}[\varphi]^1\\ \mathfrak{D}_2\\ \psi\end{array}}{\psi}\,TE_{,1}$$

[9] With respect to the problem of overgeneration, the issue with tests of a proper reduction will be discussed in Chapter 3.

[10] The assessment of his solution will be discussed in the next chapter.

The reduction procedure for $T(x)$

$$\dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1\\ \varphi\end{matrix}}{T(\ulcorner\varphi\urcorner)}\,TI \qquad \begin{matrix}[\varphi]^1\\ \mathfrak{D}_2\\ \psi\end{matrix}}{\psi}\,TE_{,1} \qquad \rhd_{T(x)} \qquad \begin{matrix}\mathfrak{D}_1\\ \varphi\\ \mathfrak{D}_2\\ \psi\end{matrix}$$

Also, we take $=E-$rule having the form of the generalized elimination rule.

$$\dfrac{t_1 = t_2 \quad \begin{matrix}\mathfrak{D}_1\\ \varphi(t_1)\end{matrix} \quad \begin{matrix}[\varphi(t_2)]^1\\ \mathfrak{D}_2\\ \psi\end{matrix}}{\psi}\,=E_{,1}$$

Then, Proposition 2.3.1 shows that there exists a full normal derivation of $\bot$ from the Liar paradox.

Proposition 2.3.1. *Let S_R be a system containing $T-$, $\neg-$, $=-$rules with their generalized form of elimination rules. S_R has a set $\mathbb{R}_R$ of reduction procedures for $T(x)$ and $\neg$, and $\trianglerighteq_{Sub}$ with permutation conversion. Suppose that, for some formula Φ, $\ulcorner\Phi\urcorner = \ulcorner\neg T(\ulcorner\Phi\urcorner)\urcorner$ is an axiom of S_R. There exists a closed full normal derivation Δ_9 of $\bot$ in S_R relative to $\mathbb{R}_R$.*

Proof. Two claims prove the result.

Claim 1. there is a closed derivation Δ_9 of $\bot$.

To begin with, we have an open derivation Δ_7 of $\bot$ from $[T(\ulcorner\Phi\urcorner)]$.

$$\dfrac{\dfrac{}{\ulcorner\Phi\urcorner = \ulcorner\neg T(\ulcorner\Phi\urcorner)\urcorner}\,Ax_1 \quad [T(\ulcorner\Phi\urcorner)]^1 \quad \dfrac{[T(\ulcorner\neg T(\ulcorner\Phi\urcorner)\urcorner)]^2 \quad \dfrac{[\neg T(\ulcorner\Phi\urcorner)]^3 \quad [T(\ulcorner\Phi\urcorner)]^1 \quad [\bot]^4}{\bot}\,\neg E_{,4}}{\bot}\,TE_{,3}}{\bot}\,=E_{,2}$$

Then, there is a closed derivation Δ_8 of $\bot$.

$$
\dfrac{
\dfrac{}{\ulcorner \neg T(\ulcorner \Phi \urcorner) \urcorner = \ulcorner \Phi \urcorner}\,Ax_2
\qquad
\dfrac{
\dfrac{
\begin{array}{c} [T(\ulcorner \Phi \urcorner)]^1 \\ \Delta_7 \\ \bot \end{array}
}{\neg T(\ulcorner \Phi \urcorner)}\,\neg I_{,1}
}{T(\ulcorner \neg T(\ulcorner \Phi \urcorner) \urcorner)}\,TI
\qquad
\begin{array}{c} [T(\ulcorner \Phi \urcorner)]^5 \\ \Delta_7 \\ \bot \end{array}
}{\bot}\,=E_{,5}
$$

Claim 2. Δ_8 is in full normal form.

All major premises in Δ_8 are assumptions or axioms. Hence, we have the result. □

The derivation Δ_8 is in full normal form. So to speak, it does not generate a non-terminating reduction sequence. For Δ_8 is not a T-paradox, it shows that TCP_E undergenerates if the Liar paradox is a genuine paradox. Tennant (2016, pp. 12–16) suggests the same result with Proposition 2.A.1 in Appendix 2.A and believes that the result shows that the Liar paradox is a genuine one. In addition, Tennant (2017, p. 110) says that he prefers to use generalized elimination rules due to the fact that the use of them affords a solution to problems that arise for the proof-theoretic criterion of paradoxicality. Unfortunately, Proposition 2.3.1 shows that the use of generalized elimination rules rather causes the problem of undergeneration. Furthermore, since no classical inference is used in Δ_8, the rejection of the use of classical inferences fails to solve the undergeneration problem.

The first and second cases seem to show that our choices of reduction procedures and forms of elimination rules cause the problem of undergeneration. There is another factor that causes the undergeneration problem: the use of axioms in natural deduction. Our derivations of the Liar paradox regards the formula $\ulcorner \neg T(\ulcorner \Phi \urcorner) \urcorner = \ulcorner \Phi \urcorner$ as an axiom. When $\ulcorner \neg T(\ulcorner \Phi \urcorner) \urcorner = \ulcorner \Phi \urcorner$ is a major premise of $=E-$rule, it is neither an assumption nor a formula derived by any rules. So, the derivation Δ_8 of $\bot$ in Proposition 2.3.1 is in (full) normal form and does not generate a non-terminating reduction sequence. We can have a similar case of Curry's paradox by using a formula, $\neg a \in a \leftrightarrow a \in \{x | x \in x \to \bot\}$ as an axiom.

As we have seen at the beginning of this chapter, the formalization of Curry's paradox needs to use $\in -$rules. Let us consider a derivation of Curry's paradox which does not employ $\in -$rule. Our natural deduction system has introduction and (standard) elimination rules for $\wedge$, $\rightarrow$, and $\neg$. We define a parameter a as a set $\{x|x \in x \rightarrow \bot\}$ and instead of using $\in -$rules we regard $(a \in a \rightarrow \bot) \leftrightarrow a \in \{x|x \in x \rightarrow \bot\}$ as an axiom of our system. For our convenience sake, we use the following abbreviations:

$$\begin{array}{c} \mathfrak{D} \\ a \in a \rightarrow \bot \\ \hline\hline a \in \{x|x \in x \rightarrow \bot\} \end{array}\; SetI$$

is an abbreviation for

$$\dfrac{\dfrac{\dfrac{\dfrac{}{(a \in a \rightarrow \bot) \leftrightarrow a \in \{x|x \in x \rightarrow \bot\}}\,Ax}{((a \in a \rightarrow \bot) \rightarrow a \in \{x|x \in x \rightarrow \bot\}) \wedge (a \in \{x|x \in x \rightarrow \bot\} \rightarrow (a \in a \rightarrow \bot))}\,def}{(a \in a \rightarrow \bot) \rightarrow a \in \{x|x \in x \rightarrow \bot\}}\,\wedge E \qquad \begin{array}{c}\mathfrak{D} \\ a \in a \rightarrow \bot\end{array}}{a \in \{x|x \in x \rightarrow \bot\}} \rightarrow E$$

Also,

$$\begin{array}{c} \mathfrak{D} \\ a \in \{x|x \in x \rightarrow \bot\} \\ \hline\hline a \in a \rightarrow \bot \end{array}\; SetE$$

is an abbreviation for

$$\dfrac{\dfrac{\dfrac{\dfrac{}{(a \in a \rightarrow \bot) \leftrightarrow a \in \{x|x \in x \rightarrow \bot\}}\,Ax}{((a \in a \rightarrow \bot) \rightarrow a \in \{x|x \in x \rightarrow \bot\}) \wedge (a \in \{x|x \in x \rightarrow \bot\} \rightarrow (a \in a \rightarrow \bot))}\,def}{a \in \{x|x \in x \rightarrow \bot\} \rightarrow (a \in a \rightarrow \bot)}\,\wedge E \qquad \begin{array}{c}\mathfrak{D} \\ a \in \{x|x \in x \rightarrow \bot\}\end{array}}{a \in a \rightarrow \bot} \rightarrow E$$

Then, there is an open derivation Δ_9 of $\bot$ from $[a \in a]$

$$\dfrac{\dfrac{\dfrac{[a \in a]^1}{a \in \{x|x \in x \to \bot\}}\,def}{a \in a \to \bot}\,SetE \qquad [a \in a]^1}{\bot} \to E$$

With the derivation Δ_9, we have a closed derivation Δ_{10} of $a \in a$.

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}[a \in a]^1\\ \Delta_9\\ \bot\end{array}}{a \in a \to \bot} \to I_{,1}}{a \in \{x|x \in x \to \bot\}}\,SetI}{a \in a}\,def$$

Now, we have a closed derivation Δ_{11} of $\bot$.

$$\dfrac{\dfrac{\begin{array}{c}[a \in a]^1\\ \Delta_9\\ \bot\end{array}}{a \in a \to \bot} \to I_{,1} \qquad \begin{array}{c}\Delta_{10}\\ a \in a\end{array}}{\bot} \to E$$

Δ_{11} has a maximum formula $a \in a \to \bot$ in the last $\to E-$rule. We apply the reduction procedure $\rhd_{\to}$ to Δ_{11} and then have the derivation Δ_{12} below.

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}[a \in a]^1\\ \Delta_9\\ \bot\end{array}}{a \in a \to \bot} \to I_{,1}}{a \in \{x|x \in x \to \bot\}}\,SetI}{a \in a \to \bot}\,SetE \qquad \begin{array}{c}\Delta_{10}\\ a \in a\end{array}}{\bot} \to E$$

Δ_{12} looks as if it is reducible, but it appears to be in normal form. $SetI-$ and $SetE-$inferences are not rules for $\in$, and they are abbreviations of derivations consisting of $\wedge E-$ and $\rightarrow E-$rules. Since the constant $\in$ is not introduced by $\in I-$rule, the reduction procedure $\rhd_{\in}$ for $\in$ cannot apply to Δ_{12}. In addition, $a \in a \rightarrow \bot$ in $SetI-$inference is a minor premise of $\rightarrow E-$rule. $a \in a \rightarrow \bot$ in the last $\rightarrow E-$rule is the conclusion of $\rightarrow E-$rule, and so it is not a maximum formula. Hence, Δ_{12} is in normal form.

The derivation Δ_{12} of Curry's paradox is a closed normal derivation of $\bot$ and it does not generate a non-terminating reduction sequence. These are similar features of what Tennant (2015, p. 589) calls, 'the clssical rub,' however any classical inferences are not involved in Δ_{12}. If Curry's paradox is a genuine paradox, Δ_{12} is a counterexample to TCP_E which represents that TCP_E undergenerates. Hence, the rejection of using classical inferences is unable to be a solution to the undergeneration problem.

In this section, we have seen two derivations of the Liar paradox and one derivation of Curry's paradox. All derivations do not use any classical inference but yield the problem of undergeneration. The next section will diagnose what the culprit of the problem is and seek to find a plausible solution.

2.4 Diagnosis

Section 2.3 put forward to three counterexamples to TCP_E which cause the undergeneration problem but does not use any classical inference. The first case explicates that when $\unrhd_{Sub}-$reduction is unavailable, the derivation Δ_4 of the Liar paradox in Proposition 2.2.1 does not generate a non-terminating reduction sequence. The second case, i.e. Proposition 2.3.1, shows that when all elimination rules are stated in generalized form, there is a derivation of the Liar paradox, such as Δ_8, which has no reduction loop. The last case represents that when we use axioms in natural deduction, there is a derivation of Curry's paradox, e.g. Δ_{12}, which does not enter into loops.

Three cases appear to show that the occurrence of a looping reduction sequence is relative to our choice of reduction procedures, forms of elimination rules, and applications of axioms. Unlike the other two cases, the use of generalized elimination rules does not seem to effect on generating a non-terminating reduction sequence. If we adopt the following reduction procedure, $\trianglerighteq_{GE_{g'}}$, proposed by Schroeder-Heister and Tranchini (2018), the derivation Δ_8 of Proposition 2.3.1 enters into loops.

$$
\dfrac{t_1 = t_2 \quad \begin{matrix}\mathfrak{D}_1 \\ \varphi(t_1)\end{matrix} \quad \dfrac{t_2 = t_1 \quad [\varphi(t_2)]^1 \quad \begin{matrix}[\varphi(t_2)]^1, [\varphi(t_1)]^2 \\ \mathfrak{D}_2 \\ \rho\end{matrix}}{\rho} = E_{,2}}{\rho} = E_{,1}
\quad \trianglerighteq_{GE_{g'}} \quad
\dfrac{t_1 = t_2 \quad \begin{matrix}\mathfrak{D}_1 \\ \varphi(t_1)\end{matrix} \quad \begin{matrix}\mathfrak{D}_1 \\ [\varphi(t_2)]^1, \varphi(t_1) \\ \mathfrak{D}_2 \\ \rho\end{matrix}}{\rho} = E_{,1}
$$

Our choice of generalized elimination rules is independent of the occurrence of a looping reduction sequence but is dependent on our choice of reduction procedures. Thus, we only consider the first and the second cases in this section.

To begin with, in order to find a solution to the problem of undergeneration, it is necessary to diagnose what the culprit of the problem is. As Tennant (2015, p. 589) said, 'The use of classical *reductio* has masked the real defect that lies at the heart of paradoxical reasoning.' He believes that $CR-$rule disguises the non-terminating reduction sequence. He does not seem to consider seriously the undergeneration problem. The use of $CR-$rule does not always block the occurrence of the non-terminating reduction sequence. So to speak, $CR-$rule is not the only culprit of the problem of undergeneration. For instance, in the beginning of this chapter, we see the derivation Σ_2 of Curry's paradox in Proposition 2.0.1 suggested by Rogerson (2006). Σ_2 uses $CR-$rule and does not initiate a non-terminating reduction sequence. Proposition 2.0.1 seems to establish that $CR-$rule masks the non-terminating reduction sequence, but it is not.

Rogerson (2006) only considered standard reduction processes suggested by Prawitz (1965). However, Schroeder-Heister and Tranchini (2017, pp. 572–573) borrow the reduction proposed by Gunnar Stålmarck (1991, pp. 131–132) and claim that her example can be further reduced. With regard to our example, Σ_2 in Proposition 2.0.1, their reduction denoted by $\trianglerighteq_{CR(\circ,\star)}$ may be depicted schematically as follows:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}[\neg\varphi]^1\\ \mathfrak{D}_1\\ \bot\end{matrix}}{\varphi}CR_{,1}}{\psi}\circ E \qquad \begin{matrix}\mathfrak{D}_2\\ \sigma\end{matrix}}{\rho}\star E \qquad \trianglerighteq_{CR(\circ,\star)} \qquad \dfrac{\begin{matrix}\dfrac{\dfrac{[\neg\rho]^2 \qquad \dfrac{\dfrac{[\varphi]^1}{\psi}\circ E \qquad \begin{matrix}\mathfrak{D}_2\\ \sigma\end{matrix}}{\rho}\star E}{\bot}\neg E}{\neg\varphi}\neg I_{,1}\\ \mathfrak{D}_1\\ \bot\end{matrix}}{\rho}CR_{,2}$$

where $\circ E-$ and $\star E-$rules are elimination rules for some operators $\circ$ and $\star$ respectively. Then, by the application of $\trianglerighteq_{CR(\circ,\star)}$, Σ_2 reduces to the derivation Σ_3 below.

$$\dfrac{\begin{matrix}\dfrac{\dfrac{[\neg\bot]^5 \qquad \dfrac{\dfrac{\dfrac{[a\in a]^4}{a\in\{x|x\in x\to\bot\}}\,def}{a\in a\to\bot}\in E \qquad \dfrac{\begin{matrix}[\neg a\in a]^3\\ \Sigma_1\\ \bot\end{matrix}}{a\in a}CR_{,3}}{\bot}\to E}{\bot}\neg E}{\neg a\in a}\neg I_{,4}\\ \Sigma_1\\ \bot\end{matrix}}{\bot}CR_{,5}$$

Interestingly, when we add $\trianglerighteq_{CR(\circ,\star)}$ to the set $\mathbb{R}_{NC}$ of reductions for S_{NC} and have $\mathbb{R}'_{NC}$, the derivation Σ_2 in Proposition 2.0.1 generates a non-terminating reduction sequence.

Proposition 2.4.1. *Let us define a parameter a as $\{x|x\in x\to\bot\}$. Then, there is a closed derivation of $\bot$ in S_{NC} with respect to $\mathbb{R}'_{NC}$ which generates a non-terminating reduction sequence and so is not normalizable.*

Proof. We borrow the derivation Σ_1 in Proposition 2.0.1 and have the following derivation

Δ of $\bot$ from $[a \in a]$.

$$\dfrac{\dfrac{\dfrac{[a\in a]^5}{a\in\{x|x\in x\to\bot\}}\,def}{a\in a\to\bot}\in E \qquad \dfrac{\begin{array}{c}[\neg a\in a]^3\\ \Sigma_1\\ \bot\end{array}}{a\in a}\,CR_{,3}}{\bot}\to E$$

Then, the following form of derivation is the same derivation with Σ_2.

$$\begin{array}{c}[\neg a\in a]^1\\ \Sigma_1\\ \dfrac{\bot}{a\in a}\,CR_{,1}\\ \Delta\\ \bot\end{array}$$

Also, the derivation Σ_3 reduced from Σ_2 is restated as below.

$$\dfrac{\dfrac{\dfrac{[\neg\bot]^6 \quad \begin{array}{c}[a\in a]^5\\ \Delta\\ \bot\end{array}}{\bot}\neg E}{\neg a\in a}\neg I_{,5} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\neg\bot]^6\quad \begin{array}{c}[a\in a]^7\\ \Delta\\ \bot\end{array}}{\bot}\neg E}{\neg a\in a}\neg I_{,7}\quad [a\in a]^2}{\bot}\neg E}{a\in a\to\bot}\to I_{,2}}{a\in\{x|x\in x\to\bot\}}\in I}{a\in a}\,def}{\bot}\neg E$$

Since Σ_3 has a maximum formula $\neg a \in a$ in $\neg E-$rule, the application of $\neg-$reduction provides the derivation Σ_4 below.

$$\dfrac{\dfrac{[\neg\bot]^6 \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\neg\bot]^6 \quad \begin{array}{c}[a\in a]^2\\ \Delta\\ \bot\end{array}}{\bot}\neg E}{a\in a\to\bot}\neg I_{,2}}{a\in\{x|x\in x\to\bot\}}\in I}{a\in a\to\bot}\in E \qquad \dfrac{\begin{array}{c}[\neg a\in a]^3\\ \Sigma_1\\ \bot\end{array}}{a\in a}CR_{,3}}{\bot}\to E}{\bot}\neg E}{\bot}CR_{,6}$$

Σ_4 still has maximum formulas, and so is reduced to Σ_5 below by $\rhd_{\in}-$ and $\rhd_{\neg}-$reductions.

$$\dfrac{\dfrac{[\neg\bot]^6 \qquad \dfrac{[\neg\bot]^6 \qquad \begin{array}{c}[\neg a\in a]^3\\ \Sigma_1\\ \dfrac{\bot}{a\in a}CR_{,3}\\ \Delta\\ \bot\end{array}}{\bot}\neg E}{\bot}\neg E}{\bot}CR_{,6}$$

Σ_5 includes the same derivation with Σ_2. Again, Σ_2 can be further reduced. Then, we have the following infinite reduction sequence.

$$
\dfrac{\dfrac{[\neg\bot]^6 \qquad \dfrac{[\neg\bot]^6 \qquad \dfrac{\dfrac{[\neg\bot]^8 \qquad \dfrac{[\neg\bot]^8 \qquad \begin{array}{c} \dfrac{\dfrac{[\neg\bot]^i \qquad \dfrac{[\neg\bot]^i \quad \begin{array}{c}\vdots\\ \bot\end{array}}{\bot}\neg E}{\bot}\neg E}{\bot}CR_{,i} \\ \vdots \\ \bot \end{array}}{\bot}\neg E}{\bot}\neg E}{\bot}CR_{,8}}{\bot}\neg E}{\bot}\neg E}{\bot}CR_{,6}
$$

where $i = 2j+4 (j > 0)$. Therefore, Σ_2 generates a non-terminating reduction sequence.

□

Having the auxiliary reduction procedure $\trianglerighteq_{CR(\circ,\star)}$, Σ_2 initiates a non-terminating reduction sequence which is not so much a looping reduction as what Tennant (2016) calls, a 'spiral reduction.'[11] Proposition 2.4.1 supports the view that not every case employing $CR-$rule disguises the occurrence of a non-terminating reduction sequence. Even though there is a derivation of a given paradox employing $CR-$rule which does not produce a non-terminating reduction sequence. It means that the application of $CR-$rule is not the only reason to disguise the non-terminating reduction sequence. On comparing Proposition 2.0.1 and 2.4.1, the two results establish that, independent of using $CR-$rule, the occurrence of a non-terminating reduction sequence is relative to our choice of reduction procedures.

[11]Tennant (1995) examines Yablo's paradox in natural deduction and considers that a derivation of Yablo's paradox produces a non-terminating reduction sequence but the sequence is different from a looping reduction. He conjectures that if a reduction procedure of a given derivation does not enter into a loop, a self-referential expression is not involved in the derivation. However, Σ_2 formalizes the Liar paradox which is a self-referential paradox but it generates a spiral reduction. Therefore, it is unconvincing that his conjecture suggested in Tennant (1995) is true.

Our choice of reduction procedure causes to produce a non-terminating reduction sequence. However, it does not mean that the application of $CR-$rule can never be one of the elements to disguise the non-terminating reduction sequence. When the conclusion of $CR-$rule is a major premise of an elimination rule, i.e. a maximum formula, the auxiliary reduction procedure for $CR-$rule does not eliminate the maximum formula but only reduces the degree of the maximum formula. When the conclusion becomes an atomic formula, the reduction procedure can no longer be applied. The reason why the derivation Σ_2 of Proposition 2.0.1 fails to generate a non-terminating reduction sequence appears to be that $CR-$rule does not have any reduction process to eliminate its conclusion as a maximum formula. Then, what if $CR-$rule can eliminate its conclusion when it is a major premise of an elimination rule?

The $CR-$rule of *classical reductio* is often regarded as an elimination rule because it eliminates the negations in a formula of the assumption. For instance, $CR-$rule is sometimes regarded as the abbreviation of $\neg I-$rule and the double negation elimination rule (DNE).

$$\dfrac{\dfrac{\begin{matrix}[\neg\varphi]^1\\ \mathfrak{D}\\ \bot\end{matrix}}{\neg\neg\varphi}\,\neg I_{,1}}{\varphi}\,DNE$$

On the other hand, Peter Milne (1994, p. 58) interprets $CR-$rule as a rule for introducing a formula φ, that is, the derivation of $\bot$ from the assumption $[\neg\varphi]$ introduces a formula φ. Then, we have the following pair of rules.

$$\dfrac{\begin{matrix}[\neg\varphi]^1\\ \mathfrak{D}_1\\ \bot\end{matrix}}{\varphi}\,CR_{,1} \qquad \dfrac{\begin{matrix}\mathfrak{D}_2\\ \varphi \quad \neg\varphi\end{matrix}}{\bot}\,CRE$$

The standard reduction procedure for $CR-$ and $CRE-$rules is below.

$$\dfrac{\dfrac{\begin{array}{c}[\neg\varphi]^1\\ \mathfrak{D}_1\\ \bot\end{array}}{\varphi}\,CR_{,1} \quad \begin{array}{c}\mathfrak{D}_2\\ \neg\varphi\end{array}}{\bot}\,CRE \qquad \rhd_{CRE} \qquad \begin{array}{c}\mathfrak{D}_2\\ \neg\varphi\\ \mathfrak{D}_1\\ \bot\end{array}$$

$CR-$ and $CRE-$rules are intrinsically harmonious. Let S_{NCE} be a natural deduction system which contains $\rightarrow -$, $\in -$, $CR-$, and $CRE-$rules. S_{NCE} has a set $\mathbb{R}_{NCE}$ of reductions having $\rhd_{\rightarrow}$, $\rhd_{\in}$, $\rhd_{CRE}$, and the following auxiliary reduction procedure $\unrhd_{\in}$.

$$\dfrac{\dfrac{\begin{array}{c}\mathfrak{D}\\ t \in \{x|\varphi(x)\}\end{array}}{\varphi[t/x]}\in E}{t \in \{x|\varphi(x)\}}\in I \qquad \unrhd_{\in} \qquad \begin{array}{c}\mathfrak{D}\\ t \in \{x|\varphi(x)\}\end{array}$$

While the derivation $\mathfrak{D}'$ includes a subderivation $\mathfrak{D}$ that has the same conclusion of $\mathfrak{D}'$, it is often desirable to reduce the length of the derivation $\mathfrak{D}'$. The auxiliary reduction $\unrhd_{\in}$ has such role of lessening the length of derivation. Moreover, instead of using $\neg-$rules, we define $\neg\varphi$ as $\varphi \rightarrow \bot$. Then, we have the following result.

Proposition 2.4.2. *Let us define a parameter a as* $\{x|x \in x \rightarrow \bot\}$. *Then, there is a closed derivation of* $\bot$ *in* S_{NCE} *with respect to* $\mathbb{R}_{NCE}$ *which generates a non-terminating reduction sequence and so is not normalizable.*

Proof. Two claims verify the result.

Claim 1. There is a closed derivation Π_3 of $\bot$ in S_{NCE}.

First, there is an open derivation Π_1 of $\bot$ from $[\neg a \in a]$.

$$\dfrac{\dfrac{\dfrac{\dfrac{[\neg a \in a]^1}{a \in a \to \bot}\,def}{a \in \{x|x \in x \to \bot\}}\in I}{a \in a}\,def \qquad [\neg a \in a]^1}{\bot}\,CRE$$

With the derivation Π_1, we have a closed derivation Π_2 of $\neg a \in a$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}[\neg a \in a]^1\\ \Pi_1\\ \bot\end{matrix}}{a \in a}\,CR_{,1}}{a \in \{x|x \in x \to \bot\}}\,def}{a \in a \to \bot}\in E}{\neg a \in a}\,def$$

Then, we have a closed derivation Π_3 of $\bot$.

$$\dfrac{\dfrac{\begin{matrix}[\neg a \in a]^1\\ \Pi_1\\ \bot\end{matrix}}{a \in a}\,CR_{,1} \qquad \begin{matrix}\Pi_2\\ \neg a \in a\end{matrix}}{\bot}\,CRE$$

Claim 2. Π_3 initiates a non-terminating reduction sequence and so is not normalizable.

Since Π_3 has a maximum formula $a \in a$, the application of the standard reduction $\rhd_{CRE}$

to Π_3 produces to the following derivation Π_4.

$$
\dfrac{
\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}[\neg a \in a]^1 \\ \Pi_1 \\ \bot\end{array}}{a \in a}\, CR_{,1}}{a \in \{x|x \in x \to \bot\}}\, def}{\dfrac{a \in a \to \bot}{a \in \{x|x \in x \to \bot\}}\, \in I}\, \in E}{a \in a}\, def
\qquad
\begin{array}{c}\Pi_2 \\ \neg a \in a\end{array}
}{\bot}\, CRE
$$

Π_4 still has a maximum formula $a \in a$ in $\in E-$rule and, by applying $\trianglerighteq_{\in}$ to Π_4, we have the same derivation with Π_3. Hence, Π_3 raises a non-terminating reduction sequence and so is not normalizable. □

A similar result can be given by using Tennant's style formalization of the Liar paradox proposed in Tennant (2015, pp. 585–590).[12] Let S_{CRE} be a natural deduction system containing $\neg-$, $T-$, $=-$, $CR-$, and $CRE-$rules which is an extension of S_{CE} used in Section 2.2. We have a set $\mathbb{R}_{CRE}$ of standard reduction procedures having $\triangleright_{\neg}$, $\triangleright_{T(x)}$, $\triangleright_{=}$, $\triangleright_{CRE}$, and the following auxiliary reduction $\trianglerighteq_{T(x)}$.

$$
\dfrac{\dfrac{\begin{array}{c}\mathfrak{D} \\ T(\ulcorner \varphi \urcorner)\end{array}}{\varphi}\, TE}{T(\ulcorner \varphi \urcorner)}\, TI
\qquad \trianglerighteq_{T(x)} \qquad
\begin{array}{c}\mathfrak{D} \\ T(\ulcorner \varphi \urcorner)\end{array}
$$

Then, we have the following result.

Proposition 2.4.3. *Suppose that, for some formula Φ, $\ulcorner \Phi \urcorner = \ulcorner \neg T(\ulcorner \Phi \urcorner) \urcorner$ is an axiom of*

[12] While Tennant (2016, pp. 12–16) claims that the Liar paradox is a genuine paradox, he employs the id est rules for the liar sentence Ψ. By using his id est rules and a derivation of the Liar paradox, we have similar results. Appendix 2.A shows that there are three closed derivations of $\bot$ which formalize the Liar paradox: (i) a T-paradox which use neither $CR-$ nor $CRE-$rules, (ii) a T-paradox using both $CR-$and $CRE-$rules, and (iii) a derivation of $\bot$ which only uses $CR-$rule and is not a T-paradox.

S_{CRE}. S_{CRE} relative to R_{CRE} has a closed derivation of $\bot$ which initiates a non-terminating reduction sequence, so is not normalizable.

Proof. Two claims verify the result.

Claim 1. There is a closed derivation Π_7 of $\bot$ in S_{CRE}.

We start with an open derivation Π_5 of $\bot$ from $[\neg T(\ulcorner\Phi\urcorner)]$.

$$\dfrac{\dfrac{\dfrac{}{\ulcorner\neg T(\ulcorner\Phi\urcorner)\urcorner = \ulcorner\Phi\urcorner}\,Ax_2 \quad \dfrac{[\neg T(\ulcorner\Phi\urcorner)]^1}{T(\ulcorner\neg T(\ulcorner\Phi\urcorner)\urcorner)}\,TI}{T(\ulcorner\Phi\urcorner)}\,{=}E \qquad [\neg T(\ulcorner\Phi\urcorner)]^1}{\bot}\,CRE$$

With the derivation Π_5, we have a closed derivation Π_6 of $\neg T(\ulcorner\Phi\urcorner)$.

$$\dfrac{\dfrac{\dfrac{}{\ulcorner\Phi\urcorner = \ulcorner\neg T(\ulcorner\Phi\urcorner)\urcorner}\,Ax_1 \quad \dfrac{\begin{matrix}[\neg T(\ulcorner\Phi\urcorner)]^2\\ \Pi_5\\ \bot\end{matrix}}{T(\ulcorner\Phi\urcorner)}\,CR_{,2}}{T(\ulcorner\neg T(\ulcorner\Phi\urcorner)\urcorner)}\,{=}E}{\neg T(\ulcorner\Phi\urcorner)}\,TE$$

Then, we have a closed derivation Π_7 of $\bot$.

$$\dfrac{\dfrac{\begin{matrix}[\neg T(\ulcorner\Phi\urcorner)]^1\\ \Pi_5\\ \bot\end{matrix}}{T(\ulcorner\Phi\urcorner)}\,CR_{,1} \quad \begin{matrix}\Pi_6\\ \neg T(\ulcorner\Phi\urcorner)\end{matrix}}{\bot}\,CRE$$

Claim 2. Π_7 initiate a non-terminating reduction sequence, and so is not normalizable.

Since Π_7 has a maximum formula $T(\ulcorner\Phi\urcorner)$ in the last $CRE-$rule, by applying $\rhd_{CRE}$ to

Π_7, we have a derivation Π_8 below.

$$
\dfrac{
  \dfrac{
    \dfrac{}{\ulcorner \neg T(\ulcorner \Phi \urcorner) \urcorner = \ulcorner \Phi \urcorner}\,Ax_2
    \qquad
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{}{\ulcorner \Phi \urcorner = \ulcorner \neg T(\ulcorner \Phi \urcorner) \urcorner}\,Ax_1
          \qquad
          \dfrac{\begin{matrix}[\neg T(\ulcorner \Phi \urcorner)]^2 \\ \Pi_5 \\ \bot\end{matrix}}{T(\ulcorner \Phi \urcorner)}\,CR_{,2}
        }{T(\ulcorner \neg T(\ulcorner \Phi \urcorner) \urcorner)}\,{=}E
      }{\neg T(\ulcorner \Phi \urcorner)}\,TE
    }{T(\ulcorner \neg T(\ulcorner \Phi \urcorner) \urcorner)}\,TI
  }{T(\ulcorner \Phi \urcorner)}\,{=}E
  \qquad
  \begin{matrix}\Pi_6 \\ \neg T(\ulcorner \Phi \urcorner)\end{matrix}
}{\bot}\,CRE
$$

There are auxiliary reduction procedures $\trianglerighteq_{T(x)}$ and $\trianglerighteq_{Sub}$ that we can apply to Π_8. We first apply $\trianglerighteq_{T(x)}$ and then $\trianglerighteq_{Sub}$ to Π_8. The same derivation with Π_7 will be given. Hence, Π_7 generates an infinite reduction sequence and is not normalizable. $\square$

Of course, it is not natural to regard $CR-$rule as an introduction rule. It should be argued whether $CR-$rule as an introduction is proper.[13] However, it is not our purpose whether $CR-$rule regarded as an introduction is acceptable. We aim to diagnose why looping or spiral reductions stop when $CR-$rule is involved. Unlike derivations in Proposition 2.4.1 and 2.4.2, the derivation Σ_2 in Proposition 2.0.1 does not generate a non-terminating reduction sequence. As Proposition 2.4.1 establishes, with the additional auxiliary reduction

[13] According to the meaning-theoretic perspective on natural deduction that the meaning of a principal constant (or an operator) is exhaustively determined by its introduction rule and determines its elimination rule, there are more than two objections to the view that $CR-$rule is a meaning-conferring introduction rule. The first is that $CR-$rule as an introduction does not confer a meaning of the conclusion φ introduced. Second, even if $CR-$rule determines the meaning of φ, it raises the circularity problem. In order to confer a meaning of φ via $CR-$rule, the rule needs to have an assumption $\neg\varphi$. If every introduction rule exhaustively determines the meaning of a principal constant, the meaning of $\neg\varphi$ is conferred by $\neg I-$rule from a derivation of $\bot$ from the assumption φ. To have the derivation, however, we should know the meaning of φ which is determined by $CR-$rule. Therefore, the circularity problem arises.

Milne (1994, pp. 59–64) argues that there is no clear proof-theoretic ground that every introduction rule should exhaustively determine the meaning of a principal constant. So to speak, some are not obliged to confer a meaning on the formula introduced. Also, he has claimed that it is impossible for $\neg I-$rule to determine the meaning of $\neg$ without circularity. It should be depply argued whether $CR-$rule could be a meaning-conferring introduction rule. However, since our purpose is not to argue whether it could be but to discuss whether the non-terminating reduction sequence stops whenever a classical inference, such as $CR-$rule, is used. We set aside the issue in this chapter.

procedure $\trianglerighteq_{CR(\circ,\star)}$, it initiates an infinite reduction sequence, such as a spiral reduction. In addition, Proposition 2.4.2 applies $CR-$ and $CRE-$rules with the standard reduction $\triangleright_{CRE}$ and shows that the derivation Π_3 enters into a looping reduction sequence. It is obvious that $CR-$rule is not always the culprit of disguising a looping or a spiral reduction procedure.

Let us diagnose whence the non-terminating reduction sequence stops. In Proposition 2.0.1, there is no process to eliminate the maximum formula, $a \in a$, which is a conclusion of $CR-$rule and simultaneously a major premise of $\in E-$rule. On the other hand, both in Proposition 2.4.1 and 2.4.2 have reduction procedures $\trianglerighteq_{CR(\circ,\star)}$ and $\triangleright_{CRE}$ which eliminate the maximum formula $a \in a$. That is, the absence of a reduction procedure to remove the maximum formula $a \in a$ derived by $CR-$rule can be considered to have prevented a non-terminating reduction sequence. For instance, the auxiliary reduction $\trianglerighteq_{CR(\circ,\star)}$ in Proposition 2.4.1 has a such role and the standard reduction $\triangleright_{CRE}$ in Proposition 2.4.2 has it. In the case of Proposition 2.0.1, as auxiliary reductions only reduce the degree of the maximum formula, the derivation Σ_2 cannot be further reduced and fails to fall into loops. Yet, the maximum formula in Σ_2 is eliminated by $\trianglerighteq_{CR(\circ,\star)}$ in Proposition 2.4.1, and Σ_2 is further reduced. Then, Σ_2 generates a non-terminating reduction sequence by yielding new maximum formulas. It is ironic that in order to have a non-terminating reduction sequence a maximum formula should have a reduction process to get rid of the maximum formula, but it appears to be that the necessary condition to generate a non-terminating reduction sequence is that every maximum formula has to have a reduction process to remove itself.

