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Which Paradox is Genuine in Accordance with the Proof-Theoretic Criterion for Paradoxicality?*

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【Abstract】Neil Tennant was the first to propose a proof-theoretic criterion for paradoxicality, a framework in which a paradox, formalized through natural deduction, is derived from an unacceptable conclusion that employs a certain form of *id est* inferences and generates an infinite reduction sequence. Tennant hypothesized that any derivation in natural deduction that formalizes a genuine paradox would meet this criterion, and he argued that while the liar paradox is genuine, Russell's paradox is not.

The present paper delves into Tennant's conjecture for genuine paradoxes and suggests that to validate the conjecture, one of two issues must be addressed. The first issue is the need for a philosophical consensus on the identification of a genuine paradox in an informal sense. The second issue is the requirement for a uniform approach to formalize paradoxes in natural deduction. If either of these issues is addressed, the conjecture could be validated, or at the very least, it could hold philosophical importance in delineating the proof-theoretic features of paradoxicality.

【Key Words】Liar paradox, Russell’s paradox, Genuine paradox, Proof-theoretic criterion for paradoxicality, Neil Tennant.

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1 Introduction

What distinguishes a genuine paradox? Numerous derivations of an absurdity ($\bot$) exist within natural deduction, but not all derivations of $\bot$ earn the label of a "paradox." In Appendix B of Prawitz (1965), it was examined how a derivation of $\bot$ from the set-theoretic paradox results in an infinite reduction sequence. While Prawitz did not openly identify the infinite reduction sequence as the distinguishing feature of paradoxes, it is often assumed that he intended to do so. For example, Schroeder-Heister and Tranchini (2017, p. 568) stated, "Prawitz proposed [the infinite reduction sequence] to be the distinguished feature of Russell's paradox." Following Prawitz's work, Tennant (1982) introduced the proof-theoretic criterion for paradoxicality, which has been further developed by Tennant (1995, 2015a, 2016, 2017). The infinite reduction sequence is thereby recognized as the key inferential feature separating a simple inconsistency from a paradox.

Tennant (1982, p. 283) initially proposed the proof-theoretic criterion for paradoxicality: a paradoxical derivation, utilizing a certain form of *id est* inferences, yields $\bot$ (or an unacceptable conclusion) and initiates an infinite reduction sequence. Tennant (2016, Sec. 1) suggested that the infinite reduction sequences are the proof-theorist's explanations of the vicious circularity (or vicious helicality) embedded in paradoxes. The criterion is perceived as a measure of the infinite reduction sequences generated by the derivation of $\bot$ linked with the paradoxes in question. Tennant (2017, p. 287) conjectured that "Genuine paradoxes are those whose associated *proofs of absurdity*, when formalized as natural deductions, cannot be converted into normal form" due to its inherently vicious circular nature, namely, the infinite reduction sequence.

Interestingly, Tennant (2016, 2017) claimed that the liar paradox qualifies as a genuine paradox, while Russell's paradox does not. The present paper will put forth an argument that one of two issues must be addressed to substantiate Tennant's conjecture for paradoxicality. The first problem is the establishment of a philosophical consensus on the definition of a genuine paradox in an informal sense. The second problem involves the need for a uniform method to formalize paradoxes in natural deduction. Following the introduction of preliminary notations, the proof-theoretic criterion for paradoxicality, and Tennant's perspective on genuine paradoxes in Section 2, Section 3 will explore examples that render Russell's paradox genuine and the liar paradox non-genuine. Section 4 contends that for the validation of the conjecture, addressing either the distinction of a genuine paradox in an informal context or the demand for a uniform method to formalize paradoxes in natural deduction is crucial.

2 Tennant's Completeness Conjecture for Genuine Paradoxes

With preliminary notions and rules detailed in Section 2.1, Section 2.2 presents the proof-theoretic criterion for paradoxicality (*PCP*), chiefly examined by Tennant (1982, 1995, 2015a, 2016, 2017), Schroeder-Heister and Tranchini (2017, 2018), and Choi (2019, 2021) along with the conjecture pertaining to genuine paradoxes.

2.1 Preliminaries: Some terminologies and natural deduction rules

The language under consideration incorporates the constant $\neg$ and $\bot$ to represent negation and absurdity, respectively. An equality symbol $=$ and a unary truth predicate $T(x)$ may be selectively employed for a specific natural deduction system. x and y are utilized as free variables, while s and t are used as closed terms. φ, ψ, and σ are designated for arbitrary formulae. A *derivation* of a natural deduction system is employed in the same context as "deduction" as per Prawitz (1965) and Tennant (2017). Furthermore, this article adheres to the following conventions: if a derivation $\mathfrak{D}$ concludes with a formula φ, it is represented as shown on the left below, and φ is referred to as an "end-formula." If it relies on the formula ψ, it is depicted as shown on the right.

$$\begin{array}{c} \\ \mathfrak{D} \\ \varphi \end{array} \qquad \begin{array}{c} \psi \\ \mathfrak{D} \\ \varphi \end{array}$$