Proposition 2.4.3 has a different aspect that when $\trianglerighteq_{Sub}-$reduction is not applicable, Tennant-style derivation of the Liar paradox using the axiom $\ulcorner\neg T(\ulcorner\Phi\urcorner)\urcorner = \ulcorner\Phi\urcorner$ neither reduce the length of the derivation nor eliminate the major premise $\ulcorner\neg T(\ulcorner\Phi\urcorner)\urcorner = \ulcorner\Phi\urcorner$. Then, a reduction loop stops but $\triangleright_{CRE}-$reduction makes the path that $\trianglerighteq_{Sub}-$reduction is applicable. It is unclear whether an axiom in natural deduction can be a maximum formula. If it is, the use of axioms in natural deduction appears to prevent a non-terminating reduction sequence. As we have discussed in Section 2.3, without $\trianglerighteq_{Sub}-$reduction, Tennant's derivation of the Liar paradox does not generate a non-terminating reduction sequence. Though the derivation Δ_3 of Proposition 2.2.1 does not use $CR-$rule, if $\trianglerighteq_{Sub}-$reduction is unacceptable, Δ_3 has no way to be further reduced by lessening the length of it and eliminating the major premise $\ulcorner \neg T(\ulcorner \Phi \urcorner) \urcorner = \ulcorner \Phi \urcorner$.

Moreover, we have examined in Section 2.3 that while $\neg a \in a \leftrightarrow a \in \{x|x \in x \rightarrow \bot\}$ is used as an axiom, we have a normal derivation Δ_{12} of $\bot$ from Curry's paradox which does not generate a non-terminating reduction sequence. Δ_{12} applies two abbreviated inferences, $SetI-$ and $SetE-$inferences, which use the formula $\neg a \in a \leftrightarrow a \in \{x|x \in x \rightarrow \bot\}$ by the axiom and, consequently, a principal constant $\in$ is introduced without $\in I-$rule. Since $SetI-$ and $SetE-$inferences are not rules for $\in$, there is no reduction procedure to eliminate a formula including $\in$. If we use $\in -$rules instead of $SetI-$ and $SetE-$inferences, we readily have a derivation of Curry's paradox which generates a non-terminating reduction sequence. The use of axioms may cause to prevent the occurrence of a non-terminating reduction sequence. That is to say, an application of an axiom, which can lead to a formula that has a principal constant but has no reduction process to remove the formula, can disguise a non-terminating reduction sequence.

Similar to the case of Curry's paradox, Tennant-style derivation Δ_3 of the Liar paradox in Proposition 2.2.1 can produce a looping reduction without $\trianglerighteq_{Sub}-$reduction when there is a standard reduction process to eliminate the formula $\ulcorner \neg T(\ulcorner \Phi \urcorner) \urcorner = \ulcorner \Phi \urcorner$. Let us consider the following rules for an equation between coded numerals.[14]

$$\dfrac{\begin{matrix}[\varphi]^1 & [\psi]^2 \\ \mathfrak{D}_1 & \mathfrak{D}_2 \\ \psi & \varphi\end{matrix}}{\ulcorner \varphi \urcorner = \ulcorner \psi \urcorner}\ulcorner\urcorner I_{,1,2} \qquad \dfrac{\ulcorner \varphi \urcorner = \ulcorner \psi \urcorner \quad \begin{matrix}\mathfrak{D}_3 \\ \varphi\end{matrix}}{\psi}\ulcorner\urcorner E_1 \qquad \dfrac{\ulcorner \varphi \urcorner = \ulcorner \psi \urcorner \quad \begin{matrix}\mathfrak{D}_4 \\ \psi\end{matrix}}{\varphi}\ulcorner\urcorner E_2$$

[14] $\ulcorner\urcorner E-$rules are what Tennant (1982, p. 289) calls 'Leibniz disquatational rule.'

Standard reductions for $\ulcorner\urcorner-$rules are described as below.

$$\dfrac{\dfrac{\begin{matrix}[\varphi]^1 & [\psi]^2\\ \mathfrak{D}_1 & \mathfrak{D}_2 \\ \psi & \varphi\end{matrix}}{\ulcorner\varphi\urcorner = \ulcorner\psi\urcorner}\,\ulcorner\urcorner I_{,1,2} \quad \begin{matrix}\mathfrak{D}_3\\ \varphi\end{matrix}}{\psi}\,\ulcorner\urcorner E_1 \qquad \rhd_{\ulcorner\urcorner_1} \qquad \begin{matrix}\mathfrak{D}_3\\ \varphi\\ \mathfrak{D}_1\\ \psi\end{matrix}$$

$$\dfrac{\dfrac{\begin{matrix}[\varphi]^1 & [\psi]^2\\ \mathfrak{D}_1 & \mathfrak{D}_2 \\ \psi & \varphi\end{matrix}}{\ulcorner\varphi\urcorner = \ulcorner\psi\urcorner}\,\ulcorner\urcorner I_{,1,2} \quad \begin{matrix}\mathfrak{D}_4\\ \psi\end{matrix}}{\varphi}\,\ulcorner\urcorner E_2 \qquad \rhd_{\ulcorner\urcorner_2} \qquad \begin{matrix}\mathfrak{D}_4\\ \psi\\ \mathfrak{D}_2\\ \varphi\end{matrix}$$

When we presume that a liar sentence Φ which is defined by $\neg T(\ulcorner\Phi\urcorner)$ is expressible in our language, there is a derivation of the Liar paradox using the formula $\ulcorner\neg T(\ulcorner\Phi\urcorner)\urcorner = \ulcorner\Phi\urcorner$ which raises a non-terminating reduction sequence.

Proposition 2.4.4. *Let S_L be a system containing $T-$, $\neg-$, and $\ulcorner\urcorner-$rules. S_L has a set $\mathbb{R}_L$ of standard reduction procedures for $T(x)$, $\neg$, and $\ulcorner\urcorner$. Let us define a formula Φ as $\neg T(\ulcorner\Phi\urcorner)$. Then, there exists a closed derivation Π_{11} of $\perp$ in S_L with respect to $\mathbb{R}_L$ which generates a non-terminating reduction sequence, and so is not normalizable.*

Proof. Two claims establish the result.

Claim 1. There is a closed derivation Π_{11} of $\perp$ in S_L with respect to $\mathbb{R}_L$.

First, there is an open derivation Π_9 of $\perp$ from $T(\ulcorner\Phi\urcorner)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{[\Phi]^1}{\neg T(\ulcorner\Phi\urcorner)}\,def \quad \dfrac{[\neg T(\ulcorner\Phi\urcorner)]^2}{\Phi}\,def}{\ulcorner\Phi\urcorner = \ulcorner\neg T(\ulcorner\Phi\urcorner)\urcorner}\,\ulcorner\urcorner I_{,1,2} \quad \dfrac{[T(\ulcorner\Phi\urcorner)]^3}{\Phi}\,TE}{\neg T(\ulcorner\Phi\urcorner)}\,\ulcorner\urcorner E_1 \quad [T(\ulcorner\Phi\urcorner)]^3}{\perp}\,\neg E$$

With the derivation Π_9, we have a closed derivation Π_{10} of $T(\ulcorner\Phi\urcorner)$.

$$
\dfrac{
\dfrac{
\dfrac{[\Phi]^4}{\neg T(\ulcorner\Phi\urcorner)}\,def
\qquad
\dfrac{[\neg T(\ulcorner\Phi\urcorner)]^5}{\Phi}\,def
}{\ulcorner\Phi\urcorner = \ulcorner\neg T(\ulcorner\Phi\urcorner)\urcorner}\,\ulcorner\urcorner I_{,4,5}
\qquad
\dfrac{\begin{array}{c}[T(\ulcorner\Phi\urcorner)]^6\\ \Pi_9\\ \bot\end{array}}{\neg T(\ulcorner\Phi\urcorner)}\,\neg I_{,6}
}{
\dfrac{\Phi}{T(\ulcorner\Phi\urcorner)}\,TI
}\,\ulcorner\urcorner E_2
$$

Having derivations Π_9 and Π_{10}, there is a closed derivation Π_{11} of $\bot$.

$$
\dfrac{
\dfrac{\begin{array}{c}[T(\ulcorner\Phi\urcorner)]^3\\ \Pi_9\\ \bot\end{array}}{\neg T(\ulcorner\Phi\urcorner)}\,\neg I_{,3}
\qquad
\begin{array}{c}\Pi_{10}\\ T(\ulcorner\Phi\urcorner)\end{array}
}{\bot}\,\neg E
$$

Claim 2. Π_{11} initiates a non-terminating reduction sequence and is not normalizable.

For simplicity, we first apply standard reduction procedures $\rhd_{\ulcorner\urcorner_1}$ and $\rhd_{\ulcorner\urcorner_2}$ to Π_9 and Π_{10} respectively. By applying $\rhd_{\ulcorner\urcorner_1}$ to Π_9, we have the derivation Π_9' below.

$$
\dfrac{
\dfrac{\dfrac{[T(\ulcorner\Phi\urcorner)]^3}{\Phi}\,TE}{\neg T(\ulcorner\Phi\urcorner)}\,def
\qquad
[T(\ulcorner\Phi\urcorner)]^3
}{\bot}\,\neg E
$$

With the derivation Π'_9, an application of $\rhd_{\ulcorner\urcorner}$ to Π_{10} yields the derivation Π'_{10}.

$$\dfrac{\begin{array}{c}\dfrac{\begin{array}{c}[T(\ulcorner\Phi\urcorner)]^6\\ \Pi'_9\\ \bot\end{array}}{\neg T(\ulcorner\Phi\urcorner)}\,\neg I_{,6}\\ \cdots\cdots\cdots\cdots\ def\\ \Phi\end{array}}{T(\ulcorner\Phi\urcorner)}\,TI$$

Then, we have a closed derivation Π'_{11}, i.e. Π_{11} reduces to Π'_{11}.

$$\dfrac{\begin{array}{c}[T(\ulcorner\Phi\urcorner)]^3\\ \Pi'_9\\ \bot\end{array}\qquad\begin{array}{c}\Pi'_{10}\\ T(\ulcorner\Phi\urcorner)\end{array}}{\bot}\,\neg E$$

By applying, $\rhd_{\neg}$ to Π'_{11}, we have a derivation Π_{12} as follows.

$$\dfrac{\begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\dfrac{\begin{array}{c}[T(\ulcorner\Phi\urcorner)]^6\\ \Pi'_9\\ \bot\end{array}}{\neg T(\ulcorner\Phi\urcorner)}\,\neg I_{,6}\\ \cdots\cdots\cdots\cdots\ def\\ \Phi\end{array}}{T(\ulcorner\Phi\urcorner)}\,TI}{\Phi}\,TE\\ \cdots\cdots\cdots\cdots\ def\\ \neg T(\ulcorner\Phi\urcorner)\end{array}\qquad\begin{array}{c}\Pi'_{10}\\ T(\ulcorner\Phi\urcorner)\end{array}}{\bot}\,\neg E$$

An application of $\rhd_{T(x)}-$reduction to Π_{12} produces the same derivation with Π'_{11}. Hence, Π_{11} initiates a non-terminating reduction sequence and so is not normalizable. □

In order for the derivation Δ_3 using the major premise $\ulcorner\neg T(\ulcorner\Phi\urcorner)\urcorner = \ulcorner\Phi\urcorner$ in Proposition 2.2.1 to produce a non-terminating reduction sequence, it needs to apply $\trianglerighteq_{Sub}-$reduction.

Proposition 2.4.4 establishes that, without $\trianglerighteq_{Sub}-$reduction, the derivation Π_{11} of the Liar paradox employing the major premise $\ulcorner\neg T(\ulcorner\Phi\urcorner)\urcorner = \ulcorner\Phi\urcorner$ can generate a looping reduction if there is a standard reduction procedure to eliminate the major premise. These results represent that every major premise in a derivation should have a reduction procedure to remove it (or to remove subderivations including it) in order for the derivation to have an infinite reduction sequence.

Our question in this section is what prevents the occurrence of a non-terminating reduction sequence. Proposition 2.4.1, 2.4.2, and 2.4.3 explicate that $CR-$rule does not always disguise a non-terminating reduction sequence. Our possible diagnosis is that the use of a major premise in a derivation which has no reduction process to get rid of it stops a non-terminating reduction sequence. Moreover, from the case using the axiom $\neg a \in a \leftrightarrow a \in \{x|x \in x \rightarrow \bot\}$ in Section 2.3, our use of a formula having a principal constant which has no reduction process to eliminate it prevents a non-terminating reduction sequence. Hence, we may summarize our diagnosis in the following way.

A Possible Diagnosis: A derivation formalizing a genuine paradox generates a non-terminating reduction sequence only if (i) every major premise in the derivation has a reduction procedure to eliminate it, or (ii) every formula including a principal constant (or operator) has a reduction procedure to eliminate it.

The derivations in Proposition 2.0.1, 2.2.2, and 2.3.1 include a major premise which has no reduction procedure to get rid of the major premise. The derivation Δ_{12} of Curry's paradox using the axiom $\neg a \in a \leftrightarrow a \in \{x|x \in x \rightarrow \bot\}$ includes a formula containing a principal constant $\in$ which has no reduction process to remove the formula. In order to have a clearer diagnosis beyond our possible diagnosis, the general condition for a non-terminating reduction sequence should be given.

From our diagnosis, we suggest a plausible solution to the problem of undergeneration. Provided that Proposition 2.0.1, 2.2.2, and 2.3.1 which raise the undergeneration problem are inappropriate counterexamples to TCP_E, we have an additional condition that a derivation only uses harmonious rules. $CR-$rule in Proposition 2.0.1 and 2.2.2 has no its corresponding harmonious rule. Although the use of the axiom $\ulcorner\neg T(\ulcorner\Phi\urcorner)\urcorner = \ulcorner\Phi\urcorner$ is regarded as an application of $=I-$rule, we cannot say that the use of the axiom $\ulcorner\neg T(\ulcorner\Phi\urcorner)\urcorner = \ulcorner\Phi\urcorner$ and $=E-$rule has a harmonious relation.

An Additional Condition: A derivation formalizing a genuine paradox only uses harmonious rules.

Harmonious $I-$ and $E-$rules automatically have a standard reduction procedure which eliminates a maximum formula. If the suggested condition is acceptable, the condition can solve the problem of undergeneration.

2.5 Conclusion

In this chapter, we have investigated derivations of Curry's and the Liar paradox which generate the problem of undergeneration. Under the assumption that Curry's and the Liar paradox are genuine paradoxes, at the beginning of this chapter, we introduce the Rogerson-type derivation of Curry's paradox employing $CR-$rule which does not generate a non-terminating reduction sequence. Section 2.2 introduces the Togerson-type derivation of the Liar paradox proposed by Tennant (2015, pp. 585–590). Some Rogerson-type derivations can be a counterexample to TCP_E which raises the problem of undergeneration that TCP_E excludes Curry's and the Liar paradox in the realm of genuine paradoxes. As we have argued in Section 2.2 and 2.3, Tennant thinks that $CR-$rule is the culprit of the problem. However, Section 2.3 discusses that not every case causing the undergeneration problem is related to $CR-$rule. Section 2.4 argues that the occurrence of a non-terminating reduction sequence relies on our choice of reduction procedures and diagnoses what the culprit of the problem of undergeneration. We have seen that the use of $CR-$rule does not always cause the problem, and so the rejection of the use of $CR-$rule is unable to be a solution. This is because, with respect to TCP_E, there are four types of closed derivations which formalize Curry's and the Liar paradox: (i) a T-paradox which does not use $CR-$rule, e.g. Proposition 2.2.1 and 2.A.1, (ii) a T-paradox using $CR-$rule, e.g. Proposition 2.4.1, 2.4.2, 2.4.3, and 2.A.3, (iii) a derivation of $\bot$ which uses $CR-$rule and is not a T-paradox, e.g. Proposition

2.0.1, 2.2.2, and 2.A.2, and (iv) a derivation of $\perp$ which neither use $CR-$rule nor is a T-paradox, e.g. Proposition 2.3.1 and the derivation Δ_{12} of Curry's paradox in Section 2.3. The occurrence of a non-terminating reduction sequence is not always dependent on the use of $CR-$rule.

We have considered that to have an infinite reduction sequence, every major premise should have a reduction procedure to eliminate it. Derivations employing $CR-$rule and axioms which has no infinite reduction sequence lacks the processes to eliminate a major premise which is a conclusion of $CR-$rule or an axiom. Hence, we propose a possible diagnosis that a derivation formalizing a genuine paradox generates a non-terminating reduction sequence only if every major premise in the derivation has a reduction procedure to eliminate it, or every formula including a principal constant (or operator) has a reduction procedure to eliminate it. From the possible diagnosis, we have proposed an additional condition to TCP_E as a plausible solution to the problem of undergeneration that a derivation formalizing a genuine paradox only uses harmonious rules.

Of course, it should be discussed why a paradoxical derivation should only use harmonious rules. However, it is rather natural to say that the use of harmonious rules is desirable in natural deduction since it is the main slogan fo Gentzen who is the founder of natural deduction that an introduction rule exhaustively defines the meaning of a principal constant (or operator) and a corresponding elimination rule is exhaustively determined by the meaning. Harmony requirement is the way to satisfy his slogan.

Furthermore, though we set aside the question of why the non-terminating reduction sequence is the main feature of genuine paradoxes, since the non-terminating reduction is dependent on a set of reduction procedures, it should be discussed how a proper reduction can be suitably evaluated. While we deal with the problem of overgeneration raised by Ekman's paradox in Chapter 3, we shall investigate tests to assess a proper reduction process.

The other remained issue is that which paradox is a genuine one. In this chapter, we presume that Curry's and the Liar paradox are genuine paradoxes. TCP_E is the proof-theoretic criterion for *genuine paradoxes*, but the term 'genuine paradox' is an informal one.

Also, even though Tennant believes that the Liar paradox is genuine, there are derivations of the Liar which do not satisfy TCP_E, e.g. Proposition 2.3.1. In Chapter 4, we will accept the different formalization of the Liar suggested by Tennant (1982) and provide the case that may represent that the Liar paradox is not a genuine paradox. We shall argue that it should be explained which paradox is a genuine paradox and which formalization is legitimate for the genuine paradox.

2.A Appendix 2.A: Tennant's *id est* Rules for a Liar Sentence and a T-Paradox Using $CR-$rule

In this appendix, we will investigate three closed derivations of $\perp$ which formalize the Liar paradox: (i) a T-paradox which use neither $CR-$ nor $CRE-$rules, (ii) a T-paradox using both $CR-$ and $CRE-$rules, and (iii) a derivation of $\perp$ which only uses $CR-rule$ and is not a T-paradox. The purpose of this appendix is to show that there is a T-paradox which the $CR-$rule is applied. Although the use of $CR-$rule is not necessary to formulate Liar paradox in natural deduction, it does not mean that the application of it makes a trouble that disguises the main feature of paradoxical derivations, i.e. the non-terminating reduction sequence. Our examples will show that our use of $CR-$rule does not always hide the occurrence of a non-terminating reduction sequence. Rather, we conclude that the occurrence of a non-terminating reduction sequence is related to a set of reduction procedures.

We already have Proposition 2.2.1, 2.2.2, and 2.4.3. However, those results are based on the formalizations of Tennant (2015) which are not stated in generalized form. TCP_E does not restrict the form of elimination rules. For our convenience, we will use in this appendix, TCP_E rather than Tennant's later criterion. As we will see in Chapter 3, the latest criterion for paradoxicality introduced by Tennant (2016) has an additional condition that all elimination rules are stated in generalized form. Moreover, while Tennant (2016, pp. 12–16) asserts that Liar paradox is a genuine paradox, he proposes the id est rules for the liar sentence Ψ.

$$\dfrac{\begin{array}{c}[T(\ulcorner\Psi\urcorner)]^1\\ \mathfrak{D}_1\\ \bot\end{array}}{\Psi}\,\Psi I_{,1} \qquad \dfrac{\Psi \qquad \begin{array}{c}[\neg T(\ulcorner\Psi\urcorner)]^1\\ \mathfrak{D}_2\\ \varphi\end{array}}{\varphi}\,\Psi E_{,1}$$

Tennant (2016, p. 12) calls $\Psi-$rules the id est rules and said,

> [$\Psi I-$ and $\Psi E-$rules] are the 'id est' rules for the Liar (so-called because of the familiar transitions '[Ψ], i.e. [$\neg T(\ulcorner\Psi\urcorner)$]'). The rules [$\Psi I-$ and $\Psi E-$rules] ensure that the sentence called [Ψ] is *interdeducible with* [$\neg T(\ulcorner\Psi\urcorner)$] - certainly a necessary (even if not sufficient) condition for the former to *be* the latter.

$\Psi-$rules have several peculiar points. First, $\Psi I-$rule introduces a seeming atomic sentence Ψ and the atomic sentence Ψ becomes a major premise in $\Psi E-$rule. Since most of introduction rules introduce a constant to formulas and make a complex formula as its conclusion, a major premise is often regarded as a complex formula. Second, $\Psi I-$rule already has Ψ in the square quote of the premise which is introduced by the rule. Lastly, as Tennant (2016, p. 13) suggests, the reduction procedure for Ψ uses $\neg I-$rule.[15]

$$\dfrac{\dfrac{\begin{array}{c}[T(\ulcorner\Psi\urcorner)]^1\\ \mathfrak{D}_1\\ \bot\end{array}}{\Psi}\,\Psi I_{,1} \qquad \begin{array}{c}[\neg T(\ulcorner\Psi\urcorner)]^2\\ \mathfrak{D}_2\\ \varphi\end{array}}{\varphi}\,\Psi E_2 \qquad \rhd_\Psi \qquad \begin{array}{c}\dfrac{\begin{array}{c}[T(\ulcorner\Psi\urcorner)]^1\\ \mathfrak{D}_1\\ \bot\end{array}}{\neg T(\ulcorner\Psi\urcorner)}\,\neg I_{,1}\\ \mathfrak{D}_2\\ \varphi\end{array}$$

Though his $\Psi-$rules have special features, because of the specificity of paradoxical derivations, he may accept $\Psi-$rules as id est rules for the liar sentence Ψ.

[15]He actually applies a particular graphical form of the reduction procedure for Ψ which fits to the proof of cut-elimination for his Core Logic suggested in Tennant (2012, 2015). However, our discussion of the paradoxes formalized in natural deduction is not confined to his Core Logic. We do not use his graphical forms of reduction processes.

Let S_Ψ be a natural deduction system containing $\neg-$, $T-$, and $\Psi-$rules. The set R_Ψ of reduction procedures have reductions for $\neg$, $T(x)$, and Ψ. We have a system $S_{\Psi C}$ by adding $CR-$rule to S_Ψ and have $S_{\Psi CE}$ by adding $CRE-$rule to $S_{\Psi C}$. Also, $\mathbb{R}_{\Psi C}$ is an extension of $\mathbb{R}_\Psi$ having $\rhd_{CRT}$. $\mathbb{R}_{\Psi CE}$ is given by supplementing $\rhd_{CRE}$ with $\mathbb{R}_{\Psi C}$. Tennant (2016, pp. 14–16) suggests the following result and claims that Liar paradox is a genuine paradox.

Proposition 2.A.1. *There is a closed derivation of $\bot$ in S_Ψ relative to $\mathbb{R}_\Psi$, which initiates a non-terminating reduction sequence and so is not fully normalizable.*

Proof. We begin with the proof of $\bot$ and show that it fails to reduce a full normal derivation. Claim 1. there is a closed derivation Σ_3 of $\bot$.

First, there is an open derivation Σ_1 of $\bot$ from $[T(\ulcorner\Psi\urcorner)]$.

$$\cfrac{[T(\ulcorner\Psi\urcorner)]^2 \qquad \cfrac{[\Psi]^4 \qquad \cfrac{[\neg T(\ulcorner\Psi\urcorner)]^1 \quad [T(\ulcorner\Psi\urcorner)]^2 \quad [\bot]^3}{\bot}\,\neg E_{,3}}{\bot}\,\Psi E_{,1}}{\bot}\,TE_{,4}$$

With Σ_1, we have a closed derivation Σ_2 of $T(\ulcorner\Psi\urcorner)$.

$$\cfrac{\cfrac{\begin{array}{c}[T(\ulcorner\Psi\urcorner)]^2\\ \Sigma_1\\ \bot\end{array}}{\Psi}\,\Psi I_{,2}}{T(\ulcorner\Psi\urcorner)}\,TI$$

Then, we have a closed derivation Σ_3 of $\bot$.

$$\cfrac{\cfrac{\begin{array}{c}[T(\ulcorner\Psi\urcorner)]^2\\ \Sigma_1\\ \bot\end{array}}{\neg T(\ulcorner\Psi\urcorner)}\,\neg I_{,2} \qquad \begin{array}{c}\Sigma_2\\ T(\ulcorner\Psi\urcorner)\end{array} \quad [\bot]^5}{\bot}\,\neg E_{,5}$$

Claim 2. Σ_3 generates a non-terminating reduction sequence and so is not fully normalizable.

$\neg T(\ulcorner\Psi\urcorner)$ in the last $\neg E-$rule of Σ_3 is not an assumption. Σ_3 reduces to the derivation Σ_4 below.

$$\dfrac{\dfrac{\dfrac{\begin{matrix}[T(\ulcorner\Psi\urcorner)]^2\\ \Sigma_1\\ \bot\end{matrix}}{\Psi}\,\Psi I_{,2}}{T(\ulcorner\Psi\urcorner)}\,TI \qquad \dfrac{[\Psi]^4 \qquad \dfrac{[\neg T(\ulcorner\Psi\urcorner)]^1 \quad \begin{matrix}\Sigma_2\\ T(\ulcorner\Psi\urcorner)\end{matrix} \quad [\bot]^3}{\bot}\,\neg E_{,3}}{\bot}\,\Psi E_{,1}}{\bot}\,TE_{,4}$$

By applying $\rhd_{T(x)}$ and $\rhd_{\Psi}$ to Σ_4, we have the same derivation with Σ_3. Σ_3 generates a non-terminating reduction sequence and thus it is not fully normalizable. □

The derivation Σ_3 above satisfies *TCP* and so is a T-paradox. From the result of Proposition 2.A.1, Tennant claims that Liar paradox is a genuine paradox.

Unlike Proposition 2.A.1, the application of $CR-$rule can provide a closed derivation of $\bot$ which does not satisfy *TCP* and so is not a T-paradox.

Proposition 2.A.2. *There is a closed non-full normal derivation Σ_6 of $\bot$ in $S_{\Psi C}$ relative to $\mathbb{R}_{\Psi C}$, and Σ_6 does not generate a non-terminating reduction sequence.*

Proof. Two claims verify the result.

Claim 1. there is a closed derivation Σ_6 of $\bot$ in $S_{\Psi C}$.

We first have an open derivation Σ_5 of $\bot$ from $[\neg\Psi]$.

$$\dfrac{[\neg\Psi]^1 \qquad \dfrac{\dfrac{[T(\ulcorner\Psi\urcorner)]^4 \qquad \dfrac{[\neg\Psi]^1 \quad [\Psi]^2 \quad [\bot]^3}{\bot}\,\neg E_{,3}}{\bot}\,TE_{,2}}{\Psi}\,\Psi I_{,4} \qquad [\bot]^5}{\bot}\,\neg E_{,5}$$

Then, we have a closed derivation Σ_6 of $\bot$.

$$\dfrac{\dfrac{\begin{matrix}[\neg\Psi]^8\\ \Sigma_5\\ \bot\end{matrix}}{\Psi}\,CR_8 \qquad \dfrac{[\neg T(\ulcorner\Psi\urcorner)]^6 \quad \dfrac{\dfrac{\begin{matrix}[\neg\Psi]^1\\ \Sigma_5\\ \bot\end{matrix}}{\Psi}\,CR_{,1}}{T(\ulcorner\Psi\urcorner)}\,TI \quad [\bot]^7}{\bot}\,\neg E_{,7}}{\bot}\,\Psi E_{,6}$$

Claim 2. Neither Σ_6 is in full normal form nor does generate a non-terminating reduction sequence.

The formula Ψ in $\Psi E-$rule of Σ_6 is the major premise which is not an assumption. Σ_6 is not in full normal form. Moreover, Ψ derived by $CR-$rule is an atomic formula. We cannot apply $\trianglerighteq_{CR(T(x))}$ to Σ_6. Hence, since there is no reduction process that we can apply to it, Σ_6 does not produce a non-terminating reduction sequence. □

The derivation Σ_6 generates the problem of undergeneration. To avoid the case that raises the problem, Tennant believes that $CR-$rule has a defect to disguise the occurrence of a non-terminating reduction sequence. Unfortunately, with his $\Psi-$rules, when we regard $CR-$rule as an introduction rule and $CRE-$rule as its corresponding elimination rule, there is a derivation of the Liar paradox employing $CR-$rule.

Proposition 2.A.3. *There is a closed derivation Σ_9 of $\bot$ in $S_{\Psi CE}$ relative to $\mathbb{R}_{\Psi CE}$, which generates a non-terminating reduction sequence and so is not fully normalizable.*

Proof. Two claims establish the result.

Claim 1. There is a closed derivation Σ_9 of $\bot$.

First, we have an open derivation Σ_7 of $\bot$ from $[\neg\Psi]$.

$$\dfrac{[\neg\Psi]^2 \qquad \dfrac{\dfrac{[T(\ulcorner\Psi\urcorner)]^4 \qquad \dfrac{[\Psi]^1 \quad [\neg\Psi]^2 \quad [\bot]^3}{\bot}\,CRE_{,3}}{\bot}\,TE_{,1}}{\Psi}\,\Psi I_{,4} \qquad [\bot]^5}{\bot}\,\neg E_{,5}$$

With Σ_7, there is a derivation Σ_8 of $\bot$ from $[\Psi]$.

$$\dfrac{[\Psi]^9 \qquad \dfrac{[\neg T(\ulcorner\Psi\urcorner)]^6 \qquad \dfrac{\dfrac{\begin{matrix}[\neg\Psi]^2\\ \Sigma_7\\ \bot\end{matrix}}{\Psi}\,CR_{,2}}{T(\ulcorner\Psi\urcorner)}\,TI \qquad [\bot]^{10}}{\bot}\,\neg E_{,10}}{\bot}\,\Psi E_{,6}$$

Then, we have a closed derivation Σ_9 of $\bot$.

$$\dfrac{\dfrac{\begin{matrix}[\neg\Psi]^2\\ \Sigma_7\\ \bot\end{matrix}}{\Psi}\,CR_{,2} \qquad \dfrac{\begin{matrix}[\Psi]^9\\ \Sigma_8\\ \bot\end{matrix}}{\neg\Psi}\,\neg I_{,9} \qquad [\bot]^{11}}{\bot}\,CRE_{,11}$$

Claim 2. Σ_9 initiates a non-terminating reduction sequence and so is not fully normalizable.

Since Ψ in $CRE-$rule of Σ_9 is the major premise which is not an assumption, we apply

$\rhd_{CRE}$ to Σ_9 and have a derivation below.

$$
\dfrac{
\dfrac{\begin{matrix}[\Psi]^{12}\\ \Sigma_8\\ \bot\end{matrix}}{\neg\Psi}\,\neg I_{,12}
\qquad
\dfrac{\dfrac{[T(\ulcorner\Psi\urcorner)]^4 \qquad \dfrac{[\Psi]^1 \quad \dfrac{\begin{matrix}[\Psi]^{9}\\ \Sigma_8\\ \bot\end{matrix}}{\neg\Psi}\,\neg I_{,9} \quad [\bot]^3}{\bot}\,CRE_{,3}}{\bot}\,TE_{,1}}{\Psi}\,\Psi I_{,4}
\qquad
[\bot]^5
}{\bot}\,\neg E_{,5}
$$

It still has a major premise $\neg\Psi$ derived by $\neg I-$rule. By applying $\rhd_{\neg}$ and $\rhd_{\Psi}$, the following derivation is achieved.

$$
\dfrac{
\dfrac{\dfrac{[T(\ulcorner\Psi\urcorner)]^4 \qquad \dfrac{[\Psi]^1 \quad \dfrac{\begin{matrix}[\Psi]^{9}\\ \Sigma_8\\ \bot\end{matrix}}{\neg\Psi}\,\neg I_{,9} \quad [\bot]^3}{\bot}\,CRE_{,3}}{\bot}\,TE_{,1}}{\neg T(\ulcorner\Psi\urcorner)}\,\neg I_{,4}
\qquad
\dfrac{\dfrac{\begin{matrix}[\neg\Psi]^{2}\\ \Sigma_7\\ \bot\end{matrix}}{\Psi}\,CR_{,2}}{T(\ulcorner\Psi\urcorner)}\,TI
\qquad
[\bot]^{10}
}{\bot}\,\neg E_{,10}
$$

Then, the applications of $\rhd_{\neg}$ and $\rhd_{T(x)}$ produce the same derivation with Σ_9. Therefore, Σ_9 generates a non-terminating reduction sequence and so is not fully normalizable. □

Σ_9 satisfies TCP_E and is a T-paradox. If Liar paradox is a genuine paradox, Σ_9 is a T-paradox using $CR-$rule.

Tennant thinks that the application of $CR-$rule disguises the occurrence of a non-terminating reduction sequence. However, the derivation Σ_9 shows that it is not. Σ_6 of Proposition 2.A.2 does not generate an infinite reduction sequence. The difference between Σ_6 and Σ_9 is whether they have a standard reduction procedure for $CR-$rule or not. It is possible to interpret that the difference between Σ_6 and Σ_9 is whether they apply a legitimate pair of introduction and elimination rules. Hence, one may say that the main reason why Σ_6 does not yield a non-terminating reduction sequence is that Σ_6 uses a rule which lacks its corresponding introduction rule. Therefore, the matter may not be the application of classical reasoning but the absence of an introduction rule.

2.B Appendix 2.B: Forms of Permutation Conversions in Natural Deduction

In this appendix, we provide forms of permutation conversion for each elimination rule. Let φ, ψ, σ, χ, and Φ be any formulas. Let Σ, Π, and Ω be any derivations. We use $\circ$ for any expression given by an introduction rule. Suppose that Φ has a form of $\circ\varphi$ or of $\varphi_1 \circ \varphi_2$. For instance, Φ may be a complex formula having the form of $\forall x\varphi$ or of $\varphi_1 \land \varphi_2$. Then the *permutation conversion* for $\circ$ has the following form:

$$\dfrac{\dfrac{\Phi \quad \begin{matrix}\Sigma_1\\ \varphi_1\end{matrix} \dots \begin{matrix}\Sigma_i\\ \varphi_i\end{matrix} \quad \begin{matrix}[\psi_1]^1\\ \Pi_1\\ \sigma\end{matrix} \quad \dots \quad \begin{matrix}[\psi_j]^j\\ \Pi_j\\ \sigma\end{matrix}}{\sigma}\circ E_{,1,\dots,j} \quad \Omega}{\chi} \quad \trianglerighteq_{per} \quad \dfrac{\Phi \quad \begin{matrix}\Sigma_1\\ \varphi_1\end{matrix} \dots \begin{matrix}\Sigma_i\\ \varphi_i\end{matrix} \quad \dfrac{\begin{matrix}[\psi_1]^1\\ \Pi_1\\ \sigma\end{matrix} \quad \Omega}{\chi} \quad \dots \quad \dfrac{\begin{matrix}[\psi_j]^j\\ \Pi_j\\ \sigma\end{matrix} \quad \Omega}{\chi}}{\chi}\circ E_{,1,\dots,j}$$

where i, j are any natural numbers. The following is particular instances of permutation conversions.