In the natural deduction, there are rules for $\wedge$, $\rightarrow$, $\neg$, and $T(x)$ that take the form of general elimination rules.[1]

[1] While Tennant (2015b,c, 2016, 2017, 2021) favored the term "parallelized" over "general," the present paper opts for "general" as general elimination rules were initially introduced by Schroeder-Heister (1984a,b) to derive a *general schema* for introduction and elimination rules of principal operators. Furthermore, Tennant (2015b,c, 2017) utilized standard (or serial) forms of the $\neg E-$rule based on his core logic. In this context, the general $\neg E-$rule will be employed to minimize unnecessary disputes. The results can be substantiated using his core logic. Lastly, the left and right corner quotes, $\ulcorner\urcorner$, are commonly used in the truth predicate $T(x)$ to encode formulae into coded expressions. For instance, if φ is a given formula, $\ulcorner\varphi\urcorner$ refers to φ. If $\psi(x)$ is a formula with one free variable x, then $\psi(\ulcorner\varphi\urcorner)$ is a formula describing that φ denoted by $\ulcorner\varphi\urcorner$ is ψ.

$$\frac{\begin{matrix}\mathfrak{D}_1 & \mathfrak{D}_2\\ \varphi_1 & \varphi_2\end{matrix}}{\varphi_1 \wedge \varphi_2}\wedge I \qquad \frac{\varphi_1 \wedge \varphi_2 \quad \begin{matrix}[\varphi_1]^1, [\varphi_2]^1\\ \mathfrak{D}_3\\ \psi\end{matrix}}{\psi}\wedge E_{,1} \qquad \frac{\begin{matrix}[\varphi]^1\\ \mathfrak{D}_1\\ \psi\end{matrix}}{\varphi \to \psi}\to I_{,1} \qquad \frac{\varphi \to \psi \quad \begin{matrix}\mathfrak{D}_2\\ \varphi\end{matrix} \quad \begin{matrix}[\psi]^1\\ \mathfrak{D}_3\\ \sigma\end{matrix}}{\sigma}\to E_{,1}$$

$$\frac{\begin{matrix}[\varphi]^1\\ \mathfrak{D}_1\\ \bot\end{matrix}}{\neg\varphi}\neg I_{,1} \qquad \frac{\neg\varphi \quad \begin{matrix}\mathfrak{D}_2\\ \varphi\end{matrix} \quad \begin{matrix}[\bot]^1\\ \mathfrak{D}_3\\ \psi\end{matrix}}{\bot}\neg E_{,1} \qquad \frac{\begin{matrix}\mathfrak{D}_1\\ \varphi\end{matrix}}{T(\ulcorner\varphi\urcorner)}TI \qquad \frac{T(\ulcorner\varphi\urcorner) \quad \begin{matrix}[\varphi]^1\\ \mathfrak{D}_2\\ \psi\end{matrix}}{\psi}TE_{,1}$$

The formulas positioned directly above the line in each rule are termed the "premise," while the formula situated directly below the line is the "conclusion." *Assumptions* that are subject to discharge are enclosed in square brackets, such as $[\varphi]$. In line with Tennant's limitations on core logic, the application of the $\neg I-$rule prohibits vacuous discharge. (Cf. Tennant (2015b,c, 2016, 2017, 2021, 2022)). The *open assumptions* of a derivation are those assumptions upon which the end formula is dependent. A derivation is classified as *closed* if it does not contain any open assumptions, and is termed *open* otherwise. A *major premise* of the elimination rule for an operator is the premise that incorporates the operator in the elimination rule. For clarity, the major premise of the $E-$rule is positioned on the far left side of the premises of elimination rules, while all other premises are designated as *minor premises*. The term *maximum formula occurrence* refers to the conclusion of an introduction rule concurrently serving as the major premise of an elimination rule. Standard reduction procedures for $\wedge$, $\to$, $\neg$ and $T(x)$ are recognized and accepted.[2]

[2] The permutation conversion, as put forth by Prawitz (1965), is also accepted as a means to eliminate the major premises derived by general elimination rules.

1. The standard reduction procedure for $\wedge$ and $\rightarrow$.

$$\dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1 & \mathfrak{D}_2\\ \varphi_1 & \varphi_2\end{matrix}}{\varphi_1\wedge\varphi_2}\wedge I \qquad \begin{matrix}[\varphi_1]^1,[\varphi_2]^1\\ \mathfrak{D}_3\\ \psi\end{matrix}}{\psi}\wedge E_{,1} \quad \rhd_{\wedge} \quad \begin{matrix}\begin{matrix}\mathfrak{D}_1 & \mathfrak{D}_2\\ \varphi_1 & \varphi_2\end{matrix}\\ \mathfrak{D}_3\\ \psi\end{matrix} \qquad \dfrac{\dfrac{\begin{matrix}[\varphi]^1\\ \mathfrak{D}_1\\ \bot\end{matrix}}{\neg\varphi}\neg I_{,1} \quad \begin{matrix}\mathfrak{D}_2\\ \varphi\end{matrix} \quad \begin{matrix}[\bot]^2\\ \mathfrak{D}_3\\ \psi\end{matrix}}{\bot}\neg E_{,2} \quad \rhd_{\neg} \quad \begin{matrix}\mathfrak{D}_2\\ \varphi\\ \mathfrak{D}_1\\ \bot\\ \mathfrak{D}_3\\ \psi\end{matrix}$$

2. The standard reduction procedure for $\neg$ and $T(x)$.

$$\dfrac{\dfrac{\begin{matrix}[\varphi]^1\\ \mathfrak{D}_1\\ \psi\end{matrix}}{\varphi\rightarrow\psi}\rightarrow I_{,1} \quad \begin{matrix}\mathfrak{D}_2\\ \varphi\end{matrix} \quad \begin{matrix}[\psi]^2\\ \mathfrak{D}_3\\ \sigma\end{matrix}}{\chi}\rightarrow E_{,2} \quad \rhd_{\rightarrow} \quad \begin{matrix}\mathfrak{D}_2\\ \varphi\\ \mathfrak{D}_1\\ \psi\\ \mathfrak{D}_3\\ \sigma\end{matrix} \qquad \dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1\\ \varphi\end{matrix}}{T(\ulcorner\varphi\urcorner)}TI \quad \begin{matrix}[\varphi]^1\\ \mathfrak{D}_2\\ \psi\end{matrix}}{\psi}TE_{,1} \quad \rhd_{T(x)} \quad \begin{matrix}\mathfrak{D}_1\\ \varphi\\ \mathfrak{D}_2\\ \psi\end{matrix}$$

Take into account the following two derivations that possess the same end formula: $\mathfrak{D}_\mathfrak{a}$ and $\mathfrak{D}_\mathfrak{b}$. An *immediate sub-derivation* of $\mathfrak{D}_\mathfrak{a}$ is an initial portion of $\mathfrak{D}_\mathfrak{a}$ that terminates with the premise of the last inference rule in $\mathfrak{D}_\mathfrak{a}$. A *sub-derivation* is the reflexive and transitive closure of an immediate sub-derivation. Furthermore, $\mathfrak{D}_\mathfrak{a} \rhd \mathfrak{D}_\mathfrak{b}$ indicates that $\mathfrak{D}_\mathfrak{a}$ *reduces* to $\mathfrak{D}_\mathfrak{b}$ by applying a single reduction procedure to a sub-derivation of $\mathfrak{D}_\mathfrak{a}$, for example, when the left-side derivation of the reduction procedure $\rhd_{T(x)}$ is named $\mathfrak{D}_\mathfrak{a}$ and the right-side derivation is named $\mathfrak{D}_\mathfrak{b}$. The last inference rule in $\mathfrak{D}_\mathfrak{a}$ is the $TE-$rule. The immediate sub-derivations of $\mathfrak{D}_\mathfrak{a}$ are as follows:

$$\dfrac{\begin{matrix}\mathfrak{D}_1\\ \varphi\end{matrix}}{T\ulcorner\varphi\urcorner}TI \qquad \begin{matrix}[\varphi]^1\\ \mathfrak{D}_2\\ \psi\end{matrix}\ .$$

Then, '$\mathfrak{D}_\mathfrak{a} \rhd_{T(x)} \mathfrak{D}_\mathfrak{b}$' means that $\mathfrak{D}_\mathfrak{a}$ reduces to $\mathfrak{D}_\mathfrak{b}$. Now, the following definitions are used in the article.[3]

Definition 2.1. A sequence $< \mathfrak{D}_1, ..., \mathfrak{D}_i, \mathfrak{D}_{i+1}, ... >$ is a *reduction sequence* relative to $\mathbb{R}$ iff $\mathfrak{D}_i \rhd \mathfrak{D}_{i+1}$ relative to $\mathbb{R}$, where $1 \leqslant i$ for any natural number i. A derivation $\mathfrak{D}_1$ is *reducible* to $\mathfrak{D}_i$ $(\mathfrak{D}_1 \succ \mathfrak{D}_i)$ relative to $\mathbb{R}$ iff there is a sequence $< \mathfrak{D}_1, \mathfrak{D}_2, ..., \mathfrak{D}_i >$ relative to $\mathbb{R}$ where for each $j < i$, $\mathfrak{D}_j \rhd \mathfrak{D}_{j+1}$; $\mathfrak{D}_1$ is *irreducible* relative to $\mathbb{R}$ iff there is no derivation $\mathfrak{D}'$ to which $\mathfrak{D}_1 \rhd \mathfrak{D}'$ relative to $\mathbb{R}$ except $\mathfrak{D}_1$ itself.

Definition 2.2. The derivation $\mathfrak{D}$ is *normal* (or in *normal form*) relative to $\mathbb{R}$ iff $\mathfrak{D}$ has no maximum formula occurrence and is irreducible to $\mathbb{R}$. A reduction sequence *terminates* iff it has a finite number of derivations and its last derivation is in normal form. A derivation $\mathfrak{D}$ is *normalizable* relative to $\mathbb{R}$ iff there is a terminating reduction sequence relative to $\mathbb{R}$ starting from $\mathfrak{D}$.