1. The case of $\land-$rule.

$$\dfrac{\dfrac{\varphi_1 \land \varphi_2 \quad \begin{matrix}[\varphi_1]^1,[\varphi_2]^1\\ \mathfrak{D}\\ \psi\end{matrix}}{\psi}\land E_{,1} \quad \Sigma}{\sigma} \quad \trianglerighteq_{per(\land)} \quad \dfrac{\varphi_1 \land \varphi_2 \quad \dfrac{\begin{matrix}[\varphi_1]^1,[\varphi_2]^1\\ \mathfrak{D}\\ \psi\end{matrix} \quad \Sigma}{\sigma}}{\sigma}\land E_{,1}$$

(i) When ψ has a form $\Psi_1 \wedge \Psi_2$,

$$
\dfrac{\dfrac{\varphi_1 \wedge \varphi_2 \quad \begin{array}{c}[\varphi_1]^1,[\varphi_2]^1\\ \mathfrak{D}_1\\ \psi_1 \wedge \psi_2\end{array}}{\psi_1 \wedge \psi_2}\wedge E_{,1} \quad \begin{array}{c}[\psi_1]^2,[\psi_2]^2\\ \mathfrak{D}_2\\ \sigma\end{array}}{\sigma}\wedge E_{,2}
\;\trianglerighteq_{per(\wedge\wedge)}\;
\dfrac{\varphi_1 \wedge \varphi_2 \quad \dfrac{\begin{array}{c}[\varphi_1]^1,[\varphi_2]^1\\ \mathfrak{D}_1\\ \psi_1 \wedge \psi_2\end{array} \quad \begin{array}{c}[\psi_1]^2,[\psi_2]^2\\ \mathfrak{D}_2\\ \sigma\end{array}}{\sigma}\wedge E_{,2}}{\sigma}\wedge E_{,1}
$$

(ii) When ψ has a form $\Psi_1 \vee \Psi_2$,

$$
\dfrac{\dfrac{\varphi_1 \wedge \varphi_2 \quad \begin{array}{c}[\varphi_1]^1,[\varphi_2]^1\\ \mathfrak{D}_1\\ \psi_1 \vee \psi_2\end{array}}{\psi_1 \vee \psi_2}\wedge E_{,1} \quad \begin{array}{c}[\psi_1]^2\\ \mathfrak{D}_2\\ \sigma\end{array} \quad \begin{array}{c}[\psi_2]^3\\ \mathfrak{D}_3\\ \sigma\end{array}}{\sigma}\vee E_{,2,3}
\;\trianglerighteq_{per(\wedge\vee)}\;
\dfrac{\varphi_1 \wedge \varphi_2 \quad \dfrac{\begin{array}{c}[\varphi_1]^1,[\varphi_2]^1\\ \mathfrak{D}_1\\ \psi_1 \vee \psi_2\end{array} \quad \begin{array}{c}[\psi_1]^2\\ \mathfrak{D}_2\\ \sigma\end{array} \quad \begin{array}{c}[\psi_2]^3\\ \mathfrak{D}_3\\ \sigma\end{array}}{\sigma}\vee E_{,2,3}}{\sigma}\wedge E_{,1}
$$

(iii) When ψ has a form $\Psi_1 \to \Psi_2$,

$$
\dfrac{\dfrac{\varphi_1 \wedge \varphi_2 \quad \begin{array}{c}[\varphi_1]^1,[\varphi_2]^1\\ \mathfrak{D}_1\\ \psi_1 \to \psi_2\end{array}}{\psi_1 \to \psi_2}\wedge E_{,1} \quad \begin{array}{c}\mathfrak{D}_2\\ \psi_1\end{array} \quad \begin{array}{c}[\psi_2]^2\\ \mathfrak{D}_3\\ \sigma\end{array}}{\sigma}\to E_{,2}
\;\trianglerighteq_{per(\wedge\to)}\;
\dfrac{\varphi_1 \wedge \varphi_2 \quad \dfrac{\begin{array}{c}[\varphi_1]^1,[\varphi_2]^1\\ \mathfrak{D}_1\\ \psi_1 \to \psi_2\end{array} \quad \begin{array}{c}\mathfrak{D}_2\\ \psi_1\end{array} \quad \begin{array}{c}[\psi_2]^2\\ \mathfrak{D}_3\\ \sigma\end{array}}{\sigma}\to E_{,2}}{\sigma}\wedge E_{,1}
$$

(iv) When ψ has a form $\neg\psi_1$,

$$
\dfrac{\dfrac{\varphi_1 \wedge \varphi_2 \quad \begin{array}{c}[\varphi_1]^1,[\varphi_2]^1\\ \mathfrak{D}_1\\ \neg\psi_1\end{array}}{\neg\psi_1}\wedge E_{,1} \quad \begin{array}{c}\mathfrak{D}_2\\ \psi_1\end{array} \quad \begin{array}{c}[\bot]^2\\ \mathfrak{D}_3\\ \sigma\end{array}}{\sigma}\neg E_{,2}
\;\trianglerighteq_{per(\wedge\neg)}\;
\dfrac{\varphi_1 \wedge \varphi_2 \quad \dfrac{\begin{array}{c}[\varphi_1]^1,[\varphi_2]^1\\ \mathfrak{D}_1\\ \neg\psi_1\end{array} \quad \begin{array}{c}\mathfrak{D}_2\\ \psi_1\end{array} \quad \begin{array}{c}[\bot]^2\\ \mathfrak{D}_3\\ \sigma\end{array}}{\sigma}\neg E_{,2}}{\sigma}\wedge E_{,1}
$$

(v) When ψ has a form $\forall x\psi_1(x)$,

$$\dfrac{\dfrac{\varphi_1\wedge\varphi_2 \qquad \begin{array}{c}[\varphi_1]^1,[\varphi_2]^1\\ \mathfrak{D}_1\\ \forall x\psi_1(x)\end{array}}{\forall x\psi_1(x)}\wedge E_{,1} \qquad \begin{array}{c}[\psi_1(t)]^2\\ \mathfrak{D}_2\\ \sigma\end{array}}{\sigma}\forall E_{,2} \quad \trianglerighteq_{per(\wedge\forall)} \quad \dfrac{\varphi_1\wedge\varphi_2 \qquad \dfrac{\begin{array}{c}[\varphi_1]^1,[\varphi_2]^1\\ \mathfrak{D}_1\\ \forall x\psi_1(x)\end{array} \quad \begin{array}{c}[\psi_1(t)]^2\\ \mathfrak{D}_2\\ \sigma\end{array}}{\sigma}\forall E_{,2}}{\sigma}\wedge E_{,1}$$

(vi) When ψ has a form $\exists x\psi_1(x)$,

$$\dfrac{\dfrac{\varphi_1\wedge\varphi_2 \qquad \begin{array}{c}[\varphi_1]^1,[\varphi_2]^1\\ \mathfrak{D}_1\\ \exists x\psi_1(x)\end{array}}{\exists x\psi_1(x)}\wedge E_{,1} \qquad \begin{array}{c}[\psi_1(y)]^2\\ \mathfrak{D}_2\\ \sigma\end{array}}{\sigma}\exists E_{,2} \quad \trianglerighteq_{per(\wedge\exists)} \quad \dfrac{\varphi_1\wedge\varphi_2 \qquad \dfrac{\begin{array}{c}[\varphi_1]^1,[\varphi_2]^1\\ \mathfrak{D}_1\\ \exists x\psi_1(x)\end{array} \quad \begin{array}{c}[\psi_1(y)]^2\\ \mathfrak{D}_2\\ \sigma\end{array}}{\sigma}\exists E_{,2}}{\sigma}\wedge E_{,1}$$

2. The case of $\vee-$rule.

$$\dfrac{\dfrac{\varphi_1\vee\varphi_2 \qquad \begin{array}{c}[\varphi_1]^1\\ \mathfrak{D}_1\\ \psi\end{array} \quad \begin{array}{c}[\varphi_2]^2\\ \mathfrak{D}_2\\ \psi\end{array}}{\psi}\vee E_{,1,2} \qquad \Sigma}{\sigma} \quad \trianglerighteq_{per(\vee)} \quad \dfrac{\varphi_1\vee\varphi_2 \qquad \dfrac{\begin{array}{c}[\varphi_1]^1\\ \mathfrak{D}_1\\ \psi\end{array} \quad \Sigma}{\sigma} \qquad \dfrac{\begin{array}{c}[\varphi_2]^2\\ \mathfrak{D}_2\\ \psi\end{array} \quad \Sigma}{\sigma}}{\sigma}\vee E_{,1,2}$$

The subcases of $\trianglerighteq_{per(\vee)}$ are similar to the subcases of $\trianglerighteq_{per(\wedge)}$.

3. The case of $\to -$rule.

$$\dfrac{\dfrac{\varphi_1\to\varphi_2 \qquad \begin{array}{c}\mathfrak{D}_1\\ \varphi_1\end{array} \quad \begin{array}{c}[\varphi_2]^1\\ \mathfrak{D}_2\\ \psi\end{array}}{\psi}\to E_{,1} \qquad \Sigma}{\sigma} \quad \trianglerighteq_{per(\to)} \quad \dfrac{\varphi_1\to\varphi_2 \qquad \begin{array}{c}\mathfrak{D}_1\\ \varphi_1\end{array} \qquad \dfrac{\begin{array}{c}[\varphi_2]^1\\ \mathfrak{D}_2\\ \psi\end{array} \quad \Sigma}{\sigma}}{\sigma}\to E_{,1}$$

The subcases of $\trianglerighteq_{per(\to)}$ are similar to the subcases of $\trianglerighteq_{per(\wedge)}$.

4. The case of $\neg-$rule.

$$\dfrac{\dfrac{\neg\varphi \quad \begin{matrix}\mathfrak{D}_1\\ \varphi\end{matrix} \quad \begin{matrix}[\bot]^1\\ \mathfrak{D}_2\\ \psi\end{matrix}}{\psi}\neg E_{,1} \qquad \Sigma}{\sigma} \quad \trianglerighteq_{per(\neg)} \quad \dfrac{\neg\varphi \quad \begin{matrix}\mathfrak{D}_1\\ \varphi\end{matrix} \quad \dfrac{\begin{matrix}[\bot]^1\\ \mathfrak{D}_2\\ \psi\end{matrix} \quad \Sigma}{\sigma}}{\sigma}\neg E_{,1}$$

The subcases of $\trianglerighteq_{per(\neg)}$ are similar to the subcases of $\trianglerighteq_{per(\wedge)}$.

5. The case of $\forall-$rule.

$$\dfrac{\dfrac{\forall x\varphi(x) \qquad \begin{matrix}[\varphi[t/x]]^1\\ \mathfrak{D}\\ \psi\end{matrix}}{\psi}\forall E_{,1} \qquad \Sigma}{\sigma} \quad \trianglerighteq_{per(\forall)} \quad \dfrac{\forall x\varphi(x) \qquad \dfrac{\begin{matrix}[\varphi[t/x]]^1\\ \mathfrak{D}\\ \psi\end{matrix} \quad \Sigma}{\sigma}}{\sigma}\forall E_{,1}$$

The subcases of $\trianglerighteq_{per(\forall)}$ are similar to the subcases of $\trianglerighteq_{per(\wedge)}$.

6. The case of $\exists-$rule.

$$\dfrac{\dfrac{\exists x\varphi(x) \qquad \begin{matrix}[\varphi[y/x]]^1\\ \mathfrak{D}\\ \psi\end{matrix}}{\psi}\exists E_{,1} \qquad \Sigma}{\sigma} \quad \trianglerighteq_{per(\exists)} \quad \dfrac{\exists x\varphi(x) \qquad \dfrac{\begin{matrix}[\varphi[y/x]]^1\\ \mathfrak{D}\\ \psi\end{matrix} \quad \Sigma}{\sigma}}{\sigma}\exists E_{,1}$$

The subcases of $\trianglerighteq_{per(\exists)}$ are similar to the subcases of $\trianglerighteq_{per(\wedge)}$.

Chapter 3

A Problem of Overgeneration: Ekman and Crabbé Cases

In Section 1.3 of Chapter 1, we introduce the early version of Tennant's criterion for paradoxicality, TCE_E. Chapter 2 deals with the problem of undergeneration and suggests an additional condition to TCP_E that a derivation formalizing a genuine paradox only employs harmonious rules. Also, we have seen that our choice of reduction procedures can raise the problem. In this chapter, we will consider the problem of overgeneration occurred by Ekman and Crabbé cases. If there is a derivation which satisfies TCP_E but is not about any genuine paradoxes, the derivation shows that TCP_E overgenerates in the sense that TCP_E makes intuitively non-paradoxical derivation paradoxical.

Tennant (1982) sets his criterion for paradoxicality, TCP_E, that genuine paradoxes are distinguished by having non-terminating reduction sequences of the derivation of $\perp$ involved. His early version of the criterion has a criticism from Schroeder-Heister and Tranchini (2017) that it is a too coarse criterion for paradoxicality. They suggest a counterexample to Tennant's early version of the criterion taken from Jan Ekman (1998), called Ekman's paradox. The case shows that Tennant's criterion overgenerates in the sense that there exists a derivation which is intuitively non-paradoxical but satisfies the criterion. To solve the problem of overgeneration, Tennant (2016, 2017) has refined his criterion and proposed an additional condition that all elimination rules stated in generalized form should always have their major premises standing proud, with no non-trivial proof work above them. On the other hand, Schroeder-Heister and Tranchini (2017) diagnose that Ekman's paradox uses too loose reduction procedure, called Ekman reduction, and suggest Triviality test to block Ekman reduction process. The present chapter aims to observe whether their solution is satisfactory

After introducing Ekman's paradox in Section 3.1, we will see Tennant's solution in Section 3.2 that the choice of generalized elimination rules can block the derivation $\perp$ from Ekman's paradox. Then, we will propose Ekman-type reductions stated in generalized form, including the cases suggested by Schroeder-Heister and Tranchini (2018), and argue that Tennant's solution is not successful. Ekman-type cases show that even Tennant's later version of the criterion overgenerates. A promising solution should restrict the use of Ekman-type reduction procedures. Section 3.3 and 3.4 deal with Ekman-type reductions and methods to evaluate them: Triviality and Translation tests. Section 3.3 introduces Schroeder-Heister and Tranchini's Triviality test for a proper reduction and discusses that their notion of a proper reduction would be relative to a based natural deduction system. Moreover, Schroeder-Heister and Tranchini (2018) introduce Crabbé's case which is unaffected by Triviality test and raises the problem of overgeneration. In Section 3.4, as an observation method for evaluating a proper reduction, we propose Translation test to detect errors in both Ekman-type and Crabbé's cases. We will suggest the requirement of a proper reduction that a proper reduction procedure must not introduce any unnecessary premise or detour. Translation test will be useful to assess a proper reduction with regard to the requirements.

3.1 Ekman's Paradox

Let us suppose that there is a derivation satisfying TCP_E which does not formalize a genuine paradox. Then, TCP_E overgenerates the scope of genuine paradoxes and the derivation can be a counterexample to TCP_E. In this section, we will see Ekman's paradox which causes the problem of overgeneration.

Schroeder-Heister and Tranchini (2017) put forward a counterexample to TCP_E taken from Ekman (1998) in order to show that TCP_E is a too coarse criterion for a (genuine) paradox. Ekman (1998) observes the following form of derivation and its reduction.

$$\dfrac{[\psi \to \varphi] \qquad \dfrac{[\varphi \to \psi] \qquad \overset{\mathfrak{D}}{\varphi}}{\psi} \to E}{\varphi} \to E \qquad \trianglerighteq_E \qquad \begin{array}{c}\mathfrak{D}\\ \varphi\end{array}$$

We will call ψ in $\to E-$rule an *Ekman maximum formula*, and this reduction will be called *Ekman reduction* process.

Schroeder-Heister and Tranchini (2017) think that Ekman reduction is a too loose reduction procedure and TCP_E has no restriction to use it. If there is a derivation of $\bot$ from $[\varphi \to \neg\varphi]$ and $[\neg\varphi \to \varphi]$ and the application of Ekman reduction to the derivation generates a looping reduction, TCP_E says that the derivation is a T-paradox. TCP_E overgenerates the scope of genuine paradoxes because it makes the non-paradoxical derivation a T-paradox. The following result is their counterexample to TCP_E.

Proposition 3.1.1. *Let S_E be a natural deduction system consisting of $\to -$ and $\neg-$rules. If the set of reductions of S_E includes Ekman reduction process, S_E has an open derivation of $\bot$ from $[\varphi \to \neg\varphi]$ and $[\neg\varphi \to \varphi]$ which generates a non-terminating reduction sequence, and is not normalizable.*

Proof. Two claims prove the result.

Claim 1. There is an open derivation of $\bot$ from $[\varphi \to \neg\varphi]$ and $[\neg\varphi \to \varphi]$ in S_E.

First, we have an open derivation $\mathfrak{D}_1$ of $\bot$ from $[\varphi \to \neg\varphi]$ and $[\varphi]$.

$$\dfrac{\dfrac{[\varphi \to \neg\varphi]^1 \quad [\varphi]^2}{\neg\varphi} \to E \qquad [\varphi]^2}{\bot} \neg E$$

With the derivation $\mathfrak{D}_1$, we have an open derivation $\mathfrak{D}_2$ of φ from $[\varphi \to \neg\varphi]$ and $[\neg\varphi \to \varphi]$.

$$\dfrac{[\neg\varphi \to \varphi]^3 \qquad \dfrac{\begin{matrix}[\varphi \to \neg\varphi]^1, [\varphi]^2 \\ \mathfrak{D}_1 \\ \bot\end{matrix}}{\neg\varphi}\,\neg I_{,2}}{\varphi}\,\neg E$$

Now, we have the following open derivation Π_1 of $\bot$ from $[\varphi \to \neg\varphi]$ and $[\neg\varphi \to \varphi]$.

$$\dfrac{\dfrac{\begin{matrix}[\varphi \to \neg\varphi]^1, [\varphi]^2 \\ \mathfrak{D}_1 \\ \bot\end{matrix}}{\neg\varphi}\,\neg I_{,2} \qquad \begin{matrix}[\varphi \to \neg\varphi]^1, [\neg\varphi \to \varphi]^3 \\ \mathfrak{D}_2 \\ \varphi\end{matrix}}{\bot}\,\neg E$$

Claim 2. If an Ekman reduction process is used for Π_1, then Π_1 generates a non-terminating reduction sequence and is not normalizable.

Π_1 has a maximum formula $\neg\varphi$ in $\neg E-$rule. By applying $\neg-$reduction, we have the following derivation Π_2.

$$\dfrac{\dfrac{[\varphi \to \neg\varphi]^1 \qquad \dfrac{[\neg\varphi \to \varphi]^3 \qquad \dfrac{\begin{matrix}[\varphi \to \neg\varphi]^1, [\varphi]^2 \\ \mathfrak{D}_1 \\ \bot\end{matrix}}{\neg\varphi}\,\neg I_{,2}}{\varphi}\,\to E}{\neg\varphi}\,\to E \qquad \begin{matrix}[\varphi \to \neg\varphi]^1, [\neg\varphi \to \varphi]^3 \\ \mathfrak{D}_2 \\ \varphi\end{matrix}}{\bot}\,\neg E$$

The derivation Π_2 has no maximum formula but it has an Ekman maximum formula. By applying Ekman reduction, we obtain the same derivation with Π_1 which we started. Therefore, if the set of reductions of S_E includes Ekman reduction, Π_1 initiates a non-terminating reduction sequence and so is not normalizable. □

We call the result of Proposition 3.1.1 Ekman's paradox. Schroeder-Heister and Tranchini (2017, pp. 570-571) appear to think that the derivation Π_1 uses id est inference since Π_1 has inferences from φ to $\neg\varphi$ and $\neg\varphi$ to φ. Also, Π_1 is a derivation of $\perp$ and generates a reduction loop. Hence, according to TCP_E, Π_1 is a T-paradox. Thus, Ekman's paradox can be a counterexample to TCP_E. Schroeder-Heister and Tranchini (2017, p. 571) diagnose the phenomenon, by saying,

> ... we take Ekman's paradox to push the question of when a certain reduction counts as acceptable: whether a derivation is normal depends on the collection of reductions adopted, and hence Tennant's criterion requires particular attention in what should be taken to be a good reduction. In particular, Ekman's phenomenon shows that on a too loose notion of reduction, one obtains a too coarse criterion of paradoxicality.

According to their diagnosis, the occurrence of a looping reduction is relative to which set of reduction procedures we accept. A wrong reduction process does not have to generate any feature of paradoxicality, such as a non-terminating reduction process. Since TCP_E has no constraint on illegitimate reduction procedures, it needs to be revised in order not to overgenerate the scope of genuine paradoxes.

3.2 The Later Version of Tennant's Criterion for Paradoxicality

Tennant (2016, 2017) proposes an additional condition to solve the problem of overgeneration: all elimination rules must be stated in the generalized form. We will see Tennant's solution to the overgeneration raised by Ekman's paradox and his later criterion for paradoxicality (TCP_L) in Section 3.2.1. Section 3.2.2 claims that the choice of generalized elimination rules offers no remedy to the overgeneration problem. The claim is already suggested by Schroeder-Heister and Tranchini (2018). However, they did not distinguish between Tennant's examination of Ekman's paradox and von Plato's. Their proposed Ekman-type reduction, which we will call '$GEkman_g$ reduction,' cannot apply to

Tennant's derivation. Therefore, Section 3.2.2 reinforces the claim that the mere adoption of generalized elimination rules is not a solution to the problem of overgeneration.

3.2.1 Tennant's Solution to the Overgeneration

As noted in Section 3.1, Ekman's paradox can be a counterexample to TCP_E since it raises the overgeneration problem. Although the phenomenon of Ekman's paradox shows that an incorrect reduction procedure produces a non-terminating reduction sequence, TCP_E does not explicitly restrict the application of a wrong reduction process. In order to solve the problem, Tennant (2016, 2017) supplements the condition to TCP_E that all elimination rules must be formulated in the generalized form.

Tennant (2016, 2017) seems to borrow von Plato's solution to Ekman's paradox. Von Plato (2000) notes that the problem of normal form from Ekman's paradox is the choice of standard elimination rules in a natural deduction system.

Let S_{EG} be a natural deduction system containing $\wedge I-$, $\to I-$, $\neg I-$rules with the generalized form of elimination rules. Plus, we define $\varphi \leftrightarrow \neg\varphi$ as $(\varphi \to \neg\varphi) \wedge (\neg\varphi \to \varphi)$. Proposition 3.2.1 is the answer of Tennant (2016, 2017) against Ekman's paradox. His solution says that a non-terminating reduction sequence does not arise if one insists on the application of the generalized form of elimination rules.

Proposition 3.2.1. *There is an open full normal derivation of* $\bot$ *from* $[\varphi \to \neg\varphi]$ *and* $[\neg\varphi \to \varphi]$ *in* S_{EG}.

Proof. The result consists of two claims.

Claim 1. There is an open derivation $\mathfrak{D}_3$ of $\bot$ from $\varphi \to \neg\varphi$ and $\neg\varphi \to \varphi$.

First, there is an open derivation $\mathfrak{D}_1$ of $\bot$ from $[\varphi \to \neg\varphi]$ and $[\varphi]$.

$$\dfrac{[\varphi \to \neg\varphi]^4 \quad [\varphi]^2 \quad \dfrac{[\neg\varphi]^1 \quad [\varphi]^2 \quad [\bot]^3}{\bot}\to E_{,3}}{\bot}\to E_{,1}$$

Then, we have an open derivation $\mathfrak{D}_2$ of φ from $[\varphi \to \neg\varphi]$ and $[\neg\varphi \to \varphi]$.

$$\dfrac{[\neg\varphi \to \varphi]^4 \qquad \dfrac{\begin{matrix}[\varphi]^2, [\varphi \to \neg\varphi]^4 \\ \mathfrak{D}_1 \\ \bot\end{matrix}}{\neg\varphi}\,\neg I_{,2} \qquad [\varphi]^5}{\varphi}\,\to E_{,5}$$

Now, we have an open derivation $\mathfrak{D}_3$ of $\bot$ from $[\varphi \to \neg\varphi]$ and $[\neg\varphi \to \varphi]$.

$$\dfrac{[\varphi \to \neg\varphi]^4 \qquad \dfrac{[\neg\varphi \to \varphi]^4 \qquad \dfrac{\begin{matrix}[\varphi]^2, [\varphi \to \neg\varphi]^4 \\ \mathfrak{D}_1 \\ \bot\end{matrix}}{\neg\varphi}\,\neg I_{,2} \qquad [\varphi]^5}{\varphi}\,\to E_{,5} \qquad \dfrac{[\neg\varphi]^7 \qquad \begin{matrix}[\varphi \to \neg\varphi]^4, [\neg\varphi \to \varphi]^4 \\ \mathfrak{D}_2 \\ \varphi\end{matrix} \qquad [\bot]^6}{\bot}\,\neg E_{,6}}{\bot}\,\to E_{,7}$$

Claim 2. $\mathfrak{D}_3$ is in full normal form.

Since all major premises in $\mathfrak{D}_3$ are assumptions, $\mathfrak{D}_3$ is in full normal form. □

Ekman reduction process is stated in the form of standard elimination rules. We cannot apply it to $\mathfrak{D}_3$, and thus $\mathfrak{D}_3$ is in full normal form. $\mathfrak{D}_3$ does not satisfy TCP_E and so is not a T-paradox. Furthermore, we readily have the following result from the derivation $\mathfrak{D}_3$ of Proposition 3.2.1.

Proposition 3.2.2. *There is a closed full normal derivation $\mathfrak{D}_4$ of $\neg(\varphi \leftrightarrow \neg\varphi)$ in S_{EG}.*

Proof. Two claims verify the result.

Claim 1. there is a closed derivation $\mathfrak{D}_4$ of $\neg(\varphi \leftrightarrow \neg\varphi)$.

From the proof of Proposition 3.2.1, S_{EG} has an open derivation $\mathfrak{D}_3$ of $\bot$ from $[\varphi \to \neg\varphi]$

and $[\neg\varphi \to \varphi]$. Then, we have a closed derivation $\mathfrak{D}_4$ of $\neg(\varphi \leftrightarrow \neg\varphi)$.

$$
\dfrac{\dfrac{[(\varphi \to \neg\varphi) \land (\neg\varphi \to \varphi)]^8 \qquad \begin{array}{c}[\varphi \to \neg\varphi]^4, [\neg\varphi \to \varphi]^4 \\ \mathfrak{D}_3 \\ \bot\end{array}}{\dfrac{\bot}{\neg((\varphi \to \neg\varphi) \land (\neg\varphi \to \varphi))}\neg I_{,8}} \land E_{,4}}{\neg(\varphi \leftrightarrow \neg\varphi)}\ def
$$

Claim 2. $\mathfrak{D}_4$ is in full normal form.

$\mathfrak{D}_3$ is a full normal derivation and every major premise in $\mathfrak{D}_4$ is an assumption. Therefore, $\mathfrak{D}_4$ is in full normal form. □

Proposition 3.2.2 seems to show that Ekman's paradox is not anymore a paradox. Tennant (2016, p. 6) explicates the result of Proposition 3.2.1 as below:

> This proof ... is in [full] normal form. ... Hence, Ekman's so called 'paradox' is no paradox at all. The inconsistency of [$\varphi \to \neg\varphi$ and $\neg\varphi \to \varphi$] has a perfectly straightforward proof in [full] normal form ... With the [generalized] form of $\to -$Elimination, as we have just seen, *there is no looping* in the resulting proof of Ekman's example. This is because it is already in [full] normal form, so there is no reduction sequence to be embarked on.
>
> That it might have been thought otherwise (i.e., that Ekman's example would resist any [full] normal-form proof) is an artefact of the mistaken presumption that a system of natural deduction ought to use the [standard] form of $\to -$Elimination ... rather than the [generalized] form used above.

He thinks that our use of the generalized elimination rules solves the problem of Ekman's paradox. Therefore, his criterion for paradoxicality has an additional condition that every elimination rule in a given derivation is to be stated in generalized form. We have the later version of Tennant's criterion for paradoxicality, TCP_L, by adding the following condition to TCP_E.

(iv) all elimination rules in $\mathfrak{D}$ are stated in generalized form.

As TCP_L uses the forms of generalized elimination rules, we will use the notion of 'full normal form' rather than 'normal form' if our discourse is on TCP_L. According to TCP_L, Ekman's paradox is not a T-paradox at all. Unfortunately, Tennant overlooks the point that we can provide a generalized form of Ekman-type reduction which is fitted to the generalized elimination rule for $\to$. In the next subsection, we will see the problem of Tennant's solution.

3.2.2 A Problem of Tennant's Solution

The main reason why Ekman reduction cannot apply to the derivation $\mathfrak{D}_3$ of Proposition 3.2.1 is that $\to E-$rule in Ekman reduction is stated in standard form. We state Ekman-type reduction procedure with respect to the generalized elimination rule for $\to$ as follows:

$$\dfrac{[\sigma \to \varphi] \qquad \dfrac{[\varphi \to \psi] \quad \begin{matrix}\mathfrak{D}'_1\\ \varphi\end{matrix} \quad \begin{matrix}[\psi]^1\\ \mathfrak{D}`_2\\ \sigma\end{matrix}}{\sigma}\to E_{,1} \qquad \begin{matrix}[\varphi]^2\\ \mathfrak{D}'_3\\ \rho\end{matrix}}{\rho}\to E_{,2} \qquad \trianglerighteq_{GE} \qquad \begin{matrix}\mathfrak{D}'_1\\ \varphi\\ \mathfrak{D}'_2\\ \sigma\end{matrix}$$

We call the process *GEkman reduction* process and the minor premise ψ in the last $\to E-$rule a *GEkman-maximum* formula. Then, $\mathfrak{D}_3$ of Proposition 3.2.1 has a GEkman-maximum formula, φ, in the last $\to E-$rule. We apply GEkman reduction to $\mathfrak{D}_3$ and have the result below.

Proposition 3.2.3. *If GEkman reduction is in the set of reduction procedures of S_{EG}, S_{EG} has an open derivation of $\bot$ from $[\varphi \to \neg\varphi]$ and $[\neg\varphi \to \varphi]$ which generates a non-terminating reduction sequence and so is not fully normalizable.*

Proof. We apply GEkman reduction to $\mathfrak{D}_3$ of Proposition 3.2.1. Then, $\mathfrak{D}_3$ yields the derivation $\mathfrak{D}_5$ below.

$$\dfrac{\dfrac{\dfrac{[\varphi \to \neg\varphi]^4 \quad [\varphi]^2 \quad \dfrac{[\neg\varphi]^1 \quad [\varphi]^2 \quad [\bot]^3}{\bot}\neg E_{,3}}{\bot}\to E_{,1}}{\neg\varphi}\neg I_{,2} \qquad \begin{matrix}[\varphi \to \neg\varphi]^4, [\neg\varphi \to \varphi]^4 \\ \mathfrak{D}_2 \\ \varphi\end{matrix} \qquad [\bot]^6}{\bot}\neg E_{,6}$$

Since the major premise $\neg\varphi$ of $\neg E-$rule is derived by $\to I-$rule and so is not an assumption, $\neg-$reduction process produces the same derivation of $\mathfrak{D}_3$. Therefore, $\mathfrak{D}_5$ generates a non-terminating reduction sequence which is not normalizable. □

We call the result of Proposition 3.2.3 GEkman paradox. Then, GEkman paradox shows that TCP_L overgenerates the scope of genuine paradoxes if GEkman reduction is allowed. Again, TCP_L has a counterexample.

Likewise, if GEkman reduction is accepted, the derivation $\mathfrak{D}_4$ of $\neg(\varphi \leftrightarrow \neg\varphi)$ in Proposition 3.2.2 cannot be converted into a full normal derivation. Tennant does not seem to take GEkman reduction into account. As long as GEkman reduction is used, there is an open derivation of $\bot$ from $[\varphi \to \neg\varphi]$ and $[\neg\varphi \to \varphi]$ and a closed derivation of $\neg(\varphi \leftrightarrow \neg\varphi)$ in S_{EG} which raise infinite reduction sequences. Therefore, our mere choice of generalized forms of elimination rules does not solve the problem of Ekman's paradox because a looping reduction is relative to the set of reduction procedures we accepted. The real issue is to be which set of proper reductions we choose and Tennant did not consider it.

Tennant may answer to the problem of GEkman paradox that if we apply *permutation conversion*, found by Gentzen (2008) and Prawitz (1965) for $\vee E-$ and $\exists E-$rules, before using GEkman reduction, we would have an open full normal derivation of $\bot$ from $[\varphi \to \neg\varphi]$ and $[\neg\varphi \to \varphi]$ because GEkman reduction is not applicable. Permutation conversion is an essential process to eliminate the major premise derived by an elimination rule. Permutation conversion for the case of $\to -$rule has the following process. Let Σ be an arbitrary derivation and $\star$ be any constant.

$$
\dfrac{\dfrac{\varphi_1 \to \varphi_2 \quad \begin{matrix} \mathfrak{D}_1 \\ \varphi_1 \end{matrix} \quad \begin{matrix} [\varphi_2]^1 \\ \mathfrak{D}_2 \\ \psi \end{matrix}}{\psi}\to E_{,1} \qquad \Sigma}{\rho}\star E
\quad \trianglerighteq_{per(\to)} \quad
\dfrac{\varphi_1 \to \varphi_2 \quad \begin{matrix} \mathfrak{D}_1 \\ \varphi_1 \end{matrix} \quad \dfrac{\begin{matrix} [\varphi_2]^1 \\ \mathfrak{D}_2 \\ \psi \end{matrix} \quad \Sigma}{\rho}\star E}{\rho}\to E_{,1}
$$

Then, by applying $\trianglerighteq_{per(\to)}$ to Tennant's derivation $\mathfrak{D}_3$ of Proposition 3.2.1, we have the derivation $\mathfrak{D}_6$ below.

$$
\dfrac{[\neg\varphi \to \varphi]^4 \qquad \dfrac{\begin{matrix} [\varphi]^2, [\varphi \to \neg\varphi]^4 \\ \mathfrak{D}_1 \\ \bot \end{matrix}}{\neg\varphi}\neg I_{,2} \qquad \dfrac{[\varphi \to \neg\varphi]^4 \quad [\varphi]^5 \qquad \dfrac{[\neg\varphi]^7 \quad \begin{matrix} [\varphi \to \neg\varphi]^4, [\neg\varphi \to \varphi]^4 \\ \mathfrak{D}_2 \\ \varphi \end{matrix} \quad [\bot]^6}{\bot}\neg E_{,6}}{\bot}\to E_{,7}}{\bot}\to E_{,5}
$$

GEkman reduction $\trianglerighteq_{GE}$ cannot apply to $\mathfrak{D}_6$ and $\mathfrak{D}_6$ is in full normal form. However, if permutation conversions were allowed, the permuted form of GEkman reduction would also be allowed.

$$
\dfrac{[\varphi \to \psi] \quad \begin{matrix} \mathfrak{D}'_1 \\ \varphi \end{matrix} \qquad \dfrac{[\psi \to \varphi] \quad [\psi]^1 \quad \begin{matrix} [\varphi]^2 \\ \mathfrak{D}'_2 \\ \rho \end{matrix}}{\rho}\to E_{,2}}{\rho}\to E_{,1}
\qquad \trianglerighteq_{GE(per)} \quad
\begin{matrix} \mathfrak{D}'_1 \\ \varphi \\ \mathfrak{D}'_2 \\ \rho \end{matrix}
$$

Although it is odd that the assumption $[\psi]$ in $\to E-$rule becomes a GEkman maximum formula of the permuted form of GEkman reduction, $\trianglerighteq_{GE(per)}$, it shares the same conclusion from the same premises with GEkman reduction $\trianglerighteq_{GE}$. There seems no reason to reject the permuted form of GEkman reduction if we accept the permutation conversion and GEkman reduction. The application of $\trianglerighteq_{GE(per)}$ to $\mathfrak{D}_6$ produces the same derivation with $\mathfrak{D}_5$ of Proposition 3.2.3 and it initiates an infinite reduction sequence.

The other possible solution to the GEkman problem is that, instead of using Tennant's derivation $\mathfrak{D}_3$ of Proposition 3.2.1, we use the derivation $\mathfrak{D}_7$ of $\bot$ from $[\varphi \to \neg\varphi]$ and $[\neg\varphi \to \varphi]$ suggested by Von Plato (2000) below.

$$\dfrac{[\neg\varphi \to \varphi]^4 \qquad \dfrac{\begin{matrix}[\varphi]^2, [\varphi \to \neg\varphi]^4 \\ \mathfrak{D}_1 \\ \bot\end{matrix}}{\neg\varphi}\,\neg I_{,2} \qquad \dfrac{[\varphi \to \neg\varphi]^4 \quad [\varphi]^5 \qquad \dfrac{[\neg\varphi]^7 \quad [\varphi]^5 \quad [\bot]^6}{\bot}\,\neg E_{,6}}{\bot}\,\to E_{,7}}{\bot}\,\to E_{,5}$$

The application of $\trianglerighteq_{GE(per)}$ to $\mathfrak{D}_7$ does not generate a non-terminating reduction sequence and it is an open full normal derivation. Von Plato (2000, p. 123) proposes a closed full normal derivation of $\neg(\varphi \leftrightarrow \neg\varphi)$ below as a solution to the problem Ekman's paradox and it may also be the solution to the problem of GEkman paradox.

$$\dfrac{\dfrac{[(\varphi \to \neg\varphi) \wedge (\neg\varphi \to \varphi)]^8 \qquad \begin{matrix}[\varphi \to \neg\varphi]^4, [\neg\varphi \to \varphi]^4 \\ \mathfrak{D}_7 \\ \bot\end{matrix}}{\bot}\,\wedge E_{,4}}{\dfrac{\neg((\varphi \to \neg\varphi) \wedge (\neg\varphi \to \varphi))}{\neg(\varphi \leftrightarrow \neg\varphi)}\,def}\,\neg I_{,8}$$

However, von Plato's solution is still unsatisfactory. As we have devised GEkman reductions, $\trianglerighteq_{GE}$ and $\trianglerighteq_{GE(per)}$, to Tennant's derivation $\mathfrak{D}_3$ and $\mathfrak{D}_6$, we can invent an Ekman-type reduction applicable to von Plato's derivation $\mathfrak{D}_7$. Schroeder-Heister and Tranchini (2018) propose the following form of reduction process:

$$
\dfrac{[\varphi \to \psi] \quad \begin{array}{c}\mathfrak{D}_1''\\ \varphi\end{array} \quad \dfrac{[\psi \to \varphi] \quad [\psi]^1 \quad \begin{array}{c}[\psi]^1, [\varphi]^2\\ \mathfrak{D}_2''\\ \rho\end{array}}{\rho}\to E_{,2}}{\rho}\to E_{,1}
\qquad \trianglerighteq_{GE_g} \qquad
\dfrac{[\varphi \to \psi] \quad \begin{array}{c}\mathfrak{D}_1''\\ \varphi\end{array} \quad \begin{array}{c}\mathfrak{D}_1''\\ [\psi]^1,\ \varphi\\ \mathfrak{D}_2''\\ \rho\end{array}}{\rho}\to E_{,1}
$$

The application of $\trianglerighteq_{GE_g}$ to $\mathfrak{D}_7$ generates a looping reduction sequence. Hence, the correct solution against the problem of overgeneration should explain why Ekman-type reductions, such as $\trianglerighteq_{GE}$, $\trianglerighteq_{GE_{(per)}}$, and $\trianglerighteq_{GE_g}$, are not proper.

Though Tennant (2016) and Von Plato (2000) think that our choice of generalized form of elimination rules can be a solution to the problem raised by Ekman-type reductions, the real issue is to be which reduction procedures are proper.

Schroeder-Heister and Tranchini (2017) have proposed a derivation of Ekman's paradox stated in standard form as a counterexample to TCP_E. Tennant's answer is to have an additional condition (iv) that all elimination rules are formulated in generalized form. We have discussed in Section 3.2.2 that, for a successful answer to the problems caused by Ekman-type reductions, the condition (iv) is not enough and Tennant needs to focus on which reduction procedures are proper. As Schroeder-Heister and Tranchini (2017, p. 571) note, whether a derivation is in (full) normal form relies on the choice of reduction procedures. They attempt to show that Ekman reduction makes two derivations which represent distinct proofs belonging to the same equivalence class and so it is a wrong process. Tennant (2016, 2017) may not wish to follow their line of thought and does believe that our mere choice of generalized elimination rules would be a more modest solution than theirs. We already have seen that Tennant's solution is not successful. A successful solution should provide a relevant criterion for a proper reduction process which can restrict illegitimate reductions, such as Ekman-type reduction procedures. In the next two sections, we shall investigate plausible tests to block the illegitimate reductions.

3.3 Schroeder-Heister and Tranchini's Triviality Test

As we have seen in Section 3.2.2, GEkman paradox shows that TCP_L overgenerates. If GEkman reduction is not proper, GEkman paradox is not a suitable counterexample to Tennant's criterion for paradoxicality. The main issue is to be which reduction process is a proper one. For instance, Schroeder-Heister and Tranchini (2017, Sec. 5) pay attention to finding a criterion for the proper reduction process. So, instead of considering Tennant's condition (iv) of TCP_L, we have the revised version of Tennant's criterion for paradoxicality:

The Revised Version of Tennant's Criterion for Paradoxicality(*RTCP*): Let S be a natural deduction system relative to a set $\mathbb{R}$ of reduction procedures. $\mathfrak{D}$ be any derivation in S. $\mathfrak{D}$ is a *T-paradox* if and only if

(i) $\mathfrak{D}$ is a (closed or open) derivation of $\bot$,

(ii) *id est* inferences (or rules) are used in $\mathfrak{D}$,

(iii) a reduction procedure of $\mathfrak{D}$ generates a non-terminating reduction sequence, such as a reduction loop,

(iv) any reduction procedure in $\mathbb{R}$ is proper.

With regard to *RTCP*, it should be asked how we find the criterion for a set of proper reduction procedures. Schroeder-Heister and Tranchini (2017, p. 575) propose a requirement of a new reduction process that the reduction must not trivialize the identity of proofs in the sense that it should be possible to show that different derivations belong to dintinct equivalence classes. They claim that Ekman reduction trivializes the identity of proofs and so is not a proper reduction. In Section 3.3.1, we shall investigate Schroeder-Heister and Tranchini's notion of 'trivialize the identity of proofs' and introduce Triviality test to restrict the application of Ekman reduction process. First, they accept Prawitz's thesis that a proper reduction should not affect the identity of proofs represented by derivations in the same equivalence class. Then, they attempt to show that when Ekman reduction is used, two derivations representing different proofs belong to the same equivalence class and any derivation of the same formula represents the same proof. That is, Ekman reduction trivializes the identity of proofs. Section 3.3.2 argues that Triviality test may not be suitable for evaluating all standard reduction procedures and it fails to block every Ekman-type reduction. Their proper reductions evaluated by Triviality test is relative to a natural deduction system.

3.3.1 Triviality Test

Prawitz (1971, p. 257) first suggests the idea that a proper reduction may not effect the identity of proofs. He conjectures that two derivations represent the same proof if and only if they are equivalent. The equivalent relation between derivations with the same assumptions and the same conclusion is the reflexive, transitive, and symmetric closure of the immediate reducibility relation.

Prawitz's equivalence relation $\sim$ is defined via the reducibility relation $\succ$ introduced in Definition 1.2.4. We borrow the notion of the equivalence relation between derivations from Prawitz (1971, p. 255). Let $\mathbb{R}$ be any set of reduction procedures.[1]

Definition 3.3.1. A derivation $\mathfrak{D}_1$ is *equivalent* to $\mathfrak{D}_i$ ($\mathfrak{D}_1 \sim \mathfrak{D}_i$) relative to $\mathbb{R}$ iff $\mathfrak{D}_1 \succ \mathfrak{D}_i$ or $\mathfrak{D}_i \succ \mathfrak{D}_1$ where $1 \leqslant i$ for any natural number i; otherwise, they are not equivalent ($\mathfrak{D}_1 \nsim \mathfrak{D}_i$). Let S be any natural deduction system. The equivalence class of $\mathfrak{D}_1$ under $\sim$ in S, denoted by $[\![\mathfrak{D}_1]\!]$, is defined as $[\![\mathfrak{D}_1]\!] = \{\mathfrak{D}_i \in S | \mathfrak{D}_1 \sim \mathfrak{D}_i\}$.

Then, the relation $\sim$ is clearly reflexive, symmetric, and transitive. Prawitz's conjecture is summarized as below.

The Conjecture for the Identity of Proofs: for any derivation $\mathfrak{D}_1$ and $\mathfrak{D}_2$, $\mathfrak{D}_1$ and $\mathfrak{D}_2$ represent the same proof iff $\mathfrak{D}_1 \sim \mathfrak{D}_2$.

When it comes to the conjecture for the identity of proofs, Prawitz (1971, p. 257) mentions two things: a proper reduction and the identity of proofs.

[1] Although Prawitz (1971) considers a set of standard reduction procedures for rules of first order intuitionistic and classical logic, since we focus on the test of a proper reduction procedure, we consider arbitrary set of reduction procedures.

> The two equivalent derivations represent the same proof seems to be a reasonable thesis. It seems evident from our discussion ... of the inversion principle that a proper reduction does not effect the identity of the proof represented. ... It should be noted that the strong normalization theorem gives a certain coherence to the conjecture. It implies that two derivations are equivalent only if the normal derivations to which they reduce are identical, and hence, that two different normal derivations are never equivalent.

His thesis says that a proper reduction does not affect the identity of proofs represented by derivations in the same equivalence class. So to speak, when two derivations represent different proofs, they are not equivalent. His conjecture is related with the strong normalization theorem. The strong normalization theorem proposed in Prawitz (1971, p. 256) states that every derivation is reducible to a unique normal derivation regardless of the order in which reductions are applied. He proposed a result that the strong normalization theorem holds in a first order minimal, an intuitionistic, and a classical natural deduction system.

Theorem 3.3.2. ***(Prawitz 1971)*** *Every derivation $\mathfrak{D}$ in a (first order) minimal, an intuitionistic, or a classical natural deduction system is reducible to a unique normal derivation $\mathfrak{D}'$ and every reduction sequence starting from $\mathfrak{D}$ terminates in $\mathfrak{D}'$.*

Let us consider two derivations $\mathfrak{D}_1$ and $\mathfrak{D}_2$ in a natural deduction system *S* which the strong normalization theorem is proved. By the strong normalization theorem, $\mathfrak{D}_1$ and $\mathfrak{D}_2$ have their unique normal derivations. If $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are equivalent, then they have the same normal derivation. Hence, when the conjecture for the identity of proofs is true, a proof represented by derivations in the same equivalence class has a unique normal derivation. In other words, derivations in the same equivalence class have the same normal derivation. In this sense, Prawitz (1971, p. 256) says, ‘two different normal derivations are never equivalent.’ Therefore, if a reduction process is proper, it does not affect the identity of proofs. We summarize his thesis in the following way.

Prawitz’s thesis A proper reduction does not affect the identity of proofs represented by derivations in the same equivalence class. (i.e. derivations representing different proofs are not equivalent.)

Schroeder-Heister and Tranchini (2017, pp. 574-575) take the conjecture for the identity of proofs and Prawitz's thesis. Especially, they suggest Triviality test to evaluate whether a newly added reduction makes derivations representing different proofs belonging an equivalent class. For instance, let us consider the following two derivations with the same assumptions and the same conclusion but the assumptions are discharged at different places.

$$\dfrac{\dfrac{\dfrac{[\varphi]^1}{\varphi \to \varphi} \to_{I,1}}{\varphi \to (\varphi \to \varphi)} \to I_{,\emptyset} \qquad \begin{matrix}\mathfrak{D} \\ \varphi\end{matrix}}{\varphi \to \varphi} \to E \qquad\qquad \dfrac{\dfrac{\dfrac{[\varphi]^1}{\varphi \to \varphi} \to I_{,\emptyset}}{\varphi \to (\varphi \to \varphi)} \to I_{,1} \qquad \begin{matrix}\mathfrak{D} \\ \varphi\end{matrix}}{\varphi \to \varphi} \to E$$

For their view, the two derivations above belong to two different equivalence classes in the sense of Prawitz' equivalent relation. In the case of the empty discharge, the reduction process for $\to$ has the following form.

$$\dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1 \\ \psi\end{matrix}}{\varphi \to \psi} \to I_{,\emptyset} \qquad \begin{matrix}\mathfrak{D}_2 \\ \varphi\end{matrix}}{\psi} \to E \qquad\qquad \rhd_{\to(\emptyset)} \qquad \begin{matrix}\mathfrak{D}_1 \\ \psi\end{matrix}$$

$\rhd_{\to(\emptyset)}$−reduction is an instance of $\rhd_{\to}$. Then, the reduced derivations below are obtained by $\rhd_{\to}$ respectively.

$$\dfrac{[\varphi]^1}{\varphi \to \varphi} \to I_{,1} \qquad\qquad \dfrac{\begin{matrix}\mathfrak{D} \\ \varphi\end{matrix}}{\varphi \to \varphi} \to I_{,\emptyset}$$

The two normal derivations are different. Provided that the equivalent derivations representing the same proof must have the same normal derivation, if two derivations are equivalent, they should have the same normal derivation, but they do not.

Let $\trianglerighteq$ be a reduction procedure not in a set $\mathbb{R}$ of reductions and $\mathbb{R}'$ be an extension of $\mathbb{R}$ by adding $\trianglerighteq$. We use an abbreviation '$S(\mathbb{R})$' for a natural deduction system S relative to

$\mathbb{R}$. Let us assume that the strong normalization theorem is provable for $S(\mathbb{R})$ and consider the case that there is a derivation Π in S such that Π has a unique normal derivation in $S(\mathbb{R})$ but has two normal derivations in $S(\mathbb{R}')$. Since, by Prawitz's thesis, a proper reduction does not affect the identity of proofs and, by strong normalization theorem, a derivation representing the same proof has a unique normal derivation, we may conclude that $\trianglerighteq$ is not a proper reduction procedure. From this perspective, Schroeder-Heister and Tranchini (2017, p. 575) propose Triviality test for a proper reduction process that a newly added reduction procedure should not trivialize the identity of proofs.

> A natural requirement of the addition of a new reduction could be that of not trivializing identity of proof, in the sense that it should always be possible to exhibit two derivations of the same conclusion belonging to two distinct equivalence classes. (Schroeder-Heister and Tranchini, 2017, p. 575)

We say that, for any derivations $\mathfrak{D}_1$ and $\mathfrak{D}_2$ which represent different proofs, i.e. $\mathfrak{D}_1 \not\approx \mathfrak{D}_2$, in $S(\mathbb{R})$, a reduction process $\trianglerighteq$ *trivializes the identity of proofs* in S iff it is not possible to show in $S(\mathbb{R}')$ that $\mathfrak{D}_1 \not\approx \mathfrak{D}_2$; otherwise, $\trianglerighteq$ does *not trivialize the identity of proofs* in S. Then, their Triviality test can be summarized as below:

Triviality Test: Let S be any natural deduction system. Let $\trianglerighteq$ be a newly added reduction procedure not in $\mathbb{R}$ and $\mathbb{R}'$ be an extension of $\mathbb{R}$ by adding $\trianglerighteq$. $\trianglerighteq$ is a *proper reduction* procedure for S iff $\trianglerighteq$ does not trivialize the identity of proofs in S.

In order to show that Ekman reduction fails to pass Triviality test, Schroeder-Heister and Tranchini (2017, pp. 575-577) provide an example that Ekman reduction trivializes the identity of proofs. Let S_T be a natural deduction system containing $\wedge-$ and $\rightarrow -$rules with their standard elimination rules. S_T has a set $\mathbb{R}_T$ of standard reductions for $\wedge$ and $\rightarrow$. We say that $\mathbb{R}'_T$ is an extension of $\mathbb{R}_T$ by adding Ekman reduction $\trianglerighteq_E$. If Ekman reduction $\trianglerighteq_E$ is a proper reduction, it does not trivialize the identity of proofs. Schroeder-Heister and Tranchini's example shows that it is not possible to show that two reduced derivations from Π_1 in $S_T(\mathbb{R}'_T)$ which represent different proofs are not equivalent.

Suppose that, for any two derivations $\mathfrak{D}_1$ and $\mathfrak{D}_2$ of φ in S_T, there is a derivation of $\varphi \wedge \varphi$ below:

$$\dfrac{\begin{matrix}\mathfrak{D}_1 & \mathfrak{D}_2 \\ \varphi & \varphi\end{matrix}}{\varphi \wedge \varphi}\wedge I$$

Schroeder-Heister and Tranchini's derivation Π_1 of $\varphi \wedge \varphi$ is as follows.

$$\dfrac{\dfrac{\dfrac{[\varphi]^1 \quad [\varphi]^1}{\varphi \wedge \varphi}\wedge I}{\varphi \to (\varphi \wedge \varphi)}\to I_{,1} \qquad \dfrac{\dfrac{\dfrac{[\varphi \wedge \varphi]^2}{\varphi}\wedge E_1}{(\varphi \wedge \varphi) \to \varphi}\to I_{,2} \quad \dfrac{\begin{matrix}\mathfrak{D}_1\mathfrak{D}_2 \\ \varphi \ \ \varphi\end{matrix}}{\varphi \wedge \varphi}\wedge I}{\varphi}\to E}{\varphi \wedge \varphi}\to E$$

$S_T(\mathbb{R}_T)$ has a derivation Π_2 of Π_1 by applying $\rhd_{\to}$ and $\rhd_{\wedge}$ with regard to $\wedge E_1-$rule.

$$\dfrac{\begin{matrix}\mathfrak{D}_1\mathfrak{D}_1 \\ \varphi \ \ \varphi\end{matrix}}{\varphi \wedge \varphi}\wedge I$$

On the other hand, $S_T(\mathbb{R}'_T)$ has not only Π_2 but also Π_3 through the application of $\trianglerighteq_E$.

$$\dfrac{\begin{matrix}\mathfrak{D}_1\mathfrak{D}_2 \\ \varphi \ \ \varphi\end{matrix}}{\varphi \wedge \varphi}\wedge I$$

A proof represented by derivations in $[\![\Pi_1]\!]$ should represent the same proof. By Definition 3.3.1, Π_2 and Π_3 are equivalent to Π_1. So Π_2 and Π_3 must represent the same proof. It means that any two derivations of φ are equivalent in $S_T(\mathbb{R}'_T)$ and, by the conjecture for the identity of proofs, they represent the same proof. Since not every derivation of φ in S_T represent the same proof, there are two distinct derivations $\mathfrak{D}_1$ and $\mathfrak{D}_2$ such that $\mathfrak{D}_1 \nsim \mathfrak{D}_2$ in $S_T(\mathbb{R}_T)$. However, because every derivation of φ in $S_T(\mathbb{R}'_T)$ is equivalent, it is not possible to show in $S_T(\mathbb{R}'_T)$ that any two derivations representing different proofs are not equivalent, i.e. $\mathfrak{D}_1 \nsim \mathfrak{D}_2$. Therefore, $\trianglerighteq_E$ trivializes the identity of proofs in S_T, and so Triviality test says that it is not a proper reduction for S_T.

As we have noted, Triviality test presumes the conjecture for the identity of proofs and Prawitz's thesis. If these theses are requisite for a correct natural deduction system, Triviality test can be a proper-reduction-checker. The condition (iv) of $RTCP$ can be claimed through Triviality test. In the next subsection, after we shall discuss that Triviality test may not evaluate the legitimacy of every type of GEkman reduction procedures because it is relative to a given system.

3.3.2 Problems of Triviality Test

Triviality test fails to block every Ekman-type reduction procedures because it works relative to a natural deduction system. Schroeder-Heister and Tranchini's derivation Π_1 is proposed in a natural deduction system with the standard elimination rules. As we have discussed in Section 3.2.2, there is an Ekman-type reduction with the generalized elimination rule for $\to$, such as GEkman reduction procedure. Let S_{TG} be a natural deduction system containing $\wedge-$ and $\to -$rules with their generalized elimination rules. A set $\mathbb{R}_{TG}$ of reductions only has the standard reduction processes for $\wedge$ and $\to$ with generalized elimination rules for $\wedge$ and $\to$. $\mathbb{R}'_{TG}$ is an extension of $\mathbb{R}_{TG}$ by adding GEkman reduction procedure. Then S_{TG} has a similar derivation with Π_1 formulated in the generalized form.

$$\dfrac{\dfrac{\dfrac{[\varphi]^1[\varphi]^1}{\varphi\wedge\varphi}\wedge I}{\varphi\to(\varphi\wedge\varphi)}\to I_{,1} \qquad \dfrac{\dfrac{\dfrac{\dfrac{[\varphi\wedge\varphi]^2[\varphi]^3}{\varphi}\wedge E_{,3}}{(\varphi\wedge\varphi)\to\varphi}\to I_{,2} \quad \dfrac{\begin{matrix}\mathfrak{D}_1\mathfrak{D}_2\\ \varphi\quad\varphi\end{matrix}}{\varphi\wedge\varphi}\wedge I \quad [\varphi]^4}{\varphi}\to E_{,4}}{} \quad [\varphi\wedge\varphi]^5}{\varphi\wedge\varphi}\to E_{,5}$$

We call the above derivation Σ_1. The minor premise φ in the last $\to E-$rule is a GEkman maximum formula. The application of GEkman reduction provides the following derivation

Σ_2.

$$\dfrac{\begin{matrix}\mathfrak{D}_1 \mathfrak{D}_2 \\ \varphi \quad \varphi\end{matrix}}{\varphi \wedge \varphi} \wedge I$$

Moreover, we apply $\rightarrow -$reduction to Σ_1 twice and have the derivation below:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1 \mathfrak{D}_2 \\ \varphi \quad \varphi\end{matrix}}{\varphi \wedge \varphi} \wedge I \quad [\varphi]^3}{\varphi} \wedge E_{,3} \quad \dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1 \mathfrak{D}_2 \\ \varphi \quad \varphi\end{matrix}}{\varphi \wedge \varphi} \wedge I \quad [\varphi]^4}{\varphi} \wedge E_{,4}}{\varphi \wedge \varphi} \wedge I$$

$\wedge I-$rule has only one generalized elimination rule for $\wedge$ and its reduction procedure uses both derivations of two conjuncts. By applying $\wedge-$reduction again, we take the derivation Σ_3 below:

$$\dfrac{\begin{matrix}\mathfrak{D}_1 \mathfrak{D}_2 \\ \varphi \quad \varphi\end{matrix}}{\varphi \wedge \varphi} \wedge I$$

Unlike Schroeder-Heister and Tranchini's case, since Σ_2 is the same derivation with Σ_3, the result does not lead to a conclusion that GEkman reduction trivializes the identity of proofs in S_{TG}. Though Σ_1 is a similar version of Π_1, it cannot be the clue to block GEkman reduction process.[2]

There are two standard elimination rules for $\wedge I-$rule and so are two reduction procedures for $\wedge$. As the derivation Π_1 uses the standard $\wedge E_1-$rule but does not have any application of $\wedge E_2-$rule, only the left conjunct φ and its derivation $\mathfrak{D}_1$ are picked by the reduction for $\wedge$. On the other hand, $\wedge I-$rule has only one generalized elimination rule and its reduction procedure uses both derivations of two conjuncts. Hence, the application of the reduction process of Σ_1 uses both derivations $\mathfrak{D}_1$ and $\mathfrak{D}_2$ of φ and provides the same derivation with Σ_2. So to speak, Schroeder-Heister and Tranchini's example stated in generalized form does not show that GEkman reduction trivializes the identity of proofs.

[2]For the permuted derivation of Σ_1, we have the same result by applying $\triangleright_{GE(per)}$.

Whether a new reduction passes Triviality test is dependent on a choice of the form of elimination rules, and so on a given system.

One may say that we can freely choose one of both derivations when we use $\rhd_\wedge-$reduction in generalized form. However, then, regardless of GEkman reduction process, we can readily show that $\rhd_\wedge-$reduction trivializes the identity of proofs because by such free choice of derivations we have the following three different derivations from Σ_1.

$$\dfrac{\overset{\mathfrak{D}_1}{\varphi} \quad \overset{\mathfrak{D}_1}{\varphi}}{\varphi \wedge \varphi}\wedge I \qquad \dfrac{\overset{\mathfrak{D}_1}{\varphi} \quad \overset{\mathfrak{D}_2}{\varphi}}{\varphi \wedge \varphi}\wedge I \qquad \dfrac{\overset{\mathfrak{D}_2}{\varphi} \quad \overset{\mathfrak{D}_2}{\varphi}}{\varphi \wedge \varphi}\wedge I$$

Thus, if the free choice of derivations in $\rhd_\wedge-$reduction is acceptable, according to Triviality test, $\rhd_\wedge-$reduction is not a proper reduction. However, Schroeder-Heister and Tranchini will not want this unwelcome conclusion.

It is not a good answer to the above approach that we take two generalized elimination rules for $\wedge$.

$$\dfrac{\varphi \wedge \psi \quad \begin{matrix}[\varphi]^1 \\ \mathfrak{D}_1 \\ \sigma\end{matrix}}{\sigma}\wedge E_{1,1} \qquad \dfrac{\varphi \wedge \psi \quad \begin{matrix}[\psi]^1 \\ \mathfrak{D}_2 \\ \sigma\end{matrix}}{\sigma}\wedge E_{2,1}$$

$\wedge I-$rule with the two generalized elimination rules above may have two distinct reduction procedures for left and right conjuncts. If $\wedge I-$rule has the two generalized elimination rules, we readily have a similar derivation with Π_1 which GEkman reduction trivializes the identity of proofs. However, there may not be a good reason why $\wedge I-$rule should have the two different generalized elimination rules. Plus, it is often said that the choice of generalized elimination rules has the advantage of having a direct translation between a natural deduction system and a sequent calculus. For example, Negri and Von Plato (2001, Ch.1 and Ch. 8) investigate that each generalized elimination rule for a logical constant in natural deduction corresponds to the left rule for the constant of sequent calculus. Since most sequent calculus systems have a single left rule for each constant, it is better to use a single generalized elimination rule for $\wedge$ in order to have a direct translation between a natural deduction and a sequent calculus.

One may rebut again that, as Gentzen (1935, p. 84) does, a sequent calculus can have two left rules for $\wedge$, and is able to be isomorphic to a certain natural deduction system. (Cf. Von Plato (2011)) Even in our choice of the two left rules, it does not change the situation that Schroeder-Heister and Tranchini's criterion for a proper reduction relies on the choice of elimination rules.

In the footnote 8 of Schroeder-Heister and Tranchini (2017), there is another example using the formulas $\varphi \to (\varphi \to \psi)$ and $\varphi \to \psi$ to show that Ekman reduction is not a proper one. $\to I-$rule has only one elimination rule in both standard and generalized forms. It can be the primary reason to establish that (G)Ekman reduction trivializes the identity of proofs. Their promising example seems to be the following.

$$
\begin{prooftree}
\AxiomC{$[\varphi \to \psi]^1$}
\AxiomC{$[\varphi]^2$}
\RightLabel{$\to E$}
\BinaryInfC{ψ}
\RightLabel{$\to I_{,2}$}
\UnaryInfC{$\varphi \to \psi$}
\RightLabel{$\to I_{,\emptyset}$}
\UnaryInfC{$\varphi \to (\varphi \to \psi)$}
\RightLabel{$\to I_{,1}$}
\UnaryInfC{$(\varphi \to \psi) \to (\varphi \to (\varphi \to \psi))$}
\AxiomC{$[\varphi \to (\varphi \to \psi)]^3$}
\AxiomC{$[\varphi]^4$}
\RightLabel{$\to E$}
\BinaryInfC{$\varphi \to \psi$}
\AxiomC{$[\varphi]^4$}
\RightLabel{$\to E$}
\BinaryInfC{ψ}
\RightLabel{$\to I_{,4}$}
\UnaryInfC{$\varphi \to \psi$}
\RightLabel{$\to I_{,3}$}
\UnaryInfC{$(\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi)$}
\AxiomC{$[\varphi]^5, [\varphi]^6$}
\noLine
\UnaryInfC{$\mathfrak{D}_*$}
\noLine
\UnaryInfC{ψ}
\RightLabel{$\to I_{,5}$}
\UnaryInfC{$\varphi \to \psi$}
\RightLabel{$\to I_{,6}$}
\UnaryInfC{$\varphi \to (\varphi \to \psi)$}
\RightLabel{$\to E$}
\BinaryInfC{$\varphi \to \psi$}
\RightLabel{$\to E$}
\BinaryInfC{$\varphi \to (\varphi \to \psi)$}
\end{prooftree}
$$

The above example has a derivation $\mathfrak{D}_*$ of ψ from assumptions $[\varphi]^5$ and $[\varphi]^6$. Two assumptions of $\mathfrak{D}_*$ have different indices and so are discharged at different places. The application of Ekman reduction $\rhd_E$ to the example provides the derivation on the left-side below and the application on $\rhd_{\to}$ gives the derivation on the right-side.

$$
\begin{prooftree}
\AxiomC{$[\varphi]^5, [\varphi]^6$}
\noLine
\UnaryInfC{$\mathfrak{D}_*$}
\noLine
\UnaryInfC{ψ}
\RightLabel{$\to I_{,5}$}
\UnaryInfC{$\varphi \to \psi$}
\RightLabel{$\to I_{,6}$}
\UnaryInfC{$\varphi \to (\varphi \to \psi)$}
\end{prooftree}
\qquad
\begin{prooftree}
\AxiomC{$[\varphi]^2, [\varphi]^2$}
\noLine
\UnaryInfC{$\mathfrak{D}_*$}
\noLine
\UnaryInfC{ψ}
\RightLabel{$\to I_{,2}$}
\UnaryInfC{$\varphi \to \psi$}
\RightLabel{$\to I_{,\emptyset}$}
\UnaryInfC{$\varphi \to (\varphi \to \psi)$}
\end{prooftree}
$$

Similar to the process proposed by Schroeder-Heister and Tranchini (2017, pp. 575-577), we consider that two derivations above are equivalent relative to $\mathbb{R}'_T$ and represent the same proof. Then, for any two derivations $\mathfrak{D}_1$ and $\mathfrak{D}_2$ of φ, two derivations can be extended by the applications of $\to E-$rule as below.

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}[\varphi]^5,[\varphi]^6\\ \mathfrak{D}_*\\ \psi\end{matrix}}{\varphi\to\psi}\to I_{,5}}{\varphi\to(\varphi\to\psi)}\to I_{,6} \quad \begin{matrix}\mathfrak{D}_1\\ \varphi\end{matrix}}{\varphi\to\psi}\to E \quad \begin{matrix}\mathfrak{D}_2\\ \varphi\end{matrix}}{\psi}\to E \qquad \dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}[\varphi]^2,[\varphi]^2\\ \mathfrak{D}_*\\ \psi\end{matrix}}{\varphi\to\psi}\to I_{,2}}{\varphi\to(\varphi\to\psi)}\to I_{,\emptyset} \quad \begin{matrix}\mathfrak{D}_1\\ \varphi\end{matrix}}{\varphi\to\psi}\to E \quad \begin{matrix}\mathfrak{D}_2\\ \varphi\end{matrix}}{\psi}\to E$$

Two derivations are extended by the same application of $\to E-$rule. They must be equivalent relative to $\mathbb{R}'_T$. Then, two derivations are reduced to the following derivations by $\rhd_{\to}-$reduction.

$$\begin{matrix}\mathfrak{D}_1\mathfrak{D}_2\\ \varphi\ \ \varphi\\ \mathfrak{D}_*\\ \psi\end{matrix} \qquad\qquad \begin{matrix}\mathfrak{D}_2\mathfrak{D}_2\\ \varphi\ \ \varphi\\ \mathfrak{D}_*\\ \psi\end{matrix}$$

It means that any two derivations $\mathfrak{D}_1$ and $\mathfrak{D}_2$ of φ are equivalent and represent the same proof. However, since not every derivation of φ represent the same proof, the result shows that it is not possible to show in $S_T(\mathbb{R}'_T)$ that any two derivations representing different proofs are not equivalent. Therefore, Ekman reduction trivializes the identity of proofs in S_T and so is not a proper reduction.

The result establishes that even when we use generalized elimination rules, Triviality test can restrict the use of GEkman reduction. The example may be a reason to reject that (G)Ekman reduction is proper. Unfortunately, it is undeniable that Triviality test relies on the choice of rules and a system.

The example applies empty (or vacuous) and multiple discharges which correspond to the applications of weakening and contraction in sequent calculus. (Cf. Negri and Von Plato (2001, p. 98).) As a weakening- and contraction-free system has been examined, one may apply the structural restriction to a natural deduction system and use it without empty and multiple discharges. Then, in such system, the suggested derivation using the formulas $\varphi \to (\varphi \to \psi)$ and $\varphi \to \psi$ is not to be the main reason to show that (G)Ekman reduction trivializes the identity of proofs since it needs to apply empty and multiple discharges.

As we have argued in this subsection, Schroeder-Heister and Tranchini's notion of a *proper reduction* with respect to Triviality test is relative to a system. Of course, it is not to say that their Triviality test is wrong. If there is a base system for a proper reduction procedure, a proper reduction can be assessed by Triviality test. However, we do not yet have a base system.

In sum, since the assessment of a proper reduction via Triviality test is relative to a (base) system, a properness of a reduction is relative to the system. Furthermore, the assessment using Triviality test only shows that there exists an example which a reduction in question trivializes the identity of proofs but does not explicate what parts of the reduction affect the identity of proofs.

In the next section, while we use generalized elimination rules and regard standard elimination rules as the special cases of the generalized eliminations, we introduce Translation test as an observation method to find substantial reasons why (G)Ekman reduction affects the identity of proofs and why it is a wrong reduction process.

3.4 Translation Test and Crabbé's case

There are two types of reduction procedures introduced by Prawitz (1965, 1971): standard and auxiliary reduction procedures. A standard reduction eliminates a maximum formula in accordance with the inversion principle. On the other hand, an auxiliary reduction process does not satisfy the inversion principle because its role is not to eliminate a maximum formula which is the conclusion of an introduction and at the same time the major premise of an elimination rule. There are at least three sorts of auxiliary reductions which

(i) lessens the degree of a major premise or (ii) lessens the length of a derivation, or (iii) changes the order of subderivations of a derivation. Standard reductions are introduced in Section 1.2 and 2.1. The examples of (i) are reductions for $CR-$rule introduced in Section 2.2 and in Prawitz (1965, p. 40). The examples of (ii) are reductions for the substitutivity of identity proposed in Section 2.2 and (G)Ekman reductions.[3] Moreover, Prawitz (1971, p. 254) introduces the following process and calls it an 'immediate simplification.'

$$\dfrac{\dfrac{\dfrac{[\neg\varphi]^1 \quad \begin{matrix}\mathfrak{D}\\ \varphi\end{matrix}}{\bot}\neg E}{\begin{matrix}\mathfrak{D}'\\ \bot\end{matrix}}}{\varphi}CR_{,1} \qquad \textit{reduces to} \qquad \begin{matrix}\mathfrak{D}\\ \varphi\end{matrix}$$

where no assumption in $\mathfrak{D}$ is closed in $\mathfrak{D}'$. The examples of (iii) are permutation conversions introduced in Appendix 2.B and in Prawitz (1971, p. 254). The target of these reductions (i), (ii), and (iii) are not to eliminate a maximum formula. Hence, standard reductions and auxiliary reductions are distinguished.

While we differentiate between standard and auxiliary reductions, the inversion principle is not the only requirement for a proper reduction, due to the fact that auxiliary reductions do not fit to the inversion principle. Although it is not clear that every standard reduction satisfying the inversion principle does not affect the identity of proofs, Prawitz's thesis for a proper reduction reflects the significant role of reduction procedures. In this way, a proper reduction has at least two roles that it should (i) preserve the identity of proofs represented by derivations in the same equivalence class and (ii) eliminate an unnecessary detour in accordance with the (generalized) inversion principle.