It is important to clarify that irreducible derivations and derivations without a maximum formula occurrence are not always synonymous. Consequently, these two categories of derivations necessitate separate discussions.[4] For any given derivation $\mathfrak{D}$, $\mathfrak{D}$ *generates an infinite reduction sequence* iff there exists a derivation $\mathfrak{D}'$ such that $\mathfrak{D} \succ \mathfrak{D}'$, but its reduction sequence does not terminate. It is evident that, in

[3] In Definition 2.1, for any term x and y, let $x \leq y$ mean that x is less than or equal to y. For Definition 2.2, the expression “relative to $\mathbb{R}$” is dropped for the sake of convenience if there is no misunderstanding in the abridged descriptions.

[4] In the course of defining a normal derivation as an irreducible derivation alone, Tranchini (2015) deliberated on a derivation employing tonk-rules that, while being irreducible, contained a maximum formula occurrence. Additionally, it is possible to have a reducible derivation without a maximum formula occurrence. For instance, in the derivation $E'g$ as outlined by Schroeder-Heister and Tranchini (2018), there is no maximum formula occurrence when “$A \supset \neg A$” and “$\neg A \subset A$” in $E'g$ are treated as assumptions or axioms. In the event of the $Ekman_g$-reduction procedure being accepted, then $E'g$ would be reducible.

the case of $\mathfrak{D}$ generating an infinite reduction sequence, $\mathfrak{D}$ will have a non-terminating reduction sequence and will not be normalizable.

The subsequent subsections will introduce the proof-theoretic criterion for paradoxicality, as delineated in Tennant (1982, 1995, 2015a, 2016, 2017). Additionally, the conjecture of Tennant regarding genuine paradoxes, as presented in Tennant (2016, 2017), will be examined.

2.2 The Completeness Conjecture for Genuine Paradoxes

With the aim of explicating the proof-theoretic criterion for paradoxicality (*PCP*) and Tennant's conjecture regarding genuine paradoxes, Tennant's stipulations for the liar sentence Φ—as recommended in Tennant (2017, pp. 298 – 302)—are employed. It is within this section that the adjustment of said rules is examined to further the initiation of the *PCP*.

Let S_L be a natural deduction system with rules for $\neg$, $T(x)$, and the following Tennant's rules for the liar sentence Φ.

$$\dfrac{\begin{matrix}[T(\ulcorner\Phi\urcorner)]^1\\ \mathfrak{D}_1\\ \bot\end{matrix}}{\Phi}\,\Phi I_{,1} \qquad \dfrac{\Phi \qquad \begin{matrix}[\neg T(\ulcorner\Phi\urcorner)]^1\\ \mathfrak{D}_2\\ \varphi\end{matrix}}{\varphi}\,\Phi E_{,1}$$

As presented by Tennant (2017, p. 299), the reduction strategy for Φ is as follows.[5]

$$
\dfrac{\dfrac{\begin{array}{c}[T(\ulcorner\Phi\urcorner)]^1\\ \mathfrak{D}_1\\ \bot\end{array}}{\Phi}\,\Phi I_{,1}\qquad \begin{array}{c}[\neg T(\ulcorner\Phi\urcorner)]^2\\ \mathfrak{D}_2\\ \varphi\end{array}}{\varphi}\,\Phi E_{,2}
\qquad \rhd_{\Phi} \qquad
\begin{array}{c}\dfrac{\begin{array}{c}[T(\ulcorner\Phi\urcorner)]^1\\ \mathfrak{D}_1\\ \bot\end{array}}{\neg T(\ulcorner\Phi\urcorner)}\,\neg I_{,1}\\ \mathfrak{D}_2\\ \varphi\end{array}
$$

S_L also incorporates a set $\mathbb{R}_L$ of reduction procedures, which involve reductions for $\neg$, $T(x)$, and Φ. Thus, the ensuing conclusion is reached:

Proposition 2.3. *There is a closed derivation of $\bot$ in S_L relative to $\mathbb{R}_L$ that generates an infinite reduction sequence and thus is not normalizable.*

Proof. Upon achieving a closed derivation of $\bot$, it becomes evident that the derivation is irreducible to a normal derivation.

Claim 1. There is a closed derivation $\mathfrak{D}_3$ of $\bot$.