Schroeder-Heister and Tranchini's Triviality test examines whether a reduction affects

[3](G)Ekman maximum formula is not a maximum formula in the standard sense because it is neither a conclusion of an introduction rule nor a major premise of an elimination rule. Plus, (G)Ekman reduction process is not a reduction which fits to the inversion principle since the principle states that applications of an immediate subderivation of an introduction rule for deriving the major premise of an elimination rule and derivations of minor premises of the same elimination rule. Therefore, we do not regard (G)Ekman maximum formula as a maximum formula in the standard sense and do consider (G)Ekman reduction to be a process to reduce the length of a derivation.

the identity of proofs. As we have discussed in Section 3.3, their notion of a 'proper reduction' with respect to Triviality test is relative to a given natural deduction system. A simpler way to test the identity of proofs represented by the same equivalence class is to check whether a reduction makes a closed derivation open or an open derivation closed. It is obvious that open and closed derivations are not able to represent the same proof though they have the same conclusion. Hence, including the second role of a reduction that eliminates an unnecessary detour with respect to the inversion principle, a proper reduction should satisfy the following requirements.

The Requirements of a Proper Reduction: **(T1)** A proper reduction should neither make an open derivation closed nor a closed derivation open in order to preserve the identity of proofs, and **(T2)** it should not introduce any unnecessary detour which causes to violate the (generalized) inversion principle.

The graphical forms of reduction procedures in natural deduction is not always suitable for evaluating a reduction through the requirements above. Rules and reductions formulated in natural deduction often have hidden assumptions of the closure under substitution for derivations and the applications of empty and multiple discharges. Unlike, natural deduction, these hidden assumptions revealed by the corresponding applications of weakening and contraction in sequent calculus. In this section, we will introduce Translation test as an observation method to examine a reduction procedure with regard to the requirement of a proper reduction process. Translation test is not a regulation of a proper reduction but a way to observe whether a reduction procedure neither affect the identity of proofs nor violate the (generalized) inversion principle by translating a reduction in natural deduction to a one in sequent calculus. Even though it does not suggest an explicit criterion for a proper reduction, when generalized elimination rules are used, Translation test can help to find the reason why (G)Ekman reduction affects the identity of proofs and violates the (generalized) inversion principle.

Section 3.4.1 argues that GEkman reductions do not satisfy the requirements The overgeneration problem occurred by GEkman's paradox will be solved. Furthermore, Schroeder-Heister and Tranchini (2018) provide Crabbé's case which raises the problem of overgeneration and say that it is unaffected by their Triviality test on a proper reduction. We shall argue in Section 3.4.2 that the overgeneration problem may not be caused by Crabbé reduction $\rhd_{\in_Z}$ but by $\in_Z I-$rule. If Crabbé reduction is the real matter, then, by Translation test, we apply the requirements of a proper reduction to Crabbé reduction and solve the problem.

3.4.1 An Ekman-Type Reduction as a Detour-Making Process

Now, we attempt to devise an alternative method for evaluating a proper reduction with regard to Prawitz's perspectives on the identity of proofs and the inversion principle. The requirements of a proper reduction propose two conditions that **(T1)** a proper reduction should neither make an open derivation closed nor a closed derivation open, and **(T2)** it should not introduce any unnecessary detour. **(T1)** is grounded on Prawitz's thesis for the identity of proofs and **(T2)** is on his inversion principle. For our purpose, we first explain what an 'unnecessary detour' in **(T2)** means. We consider that an 'unnecessary detour' refers to an application of an elimination rule having a maximum formula. As Prawitz (1965, 1971) introduces two types of reduction procedures, such as a standard reduction and an auxiliary reduction, there seems to be more than one kind of detours. Prawitz (1971, p. 258) says that a normal derivation has no detour and it represents a *direct* proof.

> With Gentzen, we may say that the proof represented by a normal derivation makes no detour ... ; or, having formulated the normal form for natural deductions, we may say somewhat more pregnantly: the proof is *direct* in the sense that it proceeds from the assumptions to the conclusions by first only using the meaning of the assumptions by breaking them down in their components ..., and then only ver[i]fying the meaning of the conclusions by building them up from the[ir] components ... .

His idea of a direct proof represented by normal derivations is connected to Gentzen's idea that introduction rules determine the meaning of an operator and elimination rules are no more than consequences of the meaning. Prawitz has proposed the inversion principle to realize the idea of Gentzen. The principle says that nothing is gained by deriving a formula from a maximum formula, but what is gained is only from the introduction rules. A standard reduction procedure describes the process which satisfies the principle by composing derivations of introduction and elimination rules. So to speak, it is the process to eliminate a maximum formula. Any derivations containing maximum formulas are not in normal form and does not represent a direct proof. Such derivations have a detour reasoning. Therefore, we say that a derivation has a *detour* reasoning if it has an application of an elimination rule with its maximum formula.

On the other hand, it is not easy to characterize a 'detour' in terms of the length of a derivation. There are cases that the length of a normal derivation is longer than the length of an original derivation. It cannot be said that a derivation of a longer length has a detour and a derivation with a shorter length has no detour. Thus, we only use a 'detour reasoning' as defined through a maximum formula. Then, **(T2)** means that a proper reduction should neither introduce an application of an elimination rule containing a maximum formula nor increase the degree of maximum formulas.

To check **(T2)**, it is useful to translate a reduction in natural deduction to a one in sequent calculus. It is often claimed that a natural deduction system is isomorphic to a sequent calculus system if the generalized eliminations are used. (Cf. Tennant (2002), Negri and Von Plato (2001) and Von Plato (2011)). Especially, Negri and Von Plato (2001, Ch.1 and Ch. 8) propose an inductive definition of the translation algorithm between an intuitionistic natural deduction system with generalized elimination rules and an intuitionistic sequent calculus system with independent contexts. Let Γ, Δ, Θ be a finite multiset, i.e. a list with multiplicity but no order, of assumptions in sequent calculus. We use a binary derivation symbol $\Rightarrow$ and '$\Gamma \Rightarrow \varphi$' means that the antecedent Γ derives the succedent φ. A sequent calculus has the following left and right rules for $\rightarrow$ and structural rules, such as cut-, weakening, and contraction rules.

$$\frac{\Gamma \Rightarrow \varphi \quad \psi, \Delta \Rightarrow \sigma}{\varphi \rightarrow \psi, \Gamma, \Delta \Rightarrow \sigma} L\rightarrow \qquad \frac{\varphi, \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \rightarrow \psi} R\rightarrow$$

$$\frac{\Gamma \Rightarrow \varphi \quad \varphi, \Delta \Rightarrow \Psi}{\Gamma, \Delta \Rightarrow \Psi}\,Cut \qquad \frac{\Gamma \Rightarrow \Psi}{\varphi, \Gamma \Rightarrow \psi}\,Wk \qquad \frac{\varphi, \varphi, \Gamma \Rightarrow \Psi}{\varphi, \Gamma \Rightarrow \Psi}\,Ctr$$

φ in $Cut-$rule is called a cut-formula. Let us remind the reduction process for $\to$ in the style of generalized elimination rule.

$$\dfrac{\dfrac{\begin{array}{c}[\varphi]^1\\ \mathfrak{D}_1\\ \psi\end{array}}{\varphi \to \psi}\to I_{,1} \quad \begin{array}{c}\mathfrak{D}_2\\ \varphi\end{array} \quad \begin{array}{c}[\psi]^1\\ \mathfrak{D}_3\\ \sigma\end{array}}{\sigma}\to E \qquad \rhd_{\to} \qquad \begin{array}{c}\mathfrak{D}_2\\ \varphi\\ \mathfrak{D}_1\\ \psi\\ \mathfrak{D}_3\\ \sigma\end{array}$$

The reduction process for $\to$, $\rhd_{\to}$, in natural deduction is translated as below by Negri and von Plato's algorithm.

$$\dfrac{\dfrac{\begin{array}{c}\mathfrak{D}_1'\\ \varphi, \Gamma \Rightarrow \psi\end{array}}{\Gamma \Rightarrow \varphi \to \psi}R\to \quad \dfrac{\begin{array}{c}\mathfrak{D}_2'\\ \Delta_1 \Rightarrow \varphi\end{array} \quad \begin{array}{c}\mathfrak{D}_3'\\ \psi, \Delta_2 \Rightarrow \sigma\end{array}}{\varphi \to \psi, \Delta_1, \Delta_2 \Rightarrow \sigma}L\to}{\Gamma, \Delta_1, \Delta_2 \Rightarrow \sigma}Cut \quad \text{reduces to} \quad \dfrac{\dfrac{\begin{array}{c}\mathfrak{D}_2'\\ \Delta_1 \Rightarrow \varphi\end{array} \quad \begin{array}{c}\mathfrak{D}_1'\\ \varphi, \Gamma \Rightarrow \psi\end{array}}{\Gamma, \Delta_1 \Rightarrow \psi}Cut \quad \begin{array}{c}\mathfrak{D}_3'\\ \psi, \Delta_2 \Rightarrow \sigma\end{array}}{\Gamma, \Delta_1, \Delta_2 \Rightarrow \sigma}Cut$$

As the maximum formula $\varphi \to \psi$ is eliminated by $\rhd_{\to}$ in the derivation having a detour, the cut-formula in the translated derivation is removed by the above reduction. If a natural deduction with generalized elimination rules has an isomorphic translation to a sequent calculus, the isomorphic translation of full normal derivations preserves the order of rules such that an introduction rule turns into corresponding right rule and an elimination rule into a left rule. The isomorphic translation guarantees the correspondence between full normal and cut-free derivations. In this sense, it appears to say that one of the main roles of reduction procedures is to eliminate the cut-formula or to lessen the degree of it. Likewise, if there is an isomorphic translation between sequent calculus and natural deduction, to lessen the degree of the maximum formula or to eliminate it can be one of the main roles for reduction procedures. We call the test to check whether a reduction process satisfies the requirements of a proper reduction by translating the reduction in natural deduction to a corresponding one in sequent calculus, *Translation test*.

Ekman-type reductions may have a problematic assumption that, for any formulas φ and ψ, if a derivation of φ (or of ψ) includes inferences $\varphi \to \psi$ and $\psi \to \varphi$ and it has a subderivation of the same conclusion, it always can be substituted for the subderivation. It is natural to think that when an open derivation reduces to a closed derivation or vice versa, the original and reduced derivation do not represent the same proof. For instance, it is the case that if every conjecture has a proof, then Goldbach conjecture has a proof. However, as Goldbach conjecture has not yet been proven, it is not the case that it has a proof. That is, an open derivation of Goldbach conjecture from the assumption that every conjecture has a proof is different from a closed derivation of Goldbach conjecture. So, it is not a proper reduction process from the open derivation to the closed derivation. Thus. for Translation test, we will examine whether a reduction makes an open derivation closed or a closed derivation open.

Translation test can discover two problems of Ekman-type reduction procedures. The one is that it makes an open (or a closed) derivation closed (or open). The other is that in the case which allows open assumptions it generates an unnecessary detour. It will be established by Translation test that Ekman-type reductions do not satisfy both requirements **(T1)** and **(T2)**.

First, for Translation test, we assume that GEkman reduction is closed under substitution of derivations for open assumptions. Standard reductions introduced in Section 1.2 and 2.1 allow open derivations and they do not make any open derivation closed or vice versa. In addition to Ekman reduction, GEkman reduction allows an open derivation. We consider a special case of GEkman reduction as below.

$$\dfrac{[\psi \to \varphi] \qquad \dfrac{\begin{matrix} & \mathfrak{D} & \\ [\varphi \to \psi] & \varphi & [\psi]^1 \end{matrix}}{\psi} \to E_{,1} \qquad [\varphi]^2}{\varphi} \to E_{,2} \quad \trianglerighteq_{GE} \quad \begin{matrix} \mathfrak{D} \\ \varphi \end{matrix}$$

Suppose that the derivation $\mathfrak{D}$ of φ is a closed derivation. GEkman reduction makes an open derivation closed by eliminating assumptions $[\varphi \to \psi]$ and $[\psi \to \varphi]$. The translation of the above derivation also shows the phenomenon clearly.

$$\dfrac{\dfrac{\begin{matrix}\mathfrak{D}'\\ \Gamma\Rightarrow\varphi\end{matrix}\quad \psi\Rightarrow\psi}{\Gamma,\varphi\to\psi\Rightarrow\psi}L\to \qquad \varphi\Rightarrow\varphi}{\varphi\to\psi,\psi\to\varphi,\Gamma\Rightarrow\varphi}L\to \qquad \textit{reduces to}\quad \begin{matrix}\mathfrak{D}'\\ \Gamma\Rightarrow\varphi\end{matrix}$$

$\trianglerighteq_{GE}-$reduction yields the closed derivation $\mathfrak{D}$ of φ from the open derivation by getting rid of the assumptions $[\varphi\to\psi]$ and $[\psi\to\varphi]$. The original derivation on the left-side says that if $\varphi\to\psi$ and $\psi\to\varphi$ are true, then φ is true. On the other hand, the reduced derivation on the right says that φ is a theorem. Of course, one may claim that the application of $Wk-$rule to the right results in $\varphi\to\psi,\psi\to\varphi,\Gamma\Rightarrow\varphi$. However, then, $\trianglerighteq_{GE}-$reduction should provide an open derivation of φ by supplementing the assumption $[\varphi\to\psi]$ and $[\psi\to\varphi]$.

Again, in the case of the permuted version of $\trianglerighteq_{GE}-$reduction, $\trianglerighteq_{GE(per)}-$reduction can make a closed derivation open. For instance, we consider the following case.

$$\dfrac{\begin{matrix}\mathfrak{D}'_4\\ \varphi\to\psi\end{matrix}\quad \begin{matrix}\mathfrak{D}'_1\\ \varphi\end{matrix}\quad \dfrac{\begin{matrix}\mathfrak{D}'_3\\ \psi\to\varphi\end{matrix}\quad [\psi]^1\qquad \begin{matrix}[\psi]^1,[\varphi]^2\\ \mathfrak{D}'_2\\ \rho\end{matrix}}{\rho}\to E_{,2}}{\rho}\to E_{,1} \qquad \trianglerighteq_{GE(per)} \quad \begin{matrix}\mathfrak{D}'_1\\ [\psi],\ \varphi\\ \mathfrak{D}'_2\\ \rho\end{matrix}$$

The above case shows that $\trianglerighteq_{GE(per)}-$reduction produces the open derivation from the closed derivation. Therefore, $\trianglerighteq_{GE}-$ and $\trianglerighteq_{GE(per)}-$reductions violate **(T1)** of the requirements of a proper reduction.

Second, GEkman reduction is not a process to reduce unnecessary detour reasoning, but is a detour-making-process. Let us consider the following case with open assumptions.

$$\dfrac{[\psi\to\varphi]\qquad \dfrac{[\varphi\to\psi]\quad \begin{matrix}\mathfrak{D}_1\\ \varphi\end{matrix}\quad [\psi]^1}{\psi}\to E_{,1}\qquad \begin{matrix}[\varphi]^2\\ \mathfrak{D}_2\\ \sigma\end{matrix}}{\sigma}\to E_{,2} \qquad \triangleright_{GE} \quad \begin{matrix}\mathfrak{D}_1\\ \varphi\\ \mathfrak{D}_2\\ \sigma\end{matrix}$$

The derivation has no maximum formula, but GEkman maximum formula ψ. It is translated as below.

$$\dfrac{\dfrac{\begin{array}{c}\mathfrak{D}_1' \\ \Gamma \Rightarrow \varphi \quad \psi \Rightarrow \psi\end{array}}{\Gamma, \varphi \to \psi \Rightarrow \psi} L\to \qquad \begin{array}{c}\mathfrak{D}_2' \\ \varphi, \Delta \Rightarrow \sigma\end{array}}{\varphi \to \psi, \psi \to \varphi, \Gamma, \Delta \Rightarrow \sigma} L\to \quad \textit{reduces to} \quad \dfrac{\begin{array}{cc}\mathfrak{D}_1' & \mathfrak{D}_2' \\ \Gamma \Rightarrow \varphi & \varphi, \Delta \Rightarrow \sigma\end{array}}{\Gamma, \Delta \Rightarrow \sigma} Cut$$

GEkman maximum formula ψ has no corresponding cut-formula. The translated version of GEkman reduction does not remove any cut-formula, and so neither it eliminates an unnecessary detour in the standard sense. The more serious problem is that the translated version shows that GEkman reduction creates an unnecessary detour though it eliminates a GEkman maximum formula ψ. The derivation on the left side has no cut-formula but after applying the reduction process the cut-formula φ appears on the right side derivation. It means that GEkman reduction fails to satisfy the requirements of a proper reduction via Translation test, because of **(T2)**.[4] If any proper reduction must not generate new cut-(or maximum) formula, GEkman reduction is not a proper reduction procedure because it is a detour-making process.

3.4.2 Does Crabbé Reduction Overgenerate?

Schroeder-Heister and Tranchini (2018) introduce an example first observed by Marcel Crabbé which arises the problem of overgeneration and says that their Triviality test cannot block the reduction used in Crabbé's case. The case has rules for Zermelo's separation axiom. For our discussions include an evaluation of Crabbé's case via Translation test, we propose the rules for Zermelo's separation axiom in generalized form.

$$\dfrac{\begin{array}{cc}\mathfrak{D}_1 & \mathfrak{D}_2 \\ t \in s & \varphi[t/x]\end{array}}{t \in \{x \in s | \varphi(x)\}} \in_Z I \qquad \dfrac{t \in \{x \in s | \varphi(x)\} \qquad \begin{array}{c}[t \in s]^1, [\varphi[t/x]]^1 \\ \mathfrak{D}_3 \\ \psi\end{array}}{\psi} \in_Z E_{,1}$$

[4]It is readily seen that $\rhd_{GE_g}$−reduction fails to pass Translation test because of **(T2)**.

The reduction procedure $\rhd_{\in_Z}$ for $\in_Z I-$ and $\in_Z E-$rules are as below:

$$\dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1 & \mathfrak{D}_2\\ t\in s & \varphi[t/x]\end{matrix}}{t\in\{x\in s|\varphi(x)\}}\in_Z I \qquad \begin{matrix}[t\in s]^1,[\varphi[t/x]]^1\\ \mathfrak{D}_3\\ \psi\end{matrix}}{\psi}\in_Z E_{,1} \qquad \rhd_{\in_Z} \qquad \begin{matrix}\mathfrak{D}_1 \quad \mathfrak{D}_2\\ t\in s \quad \varphi[t/x]\\ \mathfrak{D}_3 \\ \psi\end{matrix}$$

We all $\rhd_{\in_Z}-$reduction, ‘Crabbé reduction.’ We shall claim in this subsection that since Triviality test is not suitable for evaluating standard reductions, it does not apply to Crabbé’s case. Also, it will be discussed that the culprit of the overgeneration occurred by Crabbé’s case may not be Crabbé reduction but be the form of $\in_Z I-$rule. If Crabbé reduction is the culprit, then the requirements of a proper reduction through Translation test can solve the problem of overgeneration.

Let us investigate Crabbé’s case. For any set b, we define Z_b as a set $\{x\in b|\neg x\in x\}$. We take $\neg x\in x$ for φ in $\in_Z I-$ and $\in_Z E-$rules and for terms t and s we take Z_b and b respectively. Then, the following rules are the instances of $\in_Z I-$ and $\in_Z E-$rules.

$$\dfrac{\begin{matrix}\mathfrak{D}_1 & \mathfrak{D}_2\\ Z_b\in b & \neg Z_b\in Z_b\end{matrix}}{Z_b\in\{x\in b|\neg x\in x\}}\in_Z I \qquad\qquad \dfrac{Z_b\in\{x\in b|\neg x\in x\} \qquad \begin{matrix}[Z_b\in b]^1,[\neg Z_b\in Z_b]^1\\ \mathfrak{D}_3\\ \psi\end{matrix}}{\psi}\in_Z E_{,1}$$

Moreover, the instances of the reduction procedure $\rhd_{\in_Z}$ for $\in_Z I-$ and $\in_Z E-$rules is as below:

$$\dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1 & \mathfrak{D}_2\\ Z_b\in b & \neg Z_b\in Z_b\end{matrix}}{Z_b\in\{x\in b|\neg x\in x\}}\in_Z I \qquad \begin{matrix}[Z_b\in b]^1,[\neg Z_b\in Z_b]^1\\ \mathfrak{D}_3\\ \psi\end{matrix}}{\psi}\in_Z E_{,1} \qquad \rhd_{\in_Z} \qquad \begin{matrix}\mathfrak{D}_1 \quad \mathfrak{D}_2\\ Z_b\in b,\neg Z_b\in Z_b\\ \mathfrak{D}_3 \\ \psi\end{matrix}$$

Let $S_{\in_Z}$ be a system having $\in_Z -$ and $\neg -$rules. $\mathbb{R}_{\in_Z}$ be a set of reductions including $\rhd_{\neg}$ and $\rhd_{\in_Z}$. Then, we have the result below.

Proposition 3.4.1. *There is an open derivation of $\bot$ from $Z_b \in b$ in $S_{\in_Z}$ relative to $\mathbb{R} \in_Z$ which generates a non-terminating reduction sequence, and so is not fully normalizable.*

Proof. two claims verify the result.

Claim 1. there is an open derivation Σ_2 of $\bot$ from the assumption $[Z_b \in b]$.

We begin with an open derivation Σ_1 of $\bot$ from $[Z_b \in Z_b]$.

$$\dfrac{\dfrac{[Z_b \in Z_b]^1}{Z_b \in \{x \in b | \neg x \in x\}}\,def \qquad \dfrac{[\neg Z_b \in Z_b]^2 \qquad \dfrac{\dfrac{[Z_b \in b]^2 \quad [\neg Z_b \in Z_b]^2}{Z_b \in \{x \in b | \neg x \in x\}}\,\in_Z I}{Z_b \in Z_b}\,def \qquad [\bot]^3}{\bot}\,\neg E_{,3}}{\bot}\,\in_Z E_{,2}$$

Then, there is an open derivation Σ_2 of $Z_b \in Z_b$ from $[Z_b \in b]$.

$$\dfrac{\dfrac{[Z_b \in b]^4 \qquad \dfrac{\begin{array}{c}[Z_b \in Z_b]^1\\ \Sigma_1\\ \bot\end{array}}{\neg Z_b \in Z_b}\,\neg I_{,1}}{Z_b \in \{x \in b | \neg x \in x\}}\,\in_Z I}{Z_b \in Z_b}\,def$$

Now, we have an open derivation Σ_3 of $\bot$ from $[Z_b \in b]$.

$$\dfrac{\dfrac{\begin{array}{c}[Z_b \in Z_b]^5\\ \Sigma_1\\ \bot\end{array}}{\neg Z_b \in Z_b}\,\neg I_{,5} \qquad \begin{array}{c}[Z_b \in b]^4\\ \Sigma_2\\ Z_b \in Z_b\end{array} \qquad [\bot]^6}{\bot}\,\neg E_{,6}$$

Claim 2. Σ_3 initiates a non-terminating reduction sequence and so is not fully normalizable.

By applying $\rhd_{\neg}$ to Σ_3, we have the derivation Σ_4 below.

$$
\dfrac{\dfrac{[Z_b \in b]^4 \quad \dfrac{\begin{array}{c}[Z_b \in Z_b]^1 \\ \Sigma_1 \\ \bot\end{array}}{\neg Z_b \in Z_b}\neg I_{,1}}{Z_b \in \{x \in b | \neg x \in x\}} \in_Z I \qquad \dfrac{[\neg Z_b \in Z_b]^2 \quad \dfrac{\dfrac{[Z_b \in b]^2 \quad [\neg Z_b \in Z_b]^2}{Z_b \in \{x \in b | \neg x \in x\}} \in_Z I}{Z_b \in Z_b}\, def \quad [\bot]^3}{\bot} \neg E_{,3}}{\bot} \in_Z E_{,2}
$$

Since $Z_b \in \{x \in b | \neg x \in x\}$ in $\in_Z E-$rule is not an assumption, the application of $\rhd_{\in_Z}-$reduction produces the same derivation with Σ_3. Therefore, Σ_3 generates a non-terminating reduction sequence and so is not fully normalizable. □

We call the result of Proposition 3.4.1 Crabbé's case. Furthermore, from the derivation Σ_3, we readily obtain a closed derivation Σ_5 of $\neg\exists y(Z_y \in y)$ as follows.

$$
\dfrac{\dfrac{[\exists y(Z_y \in y)]^7 \qquad \begin{array}{c}[Z_b \in b]^4 \\ \Sigma_3 \\ \bot\end{array}}{\bot} \exists E_{,4}}{\neg\exists y(Z_y \in y)} \neg I_{,7}
$$

The result states that no set contains its own Rusell subset and is an acceptable conclusion in a consistent Zermelo's set theory. The result, i.e. Σ_5, does not represent a proof of Russell-Zermelo's paradox, however, Σ_5 contains a subderivation Σ_3 which satisfies TCP_L and so is a T-paradox. As Σ_5 does not formulate a paradox, it should neither be a genuine paradox nor a T-paradox. Thus, Crabbé's case shows that TCP_L overgenerates.

Schroeder-Heister and Tranchini (2018, Sec. 8) say that Triviality test cannot solve the problem of overgeneration caused by Crabbé's case, as they said,

> As remarked, the phenomenon observed by Crabbé is however unaffected by our proposed constraint on reductions, thus showing that further work is required for a thorough analysis of paradoxes along the lines of the Prawitz-

Tennant analysis.

It looks as if Crabbé reduction causes a non-terminating reduction sequence, the form of Crabbé reduction, $\rhd_{\in_Z}$, relies on the form of $\in_Z I-$rule. Since $\in_Z I-$rule has the premise $\neg Z_b \in Z_b$ whose degree is greater than that of the conclusion $Z_b \in \{x \in b | \neg x \in x\}$, the reduced derivation has the formula $\neg Z_b \in Z_b$ by eliminating the maximum formula $Z_b \in \{x \in b | \neg x \in x\}$. Therefore, the real problem may not be Crabbé reduction itself but be the form of $\in_Z I-$rule.

Even if Crabbé reduction is the real matter, we can solve the problem of overgeneration by applying Translation test with the requirements of a proper reduction. The translated forms of $\in_Z I-$ and $\in_Z E-$rules are as follows.

$$\dfrac{\stackrel{\mathfrak{D}'_1}{\Gamma \Rightarrow t \in s} \quad \stackrel{\mathfrak{D}'_2}{\Delta \Rightarrow \varphi[t/x]}}{\Gamma, \Delta \Rightarrow t \in \{x \in s | \varphi(x)\}} \in_Z R \qquad \dfrac{\stackrel{\mathfrak{D}'_3}{t \in s, \varphi[t/x], \Sigma \Rightarrow \psi}}{t \in \{x \in s | \varphi(x)\}, \Sigma \Rightarrow \psi} \in_Z L$$

Moreover, the translated form of the instance of $\rhd_{\in_Z}-$reduction is as below.

$$\dfrac{\dfrac{\stackrel{\mathfrak{D}'_1}{\Gamma \Rightarrow Z_b \in b} \quad \stackrel{\mathfrak{D}'_2}{\Delta \Rightarrow \neg Z_b \in Z_b}}{\Gamma, \Delta \Rightarrow Z_b \in \{x \in b | \neg x \in x\}} \in_Z R \quad \dfrac{\stackrel{\mathfrak{D}'_3}{Z_b \in b, \neg Z_b \in Z_b, \Sigma \Rightarrow \psi}}{Z_b \in \{x \in b | \neg x \in x\}, \Sigma \Rightarrow \psi} \in_Z L}{\Gamma, \Delta, \Sigma \Rightarrow \psi} Cut$$

reduces to

$$\dfrac{\stackrel{\mathfrak{D}'_2}{\Delta \Rightarrow \neg Z_b \in Z_b} \quad \dfrac{\stackrel{\mathfrak{D}'_1}{\Gamma \Rightarrow Z_b \in b} \quad \stackrel{\mathfrak{D}'_3}{Z_b \in b, \neg Z_b \in Z_b, \Sigma \Rightarrow \psi}}{\neg Z_b \in Z_b, \Gamma, \Sigma \Rightarrow \psi} Cut}{\Gamma, \Delta, \Sigma \Rightarrow \psi} Cut$$

Though the translated reduction process does not eliminate necessary premises, it increases the degree of a cut-formula. The cut-formula $Z_b \in \{x \in b | \neg x \in x\}$ is an atomic formula, but $\neg Z_b \in Z_b$ in the reduced derivation is not. The degree of the cut formula, also that of the maximum formula, is increased. $\rhd_{\in_Z}-$reduction does not satisfy the condition (**T2**), and so is not a proper reduction. Therefore, since $\rhd_{\in_Z}-$reduction is not proper, *RTCP* with respect to the requirements of a proper reduction says that Crabbé's case is not a T-paradox.

3.5 Conclusion.

Tennant (2017, pp. 109-110) proposes four reasons why he prefers to use generalized elimination rules: the uniform presentation, the efficiency of proof search, having shorter formal proofs, and giving a solution to the problem of overgeneration. As we have discussed in the present chapter, the last reason might be wrong. The mere choice of generalized elimination rules does not solve the problem. However, Tennant (2002) already claims that a natural deduction system with generalized elimination rules is isomorphic to a sequent calculus. The idea is melted in his Core Logic introduced in Tennant (2017). For his, Translation test can be a promising solution to the overgeneration.

Schroeder-Heister and Tranchini (2018) uses von Plato's derivation on Ekman's paradox and attacks Tennant's solution to the overgeneration problem. Also, they seem to reject Tennant's view on Russell's paradox that it is not a genuine paradox. Since they examined a derivation fitted to their $Ekman_g^{=}-$ reduction, Section 4.1 argues that if an Ekman-type reduction for set-abstraction is adopted, Russell's paradox formalized in a system S_F for the free logic of sets becomes a T-paradox. Fortunately, Translation test is not only applicable to GEkman reduction but also other Ekman-type reductions. We introduce rules for a set-forming operator $\{\}$ and Ekman-type reduction for $\{\}$. $\{\}I-$ and $\{\}E-$rules have the corresponding $L\{\}_i$ and $R\{\}-$rules where $i = 1, 2$.

$$\dfrac{\begin{matrix}\mathfrak{D}'_1 \\ \Gamma, \varphi[a/x], \exists! a \Rightarrow a \in t\end{matrix} \quad \begin{matrix}\mathfrak{D}'_2 \\ \Delta \Rightarrow \exists! t\end{matrix} \quad \begin{matrix}\mathfrak{D}'_3 \\ \Theta, a \in t \Rightarrow \varphi[a/x]\end{matrix}}{\Gamma, \Delta, \Theta \Rightarrow t = \{x|\varphi(x)\}} R\{\}$$

$$\dfrac{\begin{matrix}\mathfrak{D}'_4 \\ \Gamma \Rightarrow \varphi[u/x]\end{matrix} \quad \begin{matrix}\mathfrak{D}'_5 \\ \Delta \Rightarrow \exists! u\end{matrix} \quad \begin{matrix}\mathfrak{D}'_6 \\ \Theta, u \in t \Rightarrow \psi\end{matrix}}{t = \{x|\varphi(x)\}, \Gamma, \Delta, \Theta \Rightarrow \psi} L\{\}_1 \quad \dfrac{\begin{matrix}\mathfrak{D}'_7 \\ \Gamma \Rightarrow u \in t\end{matrix} \quad \begin{matrix}\mathfrak{D}'_8 \\ \varphi[u/x] \Rightarrow \psi\end{matrix}}{t = \{x|\varphi(x)\}, \Gamma, \Delta \Rightarrow \psi} L\{\}_2$$

Also, Ekman-type reduction for Ekman-type maximum formula $u \in t$ is translated as follows.

$$\dfrac{\dfrac{\begin{array}{c}\mathfrak{D}'_4\\ \Gamma \Rightarrow \varphi[x/u]\end{array} \quad \begin{array}{c}\mathfrak{D}'_5\\ \Delta \Rightarrow \exists! u\end{array} \quad \begin{array}{c}\mathfrak{D}'_6\\ \Theta, u \in t \Rightarrow u \in t\end{array}}{t = \{x|\varphi(x)\}, \Gamma, \Delta, \Theta \Rightarrow u \in t} L\{\}_1 \quad \begin{array}{c}\mathfrak{D}'_8\\ \Delta, \varphi[x/u] \Rightarrow \psi\end{array}}{t = \{x|\varphi(x)\}, t = \{x|\varphi(x)\}, \Gamma, \Delta, \Theta \Rightarrow \psi} L\{\}_2$$

reduces to

$$\dfrac{\begin{array}{c}\mathfrak{D}'_4\\ \Gamma \Rightarrow \varphi[x/u]\end{array} \quad \begin{array}{c}\mathfrak{D}'_8\\ \Delta, \varphi[x/u] \Rightarrow \psi\end{array}}{\Gamma, \Delta \Rightarrow \psi} Cut$$

As the translated GEkman reduction does, an Ekman-type reduction for $u \in t$ generates a cut-formula $\varphi[x/u]$. So it is a detour-making reduction process. The Ekman-type reduction for set-abstraction is canceled by the requirements of a proper reduction via Translation test. If Tennant accepts *RTCP* with respect to the requirements, Russell's paradox formulated in S_F is still not a genuine paradox.

Although Translation test is not a regulation of a proper reduction but an observational method to assess a proper reduction, it has some advantages over Triviality test. Triviality test does not restrict every Ekman-type reduction due to the fact that it is relative to our choice of the form of rules and a system. While we regard standard elimination rules as special cases of generalized elimination rules, Translation test will be considered to be a relatively system-independent method to inspect a reduction procedure. Moreover, Schroeder-Heister and Tranchini (2018) say that Triviality test is unable to solve the problem generated by Crabbé's case. If Crabbé reduction causes the problem, Translation test restricts the application of $\rhd_{\in_Z}$ -reduction and solves the problem.

In sum, we have argued that the problem of overgeneration caused by (G)Ekman's paradox and Crabbé's case reminds us that there must be a method to evaluate which reduction procedures are proper. Two tests are introduced: Triviality test and Translation test. Triviality test only can restrict Ekman-type reductions depending on a base system and is not a remedy for the problem raised by Crabbé's case. Then, Translation test can help to solve those problems with the same perspectives of Prawitz's thesis and the inversion principle.

Chapter 4

Can the Requirement of a Normal Derivation be a Solution to the Paradoxes?

When a doctor finds a sick person, he diagnoses what the illness the person has and prescribes it in accordance with his diagnosis. Likewise, when a logician faces a problematic argument (or proof), (s)he characterizes the problem and solves it on the basis of her characterization. Paradoxes which raise a contradiction have been a significant issue to the foundations of logic and mathematics. It is often believed that solutions to the paradoxes are closely tied with the characterization of the paradoxes. For instance, an informal characterization of a paradox proposed by Sainsbury (2009, p. 1) says that it is an unacceptable conclusion elicited from the acceptable premises via acceptable reasoning. A diagnosis of the paradoxes through the characterization can be that it is a trouble that acceptability leads to unacceptability. Thus, from the diagnosis with the characterization, three responses to the paradoxes can be proposed such that either the premises or the reasoning is not in fact acceptable, or else the conclusion is acceptable. As noted in Section 1.1, we shall call the first response the premise-rejection, the second the reasoning-rejection, and the last the conclusion-acceptance.

It is not to say that every solution to the paradoxes is understood as one of three solutions, but it is the case that Sainsbury's characterization is the simplest way to grasp the informal notion of a 'paradox.'

Sainsbury regards a *sentence* (or a *formula*) with special characteristics a paradox. However, Tennant's criterion for paradoxicality *TCP* understands a paradox as a *derivation* (or an *argument*) which might have an unacceptable conclusion from the acceptable premises by the acceptable inference rules. As Sainsbury's characterization of a paradox is connected to three solutions, we may have a proof-theoretic solution to the paradoxes on the perspectives of *TCP*. Tennant (1982) proposes *TCP* as a criterion for genuine paradoxes and regards it as a conjecture namely that for any derivation $\mathfrak{D}$, $\mathfrak{D}$ formalizes a genuine paradox iff $\mathfrak{D}$ is a T-paradox. Focussing on normalizability, Tennant (2017, pp. 286–287) suggests a similar conjecture which is linked to his criterion for paradoxicality.

> How ... are we to solve the paradoxes? It is not from this study to venture any new suggestions beyond those of Tennant (1982) and Tennant (1995). Those works provided ... proofs, formalized as natural deductions, for all the major paradoxes ... . They showed that all these ... proofs ... cannot be converted into normal form. The original proof-theoretic thesis stands:
>
> > Genuine paradoxes are those whose associated *proofs of absurdity*, when formalized as natural deductions, cannot be converted into normal form.
>
> This conjecture provides a proof-theoretic criterion for the identification of genuine paradoxes ... .

The conjectures can be his diagnoses of genuine paradoxes. If the conjectures are true, every derivation of genuine paradoxes, such as T-paradoxes, generates a non-terminating reduction sequence and so is not normalizable. His diagnoses, if true, provide a proof-theoretic solution to the paradoxes.

As an anti-realist and a constructivist, Tennant (2015, p. 578) believes that 'every truth is knowable, and its truth consists in the existence of a(n in principle) surveyable truthmaker, also called a (canonical) proof.' Moreover, he thinks that a (constructive) proof must be convertible into normal form, and so he suggests the proof-theoretic principle for constructive mathematics.

> The following principle is a cornerstone of proof-theoretic foundations for constructive mathematics:
>
> For every proof Π that we may provide for a mathematical theorem φ, it must be possible, in principle, to transform Π, *via* a finite sequence of applicable reduction procedures, into a *canonical* proof of φ, that is, a proof of φ that is *in normal form*, so that none of the reduction procedure is applicable to it. (Tennant, 2015, p. 579)

Though he proposes the proof-theoretic principle for *constructive mathematics*, the principle can be extended to a general case. He appears to think that any derivation representing a proof of the true statement must be, in principle, able to be brought into (full) normal form. Also, when Tennant (1982) proposes his earlier criterion, TCP_E, he stresses the importance of normalizability as below:

> The general loss of normalizability, confined as it is according to [TCP_E] to just the paradoxical part of the semantically closed language, is a small price to pay for the protection it gives against paradox itself. Logic plays its role as an instrument of knowledge only insofar as it keeps proofs in sharp focus, through the lens of normality. Normali[z]ability, in the context of semantically closed languages, is not to be pressed as a general pre-condition for the very possibility of talking sense; rather, normality of proof is to be pressed as a general pre-condition for the very possibility of telling the truth. (Tennant, 1982, p. 284)

Provided that there is a requirement that every derivation representing a proof of the true statement should be, in principle, reducible to a (full) normal derivation, the requirement can block T-paradoxes and be a proof-theoretic solution to the paradoxes. Even though Tennant did not explicitly propose the requirement as the proof-theoretic solution, from *TCP*, we may interpret the following principle as a plausible proof-theoretic solution to the paradoxes.

The Requirement of a (Full) Normal Derivation(*RND*): For any derivation $\mathfrak{D}$ in natural deduction, $\mathfrak{D}$ is acceptable only if $\mathfrak{D}$ is (in principle) convertible into a normal derivation.

Furthermore, while he compares his natural deduction system for naive set-theory and Fitch's, Prawitz (1965, p. 95) introduces a similar requirement with *RND* by saying, 'the set-theoretical paradoxes ruled out by the requirement that the [derivations] shall be normal.' Prawitz (1965, p. 96) also claims that his requirement is less *ad hoc* than Fitch's simple/special restrictions introduced by Fitch (1952, Sec. 18 and 20). Even though both Prawitz and Tennant did not explicitly claim that *RND* (or a similar requirement) could be a solution to the paradoxes, it is possible from their views that they would have in mind that *RND* could be the solution. Hence, our question is whether *RND* can really be a solution to the paradoxes.

In order for *RND* to be the proof-theoretic solution, three things must be answered: (i) which paradox is a genuine paradox and which formalization is legitimate for the genuine paradox, (ii) why the only normalizable derivations are acceptable, and (iii) why the only propositional constant $\perp$ for absurdity is an unacceptable conclusion. This chapter aims to discuss that there are some obstacles to claim that *RND* is the proof-theoretic solution to the paradoxes.

For the first question (i), since *RND* is proposed on the perspectives on *TCP*, the following conjecture for genuine paradoxes should be true.

Tennant's Conjecture for Genuine Paradoxes: For any derivation $\mathfrak{D}$ in natural deduction, $\mathfrak{D}$ formalizes a genuine paradox iff $\mathfrak{D}$ is a T-paradox with respect to *TCP*.

As we have noted, *TCP* itself assumes the conjecture for genuine paradoxes. However, since the notion of a 'genuine paradox' is informal, it is unclear what kinds of paradoxes are genuine paradoxes. Tennant has claimed that Russell's paradox is not a genuine paradox, whereas the Liar paradox is a genuine one. In Section 4.1, we shall introduce his argument on why Russell's paradox is not a genuine paradox, and argue that by following his argument, if Russell's paradox is not a genuine paradox, neither is the Liar paradox. Tennant has no standard for genuine paradoxes. Our discussion comes into a question of which formalization is legitimate for the genuine paradox. *RND* only blocks non-normalizable derivations, such as T-paradoxes. If *RND* is regarded as a promising proof-theoretic solution to *genuine paradoxes*, it should be answered to the first question of which paradoxes are genuine paradox.

The second question asks, even though Tennant's conjecture for genuine paradoxes is true and *RND* can restrict the use of T-paradoxes, why should we consider that *RND* is convincing? A plausible answer is that every non-normalizable derivation is (proof-theoretically) invalid. For the second question, we shall consider a relation between proof-theoretic validity and normalizability in Section 4.2. If non-normalizable derivations were proof-theoretically invalid, paradoxical derivations which generate a non-terminating reduction sequence would be invalid. It will be discussed that in a restricted system proof-theoretic validity implies normalizability. However, it is not clear that the relation would be extended to a general case. Therefore, it should be established that proof-theoretic validity *generally* implies (strong) normalizability, or another answer to the second question (ii) should be proposed.

Apart from our three questions for *RND* as a proof-theoretic solution to the paradoxes, Section 4.3 deals with the question of whether *RND* is a reasoning-reduction solution which restricts the use of inference rules. Under the assumption that proof-theoretic validity implies (full) normalizability, we introduce Prawitz's definition of valid inferences via his notion of proof-theoretic validity and find invalid rules in paradoxical derivations. The restriction of an application of invalid rules would be the reasoning-rejection solution to the paradoxes. However, we will argue that to limit the application of inference rules through Prawitz's definition of valid inferences can be used independently of *RND*. Thus, it does not support the view that *RND* is a reasoning-rejection solution in the sense that it restricts the application of a single inference rule.

With regard to the last question (iii), Section 4.4 shall consider a case of a normal derivation of a formula having the form $\varphi \wedge \neg\varphi$, suggested by Petrolo and Pistone (2018). Acontradiction is often regarded as a formula of the form $\varphi \wedge \neg\varphi$. If a contradiction, separated from absurdity ($\bot$), can be an unacceptable conclusion of a paradoxical derivation, *RND* neither block the paradoxical derivation nor be a proof-theoretic solution to the paradoxes. Thus, the third question (iii) must be answered in order to assess whether *RND* can be a proof-theoretic solution to the paradoxes.

4.1 Which Paradoxes Are Genuine Paradoxes?

Tennant (1982, 1995, 2015, 2017) suggests *TCP* as a criterion for genuine paradoxes and *TCP* implicitly assumes his conjecture for genuine paradoxes.

Tennant's Conjecture for Genuine Paradoxes: For any derivation $\mathfrak{D}$ in natural deduction, $\mathfrak{D}$ formalizes a genuine paradox iff $\mathfrak{D}$ is a T-paradox with respect to *TCP*.

Although we only consider Russell's (or Curry's) and the Liar paradox in this dissertation, there are more than one formalization of each paradox. For instance, as we have seen in Chapter 2, eight derivations of the Liar paradox and not every derivation is a T-paradox.

Proposition	Is It a T-Paradox	Proposition	Is It a T-Paradox?
2.2.1	Yes	2.4.4	Yes
2.2.2	No	2.A.1	Yes
2.3.1	No	2.A.2	No
2.4.3	Yes	2.A.3	Yes

Table 4.1: Derivations Formalizing the Liar Paradox in Chapter 2

If it is not assumed that the Liar paradox is a genuine paradox, we cannot evaluate whether *TCP* is a correct criterion for genuine paradoxes from the different formalizations of the Liar paradox and neither can *RND* be a promising proof-theoretic solution to the genuine paradoxes. Therefore, it should be answered to the question of which paradoxes are genuine paradoxes.

The First Question (i): Which paradoxes are genuine paradoxes?

In this section, we argue that Tennant did not have any clear answer to the question. Tennant (2016, pp. 12–16) proposes the same result with Proposition 2.A.1 in Appendix 2.A and believes that the result shows that the Liar paradox is a genuine paradox. In 2016 year paper, he may believe that every T-paradox is a genuine paradox. Unlike the Liar paradox, Tennant (2016, pp. 8–12) asserts that Russell's paradox is not a genuine one. The derivation of $\bot$ from Russell's paradox begins with the assumption that there is a set of all sets not members of themselves. He proves in his natural deduction system for the free logic of sets that there is a closed full normal derivation of the rejection of the assumption, i.e. there is no such set which contains all sets not members of themselves. He uses the result in order to support his view that Russell's paradox is not a genuine paradox.

We first see the derivation in his free logic of sets which supports the view that Russell's paradox is not a genuine paradox. We show that, by the adoption of Ekman-type reductions introduced in Chapter 3, the derivation enters into loops and satisfies TCP_E and TCP_L. If every T-paradox is a genuine paradox, Russell's paradox becomes a genuine paradox. Furthermore, similar to his argument that Russell's paradox is not genuine, we use his early formalization of the Liar paradox in Tennant (1982, p. 271) with generalized elimination rules and put forward a derivation which represents that the Liar paradox is not a genuine paradox.

Tennant (2016, Sec. 3) and Tennant (2017, pp. 294-298) show in his natural deduction system S_F for the free logic of sets that there is a full normal closed derivation of $\neg\exists y(y = \{x|\neg x \in x\})$. His result will be introduced in Proposition 4.1.1. The derivation Σ_3 of $\neg\exists y(y = \{x|\neg x \in x\})$ in Proposition 4.1.1 does not generate a looping reduction, and so is not a T-paradox. Since he believes that every genuine paradox is a T-paradox, he claims that Russell's paradox is not a genuine paradox.

Tennant's natural deduction system S_F differs from the system S_N for the naive set theory discussed in Section 1.3 of Chapter 1.[1] Free logic is one whose quantifiers are interpreted in the usual way, but whose singular terms may denote objects outside of a domain or fail to denote at all. Therefore, unlike other logics, it has the rule of denotation. We abbreviate $\exists x(x = t)$ as $\exists !t$ which means that t exists. Let t and u be closed terms and a be a parameter. A natural deduction system S_F for the free logic of sets has the following rules for a set-forming operator and the rule of denotation (RD) with $\neg-$ and $\exists-$rules stated in generalized form.

$$\dfrac{\begin{matrix} [\varphi[a/x], \exists !a]^1 & & [a \in t]^1 \\ \mathfrak{D}_1 & \mathfrak{D}_2 & \mathfrak{D}_3 \\ a \in t & \exists !t & \varphi[a/t] \end{matrix}}{t = \{x|\varphi(x)\}} \{\}I_{,1}$$

where a does not occur in $t = \{x|\varphi(x)\}$ nor in any undischarged assumptions of the subordinate derivations other than those of the form of rules displayed

$$\dfrac{\begin{matrix} & & & [u \in t]^1 \\ & \mathfrak{D}_4 & \mathfrak{D}_5 & \mathfrak{D}_6 \\ t = \{x|\varphi(x)\} & \varphi[u/x] & \exists !u & \psi \end{matrix}}{\psi} \{\}E_{1,1} \qquad \dfrac{\begin{matrix} & & [\varphi[u/x]]^1 \\ & \mathfrak{D}_7 & \mathfrak{D}_8 \\ t = \{x|\varphi(x)\} & u \in t & \psi \end{matrix}}{\psi} \{\}E_{2,1}$$

Also, RD is stated as follows

$$\dfrac{\varphi(\ldots t \ldots)}{\exists !t} RD$$

where φ is atomic. For our result of the derivation of $\perp$ from $[a = \{x|\neg x \in x\}]$, we take $\neg x \in x$ for φ in $\{\}E_1-$ and $\{\}E_2-$rules, and for both terms t and u we take the parameter a. Then the following rules are the instances of $\{\}E_1-$ and $\{\}E_2-$rules.

[1]For the detailed introduction of his system for the free logic of sets, the reader can consult Section 7.10 of Tennant (1978).

$$\dfrac{a = \{x|\neg x \in x\} \quad \begin{array}{c} \\ \mathfrak{D}_4 \\ \neg a \in a \end{array} \quad \begin{array}{c} \\ \mathfrak{D}_5 \\ \exists! a \end{array} \quad \begin{array}{c} [a \in a]^1 \\ \mathfrak{D}_6 \\ \psi \end{array}}{\psi}\{\}E_{1,1} \qquad \dfrac{a = \{x|\neg x \in x\} \quad \begin{array}{c} \\ \mathfrak{D}_7 \\ a \in a \end{array} \quad \begin{array}{c} [\neg a \in a]^1 \\ \mathfrak{D}_8 \\ \psi \end{array}}{\psi}\{\}E_{2,1}$$

Then, we have a closed full normal derivation of $\neg\exists y(y = \{x|\neg x \in x\})$.

Proposition 4.1.1. *S_F has a closed full normal derivation of* $\neg\exists y(y = \{x|\neg x \in x\})$

Proof. We start to have a closed derivation Σ_3 of $\neg\exists y(y = \{x|\neg x \in x\})$, and show that Σ_3 is in full normal form.

Claim 1. there is a closed derivation Σ_3 of $\neg\exists y(y = \{x|\neg x \in x\})$.

There is an open derivation Σ_1 of $\perp$ from $[a = \{x|\neg x \in x\}]$ and $[a \in a]$.

$$\dfrac{[a = \{x|\neg x \in x\}]^1 \quad [a \in a]^5 \qquad \dfrac{[a = \{x|\neg x \in x\}]^1 \quad [\neg a \in a]^2 \quad \dfrac{[a = \{x|\neg x \in x\}]^1}{\exists! a}RD \quad \dfrac{[\neg a \in a]^2 \quad [a \in a]^3 \quad [\perp]^4}{\perp}\neg E_{,4}}{\perp}\{\}E_{1,3}}{\perp}\{\}E_{2,2}$$

With the open derivation Σ_1, we have an open derivation Σ_2 of $\perp$ from $[a = \{x|\neg x \in x\}]$.

$$\dfrac{[a = \{x|\neg x \in x\}]^1 \qquad \dfrac{\begin{array}{c} [a = \{x|\neg x \in x\}]^1, [a \in a]^5 \\ \Sigma_1 \\ \perp \end{array}}{\neg a \in a}\neg I_{,5} \qquad \dfrac{[a = \{x|\neg x \in x\}]^1}{\exists! a}RD \qquad \begin{array}{c} [a = \{x|\neg x \in x\}]^1, [a \in a]^6 \\ \Sigma_1 \\ \perp \end{array}}{\perp}\{\}E_{1,6}$$

Now, we have a closed derivation Σ_3 of $\neg\exists y(y = \{x|\neg x \in x\})$.

$$\dfrac{\dfrac{[\exists y(y = \{x|\neg x \in x\})]^7 \qquad \begin{array}{c} [a = \{x|\neg x \in x\}]^1 \\ \Sigma_2 \\ \perp \end{array}}{\perp}\exists E_{,1}}{\neg\exists y(y = \{x|\neg x \in x\})}\neg I_{,7}$$

Claim 2. Σ_3 is in full normal form.

Since all major premises in Σ_1, Σ_9, and Σ_3 are assumptions, Σ_3 is in full normal form.

□

The derivation Σ_3 of $\neg\exists y(y = \{x|\neg x \in x\})$ in Proposition 4.1.1 is in full normal form. However, if we accept an Ekman-type reduction introduced in Section 3.1 of Chapter 3, S_F has an open non-full normal derivation of $\bot$ from $[a = \{x|\neg x \in x\}]$ which generates a looping reduction. When the derivation employs the id est inference from $a \in a$ to $\neg a \in a$ and $\neg a \in a$ to $a \in a$, by TCP_E and TCP_L, the derivation in question would be a T-paradox. If every T-paradox is a genuine paradox, Russell's paradox becomes a genuine paradox.

We state an Ekman-type reduction process in generalized form for set-abstraction below:

$$\dfrac{\begin{matrix}\mathfrak{D}_1\\ t=\{x|\varphi(x)\}\end{matrix} \quad \dfrac{\begin{matrix}\mathfrak{D}_1\\ t=\{x|\varphi(x)\}\end{matrix}\quad \begin{matrix}\mathfrak{D}_2\\ \varphi[x/u]\end{matrix}\quad \begin{matrix}\mathfrak{D}_3\\ \exists! u\end{matrix}\quad [u\in t]^1}{u\in t}\{\}E_{1,1} \quad \begin{matrix}[\varphi[x/u]]^2\\ \mathfrak{D}_4\\ \psi\end{matrix}}{\psi}\{\}E_{2,2} \quad \trianglerighteq_{GEF} \quad \begin{matrix}\mathfrak{D}_2\\ \varphi[x/u]\\ \mathfrak{D}_4\\ \psi\end{matrix}$$

We call the minor premise $u \in t$ of $\{\}E_2-$rule a $GEkman_F$ maximum formula. Then, we have the following result.

Proposition 4.1.2. *If the set of reductions of S_F includes an Ekman-type reduction process in generalized form for set-abstraction, $\trianglerighteq_{GEF}$, S_F has an open derivation of $\bot$ from $[a = \{x|\neg x \in x\}]$ which generates a non-terminating reduction sequence and is not fully normalizable.*

Proof. Two claims justify the result.

Claim 1. there is an open derivation Σ_6 of $\bot$ from $[a = \{x|\neg x \in x\}]$ in S_F.

We begin with the open derivation Σ_4 of $\bot$ from $[a \in a]$ and $[a = \{x|\neg x \in x\}]$.

$$\dfrac{[a=\{x|\neg x\in x\}]^1 \quad [a\in a]^2 \quad \dfrac{[\neg a\in a]^3\quad [a\in a]^2\quad [\bot]^4}{\bot}\neg E_{,4}}{\bot}\{\}E_{2,3}$$

With the derivation Σ_4, we have an open derivation Σ_5 of $a \in a$ from $[a = \{x|\neg x \in x\}]$.

$$\dfrac{[a=\{x|\neg x\in x\}]^1 \qquad \dfrac{\begin{array}{c}[a=\{x|\neg x\in x\}]^1,[a\in a]^2\\ \Sigma_4\\ \bot\end{array}}{\neg a\in a}\,\neg I_{,2} \qquad \dfrac{[a=\{x|\neg x\in x\}]^1}{\exists! a}\,RD \qquad [a\in a]^5}{a\in a}\,\{\}E_{1,5}$$

Then, we have an open derivation Σ_6 of $\bot$ from $[a = \{x|\neg x \in x\}]$.

$$\dfrac{\dfrac{\begin{array}{c}[a=\{x|\neg x\in x\}]^1,[a\in a]^2\\ \Sigma_4\\ \bot\end{array}}{\neg a\in a}\,\neg I_{,2} \qquad \begin{array}{c}[a=\{x|\neg x\in x\}]^1\\ \Sigma_5\\ a\in a\end{array} \qquad [\bot]^6}{\bot}\,\neg E_{,6}$$

Claim 2. if an Ekman-type reduction, $\triangleright_{GEF}$, applies to Σ_6, then Σ_6 generates a non-terminating reduction sequence and so is not fully normalizable.

Since Σ_6 has a major premise $\neg a \in a$ which is not an assumption, it reduces to the following derivation Σ_7.

$$\dfrac{[a=\{x|\neg x\in x\}]^1 \qquad \dfrac{[a=\{x|\neg x\in x\}]^1 \qquad \dfrac{\begin{array}{c}[a=\{x|\neg x\in x\}]^1,[a\in a]^2\\ \Sigma_4\\ \bot\end{array}}{\neg a\in a}\,\neg I_{,2} \qquad \begin{array}{c}\vdots\\ \exists! a\end{array} \quad [a\in a]^5}{a\in a}\,\{\}E_{1,5} \qquad \dfrac{[\neg a\in a]^3 \quad \begin{array}{c}\vdots\\ \Sigma_5\\ a\in a\end{array} \quad [\bot]^4}{\bot}\,\neg E_{,4}}{\bot}\,\{\}E_{2,3}$$

The minor premise $a \in a$ in $\{\}E_2-$rule is a $GEkman_F$ maximum formula. By applying $\triangleright_{GEF}$ to Σ_7, we have the same derivation with Σ_6. Therefore, Σ_6 initiates a non-terminating reduction sequence and cannot be reduced to full normal form. $\square$

If Σ_6 employs the id est inferences, then, by TCP_E and TCP_L, Σ_6 is a T-paradox. Therefore, unlike Tennant's view, Russell's paradox becomes a genuine paradox if $\trianglerighteq_{GEF}$ is acceptable. As we have discussed in Section 3.2.2, $\trianglerighteq_{GEF}$ does not apply to a derivation given by permutation conversion. However, we readily give a permuted version of $\trianglerighteq_{GEF}$ and have a looping reduction.[2]

On the other hand, by following Tennant's argument that Russell's paradox is not a genuine paradox, we can propose a derivation which represents that the Liar paradox is not a genuine paradox.

Let S_L be a system having the rules for $\wedge$, $\rightarrow$, $\neg$, and $T(x)$. The set R_L of reduction procedures for $\wedge$, $\rightarrow$, $\neg$, and $T(x)$ are given. We define $\varphi \leftrightarrow \psi$ as $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$. Then, the liar sentence Φ satisfies the relation $\Phi \leftrightarrow \neg T(\ulcorner \Phi \urcorner)$. Proposition 4.1.3 supports the view that Liar paradox is not a genuine paradox.

Proposition 4.1.3. *S_L relative to $\mathbb{R}_L$ has a closed full normal derivation of* $\neg(\Phi \leftrightarrow \neg T(\ulcorner \Phi \urcorner))$.

Proof. We start to show the closed derivation Σ_9 of $\neg(\Phi \leftrightarrow \neg T(\ulcorner \Phi \urcorner))$ and to establish that Σ_9 is in full normal form.

Claim 1. there is a closed derivation Σ_9 of $\neg(\Phi \leftrightarrow \neg T(\ulcorner \Phi \urcorner))$.

We first have an open derivation Σ_7 of $\perp$ from $[T(\ulcorner \Phi \urcorner)]$ and $[\Phi \leftrightarrow \neg T(\ulcorner \Phi \urcorner)]$

$$\dfrac{\dfrac{[\Phi \leftrightarrow \neg T(\ulcorner \Phi \urcorner)]^6}{(\Phi \rightarrow \neg T(\ulcorner \Phi \urcorner)) \wedge (\neg T(\ulcorner \Phi \urcorner) \rightarrow \Phi)}\,def \qquad \dfrac{[\Phi \rightarrow \neg T(\ulcorner \Phi \urcorner)]^5 \quad \dfrac{[T(\ulcorner \Phi \urcorner)]^1 \quad [\Phi]^2}{\Phi}\,TE_{,2} \quad \dfrac{[\neg T(\ulcorner \Phi \urcorner)]^3 \quad [T(\ulcorner \Phi \urcorner)]^1 \quad [\perp]^4}{\perp}\,\neg E_{,4}}{\perp}\,\rightarrow E_{,3}}{\perp}\,\wedge E_{,5}$$

[2] With regard to the problem of overgeneration in Chapter 3, the result shows that the adoption of Ekman-type reduction affects Tennant's view that Russell's paradox is not a genuine paradox. In his 1982 paper, "Proof and Paradox," Tennant used the standard form of the elimination rule for set-abstraction. Tennant (1982, p. 276) claimed that the derivation of $\perp$ from the assumption $\exists! a$ where $a = \{x | \neg x \in x\}$ enters a looping reduction and said, 'Russell's [paradox] remains an intrinsically troublesome case of paradox.' Later, from the result of Proposition 4.1.1, Tennant (2016, Sec. 3) claims that Russell's is not a genuine paradox because the derivation in question does not enter into loops. He diagnoses that the standard form of the elimination rule for set-abstraction creates an artefact feature of the looping reduction sequence. Unfortunately, as we have seen in Σ_6 of Proposition 4.1.2, the generalized form of the elimination rule either creates a reduction loop. The real issue is not which form of elimination rules we choose, but which set of reduction procedures we accept. Moreover, even when we use standard elimination rules, it is readily proved that there is a normal derivation of $\neg \exists y(y = \{x | \neg x \in x\})$. His assessment of the genuineness of Russell's paradox was wrong in the first place.

Then, we have an open derivation Σ_8 of Φ from $[\Phi \leftrightarrow \neg T(\ulcorner\Phi\urcorner)]$.

$$
\dfrac{\dfrac{[\Phi \leftrightarrow \neg T(\ulcorner\Phi\urcorner)]^6}{(\Phi \to \neg T(\ulcorner\Phi\urcorner)) \wedge (\neg T(\ulcorner\Phi\urcorner) \to \Phi)}\,def \qquad \dfrac{[\neg T(\ulcorner\Phi\urcorner \to \Phi)]^5 \quad \dfrac{\begin{array}{c}[T(\ulcorner\Phi\urcorner)]^1\\ \mathfrak{D}_{13}\\ \bot\end{array}}{\neg T(\ulcorner\Phi\urcorner)}\neg I_{,1} \quad [\Phi]^7}{\Phi}\to E_{,7}}{\Phi}\wedge E_{,5}
$$

With Σ_7 and Σ_8, we finally have a closed derivation $\mathfrak{t}_9$ of $\neg(\Phi \leftrightarrow \neg T(\ulcorner\Phi\urcorner))$.

$$
\dfrac{\dfrac{\dfrac{[\Phi \leftrightarrow \neg T(\ulcorner\Phi\urcorner)]^6}{(\Phi \to \neg T(\ulcorner\Phi\urcorner)) \wedge (\neg T(\ulcorner\Phi\urcorner) \to \Phi)}\,def \qquad \dfrac{[\Phi \to \neg T(\ulcorner\Phi\urcorner)]^7 \quad \begin{array}{c}[\Phi \leftrightarrow \neg T(\ulcorner\Phi\urcorner)]^6\\ \Sigma_7\\ \Phi\end{array} \quad \dfrac{[\neg T(\ulcorner\Phi\urcorner)]^8 \quad \dfrac{\begin{array}{c}[\Phi \leftrightarrow \neg T(\ulcorner\Phi\urcorner)]^6\\ \Sigma_7\\ \Phi\end{array}}{T(\ulcorner\Phi\urcorner)}TI \quad [\bot]^{10}}{\bot}\neg E_{,10}}{\bot}\to E_{,8}}{\bot}\wedge E_{,7}}{\neg(\Phi \leftrightarrow \neg T(\ulcorner\Phi\urcorner))}\neg I_{,6}
$$

Claim 2. Σ_9 is in full normal form.

Since all major premises in Σ_7, Σ_8, Σ_9 are assumptions, Σ_9 is in full normal form. □

As Tennant claims that Russell's paradox is not a genuine paradox with the full normal derivation of the formula that there is no set of all sets not members of themselves, the full normal derivation of Σ_9 of $\neg(\Phi \leftrightarrow \neg T(\ulcorner\Phi\urcorner))$ supports that Liar paradox is not a genuine one.

Including Proposition 4.1.3, there are nine derivations formalizing the Liar paradox. Among nine derivations, Tennant only considers that the derivation Σ_3 of Proposition 2.A.1 is a ground to make the Liar paradox genuine. However, he does not have a good reason to repudiate that the derivation Σ_9 of Proposition 4.1.3 supports the view that the Liar paradox is not a genuine paradox. In a similar vein, if we choose Prawitz's derivation $\mathfrak{D}_4$ of Russell's paradox in Proposition 1.3.1 (or the derivation Σ_6 of Proposition 4.1.2), then $\mathfrak{D}_4$ becomes a ground for claiming that Russell's paradox is a genuine paradox. Since Tennant has never spoken about the ground for genuine paradoxes in the perspectives on proof-theory, in order to evaluate *TCP* and *RND*, it should be explained which paradoxes are genuine paradoxes and which derivation is a legitimate one for genuine paradoxes.

Even if these questions are answered, it is not enough to claim that *RND* is a proper proof-theoretic solution to genuine paradoxes. In the next section, we move on to the second question on why *RND* is a convincing requirement and claim that a clear answer is not yet given.

4.2 Why Should We Accept Only a Normalizable Derivation?

Our proof-theoretic analysis of the paradoxes uses a natural deduction system developed by Prawitz (1965, 1971) but first introduced by Gentzen (1935, 2008). When Gentzen (1936) attempted to show the consistency of arithmetic, he believed that if there was a fault in the paradoxes, it must be sought in the logical reasoning employed. One of the main purposes of the proof-theoretic analysis of the paradoxes is to find any errors in the reasoning. It appears to be convincing that a suitable proof-theoretic solution to the paradoxes can be the reasoning-rejection solution. *RND* may be the reasoning-rejection solution in a broad sense. However, as our second question asks, it should be explained why a non-normalizable derivation is unacceptable.

The Second Question (ii): Why should we accept only a normalizable derivation?

A promising answer is that a non-normalizable derivation would not be proof-theoretically valid. In this section, we will briefly investigate Prawitz's idea of proof-theoretic validity and the relation between proof-theoretic validity and normalizability. If proof-theoretic validity implies (full) normalizability, then paradoxical derivations, i.e. T-paradoxes which are not (fully) normalizable, are not proof-theoretically valid. Then, since there is no non-normalizable derivation which is proof-theoretically valid, it can be explained why any non-normalizable derivations are unacceptable.

The name 'proof theory' was originally coined by David Hilbert. The aim of his proof theory is to obtain a reduction of mathematics to some more elementary part of it, such as finitistic or constructive mathematics, by analyzing the proofs of mathematical theories. Prawitz (1971, 1973) thinks that Hilbert's proof theory is only a tool to obtain the reduction because it does not aim at studying the very proofs. He calls it *reductive proof theory*. He suggests *general proof theory* as the study of the notion of proof.

> The subject matter of *general proof theory* is thus proofs considered as a process by which we get to know the theorems of a theory or the validity of an argument, and this process is studied here in its own right. (Prawitz, 1971, p. 237)

> In general proof theory, we are ... interested in understanding the very proofs themselves, i.e., in understanding not only *what* deductive connections hold but also *how* they are established, ... (Prawitz, 1973, p. 225)

One of the main topics of general proof theory is the validity of an argument.

In this section, we briefly introduce Prawitz's notion of 'proof-theoretic validity' and consider a possible answer to the second question that every non-normalizable derivation is not proof-theoretically valid. If proof-theoretic validity implies normalizability then *RND* can be a promising proof-theoretic solution to the paradoxes. So to speak, if every paradoxical derivation, i.e. a T-paradox, generates a non-terminating reduction sequence and so is not normalizable, it cannot be a proof-theoretically valid derivation. Hence, we do not need to accept non-normalizable derivation. However, we shall argue that although it can be shown that in a particular system proof-theoretic validity implies normalizability, there should be a further research to extend the result in a general case.

Prawitz (1971, Appendix A) introduces a programme of defining a general notion of 'validity' based on Gentzen's idea that an introduction rule determines the meaning of a logical constant in terms of which an elimination rule is determined. That is, a derivation (or an argument) is valid if it can be built up by introduction rules. Since the programme generalizes the result of normalization theorem. He considers that proof-theoretic validity is not only a property of formal derivations in a particular system but also of more arbitrary natural deduction system. To prove the theorem, we first distinguish certain individual rules and then compose derivations from the rules. On the other hand, Prawitz's proof-theoretic validity deals with derivations (or arguments) in the first place and regards rules as steps which preserve the validity of derivations. So to speak, the investigation of the relation between Prawitz's notion of proof-theoretic validity and normalizability does not seem appropriate. However, when we restrict our concern on the proof-theoretic validity as a property of formal derivations in a particular system, we can consider an expected result that every proof-theoretically valid derivation in a specific system is normalizable. Most of our examples of paradoxical derivations use standard rules for minimal logic. If proof-theoretic validity implies normalizability in a minimal natural deduction system, at least in restricted sense, one might see the possibility that *RND* would be a method to single out proof-theoretically invalid derivations. Since we shall regard in this chapter a normal derivation as a formal one in a natural deduction system, we restrict our concern on the proof-theoretic validity as a property of formal derivations in the system. We will borrow Prawitz's notion of proof-theoretic validity introduced in Prawitz (1971, 1973, 1974, 2006), and it will be applied to formal derivations relative to a set of reduction procedures but not to his notion of an argument.

Prawitz (1965) shows that by iterated application of reduction processes, every derivation in an intuitionistic natural deduction system can be converted into a normal derivation. It has a corollary that every closed derivation in the system can be restated to one using an introduction rule in the last step. Prawitz (1971, 1973, 1974, 2006) interprets the collorary as the requirement of a valid inference that a valid closed derivation is able to be reduced to one using an introduction rule in the last step.

The results are connected to Prawitz's proof-theoretic validity through his inversion principle that whatever follows from a formula must follow from the direct ground for deriving that formula. As we have seen in Section 1.2, standard reduction procedures for $\wedge$, $\vee$, $\rightarrow$, $\neg$, $\forall$, and $\exists$ show that a pair of introduction and elimination rules of each constant satisfy the inversion principle. The inversion principle reflects Gentzen's idea that the meaning of an operator (or a constant) is determined by an introduction rule and determines an elimination rule. The idea gives a semantic interpretation of an introduction rule that nothing is gained by an application of an elimination rule when its major premise has been derived by means of an introduction rule which confers a meaning of a constant.

With the semantic role of an introduction rule, Prawitz (1971, 1973, 1974, 2006) introduces his definitions of proof-theoretic validity based on introduction rules.[3] The main idea is that introduction rules preserve proof-theoretic validity and elimination rules are justified by (standard) reduction procedures. Simply put, a derivation is proof-theoretically valid if either it reduces to a derivation of an atomic formula, or it reduces to a derivation whose last step is an introduction rule and whose immediate subderivations are proof-theoretically valid.

To introduce Prawitz's definition of proof-theoretic validity more precisely, we borrow some terminologies from Schroeder-Heister (2006). Let $\mathfrak{D}_1, \ldots, \mathfrak{D}_n$ be derivations where n is a natural number, $\mathbb{P}$ be a set of production rules which derives an atomic formula from one or more other atomic formulas, and $\mathbb{R}$ be a set of reduction procedures. We say that a derivation $\mathfrak{D}$ is *canonical* (or in *canonical form*) if it uses an introduction rule in the last step. Prawitz's proof-theoretic validity of a given derivation $\mathfrak{D}$ depends not only on the set $\mathbb{P}$ of production rules but also on the set $\mathbb{R}$ of reduction procedures. For our purpose of investigating the relation between proof-theoretic validity and normalizability, we only consider standard reduction procedures. Every extension of $\mathbb{R}$ will consist of standard reductions. We provide his definition of proof-theoretic validity of $\mathfrak{D}$ relative to $\mathbb{P}$ and $\mathbb{R}$ in the following way.

Definition 4.2.1. (Inductive Definition of $\mathbb{P}-$Validity) Let $\mathbb{P}$ be a set of production rules and $\mathbb{R}$ be a set of reduction procedures.

(1) Every closed derivation in $\mathbb{P}$ is $\mathbb{P}-$*valid* relative to $\mathbb{R}$ (for every $\mathbb{R}$).

(2) A closed canonical derivation $\mathfrak{D}$ is $\mathbb{P}-$*valid* relative to $\mathbb{R}$ iff all immediate subderivations of $\mathfrak{D}$ are $\mathbb{P}-$*valid* relative to $\mathbb{R}$.