Initially, an open derivation $\mathfrak{D}_1$ of $\bot$ from $[T(\ulcorner\Phi\urcorner)]$ is observed below toward the left. Accompanying $\mathfrak{D}_1$, a closed derivation $\mathfrak{D}_2$ of $T(\ulcorner\Phi\urcorner)$ can be found below toward the right.

$$
\dfrac{[T(\ulcorner\Phi\urcorner)]^1 \qquad \dfrac{[\Phi]^2 \qquad \dfrac{[\neg T(\ulcorner\Phi\urcorner)]^3 \quad [T(\ulcorner\Phi\urcorner)]^1[\bot]^4}{\bot}\,\neg E_{,4}}{\bot}\,\Phi E_{,3}}{\bot}\,TE_{,2}
\qquad\qquad
\dfrac{\dfrac{\begin{array}{c}[T(\ulcorner\Phi\urcorner)]^1\\ \mathfrak{D}_1\\ \bot\end{array}}{\Phi}\,\Psi I_{,1}}{T(\ulcorner\Phi\urcorner)}\,TI
$$

[5] The reduction procedure for $T(x)$ is depicted graphically in a separate manner by Tennant (2017, p. 299). The fundamental notion that underlies this reduction, however, stays in line with the ongoing discussion, upholding a compatible perspective on the matter.

Following that, there exists a closed derivation $\mathfrak{D}_3$ of $\bot$.

$$\dfrac{\dfrac{\begin{matrix}[T(\ulcorner\Phi\urcorner)]^1\\ \mathfrak{D}_1\\ \bot\end{matrix}}{\neg T(\ulcorner\Phi\urcorner)}\neg I_{,1} \qquad \begin{matrix}\mathfrak{D}_2\\ T(\ulcorner\Phi\urcorner)\end{matrix} \quad [\bot]^5}{\bot}\neg E_{,5}$$

Claim 2. $\mathfrak{D}_3$ generates an infinite reduction sequence and thus is not normalizable.

Within the final $\neg E-$rule of $\mathfrak{D}_3$, $\neg T(\ulcorner\Phi\urcorner)$ serves as the maximum formula. The reduction of $\mathfrak{D}_3$ yields the derivation $\mathfrak{D}_4$ as displayed below.

$$\dfrac{\dfrac{\dfrac{\begin{matrix}[T(\ulcorner\Phi\urcorner)]^1\\ \mathfrak{D}_1\\ \bot\end{matrix}}{\Phi}\Phi I_{,1}}{T(\ulcorner\Phi\urcorner)}TI \qquad \dfrac{[\Phi]^2 \qquad \dfrac{[\neg T(\ulcorner\Phi\urcorner)]^3 \quad \begin{matrix}\mathfrak{D}_2\\ T(\ulcorner\Phi\urcorner)\end{matrix} \quad [\bot]^4}{\bot}\neg E_{,4}}{\bot}\Phi E_{,3}}{\bot}TE_{,2}$$

By applying $\rhd_{T(x)}$ followed by $\rhd_{\Phi}$, the reduction of $\mathfrak{D}_4$ to $\mathfrak{D}_3$ occurs. As $\mathfrak{D}_3$ generates an infinite reduction sequence, it remains non-normalizable. □

The process of reducing $\mathfrak{D}_3$ continually alternates among the following three reductions: $\rhd_{\neg}$, $\rhd_{T(x)}$, and $\rhd_{\Phi}$. As this procedure results in maximum formulae, including $\neg T(\ulcorner\Phi\urcorner)$, $T(\ulcorner\Phi\urcorner)$, and Φ, it is incapable of removing every maximum formula. This endless oscillation has been characterized by Tennant (1982, pp. 270–271) as a *falling into a looping reduction sequence*, and the completeness conjecture on paradoxicality was put forth as the proof-theoretic criterion for identifying paradoxical derivations.

> The completeness conjecture is then that [a] set of sentences is paradoxical ... iff there is some proof of [$\bot$] ... , involving those sentences in *id est* inferences that [have] a looping reduction sequence. (Tennant, 1982, p. 283)

The term *id est* inferences has been applied to instances where a formula can be interdeduced with its own negation (or its predication). Serving as the *id est* rules for the liar sentence Φ are the $\Phi I-$ and $\Phi E-$rules. The infinite reduction sequence was also deemed the defining aspect of paradoxes. Tennant (2016) outlined the criterion for paradoxicality accordingly.[6]:

> Tennant (1982) proposed a proof-theoretic criterion, or test, for paradoxicality—that of [an *infinite*] *reduction sequence* initiated by the "proofs of $\bot$" associated with the paradoxes in question (p. 271).

The initial criterion offered declares that a derivation is characterized as *paradoxical* provided it succeeds in deriving $\bot$, relies on *id est* inferences, and leads to an infinite reduction sequence. A summary of the proof-theoretic criterion for paradoxicality presents itself thus:

The Proof-Theoretic Criterion for Paradoxicality(*PCP*): Let $\mathfrak{D}$ be any derivation of a natural deduction system S and $\mathbb{R}$ be a set of reduction procedures of S. $\mathfrak{D}$ is *paradoxical* iff

(i) $\mathfrak{D}$ is a (open/closed) derivation of $\bot$,

[6] The term "non-terminating reduction sequence" utilized by Tennant (1982, 1995, 2015a, 2016, 2017) corresponds to the infinite reduction sequence in the present context. Both looping and spiral reduction sequences exemplify not merely non-terminating reduction sequences, but also infinite reduction sequences.