[3] Prawitz (1971, 2007) either suggests proof-theoretic validity starting with elimination rules. However, for our purpose to propose a perspective of proof-theoretic solution, we only consider proof-theoretic validity based on introduction rules.

(3) A closed non-canonical derivation $\mathfrak{D}$ is $\mathbb{P}-$*valid* relative to $\mathbb{R}$ iff $\mathfrak{D}$ reduces to a $\mathbb{P}-$*valid* canonical derivation relative to $\mathbb{R}$.

(4) An open derivation $\begin{matrix}\varphi_1,\ldots,\varphi_n\\ \mathfrak{D}\\ \psi\end{matrix}$, where all open assumptions of $\mathfrak{D}$ are among $\varphi_1,\ldots,\varphi_n$, is $\mathbb{P}-$*valid* relative to $\mathbb{R}$ iff for every $\mathbb{P}' \supseteq \mathbb{P}$ and $\mathbb{R}' \supseteq \mathbb{R}$, and for every list of closed derivation $\begin{matrix}\mathfrak{D}_i\\ \varphi_i\end{matrix}$ $(1 \leqslant i \leqslant n)$, which are $\mathbb{P}'-$*valid* relative to $\mathbb{R}'$, $\begin{matrix}\mathfrak{D}_1 & & \mathfrak{D}_n\\ \varphi_1 & ,\ldots, & \varphi_n\\ & \mathfrak{D} & \\ & \psi & \end{matrix}$ is $\mathbb{P}'-$*valid* relative to $\mathbb{R}'$.[4]

Proof-theoretic validity is proposed by Prawitz's interpretation of normalization results through the inversion principle. It is thus natural to think that proof-theoretic validity implies normalizability in a particular system. We consider a system that its language only has constants $\wedge$, $\vee$, $\rightarrow$, $\forall$, and $\exists$. The system does not use any formula including $\perp$ and $\neg$. Then, by induction on the degree of the end formula of a given derivation, we obtain the desired result in the system.

Theorem 4.2.2. *Let $\mathscr{L}$ be a language which has constants $\wedge$, $\vee$, $\rightarrow$, $\forall$, and $\exists$, but does not have $\neg$ and $\perp$. Let S be a natural deduction system in $\mathscr{L}$ which has rules for $\wedge$, $\vee$, $\rightarrow$, $\forall$, and $\exists$. Let $\mathbb{P}$ be a set of production rules for S which only consists of closed derivations, and $\mathbb{R}$ be a set of reduction procedures for $\wedge$, $\vee$, $\rightarrow$, $\forall$, and $\exists$. For every derivation $\mathfrak{D}$ in S, if $\mathfrak{D}$ is $\mathbb{P}-$valid relative to $\mathbb{R}$, $\mathfrak{D}$ is normalizable relative to $\mathbb{R}$.*

Proof. Let $\mathfrak{D}$ be any derivation in S and φ be any end-formula of $\mathfrak{D}$. Suppose that $\mathfrak{D}$ is $\mathbb{P}-$valid relative to $\mathbb{R}$. The proof is by induction over the degree of φ. Since Definition 4.2.1 has four conditions, there are four cases that we should consider: (1) $\mathfrak{D}$ is a closed derivation in $\mathbb{P}$, (2) $\mathfrak{D}$ is a closed canonical derivation, (3) $\mathfrak{D}$ is a closed non-canonical derivation, and (4) $\mathfrak{D}$ is an open derivation.

[4] Although Prawitz (1973, p. 236; 1974, p. 73; 2006, p. 515) does not consider extensions of $\mathbb{R}$, we follow Schroeder-Heister (2006, 2015) and consider both extensions of $\mathbb{R}$ and $\mathbb{P}$. Extensions of $\mathbb{P}$ and $\mathbb{R}$ are required because when the open derivation contain any expressions and inference rules that are not already given by $\mathbb{P}$ and $\mathbb{R}$, we need to add the expressions and to assign reductions to the inference rules substituted for the open assumption of the derivation.

Induction basis: if $d(\varphi) = 0$, then φ is $\bot$ or φ is α for an atomic formula α. It means that $\mathfrak{D}$ is a closed derivation in $\mathbb{P}$.

Case (1) if $\mathfrak{D}$ is a closed derivation in $\mathbb{P}$, then it is in normal form. Hence, by Definition 1.2.5, $\mathfrak{D}$ is normalizable relative to $\mathbb{R}$.

Induction hypothesis: let $\mathfrak{D}'$ be any immediate subderivation of $\mathfrak{D}$ and φ' is an end-formula of $\mathfrak{D}'$. Suppose that $\mathfrak{D}'$ is normalizable relative to $\mathbb{R}$ with $d(\varphi') \leqslant n$. We have to show that $\mathfrak{D}$ is normalizable relative to $\mathbb{R}$ with $d(\varphi) \leqslant n+1$.

Case (2) if $\mathfrak{D}$ is a closed canonical derivation, since $\mathfrak{D}'$ is normalizable, trivially $\mathfrak{D}$ is normalizable relative to $\mathbb{R}$.

Case (3) if $\mathfrak{D}$ is a closed non-canonical derivation, then, by Definition 4.2.1, $\mathfrak{D}$ reduces to a $\mathbb{P}-$valid canonical derivation relative to $\mathbb{R}$. By the case (2), $\mathfrak{D}$ is normalizable relative to $\mathbb{R}$.

Case (4) if $\mathfrak{D}$ is an open derivation, then, by Definition 4.2.1, for any $\mathbb{P}' \supseteq \mathbb{P}$ and $\mathbb{R}' \supseteq \mathbb{R}$, a closed derivation

$$\begin{array}{c} \mathfrak{D}_1 \quad\ \ \mathfrak{D}_n \\ \varphi_1 , \ldots , \varphi_n \\ \mathfrak{D} \\ \psi \end{array}$$

is $\mathbb{P}'-$valid relative to $\mathbb{R}'$. By the cases (1), (2), and (3), since every list of closed derivation $\begin{array}{c}\mathfrak{D}_i \\ \varphi_i\end{array}$ $(1 \leqslant i \leqslant n)$ is normalizable relative to $\mathbb{R}'$, $\mathfrak{D}$ is normalizable relative to $\mathbb{R}$.

□

The result shows that at least in a particular system containing rules for $\wedge$, $\vee$, $\rightarrow$, $\forall$, and $\exists$, proof-theoretic validity implies normalizability. There is a further strengthened concept, *strong*-$\mathbb{P}-$*validity*, which implies strong normalizability.

Definition 4.2.3. (Inductive Definition of Stong $\mathbb{P}-$Validity) Let $\mathbb{P}$ be a set of production rules and $\mathbb{R}$ be a set of reduction procedures.

(1) Every closed derivation in $\mathbb{P}$ is *strongly* $\mathbb{P}-$*valid* relative to $\mathbb{R}$ (for every $\mathbb{R}$).

(2) A closed canonical derivation $\mathfrak{D}$ is *strongly* $\mathbb{P}-$*valid* relative to $\mathbb{R}$ iff all immediate subderivations of $\mathfrak{D}$ are *strongly* $\mathbb{P}-$*valid* relative to $\mathbb{R}$.

(3) A closed non-canonical derivation $\mathfrak{D}$ is *strongly* $\mathbb{P}-$*valid* relative to $\mathbb{R}$ iff every $\mathfrak{D}'$, such that $\mathfrak{D} \rhd \mathfrak{D}'$, is *strongly* $\mathbb{P}-$*valid* relative to $\mathbb{R}$.

(4) An open derivation $\begin{array}{c}\varphi_1,\ldots,\varphi_n\\ \mathfrak{D}\\ \psi\end{array}$, where all open assumptions of $\mathfrak{D}$ are among $\varphi_1,\ldots,\varphi_n$, is *strongly* $\mathbb{P}-$*valid* relative to $\mathbb{R}$ iff for every $\mathbb{P}' \supseteq \mathbb{P}$ and $\mathbb{R}' \supseteq \mathbb{R}$, and for every list of closed derivation $\begin{array}{c}\mathfrak{D}_i\\ \varphi_i\end{array}$ $(1 \leqslant i \leqslant n)$, which are *strongly* $\mathbb{P}'-$*valid* relative to $\mathbb{R}'$, $\begin{array}{c}\mathfrak{D}_1 \quad \mathfrak{D}_n\\ \varphi_1,\ldots,\varphi_n\\ \mathfrak{D}\\ \psi\end{array}$ is *strongly* $\mathbb{P}'-$*valid* relative to $\mathbb{R}'$.

Theorem 4.2.4. *Let S be a natural deduction system having rules for* $\wedge$, $\vee$, $\rightarrow$, $\forall$, *and* $\exists$. *Let* $\mathbb{P}$ *be a set of production rules for S which only consists of closed derivations and has no closed derivation of* $\perp$ *and* $\mathbb{R}$ *be a set of reduction procedures for* $\wedge$, $\vee$, $\rightarrow$, $\forall$, *and* $\exists$. *For every derivation* $\mathfrak{D}$ *in S, if* $\mathfrak{D}$ *is strongly* $\mathbb{P}-$*valid relative to* $\mathbb{R}$, $\mathfrak{D}$ *is strongly normalizable relative to* $\mathbb{R}$.

In analogy with the proof of Theorem 4.2.2, by induction over the degree of the end-formula, Theorem 4.2.4 can be proved.

It is obvious in a particular system that strong $\mathbb{P}-$validity implies $\mathbb{P}-$validity and strong normalizability implies normalizability. In addition, the results may be extended to some systems which have legitimate pairs of introduction and elimination rules for an operator. There seems to be a good reason to believe that proof-theoretic validity implies normalizability. Then, if a paradoxical derivation generates a non-terminating reduction sequence and so is not normalizable, by applying modus tollens, it is not a proof-theoretically valid derivation. In this sense, *RND* may rule out non-normalizable derivations and be a promising proof-theoretic solution to the paradoxes.

Of course, it must be argued whether the results can be extended to a system having additional rules and their reductions. Especially, if a system includes formulas containing a propositional constant $\perp$, there exists a case which is proof-theoretically valid but not normalizable. Schroeder-Heister (2006, p. 547) argues that when we regard $\perp$ as the only open assumption, $\begin{array}{c}\perp\\ \mathfrak{D}\end{array}$ is vacuously $\mathbb{P}-$valid for any $\mathfrak{D}$, even if $\mathfrak{D}$ is not normalizable. So he adds an additional condition for open derivations to Prawitz's definition of $\mathbb{P}-$validity that an open reducible derivation is $\mathbb{P}-$valid, if it reduces to a $\mathbb{P}-$valid derivation. He calls it $strict\mathbb{P}-validity$ and proposes the result in implicational logic that proof-theoretic validity implies (strong) normalizability. However, since his investigation does not include the case that allows open derivations in general, it is still arguable whether the results can extend to a general case. The purpose of the present section is to investigate how *RND* can be a proof-theoretic solution to the paradoxes. To be the solution, it should at least be established that proof-theoretic validity *generally* implies (strong) normalizability.

RND appears to be a stronger restriction than other reasoning-rejection solutions because a usual reasoning-rejection restricts the use of a specific inference rule but *RND* restricts every derivation in a natural deduction system. In the next section, we shall introduce Prawitz's definition of a $\mathbb{P}-$valid inference rule and see which rules are not $\mathbb{P}-$valid in a paradoxical derivation. Moreover, we shall discuss that if a reasoning-rejection solution to the paradoxes is a restriction of an inference rule, *RND* cannot be a reasoning-rejection solution.

4.3 Is *RND* a Reasoning-Rejection Solution?

RND demands every derivation in an intended system to be a normalizable derivation and it is not a restriction of a particular inference rule. Let us remind a closed derivation of Russell's paradox in Proposition 1.3.1 of Section 1.3

$$
\dfrac{
\dfrac{\dfrac{\dfrac{\dfrac{[a \in a]^1}{a \in \{x|\neg x \in x\}}\,def}{\neg a \in a}\in E \qquad [a \in a]^1}{\bot}\neg E}{\neg a \in a}\neg I_{,1}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[a \in a]^2}{a \in \{x|\neg x \in x\}}\,def}{\neg a \in a}\in E \qquad [a \in a]^2}{\bot}\neg E}{\neg a \in a}\neg I_{,2}}{a \in \{x|\neg x \in x\}}\in I}{a \in a}\,def
}{\bot}\neg E
$$

We call the immediate subderivation of the major premise $\neg a \in a$, Σ_{10}, the immediate subderivation of the minor premise $a \in a$, Σ_{11}, and the whole derivation Σ_{12}. Then, the derivation Σ_3 is abbreviated as

$$
\dfrac{\begin{matrix}\Sigma_{10} & \Sigma_{11} \\ \neg a \in a & a \in a\end{matrix}}{\bot}\neg E
$$

$\neg-$ and $\in -$rules used in Σ_{12}, i.e. $\mathfrak{D}_3$ in Proposition 1.3.1, are intrinsically harmonious but, as Proposition 1.3.1 shows, Σ_{12} is not normalizable. *RND* requests the weak normalization that every derivation in a natural deduction system can be reduced to a normalizable derivation. Stephen Read (2010, p. 574) already notes that the intrinsic harmony requirement does not guarantee the weak normalization result. Similarly, it is not to say that any system which only contains intrinsically harmonious rules satisfies *RND*. The intrinsic harmony is the requirement for a pair of *rules* but *RND* is not. If the intrinsic harmony is a requirement for a legitimate pair of rules, *RND* cannot be a requirement for a proof-theoreically valid *inference rule*. Moreover, though *RND* can block the derivation Σ_3 of $\bot$, it does not single out a rule that is invalid in Σ_3. If a reasoning-rejection solution to the paradoxes is a constraint on a specific rule in a paradoxical derivation, *RND* is not a reasoning-rejection but a structural restriction of all derivations in an intended system. Our question in this section is how *RND* can single out an inference rule which makes a derivation non-normalizable. If it cannot, then it will be regarded as a different kind of solution than a reasoning-rejection in a usual sense.

Interestingly, when Prawitz (1965, p. 95) mentions that the set-theoretical paradoxes are ruled out by the requirement that the derivations should be in normal form, he seems to request that the application of $\rightarrow E-$rule, in our case $\neg E-$rule, has to be restricted. While defining $\neg\varphi$ as $\varphi \rightarrow \bot$, the derivation Σ_{12} is restated as the following derivation Σ'_{12}.

$$\dfrac{\begin{matrix}\Sigma'_{10} & \Sigma'_{11} \\ \neg a \in a & a \in a\end{matrix}}{\bot} \rightarrow E$$

With the derivation Σ'_{12} in his natural deduction system for the naive set theory, he remarked on the application of the last $\rightarrow E-$rule in Σ'_{12}.

> ... the system [for the naive set theory] has serious disadvantages. Thus, although [$\rightarrow E-$rule] is a rule of the system, one cannot in general infer that [ψ] is provable given that [φ] and [$\varphi \rightarrow \psi$] are provable, since there may be only a [derivation] of ψ [but not a normal derivation of ψ]. This renders investigations of the system rather difficult as it is not sufficient to derive the axioms of an ordinary mathematical theory in the system in order to conclude that also its theorems are provable in the system. (Prawitz, 1965, p. 95)

Immediate subderivations Σ'_{10} and Σ'_{11} are in normal form, but Σ'_{12} is not. Since Σ'_{10} and Σ'_{11} are canonical derivations, by Definition 4.2.1 of $\mathbb{P}-$validity, they are $\mathbb{P}-$valid but Σ'_{12} is not. Therefore, he may think that $\neg E-$rule (or in his case $\rightarrow E-$rule) is problematic. However, Definition 4.2.1 is not about the proof-theoretic validity of an inference rule but about that of a derivation. We cannot yet claim that $\neg E-$rule is not $\mathbb{P}-$valid.

Prawitz (1974) has developed an idea of logical consequence via the notion of proof-theoretic validity. Unlike the Tarskian notion of logical consequence understood as truth-preservation relation, Prawitz (1974, 1985) has proposed the notion of logical consequence as the preservation of proof-theoretic validity of arguments. A validity of an inference rule is defined in a similar way.

> An *inference rule* may be said to be *valid* when each application of it preserves *validity* of arguments. An introduction rule is then trivially valid ... , which is as it should be, if they are thought of as producing canonical forms of arguments. An elimination rule R is valid depending on whether there exists a justifying operation ϕ such that if $\mathfrak{D}$ is any argument whose last inference is R and whose immediate subarguments are valid with respect to the justifying procedure $\mathfrak{J}$, then $\mathfrak{D}$ is also valid with respect to $\mathfrak{J} \cup \{\phi\}$. If ϕ is independent of the system of canonical arguments for atomic formulas, R may be said to be logically valid. (Prawitz, 1985, p. 165)

He defines an argument as a pair $(\mathfrak{D}, \mathfrak{J})$ of a derivation $\mathfrak{D}$ and a justifying operation $\mathfrak{J}$. '$(\mathfrak{D}, \mathfrak{J})$ is valid' is read as '$\mathfrak{D}$ is valid with respect to $\mathfrak{J}$.' As we restrict our concern on the proof-theoretic validity as a property of (formal) derivations in a natural deduction system, we will consider Prawitz's notion of logical consequence to be the preservation of proof-theoretic validity of derivations. Prawitz's justifying operation is a reduction procedure in our terminology. Instead of the justifying procedure $\mathfrak{J}$, we will use a set $\mathbb{P}$ of reduction procedures and define a proof-theoretic validity of an inference as below.

Definition 4.3.1. Let $\mathbb{P}$ be a set of production rules and $\mathbb{R}$ be a set of reduction procedures. Let $\mathfrak{D}$ be any derivation whose immediate subderivations are $\mathbb{P}-$valid relative to $\mathbb{R}$ and $\mathfrak{F}$ be a last inference rule of $\mathfrak{D}$. $\mathfrak{F}$ is $\mathbb{P}-$*valid relative to* $\mathbb{R}$ iff either (i) $\mathfrak{F}$ is an introduction rule, or (ii) $\mathfrak{F}$ is an elimination rule and $\mathfrak{D}$ is $\mathbb{P}-$valid relative to $\mathbb{R}$. $\mathfrak{F}$ is *logically valid* iff, for every $\mathbb{P}' \supseteq \mathbb{P}$ and $\mathbb{R}' \supseteq \mathbb{R}$, $\mathfrak{F}$ is $\mathbb{P}'-$valid relative to $\mathbb{R}'$.

Let S_N be a natural deduction system for the naive set theory which only contains the rules for $\wedge$, $\rightarrow$, $\neg$, and $\in$. S_N has a set $\mathbb{R}$ of standard reductions for $\wedge$, $\rightarrow$, $\neg$, and $\in$, and its set $\mathbb{P}$ of production rules. Then, S_N has the derivation Σ_{12} (or Σ'_{12}) of Russell's paradox. With respect to Definition 4.3.1, since Σ_{10} and Σ_{11} are $\mathbb{P}-$valid relative to $\mathbb{R}$ but Σ_{12} is not, the last inference rule, i.e. $\neg E-$rule, is not $\mathbb{P}-$valid relative to $\mathbb{R}$.

Tranchini (2016) accepts Prawitz's definition of $\mathbb{P}-$validity of an inference rule as the correctness of the rule and argues that the correctness of the rule is different from the validity of it. He examines a similar phenomenon and says,

> It is not [$\neg E-$rule] to be blamed for not preserving validity. The source of the problem should rather be identified with the presence of [the parameter a defined as $a \in \{x|\neg x \in x\}$]. How can this intuition be spelled out?
>
> ... the availability of reduction procedures usually suffices to warrant the correctness of the elimination rule to which they are associated. It should now be clear that, when the language contains [the parameter a which raises paradoxical derivations], this is no more the case. To repeat, while in standard cases the existence of reduction procedures associated to the rule is enough to show that the rule preserves validity, this is not so in general. (Tranchini, 2016, pp. 505–506)

Though he thinks that $\neg E-$rule is correct, when the parameter a as $a \in \{x|\neg x \in x\}$ is associated, he agrees that $\neg E-$rule is not $\mathbb{P}-$valid relative to $\mathbb{R}$. Prawitz (1965, p. 95) already has considered the similar phenomenon. So, we conclude that in the case of Russell's paradox, $\neg E-$rule fails to preserve validity of derivations. If the application of $\neg E-$rule is restricted when the parameter a as $a \in \{x|\neg x \in x\}$ is involved in the language of the system S_N for the naive set theory, the restriction can block the derivation Σ_{12} of Russell's paradox. From Prawitz's perspectives on $\mathbb{P}-$validity, the restriction can be a reasoning-rejection solution which restricts a particular rule but not a system.

Unfortunately, in this section, we attempt to find a way to prevent a particular rule through *RND* with the assumption that proof-theoretic validity implies normalizability. The restriction of $\neg E-$rule via Definition 4.3.1 is executed by the requirement of a $\mathbb{P}-$valid inference that only $\mathbb{P}-$valid inferences are to be used, but not by *RND*. The requirement of a $\mathbb{P}-$valid inference can be requested independently of *RND*. Therefore, if a reasoning-rejection solution is a constraint on a particular inference rule, *RND* is not a reasoning-rejection solution.

4.4 Should We Consider Only $\bot$ as an Unacceptable Conclusion?

In this section, by examining Petrolo and Pistone's case of a (full) normal derivation of a contradiction, we shall deal with the last question of this chapter, 'should we consider only $\bot$ as an unacceptable conclusion?'

The Third Question (iii): Should we consider only $\bot$ as an unacceptable conclusion?

As we have examined in the last section 4.3, a derivation of $\bot$ formalizing Russell's paradox consists of two normal derivations Σ_{10} and Σ_{11}.

$$\dfrac{\begin{matrix}\Sigma_{10} & \Sigma_{11} \\ \neg a \in a & a \in a\end{matrix}}{\bot}\,\neg E$$

Instead of applying $\neg E-$rule, we can apply $\wedge I-$rule and have

$$\dfrac{\begin{matrix}\Sigma_{10} & \Sigma_{11} \\ \neg a \in a & a \in a\end{matrix}}{\neg a \in a \wedge a \in a}\,\wedge I$$

Since Σ_{10} and Σ_{11} are normal derivations, either the above derivation is in normal form. One of interesting points is that we often call the formula $\neg a \in a \wedge a \in a$ a contradiction. Let us distinguish between a contradiction and an absurdity. For any formula having the form $\neg\varphi \wedge \varphi$ a contradiction. A contradiction is often regarded as an unacceptable conclusion. Then, it seems convincing that the above derivation is paradoxical because it derives an unacceptable conclusion through acceptable reasoning with acceptable premises. *RND* imposes a constraint on the application of non-normalizable derivations. If there is a paradoxical derivation which is a (closed) full normal derivation of an unacceptable conclusion, such as $\neg\varphi \wedge \varphi$, then *RND* cannot be the proof-theoretic solution to the paradoxes. Petrolo and Pistone (2018) have considered the very case.

Under the distinction between a contradiction and an absurdity, Petrolo and Pistone (2018) consider the possibility of a normal derivation of the paradoxes. Unlike Tennant (1982, 2016, 2017), they call a derivation $\mathfrak{D}$ in a given system S a *normal paradox* (N-paradox) iff (i) $\mathfrak{D}$ is closed, (ii) $\mathfrak{D}$ is involved in id est rules, (iii) $\mathfrak{D}$ is in full normal form, and (iv) if a formula φ is the conclusion of $\mathfrak{D}$, then either $\varphi \to \bot$ or $\neg\varphi$ can be derived in S. Similar to Sainsbury's notion of a paradox, one may regard a paradox as a derivation of the unacceptable conclusion from the acceptable premises by the acceptable reasoning. Petrolo and Pistone (2018) regard both a contradiction and an absurdity as unacceptable conclusions and suggest a closed normal derivation of a contradiction from Russell's paradox. We borrow from Prawitz (1965, Appendix B) the natural deduction system S_N for the naive se theory which contains the rules for $\wedge$, $\to$, $\neg$, and $\in$. Then, we have a closed normal derivation of $a \in a \wedge \neg a \in a$.

Proposition 4.4.1. *Let us define a parameter a as $\{x|\neg x \in x\}$. Then, there is a closed normal derivation of $\neg a \in a \wedge a \in a$ in S_N.*

Proof. Two claims verify the result.

Claim 1. there is a closed derivation Σ_{15} of $\neg a \in a \wedge a \in a$.

First, we have a closed derivation Σ_{13} of $\neg a \in a$.

$$\dfrac{\dfrac{\dfrac{\dfrac{[a \in a]^1}{a \in \{x|\neg x \in x\}}\,def}{\neg a \in a}\in E \qquad [a \in a]^1}{\bot}\neg E}{\neg a \in a}\neg I_{,1}$$

With the derivation Σ_{13}, we have a closed derivation Σ_{14} of $a \in a$.

$$\dfrac{\dfrac{\begin{array}{c}\Sigma_{13}\\ \neg a \in a\end{array}}{a \in \{x|\neg x \in x\}}\in I}{a \in a}\,def$$

Then, we have a closed derivation Σ_{15} of $\neg a \in a \wedge a \in a$.

$$\dfrac{\begin{matrix}\Sigma_{13} & \Sigma_{14} \\ \neg a \in a & a \in a\end{matrix}}{\neg a \in a \wedge a \in a} \wedge I$$

Claim 2. Σ_{15} is in normal form.

Since Σ_{15} has no maximum formula, Σ_{15} is in normal form. □

According to Petrolo and Pistone (2018), the derivation Σ_{15} of Proposition 4.4.1 is an N-paradox. As they do, if a contradiction, $\neg a \in a \wedge a \in a$, is an unacceptable conclusion, there is a normal derivation of a contradiction. Then, *RND* fails to put a constraint on paradoxical derivations. That is, it cannot be a proof-theoretic solution to the paradoxes.

Then, should any form of $\neg\varphi \wedge \varphi$ be proof-theoretically unacceptable?

While we use natural deduction, the answer may not be always 'yes.' $\perp$ is a propositional constant for absurdity (or falsity) and is often regarded as the only constant not derived by an introduction rule. One of the main roles of the normalization theorem is to show the consistency of a given system. In accordance with standard practice, we write '$S \vdash \varphi$' to mean that a given system S derives φ and '$S \nvdash \varphi$' means that S does not derive φ. Then S is consistent iff $S \nvdash \perp$; otherwise, inconsistent. Prawitz (1965, Ch. 4) shows that every derivation in an intuitionistic natural deduction system can be converted into a normal derivation. One fundamental corollary of the result is that every derivation in the intuitionistic system can be reduced to one using an introduction rule in the last step.[5] Since there is no introduction rule for $\perp$, from the fundamental corollary, we soon have the consistent result of the intuitionistic system. On the other hand, any form of $\neg\varphi \wedge \varphi$, i.e. a contradiction, can be derived by $\wedge I-$rule if there are derivations of φ and $\neg\varphi$. In the sense of normalization result, it might not be a special case that $\neg\varphi \wedge \varphi$ is derived by $\wedge I-$rule when there are normal derivations of φ and $\neg\varphi$.

Unlike the derivation of $\neg\varphi \wedge \varphi$, any derivation of $\perp$ is considered to be an unacceptable

[5] His weak normalization result is Theorem 1 of Chapter 4 of Prawitz (1965) and the suggested corollary is Theorem 2 of the same chapter.

conclusion. Prawitz's inversion principle is the main principle to develop natural deduction system and the principle is based on Gentzen's idea that the meaning of an operator (or a constant) is determined by an introduction rule and determines an elimination rule. Normalization theorem and its fundamental corollary may support the idea that an introduction rule is the meaning-conferring inference. In this perspective, since it is naturally acceptable that there is no introduction rule for $\perp$, it is unacceptable that there exists a closed (full) normal derivation of $\perp$, whereas a closed (full) normal derivation of $\neg\varphi \wedge \varphi$ from normal derivations of φ and $\neg\varphi$ via $\wedge I-$rule is not so proof-theoretically unacceptable.

The second reason why one may accept $\neg\varphi \wedge \varphi$ as an acceptable conclusion is that in some natural deduction system $\neg\varphi \wedge \varphi$ is not logically equivalent to an absurdity, $\perp$. Petrolo and Pistone (2018, Sec. 4) think that a contradiction, $\neg\varphi \wedge \varphi$ is logically equivalent to $\perp$. As is shown in Proposition 1.3.1 in Chapter 1, S_N has a closed derivation of $\perp$ which generates a non-terminating reduction sequence and is a T-paradox with respect to TCP_E. They say that the structure of the T-paradox and their N-paradox, e.g. Σ_{15} of Proposition 4.4.1, looks *morally* the same.

> ... it might seem that if [the T-paradox] is regarded as a paradox, then the [N-paradox] should be regarded as a paradox as well: first, the structure of [the T-paradox and the N-paradox] looks "morally" the same, second, even if the [N-paradox] does not correspond to a closed derivation of the absurdity, it still corresponds to a closed derivation of a contradiction. (Petrolo and Pistone, 2018, Sec. 4)

However, two structures are definitely different. Especially, the main feature of the T-paradox is the non-terminating reduction sequence but their N-paradox does not produce the feature. Also, it is a nonsensical claim that a non-normalizable derivation and a normal derivation are structurally the same.

At last, their argument presumes that a contradiction is logically equivalent to an absurdity. With the rules for $\wedge$ and $\neg$, it is easily seen that every contradiction implies an absurdity but not vice versa. To derive the equivalence between them, there must be a rule for *Ex Falso Quodlibet* (EFQ) which means in this case that an absurdity (or falsity) implies every formula.

$$\frac{\bot}{\varphi}\,EFQ$$

$(\varphi \wedge \neg\varphi) \leftrightarrow \bot$ is readily derived by $\wedge-$, $\neg-$, and $EFQ-$rules.

$$\dfrac{\dfrac{\dfrac{[\varphi\wedge\neg\varphi]^1 \quad \dfrac{[\neg\varphi]^3 \quad \dfrac{[\varphi\wedge\neg\varphi]^1 \quad [\varphi]^2}{\varphi}\wedge E_{,2} \quad [\bot]^4}{\bot}\neg E_{,4}}{\bot}\wedge E_{,3}}{(\varphi\wedge\neg\varphi)\rightarrow\bot}\rightarrow I_{,1} \qquad \dfrac{\dfrac{[\bot]^5}{\varphi\wedge\neg\varphi}EFQ}{\bot\rightarrow(\varphi\wedge\neg\varphi)}\rightarrow I_{,5}}{\dfrac{((\varphi\wedge\neg\varphi)\rightarrow\bot)\wedge(\bot\rightarrow(\varphi\wedge\neg\varphi))}{(\varphi\wedge\neg\varphi)\leftrightarrow\bot}\,def}\wedge I$$

However, the equivalence holds only if $EGQ-$rule is applied. Their examination of Russell's paradox, e.g. Proposition 4.4.1, does not apply $EFQ-$rule. On formalizing Russell's paradox, it is not necessary that a contradiction is logically equivalent to an absurdity. Therefore, it is not to say that $\neg\varphi \wedge \varphi$ must be proof-theoretically unacceptable conclusion.

We do not claim that $\neg\varphi \wedge \varphi$ has to be a proof-theoretically acceptable conclusion. It may rely on our choice of the definition of inconsistency. We may allow the definition of consistency that, for any φ, S is consistent iff $S \nvdash \neg\varphi \wedge \varphi$; otherwise, inconsistent. If any formula which renders a system inconsistent were proof-theoretically unacceptable, $\neg\varphi \wedge \varphi$ would be an unaccpetable conclusion. However, the investigation of the proof-theoretic criterion for paradoxicality is directly related to the proof-theoretic solution to the paradoxes, such as our plausible solution *RND*. It is unclear how Petrolo and Pistone's criterion of N-paradox is relative to the proof-theoretic solution. For exploring the proof-theoretic structure of paradoxicality, their N-paradox overlooks the fact that the non-terminating reduction sequences are the key features of the structure. Tennant (2016) has allegedly described the feature that 'these are the proof-theorist's explication of the *vicious circularity* involved in paradoxes.' The vicious circularity is often considered to be the primary characteristic of the self-referential paradoxes. It should not be ignored when investigating the proof-theoretic structure of the paradoxes. Petrolo and Pistone's criterion of N-paradox loses this point.

Nevertheless, as Petrolo and Pistone (2018) claim, any formula with the form $\neg\varphi \wedge \varphi$ can be a proof-theoretically acceptable conclusion. If it is, *RND* fails to be a proof-theoretic solution to the paradoxes. In order for evaluating whether *RND* can be a legitimate proof-theoretic solution to the paradoxes, the three questions in this chapter still need to be resolved.

4.5 Summary

In this chapter, under the assumption that the non-terminating reduction sequence is the proof-theoretic feature of the paradoxes, we have dealt with three questions: (i) 'Which paradoxes are genuine paradoxes', (ii) 'Why should we accept only a normalizable derivation?', and (iii) 'Should we consider only $\perp$ as an unacceptable conclusion?' In order for *RND* to be a proof-theoretic solution to the paradoxes, at least three questions must be explicated.

Section 4.1 deals with the first question and argues that Tennant has no clear ground for genuine paradoxes. Although Tennant believes that the Liar paradox is a genuine paradox but Russell's paradox is not, the similar argument of him suggests the opposite results that Russell's paradox is a genuine one but the Liar is not.

Since he ignores the fact that a non-terminating reduction sequence is relative to our choice of reduction procedure, we have shown in Proposition 4.1.2 that his derivation of Russell's paradox generates a looping reduction sequence by applying the Ekman-type reduction $\trianglerighteq_{GEF}$. There are two points of Tennant's argument. First, his derivation of Russell's paradox does not satisfy TCP_L (or TCP_E). Second, as Proposition 4.1.1 shows, there is a closed full normal derivation of $\neg\exists y(y = \{x|\neg x \in x\})$ which states the rejection of the formula that there is a set a such that $a = \{x|\neg x \in x\}$. Since $a = \{x|\neg x \in x\}$ is often considered to be the main reason that generates Russell's paradox, he believes that Proposition

4.1.1 can be the ground for claiming that Russell's paradox is not a genuine paradox. In a similar way, we have shown in Proposition 4.1.3 that there is a full normal derivation of $\neg(\Phi \leftrightarrow \neg T(\ulcorner \Phi \urcorner))$. If the full normal derivation of $\neg\exists y(\{x|\neg x \in x\})$ supports the view that Russell's paradox is not genuine, either Proposition 4.1.3 supports the view that the Liar is not. These results establish that Tennant has no clear ground for genuine paradoxes. If *TCP* and *RND* are about genuine paradoxes, it should be explained which paradoxes are genuine paradoxes.

For the second question, Section 4.2 considers a possible explanation that when proof-theoretic validity implies normalizability, *RND* can be a proof-theoretic solution to the paradoxes. Since a paradoxical derivation, such as T-paradox, initiates a non-terminating reduction sequence and is not normalizable, it is not proof-theoretically valid. Hence, *RND* can block the proof-theoretically invalid derivation and so it can be a solution to the paradoxes. To support this view, we have suggested Theorem 4.2.2 and 4.2.4 that in a particular system, proof-theoretic validity implies (strong) normalizability. However, it must be shown that the result is able to be extended to a general case.

If *RND* is a plausible proof-theoretic solution to the paradoxes, it may be regarded as a reasoning-rejection solution. Section 4.3 argues that *RND* is a stronger restriction than other reasoning-rejection solutions due to the fact that it does not restrict a specific inference rule but restricts every derivation in an intended system.

With regard to the third question, Section 4.4 investigates a posibility that there exists a normal derivation of an unacceptable conclusion and argues that it a contradiction, $\neg\varphi \wedge \varphi$, is regarded as a proof-theoretically unacceptable conclusion, then *RND* cannot be a *general* solution to the paradoxes.

Chapter 5

Conclusion

In this dissertation, we have investigated a proof-theoretic criterion for and solution to the paradoxes. After having preliminary notations and natural deduction rules, Chapter 1 introduces the early version of Tennant's criterion for paradoxicality *TCP* and the requirement of a normal derivation *RND*. As a doctor treats the disease in accordance with her diagnosis, a logician solves the paradoxes on the basis of her characterization of the paradoxes. When *TCP* is regarded as a proof-theoretic criterion for paradoxicality, *RND* can be a possible proof-theoretic solution to the paradoxes.