(ii) *id est* inferences (or rules) are used in $\mathfrak{D}$,

(iii) $\mathfrak{D}$ generates an infinite reduction sequence.

Some aspects necessitate philosophical deliberation. Regarding condition (i), $\perp$ does not represent the exclusive conclusion derived from paradoxical derivations. A propositional variable p could be employed during the formalization of Curry's paradox. (See Tennant (1982)).

Pertaining to condition (ii), infinite reduction sequences can be classified into two categories: loops and spirals. Tennant (1995, p. 207) postulated that the primary characteristic of self-referential paradoxes is a looping reduction sequence, while non-self-referential paradoxes predominantly exhibit a spiraling reduction sequence.[7] The analysis within the present paper remains confined to self-referential paradoxes.

The criterion, as suggested by Tennant (1982, p. 285), reflects the completeness conjecture for genuine paradoxes, prompting a summarization in the ensuing manner.

The Completeness Conjecture for Genuine Paradoxes: For any derivation $\mathfrak{D}$ in a natural deduction, $\mathfrak{D}$ formalizes a *genuine paradox* iff $\mathfrak{D}$ is paradoxical.

While the expression "genuine paradox" was not utilized in Tennant (1982), Tennant (2016), in a later exploration of the criterion for paradoxicality, designates the Liar paradox derivation in compliance with *PCP* as genuinely paradoxical. Additionally, in Tennant (2017, p. 288), it is mentioned, "Genuine paradoxes are those whose associated

[7] For additional discourse surrounding Tennant's conjecture on self-referential paradoxes, refer to Choi (2021).

proofs of absurdity, when formalized as natural deductions, cannot be converted into normal form." Consequently, it naturally follows that the completeness conjecture pertains to genuine paradox.

Notably, Tennant (2016) maintained that the Liar paradox is genuine while Russell's paradox is not. In the ensuing two sections, an examination of this perspective will be conducted, and two cases will be presented, suggesting that Russell's paradox can be considered a genuine paradox and the Liar paradox a non-genuine one, based on the same rationale employed by Tennant (2016, 2017).

3 The Problem of the Completeness Conjecture

The contention in this section is that Tennant falls short of providing a definitive response to the matter of identifying genuine paradoxes in line with *PCP*. In presenting a result concurrent with Proposition 2.3 in Section 2.2, Tennant (2016, pp. 12–16) posits that the Liar paradox is indeed genuine while Russell's paradox does not merit the distinction of being a genuine paradox. As Tennant (2016, Ch. 5) said,

> We find also that Russell's Paradox enjoys a proof in normal form, so that it is not genuinely paradoxical. ... [The logico-semantical paradoxes such as the Liar] call for a proof-theoretic clarification of the vicious circles and helices within them—which is what I am seeking to provide. ... In the latter case (for example, the Liar Paradox) we cannot have normality of (dis)proof. In the former case (for example, Russell's Paradox) we can; and we thereby obtain important negative existentials.

In order to arrive at $\perp$ from Russell's paradox, an assumption must be entertained that there exists a set of all sets not members of themselves. Tennant ascertains within the natural deduction system for the free logic of sets, that a closed normal derivation aims to disprove the initial assumption, establishing that no such set exists. This outcome buttresses the argument that Russell's paradox is not a genuine paradox. However, Section 3.1 argues that his derivation can generate an infinite reduction sequence, by adding a special form of a reduction procedure, which makes the derivation non-normal. In addition, it shall be contended that, should the derivation be verified to encompass an *id est* inference, the derivation fulfills the *PCP* criterion and may be deemed a genuine paradox.

3.1 Is Russell's Paradox Not a Genuine Paradox?

The analysis commences by scrutinizing the derivation within the free logic of sets, which bolsters the stance that Russell's paradox does not constitute a genuine paradox. His derivation states that there is no set of all and only those sets that do not contain themselves. Unfortunately, the introduction of a new reduction technique, known as Ekman reduction, initially offered by Ekman (1998) and later examined by Schroeder-Heister and Tranchini (2017, 2018) and Choi (2019), causes the derivation to be characterized as non-normal by generating an infinite reduction sequence. Furthermore, the derivation features an *id est* inference that moves from $a \in a$ to $\neg a \in a$ and from $\neg a \in a$ to $a \in a$. Should all paradoxical derivations indeed be genuine paradoxes, then Russell's paradox could be recognized as a genuine paradox.

Tennant (2016, Sec. 3) and Tennant (2017, pp. 294-298) lay out the groundwork for a normal closed derivation of $\neg\exists y(y = \{x|\neg x \in$

$x\})$ in the natural deduction system S_F for free logic of sets, to be detailed further in Proposition 3.1. It is made clear that the derivation Σ_3 of $\neg\exists y(y = \{x|\neg x \in x\})$ in Proposition 3.1 does not generate an infinite reduction sequence, indicating absence of paradoxicality. Russell's paradox does not fit the mold of a genuine paradox, as it does not meet *PCP*.