The Early Version of Tennant's Criterion for Paradoxicality(TCP_E): Let $\mathfrak{D}$ be any derivation of a given natural deduction system *S*. $\mathfrak{D}$ is a *T-paradox* if and only if

(i) $\mathfrak{D}$ is a (closed or open) derivation of $\bot$,

(ii) *id est* inferences (or rules) are used in $\mathfrak{D}$,

(iii) a reduction procedure of $\mathfrak{D}$ generates a non-terminating reduction sequence, such as a reduction loop.

The Requirement of a (Full) Normal Derivation(*RND*): For any derivation $\mathfrak{D}$ in natural deduction, $\mathfrak{D}$ is acceptable only if $\mathfrak{D}$ is (in principle) convertible into a normal derivation.

There are two types of counterexamples to *TCP*. The one generates the problem of undergeneration in the sense that *TCP* makes a paradoxical derivation non-paradoxical. The other counterexample causes the overgeneration problem that *TCP* includes non-paradoxical derivations in the realm of paradoxical derivations. Chapter 2 and 3 deal with the problems of under- and overgeneration.

In Chapter 2, we have focuses on Rogerson-type counterexample which employs the rule for *Classical Reductio*, such as *CR*−rule, and does not generate a non-terminating reduction sequence. Rogerson-type counterexamples raise the problem of undergeneration. In order to solve the undergeneration problem, Tennant (2015) thinks that the application of *CR*−rule has a defect that it conceals the primary feature of the paradoxes, i.e. a non-terminating reduction sequence. However, as we have argued in Section 2.2, *CR*−rule is not the only culprit of the defect. We have seen in 2.3 that there exist cases which do not use *CR*−rule but raise the problem of undergeneration. Furthermore, we have suggested examples employing *CR*−rule which generate a non-terminating reduction sequence. From our observations in Section 2.3 and 2.4, our diagnosis says that a non-terminating reduction sequence does not occur if a derivation in question includes (i) a major premise which has no reduction process to eliminate it or (ii) a formula having a principal constant which has no reduction procedure to get rid of it. We have suggested an additional condition to TCP_E that a derivation formalizing a genuine paradox only uses harmonious rules. With the additional condition, any counterexample in Section 2.2 and 2.3 which causes the problem of undergeneration can be singled out.

In Chapter 3, we have focused on the problem of overgeneration and examined (G)Ekman's paradox. Tennant (2016) claims that the overgeneration problem raised by Ekman's paradox is solved by using generalized elimination rules. However, we have argued that even when we use generalized elimination rules, there is a (G)Ekman's paradox which shows that TCP_L overgenerates. He overlooks the fact that an application of an auxiliary reduction sometimes raises a non-terminating reduction sequence. So we have explored methods to evaluate a proper reduction process, such as Triviality and Translation tests. An assessment of a proper reduction via Triviality test is relative to our choice of rules and a system.

Since the properness of Ekman reduction through Triviality test can be dependent on our choice of natural deduction systems, Triviality test does not block every Ekman-type reduction process. For a system-independent method to evaluate a proper reduction, we have proposed Translation test. While Ekman-type reductions in natural deduction can be translated to one in sequent calculus, the test diagnoses that they are detour-making processes and so is not proper. Eventually, from our discussions in Chapter 2 and 3, we have the following criterion for paradoxicality.

The Revised Version of Proof-Theoretic Criterion for Paradoxicality: Let S be a natural deduction system relative to a set $\mathbb{R}$ of reduction procedures. $\mathfrak{D}$ be any derivation in S. $\mathfrak{D}$ is a *T-paradox* if and only if

(i) $\mathfrak{D}$ is a (closed or open) derivation of $\perp$,

(ii) *id est* inferences (or rules) are used in $\mathfrak{D}$,

(iii) a reduction procedure of $\mathfrak{D}$ generates a non-terminating reduction sequence, such as a reduction loop,

(iv) any reduction procedure in $\mathbb{R}$ is proper,

(v) only harmonious rules are applied in $\mathfrak{D}$.

Chapter 4 centers on the question, 'Can the requirement of a normal derivation be a proof-theoretic solution to the paradoxes?' There are three questions which should be explicated in order for *RND* to be a proof-theoretic solution to the paradoxes: (i) which paradox is a genuine paradox and which formalization is legitimate for the genuine paradox, (ii) why the only normalizable derivation is acceptable, and (iii) why the only propositional constant $\perp$ for absurdity is a proof-theoretically unacceptable conclusion.

For the first question, we have claimed that Tennant has no clear ground for genuine paradoxes. We have attempted to find an answer to the second question that if proof-theoretic validity *generally* implies normalizability, *RND* can be a plausible proof-theoretic solution to the paradoxes. With regard to the third question, we have investigated a closed (full) normal derivation of $\neg\varphi \wedge \varphi$. If any formula having the form $\neg\varphi \wedge \varphi$ is considered to be a proof-theoretically unacceptable conclusion, *RND* cannot be a *general* solution to the paradoxes.

When exploring proof-theoretic criterion for and solution to the paradoxes, there are two important questions that are not discussed in this paper. Why should the non-terminating reduction sequence be the main feature of the paradoxes? What is a legitimate formalization of genuine paradoxes? We have proposed four formalizations of Russell's paradox, four formalizations of Curry's paradox, nine formalizations about the Liar paradox, four formalizations of Ekman's paradox, and one formalization of Crabbé's case. Table 5.1, 5.2 5.3 summarize the characteristics of derivations of the Liar, Curry's, and Russell's paradox.

Proposition	Derivation (Conclusion)	Loop	$CR-$rule	Aux. Reduction
2.2.1	Non-normalizable ($\bot$)	Occurred	·	Used
2.2.2	Normal ($\bot$)	·	Used	Used
2.3.1	Full normal ($\bot$)	·	·	·
2.4.3	Non-normalizable ($\bot$)	Occurred	Used	Used
2.4.4	Non-normalizable ($\bot$)	Occurred	·	·
4.1.3	Full normal ($\neg(\Phi \leftrightarrow \neg T(\ulcorner\Phi\urcorner))$)	·	·	·
2.A.1	Non-fully normalizable ($\bot$)	Occurred	·	·
2.A.2	Non-full normal ($\bot$)	·	Used	·
2.A.3	Non-fully normalizable ($\bot$)	Occurred	Used	·

Table 5.1: Derivations Formalizing the Liar Paradox

Proposition	Derivation (Conclusion)	Loop	$CR-$rule	Aux. Reduction
1.3.1	Non-normalizable ($\bot$)	Occurred	·	·
4.1.1	Full normal ($\neg\exists y(y = \{x\neg x \in x\})$)	·	·	·
4.1.2	Non-fully normalizable ($\bot$)	Occurred	·	Used
4.4.1	Normal ($\neg a \in a \wedge a \in a$)	·	·	·

Table 5.2: Derivations Formalizing Russell's Paradox

Proposition	Derivation (Conclusion)	Loop	$CR-$rule	Aux. Reduction
2.0.1	Non-normal ($\perp$)	·	Used	·
pp. 60–62	Normal ($\perp$)	·	·	·
2.4.1	Non-normalizable ($\perp$)	Occurred	Used	Used
2.4.2	Non-normalizable ($\perp$)	Occurred	Used	Used

Table 5.3: Derivations Formalizing Curry's Paradox

Three tables do not show which feature yields a looping reduction sequence since the reduction loop occurs independently of the application of $CR-rule$ and auxiliary reduction procedures. An application of auxiliary reductions sometimes generates a non-terminating reduction sequence, but it is not always the case. As the self-referential paradoxes has a characteristic of vicious circularity, some forms of formalization of the paradoxes in natural deduction may have such characteristics and causes the non-terminating reduction sequence.

Moreover, even though Tennant believes that the Liar paradox is a genuine one but Russell's is not, each can be both genuine and ingenuine. Proposition 2.A.2 and 1.3.1 make them genuine paradoxes. Proposition 4.1.1 and 4.1.3 make them ingenuine. No one, including Tennant, explains which way of formalization is legitimate for genuine paradoxes. In order to have a fruitful investigation of the proof-theoretic criterion for and solution to the paradoxes, including three things in Chapter 4, these things should be explained. A thorough investigation of these issues must be left for another occasion.

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역설에 대한 증명론적 접근: 프라위츠-테넌트 분석에서 과소생성 그리고 과잉생성 문제를 중심으로

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지도교수: 정인교

아픈 환자를 발견했을 때, 의사는 환자의 병이 무엇인지를 진단하고 그 진단에 따라 처방을 한다. 유사한 방향에서 어떤 종류의 논변이 문제가 있다고 여겨질 때, 그 문제를 어떻게 규정하느냐에 따라서 그에 대한 해결책이 달라질 수 있다. 역설의 해결책도 ‘역설’을 어떻게 규정하느냐에 따라 달라질 수 있다.

Richard M. Sainsbury (2009)는 ‘역설’을 명백히 수용할 수 있는 전제와 추론으로 부터 도출된 명백히 수용할 수 없는 결론이라고 정의한다. 명백히 수용할 수 있는 것들이 명백히 수용할 수 없는 것을 도출하기에 이를 ’역설’이라고 부른 것이다. 이러한 역설에 대한 정의를 따를 때, 우리는 세 가지 방향의 역설에 대한 해결책을 지닐 수 있다. 첫째는 전제가 명백히 수용가능한 것이 아니라고 주장하는 것이고 둘째는 추론 규칙이 명백히 수용가능하지 않다고 주장하는 것이다. 마지막으로 결론이 명백히 수용가능하지 않음을 부정함으로써 역설에 대한 해결책이 제시될 수 있다. 첫 번째 해결책을 우리는 ‘전제-부정’의 해결책이라고 부를 것이며 두 번째는 ‘추론-부정’, 마지막으로 세 번째는 ‘결론-수용’의 해결책이라고 부를 것이다. 물론, 역설에 대한 규정이나 전통적인 해결책이 Sainsbury의 정의에 완전히 부합한다고는 할 수 없을 것이다. 하지만 그의 비형식적 정의는 ‘역설’과 역설의 해결책을 가장 쉽게 이해할 수 있는 방향이기도 하다. 그렇기에 이 글에서 우리는 ‘역설’에 대한 전통적인 이해가 Sainsbury의 정의에 상당히 부합함을 전제할 것이다.

이 논문은 역설에 대한 증명론적 기준과 해결책에 대한 것이다. 역설에 대한 증명론적 해결책은 ‘역설’을 증명론에서 어떻게 규정하는가에 의존해 있다고 볼 수 있다. 그런 점에서 역설에 대한 증명론적 기준과 해결책은 Sainsbury의 정의와는 차이가 있을 수 있다.

먼저 우리는 19세기 후반에서 20세기 초반 수학기초론을 중심으로 논의되었던 집합론적/의미론적 역설로 주로 다룰 것이다. 다시 말해, 주로 '자기지시적 역설'(self-referential paradoxes)이라 불리는 역설들을 대상으로 할 것이다. 이 글은 모두 5개의 장으로 구성되어 있으며 1장에서는 역설에 대한 전통적인 입장을 집합론적 역설과 의미론적 역설의 경우로 나누어 요약할 것이다. 그리고 이러한 전통적인 입장은 앞서 언급한 역설에 대한 세 가지 해결책, 다시 말해, 전제-부정, 추론-부정, 그리고 결론-긍정의 방식으로 고려될 수 있음을 언급할 것이다. 이러한 전통적인 입장은 모형론적인 방식을 차용하고 구성적이지 않은 (고전적) 규칙을 사용하는 경우가 있기 때문에 구성주의자들의 입장에서 수용하기 어려운 부분이 있다. 역설에 대한 증명론적 탐구는 구성주의의 정신에 부합할 뿐만 아니라 집합론적/의미론적 역설을 한꺼번에 해결하는 단일한 해결책을 제시할 가능성을 열어준다는 측면에서 연구의 가치가 있을 것이다.

1장의 말미인 1.3절에서는 Dag Prawitz와 Neil Tennant의 역설에 대한 증명론적 분석 소개할 것이다. Prawitz (1965, Appendix B)는 자연연역에서 집합론의 역설을 탐구하며 (모든) 도출이 정형화되어야 한다는 요구가 역설을 막을 수 있는지를 고려했다. 정형도출의 요구는 다음과 같이 요약된다.

정형 도출의 요구(the Requirement of a Normal Derivation, ***RND*****):** 자연연역에서의 임의의 도출 $\mathfrak{D}$ 에 대해, $\mathfrak{D}$ 가 수용가능하다는 것은 오직 $\mathfrak{D}$ 가 (원리상으로라도) 정형도출로 전환가능할 경우이다.

그는 집합론의 역설을 자연연역에서 형식화하며 정형 도출만을 사용하는 것이 역설적 도출로 부터 모순이 도출되는 것을 막을 수 있는가를 탐구했는데 그에 따르면 이러한 도출을 정형도출로 환원하려고 하면 끊이지 않는 환원열(the non-terminating reduction sequence)에 빠지게 된다고 말한다. Tennant (1982, 1995, 2016, 2017)는 이러한 입장을 받아들여 다음과 같은 역설에 대한 증명론적 기준(Tennant's Proof-Theoretic Criterion for Paradoxicality, *TCP*)을 제시한다.

테넌트의 역설에 대한 증명론적 기준(***TCP*****):** $\mathfrak{D}$ 를 자연연역체계에서의 임의의 도출이라고 하자. $\mathfrak{D}$ 가 *T-역설*이라는 것은 만약에 그리고 오직 그 경우에

(i) $\mathfrak{D}$ 가 모순에 대한 (닫힌 혹은 열린) 도출이다,

(ii) $\mathfrak{D}$는 바꿔말하기(id est) 추론(혹은 규칙)을 사용한다,

(iii) $\mathfrak{D}$의 환원절차는 끊이지 않은 환원열을 일으킨다.

Tennant (1982)가 처음 *TCP*를 제안할 때, 그는 이 기준이 모든 진정한 역설(genuine paradoxes)에 적용된다고 여겼으며 이를 일종의 가설로 고려했다. 다시 말해, 임의의 도출 $\mathfrak{D}$에 대해, $\mathfrak{D}$가 진정한 역설을 형식화한다는 것은 만약에 그리고 오직 그 경우에 $\mathfrak{D}$가 T-역설이다라는 관계가 성립한다고 여긴 것이다. 만약 가설이 참이라면 진정한 역설은 T-역설일 것이고 T-역설은 정형식(normal form)으로 환원될 수 없기 때문에 정형 도출의 요구가 역설을 막을 수 있을 것이다.

이 글의 목적은 *RND*가 증명론적 해결책이 될 수 있는지 그리고 이러한 입장에 기초하여 제시된 *TCP*가 (진정한) 역설의 기준으로서 적절한지에 대해 탐구하는 것이다. *TCP*는 두 가지 방향에서 반례가 제시될 수 있는데 하나는 과잉생성(overgeneration)의 문제를 야기하는 것이며 다른 하나는 과소생성(undergeneration)의 문제를 야기하는 것이다. 과잉생성의 문제는 역설이 아닌 것 같은 도출을 *TCP*가 T-역설로 만드는 것이다. 반면 과소생성의 문제는 (진정한) 역설로 보이는 도출을 *TCP*가 T-역설이 아닌 것으로 만드는 것이다. 2장과 3장은 *TCP*가 진정한 역설에 대한 기준이 될 수 있는가를 다루며 2장에서는 과소생성 문제를 그리고 3장에서는 과잉생성 문제를 다룬다. 과소생성 문제를 다루는 2장에서는 과소생성 문제를 야기하는 고전적 귀류법(Classical Reduction)이 사용된 예시에 대한 Tennant의 진단이 옳지 않음을 논하고 과소생성 문제의 해결책을 제시한다. 또한 3장에서도 과잉생성 문제를 야기하는 예시에 대한 Tennant의 진단이 옳지 않음을 논하고 이에 대한 해결책 역시 제시할 것이다. 마지막으로 4장은 *RND*가 역설에 대한 증명론적 해결책이 되기 위해서 설명되어야 할 바에 대해 다룬다. 그래서 2–4장의 내용은 다음과 같이 요약될 수 있다.

2장 요약. Tennant (2015)는 고전적 귀류법에 관한 규칙인 *CR*−규칙이 사용된 거짓말쟁이 역설의 도출을 고려하며 *CR*−규칙이 사용될 경우, 끊이지 않는 환원열인 '환원고리'(reduction loop)가 생성이 되지 않으며 $\perp$에 관한 정형도출(normal derivation)이 제시되는 것으로 보인다고 말한다. 그리고 자연연역에서 역설적 도출을 형식화함에 있어 *CR*−규칙이 꼭 사용될 필요는 없음을 들어 진정한 역설은 엄밀히 고전적일 수 없다는 방법론적 가설(the methodological conjecture)을 제시한다. 테넌트는

아마도 과소생성 문제를 심각하게 고려하지 않은 것 같으며 그의 입장에서 유추할 수 있는 해결책은 $CR-$ 규칙을 사용하지 않는 것이다. **2장에서는 $CR-$ 규칙을 사용하지 않는 것이 과소생성의 문제를 해결할 수 없음을 주장하고 환원고리가 어떠한 경우에 멈추게 되는지를 진단한후 가능한 해결책을 제시할 것이다.** 먼저 우리는 환원고리의 생성유무가 $CR-$ 규칙의 사용에 독립적임을 보이기 위해 $CR-$ 규칙이 사용되었음에도 환원고리가 생성되는 거짓말쟁이 역설의 예시와 $CR-$ 규칙을 사용하지 않음에도 환원 고리가 생성되지 않는 예시를 제시할 것이다. 그리고 환원고리가 생성되지 않는 이유를 진단한 후 **진정한 역설을 형식화하는 도출은 오직 조화로운 규칙만을 사용해야 한다**는 조건을 역설에 대한 증명론적 기준에 추가할 경우 과소생성 문제가 해결됨을 논할 것이다.

3장 요약. Tennant는 에크만 사례(Ekman's case)를 그의 역설에 대한 초기 기준의 반례로 여기고 이에 대한 해결책으로 역설적 도출이 일반화된 제거규칙(generalized elimination rule)을 사용해야 할 것을 요구한다. 다시 말해, 에크만 사례가 야기하는 과잉생성 문제의 해결책으로 일반화된 제거규칙의 사용을 요구한 것이다. 하지만 **3장에서는 일반화된 제거규칙의 사용이 에크만 사례가 야기하는 과잉생성 문제를 해결하지는 못하며 그의 기준이 진정한 역설에 대한 기준이 되기 위해서는 최소한 적합한 환원절차에 대한 기준이 추가되어야 함을 주장할 것이다.** 적절한 환원 절차에 대한 기준으로 우리는 Schroeder-Heister와 Tranchini의 사소성 테스트(Triviality test)를 살펴 볼 것이고 이것이 체계에 상대적이라 모든 종류의 에크만-유형 환원 절차를 막지는 못함을 주장할 것이다. 그리고 대안으로 번역 테스트(Translation test)를 제안할 것이다.

4장 요약. 4장에서는 정형도출의 요구(RND)가 역설에 대한 증명론적 해결책이 되기 위해서는 (i) '어떠한 역설이 진정한 역설인가?', (ii) '왜 정형화가능한 도출만을 받아들여야 하는가?', 그리고 (iii) '왜 $\perp$만이 수용가능하지 않은 결론이어야 하는가?'에 대한 대답이 제시되어야 할 것임을 논할 것이다. 첫 번째 질문 (i)과 관련하여 우리는 Tennant가 진정한 역설에 대한 분명한 기준을 지니지 않음을 논할 것이다. 또한 (ii)와 관련하여 만약 증명론적 타당성 개념이 정형화가능성을 함축한다면 RND가 증명론적 해결책이 될 수 있겠지만 이 함축 관계가 보편적으로 성립해야 함을 논할 것이다. 그리고 RND가 증명론적 해결책이 된다면 특정 추론규칙을

제약하는 일반적인 추론-부정 해결책과는 다른 종류의 해결책이 될 것임을 논할 것이다. 마지막으로 (iii)과 관련하여 $\bot$ 이외에도 $\neg\varphi\wedge\varphi$를 결론으로 가지는 정형 도출을 고려하고 이 경우에는 *RND*가 이 도출을 막을 수 없기 때문에 증명론적으로 받아들일 수 없는 결론이 무엇인지에 대한 설명이 필요함을 논할 것이다.

보다 세부적으로 2장에서는 Tennant의 기준과 관련해 Rogerson (2006)의 반례를 소개할 것이다. Rogerson의 예시는 Tennant가 진정한 역설로 고려하는 커리의 역설(Curry paradox)을 형식화 함에도 고전적 규칙을 사용함으로써 끊임없는 환원열을 야기하지 않는다. 다시 말해, 진정한 역설임에도 Tennant의 기준에 따르면 T-역설이 되지 않는 과소생성의 문제가 발생하는 것이다. 2장 1절에서는 Tennant의 기준(Tennant's Criterion for Paradoxicality, *TCP*)에 대한 반례로 일반화된 제거규칙을 사용하는 사례도 있을 것이기 때문에 예비적으로 이에 대한 규칙을 먼저 소개할 것이다. 그리고 논의를 위해 도입규칙과 제거규칙 간의 조화로운 관계(harmony relation)가 어떠한 관계인지를 소개할 것이다.

2장 2절에서는 고전적 귀류법(classical reductio)에 관한 규칙인 $CR-$규칙을 사용한 과소생성의 문제를 야기하는 예시에 대한 Tennant의 대응에 대해 소개하고 그의 진단이 옳지 않음을 논할 것이다. 아마도 그는 $CR-$규칙이 모순($\bot$)에 관한 정형도출을 양산하는 것으로 보일뿐만 아니라 역설적 도출이 지닌 주요한 특징을 감춘다고 전제하는 듯하다. 그리고 그는 이를 '고전적 문제'(the classical rub)라고 소개한다. 그리고 이러한 도출을 피하는 방향에서 '진정한 역설은 고전적일 수 없다'는 방법론적인 가설(Methodolodical Conjecture)을 제시한다. 방법론적 가설이 옳은가에 대한 문제를 차치하더라도 Tennant가 고려하는 과소생성 문제에 대한 해결책은 고전적 추론규칙을 사용하지 말아야 한다는 것으로 이해될 수 있다.

2장 3절에서는 고전적 귀류법의 규칙인 $CR-$규칙이 사용되지 않더라도 과소생성 문제가 발생함을 보이고 고전적 추론규칙을 사용하지 않는 것이 과소생성 문제의 해결책이 될 수 없음을 논할 것이다. 2장 4절에서는 과소생성 문제의 해결책을 제시하기 위해 무엇이 환원 고리를 멈추는가에 대한 진단을 시작할 것이다. 그리고 끊이지 않는 무한한 환원의 연속은 어떠한 환원 절차를 차용하느냐에 의존적임을 보이고 과소생성 문제를 야기하는 예시들이 두 가지 특징이 있음을 살펴 볼 것이다. 말하자면, 끊임 없는 환원의 연속을 멈추는 예시들은 주전제를 제거하는 환원 절차가 없거나 주요 상항(a principal constant)을 포함하는 식을 제거하는 환원 절차가 없을 경우들임을 살펴 볼 것이다. 그리고 이러한 사례들을 통해 다음과 같은 가능한 진단을 제시할 것이다.

가능한 진단: 진정한 역설을 형식화하는 도출이 끊임 없는 환원의 연속을 야기한다는 것은 오직 (i) 그 도출에서의 각 주전제를 제거하는 환원 절차가 존재할 경우 혹은 (ii) 주요한 상항을 포함하는 각 식을 제거하는 환원 절차가 존재할 경우이다.

그리고 이러한 진단을 통해 다음과 같은 조건을 *TCP*에 추가할 경우 2장에서 살펴 보았던 과소생성 문제는 일어나지 않을 것임을 주장할 것이다.

추가 조건: 진정한 역설을 형식화하는 도출은 오직 조화로운 규칙만을 사용한다.

3장에서는 Tennant의 기준에 대한 과잉생성에 대해 다룰 것이다. 특히 Schroeder-Heister and Tranchini (2017)가 제시한 에크만의 역설(Ekman's paradox)를 소개할 것이다. 에크만의 역설은 옳지 않은 환원절차를 포함하기에 진정한 역설로 고려되어서는 안되나 Tennant의 기준에 따르면 T-역설이 되기 때문에 과잉생성의 문제를 야기하는 것으로 여겨진다. 먼저 우리는 Tennant (2016)의 에크만 역설에 대한 대응을 살펴 볼 것이다. 그는 도출의 모든 제거규칙을 일반화된 제거규칙(generalized elimination rule)로 제시할 경우 과잉생성의 문제는 일어나지 않을 것이라고 주장한다. 하지만 3장 2절에서는 Tennant의 대응이 적절하지 않으며 일반화된 제거규칙만을 사용하더라도 과잉생성의 문제는 여전히 일어날 것임을 논할 것이다. 그리고 Tennant의 기준에는 적절한 환원절차가 무엇인가에 대한 논의가 필요함을 주장할 것이다. 3장 3절에서는 적절한 환원절차에 대한 Schroeder-Heister and Tranchini (2017)의 사소성 테스트(Triviality Test)를 소개하고 그들의 사소성 테스트는 체계에 의존적이기 때문에 적절한 환원 절차 자체를 테스트하기에는 무리가 있음을 논할 것이다. 이어서 3장 4절에는 필자의 번역 테스트(Translation Test)를 제시할 것이다. 번역 테스트에 따르면 에크만 유형의 환원절차는 우회를 만들어내는 환원절차이기에 적절하지 않으며 번역 테스트는 사소성 테스트에 비해서 환원절차 그 자체를 테스트할 수 있는 장점이 있음을 논할 것이다.

4장에서는 정형도출의 요구(*RND*)가 역설에 대한 해결책이 될 수 있는지에 대해 살펴 볼 것이다. 이를 위해서 우리는 세 가지 물음을 고려할 것이다. (i) '어떠한 역설이 진정한 역설인가?', (ii) '왜 정형화가능한 도출만을 받아들여야 하는가?', 그리고 (iii) '왜 ⊥만이

수용가능하지 않은 결론이어야 하는가?' *RND*가 진정한 역설에 대한 해결책이라면 무엇이 진정한 역설인지에 대한 설명이 필요하다. 그리고 *RND*가 역설적 도출을 막는다고 하더라도 왜 정형화가능한 도출만을 받아들여야 하는지가 설명이 되지 않는다면 *RND*는 역설에 대한 증명론적 해결책으로 채택될 수 없을 것이다. 마지막으로 $\perp$ 이외의 문장이 받아들일 수 없는 결론이 될 수 있고 그러한 결론을 지닌 역설적 도출이 정형 도출이라면 *RND*는 그러한 역설적 도출을 막지 못 하니 역설에 대한 해결책이 될 수 없을 것이다.

4.1절에서는 첫 번째 물음에 대한 논의를 진행한다. Tennant는 러셀의 역설은 진정한 역설이 아니나 거짓말쟁이 역설은 진정한 역설이라고 주장한다. 먼저 우리는 러셀의 역설이 진정한 역설이 아니라는 그의 논변을 살펴 보고 같은 논리로 거짓말 쟁이 역설도 진정한 역설이 아니게 됨을 논할 것이다. 그리고 이러한 논의를 통해 Tennant는 진정한 역설에 대한 명확한 기준을 지니지는 못 했음을 논할 것이다.

4.2 절에서는 두 번째 질문인 정형도출만을 사용해야 하는 이유에 대해 탐구할 것이다. 가능성있는 한 가지는 증명론적 타당성이 정형화가능성을 함축한다는 것이다. 다시 말해, 역설적 도출이 정형화가능하지 않다면 이는 증명론적으로 타당하지 않은 도출이 되기 때문에 이를 배제할 수 있고 *RND*는 역설에 대한 해결책이 될 수 있다. 4.2절에서는 특정 체계에서 증명론적 타당성이 정형화가능성을 함축함을 보일 것이다. 하지만 *RND*가 증명론적 해결책이 되기 위해서는 위의 결과가 일반적인 사례에까지 확장되어야 할 것이다. 4.3절에서는 정형도출의 요구가 통상적으로 말하는 역설에 대한 추론-부정의 해결책과는 다름을 논할 것이다. 그리고 만약 추론-부정의 해결책이 특정 추론규칙을 제약함으로써 역설을 해결하는 것이라면 *RND*는 모든 도출을 제약하는 것이기 때문에 추론-부정의 해결책이 아님을 논할 것이다. 4.4절에서는 Petrolo and Pistone (2018)가 제시한 $\neg\varphi \wedge \varphi$에 대한 정형도출을 소개하고 $\perp$ 뿐만 아니라 $\neg\varphi \wedge \varphi$ 형식의 문장도 받아들일 수 없는 결론(unacceptable conclusion)으로 여길 경우 *RND*는 역설에 대한 증명론적 해결책이 될 수 없음을 논할 것이다. 결론적으로 증명론적 타당성이 보편적으로 정형화가능성을 함축하며 $\neg\varphi \wedge \varphi$와 같은 식이 증명론적으로 받아들일 수 없는 결론이 되지 않을 때, *RND*는 역설에 대한 증명론적 해결책이 될 수 있을 것이다.

감사의글

2006년 9월 군대 말년휴가를 나와 정인교 교수님의 *Computability and Logic* 수업을 수강하며 괴델의 불완전성 정리를 배웠던 순간이 기억에 남습니다. 지금은 사라진 중앙광장 3열람실에서 산술적 진술들을 코딩을 통해 나열하고 형식적 구조물을 쌓는 작업에 푹 빠져 있었습니다. 완성된 구조물을 볼 수 있다는 기대감이 고조될 쯤, 고정점 정리(Fixpoint theorem)를 맞이하게 됐고 제 머릿속에 그려졌던 무한한 산술적 진리의 구조물은 위에서부터 아래로 도미노처럼 무너져 내렸습니다. 무너지는 구조물을 상상하며 허탈함도 느꼈지만 왠지 모르게 아름답다는 생각도 했었습니다. 정교하게 짜여진 도미노가 무너지는 모습이 아름답듯 제 머릿속에는 비슷한 이미지를 그렸던 것 같습니다. 어찌되었건 당시의 저는 흥분 보다는 약간의 공포감을 느꼈었고 논리학이나 수리논리학은 제 수준의 사람이 할 수 있는 분야가 아니라고 생각했습니다. 그래서 괴델 정리 정도만 어느 정도 이해하고 취직을 준비하기로 마음먹었습니다. 그렇게 하루하루를 지내고 어쩌다 석사 과정에 진학하고 보니 어느새 박사학위를 목전에 두고 강의나 연구직 자리를 찾으며 취직을 준비하고 있습니다.

감사의 말을 쓰기에 앞서 쿠르트 괴델(Kurt Gödel)이 떠오른 것은 괴델 정리를 알고 싶다는 마음이 없었다면 논리학 공부도 시작하지 않았을 것이란 생각 때문입니다. 사실 괴델 정리는 어떤 수학적 구조물을 제시하고 그것을 무너뜨리는 목적을 지닌다고 할 수는 없을 것이고 처음 괴델 정리를 접했을 당시 제가 불완전성 정리를 대해 제대로 이해했다고도 말할 수 없을 것입니다. 다만 저는 그러한 상상을 했고 당시의 감정은 논리학의 문제들을 연구하는 지금의 저를 있게 했습니다. 논리학은 타당한 논변이 무엇인지를 탐구하는 학문이며 '타당성'은 논리적 사고의 근간이고 이는 인간의 합리성을 설명하는데 빼놓을 수 없는 개념입니다. 그렇다보니 가장 이성적인 분야를 공부하기 시작한 이유가 가장 감성적인 경험이라는 것에 아이러니함을 느끼기도 합니다.

괴델의 불완전성 정리는 증명 된지 곧 100년을 바라보고 있고 이제는 논리학 분야에서는 특수하다기 보다는 기초적으로 알아야 할 정리로 고려되고 있습니다. 철학을 전공으로 하기에 괴델 정리에 대한 여러 철학적 해석들을 공부했었고 석사 과정에서는 '괴델 정리는 사용의미론의 반례인가?'를 주제로 학위 논문을 쓰기도 했습니다. 『논리연구』에 처음 게재된 제 논문도 괴델 정리가 양진주의의 근거가 되는가에 대한 것이었고 박사

학위 논문 역시 괴델 정리의 증명론적 구조를 탐구하다가 역설에 대한 증명론적 구조의 탐구에 집중하게 되었습니다. 기초적인 부분에 지나치게 몰두하고 있다는 느낌도 지울 수 없고 가끔 '내가 왜 이러고 있지?'라는 생각을 할 때도 있습니다만 학문의 즐거움을 알게 해준 쿠르트 괴델(Kurt Gödel)에게 먼저 감사의 마음을 전하고 싶습니다.

대학원에 진학해서 지금까지 공부를 꾸준히 해 올 수 있었던 것은 많은 교수님들의 도움과 조언이 있었기 때문이라 생각합니다. 2009년 9월부터 지금까지 한국논리학회의 총무 및 편집간사 일을 하며 학자들의 요람인 학회가 어떻게 돌아가는지 연구 프로젝트는 어떤 방식으로 진행되는지 저널은 어떤 방식으로 출판되는지에 대해 꾸준히 배울 수 있었습니다. 무엇보다 논리학회에 참석하시는 열정적인 교수님들의 모습을 바라보며 학자의 꿈을 키울 수 있었던 것이 제게는 가장 값진 선물이었다고 생각합니다. 특히 프로젝트에 참석시켜 주시거나 연구하는데 조언을 아끼지 않아 주신 이초식 교수님, 박창균 교수님, 송하석 교수님, 최원배 교수님, 이병덕 교수님, 박준용 교수님, 여영서 교수님, 양은석 교수님, 김신 교수님, 김명석 교수님, 김준걸 교수님, 이진희 교수님, 박일호 교수님, 정재민 교수님께 감사드립니다. 그리고 학회 일을 함께 하며 도움을 많이 주신 강수연 선생님, 백송이 선생님, 신소혜 선생님께도 감사드립니다. 수업이나 세미나를 통해 많은 가르침을 주신 김병한 교수님, 하종호 교수님, 선우환 교수님, 손병석 교수님, 김창래 교수님, 성창원 교수님께도 감사드립니다. 2014년부터 한국연구재단에서 지원해 주는 글로벌박사 펠로우십에 참석하게 되면서 연세대학교 송도 국제캠퍼스의 교수님들과 미네소타 대학 및 오하이오 주립대 교수님들께도 많은 도움을 받았었습니다. 특히 Geoffrey Hellman 교수님, Stewart Shapiro 교수님, Roy Cook 교수님, Nikolaj Pedersen 교수님, Colin Caret 교수님, 그리고 Jeremy Wyatt 교수님께 감사드립니다. 또한 *Asian Logic Conference*, *Pluralism Week* 그리고 미네소타 과학철학 연구소에서 발표를 하며 연구와 관련 도움을 받았던 Teresa Kouri 교수님, Stella Moon 선생님, 김동우 선생님께도 감사드립니다. 박사 학위 논문의 주제인 역설에 대한 증명론적 탐구를 하는데 있어 심사를 통해 많은 조언을 주신 이종권 교수님, 최재웅 교수님, 최진영 교수님, 그리고 이계식 교수님께도 감사드리며 이메일을 통해 조언을 받을 수 있었던 Neil Tennant 교수님과 영어 교정에 도움을 주신 김지원 선생님 그리고 틈틈이 증명론적 의미론과 추론주의에 관해 의견을 교환해 주신 이종현 선생님께도 감사드립니다.

마지막으로 항상 끊임없는 격려로 저를 더 나은 사람이 될 수 있게 도와주신 조수지

작가님께 감사드립니다. 그리고 지난 11년간 많이 모자랐던 저를 끝까지 격려해 주시고 지도해 주신 정인교 교수님과 꾸준히 하면 안 될 것이 없다는 것을 삶으로써 보여주신 부모님께도 감사드립니다. 이제 산을 등정하기 위한 베이스캠프에 도달한 만큼 앞으로 더 열심히 연구해서 더 나은 학자가 되도록 노력하겠습니다. 감사합니다.

2019년 6월 27일
안암동 중앙광장에서
최승락.