The natural deduction system S_F conceived by Tennant bears distinctions from a natural deduction system purposed for naive set theory, as examined in Prawitz (1965, Appendix B).[8] Let $\{x|\varphi(x)\}$ be a set of objects that satisfies $\varphi(x)$ for some φ. $\in$ be a two-place relation for set membership. In free logic, the singular terms can denote entities outside a domain or can fail to denote at all, even while the quantifiers retain conventional interpretation, thus introducing the rule of denotation. Here, $\exists x(x = t)$ is denoted as $\exists!t$, indicating that t exists. Given that t and u are closed terms and a represents a parameter, the natural deduction system S_F for the free logic of sets prescribes the rules for a set-forming operator and the rule of denotation (*RD*), complete with $\neg-$ and $\exists-$rules stated in generalized form.

$$\dfrac{\begin{matrix}[\varphi[a/x]]^1, [\exists!a]^1 \\ \mathfrak{D}_1 \\ a \in t\end{matrix} \qquad \begin{matrix} \\ \mathfrak{D}_2 \\ \exists!t\end{matrix} \qquad \begin{matrix}[a \in t]^1 \\ \mathfrak{D}_3 \\ \varphi[a/x]\end{matrix}}{t = \{x|\varphi(x)\}} \{\}I_{,1}$$

where a does not occur in $t = \{x|\varphi(x)\}$ nor in any undischarged assumptions of the subderivations other than those of the form of rules displayed.

[8] For a detailed understanding of the system for the free logic of sets, it is recommended to consult Section 7.10 of Tennant (1978).

$$\frac{t=\{x|\varphi(x)\} \quad \begin{matrix}\mathfrak{D}_4\\ \varphi[u/x]\end{matrix} \quad \begin{matrix}\mathfrak{D}_5\\ \exists!u\end{matrix} \quad \begin{matrix}[u\in t]^1\\ \mathfrak{D}_6\\ \psi\end{matrix}}{\psi}\{\}E_{1,1} \qquad \frac{t=\{x|\varphi(x)\} \quad \begin{matrix}\mathfrak{D}_7\\ u\in t\end{matrix} \quad \begin{matrix}[\varphi[u/x]]^1\\ \mathfrak{D}_8\\ \psi\end{matrix}}{\psi}\{\}E_{2,1}$$

where both $\{\}E$-rules prohibit vacuous discharge. In addition, RD is articulated in the following manner:

$$\frac{\varphi(\ldots\, t\, \ldots)}{\exists!t}RD$$

where φ is atomic. S_F has a set $\mathbb{R}_F$ of reduction procedures for $\neg$, $\exists$, $\{\}$. The reduction procedures for $\{\}$ are as follows:

$$\dfrac{\dfrac{\begin{matrix}[\varphi[a/x]]^1,[\exists!a]^1\\ \mathfrak{D}_1\\ a\in t\end{matrix} \quad \begin{matrix}\mathfrak{D}_2\\ \exists!t\end{matrix} \quad \begin{matrix}[a\in t]^1\\ \mathfrak{D}_3\\ \varphi[a/x]\end{matrix}}{t=\{x|\varphi(x)\}}\{\}I_{,1} \quad \begin{matrix}\mathfrak{D}_4\\ \varphi[u/x]\end{matrix} \quad \begin{matrix}\mathfrak{D}_5\\ \exists!u\end{matrix} \quad \begin{matrix}[u\in t]^2\\ \mathfrak{D}_6\\ \psi\end{matrix}}{\psi}\{\}E_{1,2} \quad \triangleright_{\{\}_1} \quad \begin{matrix}\mathfrak{D}_4 \qquad \mathfrak{D}_5\\ \varphi[u/x][a/u] \quad \exists!u[a/u]\\ \mathfrak{D}_1\\ a\in t[u/a]\\ \mathfrak{D}_6\\ \psi\end{matrix}$$

$$\dfrac{\dfrac{\begin{matrix}[\varphi[a/x]]^1,[\exists!a]^1\\ \mathfrak{D}_1\\ a\in t\end{matrix} \quad \begin{matrix}\mathfrak{D}_2\\ \exists!t\end{matrix} \quad \begin{matrix}[a\in t]^1\\ \mathfrak{D}_3\\ \varphi[a/x]\end{matrix}}{t=\{x|\varphi(x)\}}\{\}I_{,1} \quad \begin{matrix}\mathfrak{D}_7\\ u\in t\end{matrix} \quad \begin{matrix}[\varphi[u/x]]^2\\ \mathfrak{D}_8\\ \psi\end{matrix}}{\psi}\{\}E_{2,2} \quad \triangleright_{\{\}_2} \quad \begin{matrix}\mathfrak{D}_7\\ u\in t[a/u]\\ \mathfrak{D}_3\\ \varphi[a/x][u/a]\\ \mathfrak{D}_8\\ \psi\end{matrix}$$

In pursuit of a derivation of $\perp$ from $[a=\{x|\neg x\in x\}]$, $\neg x\in x$ is appropriated for φ in the E_1- and E_2-rules, and the parameter a is employed for both terms t and u. The rules that follow, therefore, are instances of E_1- and E_2-rules. Consequently, a closed normal derivation of $\neg\exists y(y=\{x|\neg x\in x\})$ is attained.

$$\dfrac{a=\{x|\neg x\in x\} \quad \begin{matrix}\mathfrak{D}_4\\ \neg a\in a\end{matrix} \quad \begin{matrix}\mathfrak{D}_5\\ \exists! a\end{matrix} \quad \begin{matrix}[a\in a]^1\\ \mathfrak{D}_6\\ \psi\end{matrix}}{\psi}\{\}E_{1,1} \qquad \dfrac{a=\{x|\neg x\in x\} \quad \begin{matrix}\mathfrak{D}_7\\ a\in a\end{matrix} \quad \begin{matrix}[\neg a\in a]^1\\ \mathfrak{D}_8\\ \psi\end{matrix}}{\psi}\{\}E_{2,1}$$

Proposition 3.1. *S_F has a closed normal derivation of $\neg\exists y(y = \{x|\neg x\in x\})$ relative to $\mathbb{R}_F$.*

Proof. The beginning point involves establishing a closed derivation Σ_4 of $\neg\exists y(y = \{x|\neg x\in x\})$, and demonstrating that Σ_4 is in normal form.

Claim 1. there is a closed derivation Σ_4 of $\neg\exists y(y = \{x|\neg x\in x\})$.

There is an open derivation Σ_1 of $\bot$ from $[a\in a]$ and $[a = \{x|\neg x\in x\}]$.

$$\dfrac{[a=\{x|\neg x\in x\}]^1 \quad [a\in a]^2 \quad \dfrac{[\neg a\in a]^3 \quad [a\in a]^2 \quad [\bot]^4}{\bot}\neg E_{,4}}{\bot}\{\}E_{2,2}$$

Utilizing derivation Σ_1 results in the emergence of an open derivation Σ_2 of $a\in a$ from $[a = \{x|\neg x\in x\}]$.

$$\dfrac{[a=\{x|\neg x\in x\}]^1 \quad \dfrac{\begin{matrix}[a=\{x|\neg x\in x\}]^1, [a\in a]^2\\ \Sigma_1\\ \bot\end{matrix}}{\neg a\in a}\neg I_{,2} \quad \dfrac{[a=\{x|\neg x\in x\}]^1}{\exists! a}RD \quad [a\in a]^5}{a\in a}\{\}E_{1,5}$$

Given the open derivation Σ_2, an open derivation Σ_3 of $\bot$ from $[a =$

$\{x|\neg x \in x\}]$ is procured.

$$\dfrac{[a=\{x|\neg x\in x\}]^1 \qquad \dfrac{[a=\{x|\neg x\in x\}]^1 \qquad \dfrac{\begin{matrix}[a=\{x|\neg x\in x\}]^1,[a\in a]^2\\ \Sigma_1\\ \bot\end{matrix}}{\neg a\in a}\neg I_{,2} \qquad \begin{matrix}\vdots\\ \exists!a\end{matrix} \qquad [a\in a]^5}{a\in a}\{\}E_{1,5} \qquad \dfrac{[\neg a\in a]^3 \qquad \begin{matrix}\vdots\\ \Sigma_2\\ a\in a\end{matrix} \qquad [\bot]^4}{\bot}\neg E_{,4}}{\bot}\{\}E_{2,3}$$

As it stands, a closed derivation Σ_4 of $\neg\exists y(y=\{x|\neg x\in x\})$ is in possession.

$$\dfrac{\dfrac{[\exists y(y=\{x|\neg x\in x\})]^6 \qquad \begin{matrix}[a=\{x|\neg x\in x\}]^1\\ \Sigma_3\\ \bot\end{matrix}}{\bot}\exists E_{,1}}{\neg\exists y(y=\{x|\neg x\in x\})}\neg I_{,6}$$

Claim 2. Σ_4 is in normal form.

Considering all major premises in Σ_1, Σ_2, Σ_3 and Σ_4 are assumptions, and no reduction procedure in $\mathbb{R}_F$ is applicable to Σ_4, Σ_4 has been established in normal form. □

The derivation Σ_4 of $\neg\exists y(y=\{x|\neg x\in x\})$ in Proposition 3.1 is in normal form. Nevertheless, when acknowledging an Ekman-type reduction, as presented in Ekman (1998), S_F furnishes an open non-normal derivation of $\bot$ from $[a=x|\neg x\in x]$ that generates an infinite reduction sequence. Although Tennant (2016) considered that Proposition 3.1 is Russell's paradox made unparadoxical, the following Ekman-type reduction procedure in generalized form for set-abstraction devastates his view.

$$\dfrac{\begin{matrix}\mathfrak{D}_1\\ t=\{x|\varphi(x)\}\end{matrix} \quad \dfrac{\begin{matrix}\mathfrak{D}_1\\ t=\{x|\varphi(x)\}\end{matrix} \quad \begin{matrix}\mathfrak{D}_2\\ \varphi[x/u]\end{matrix} \quad \begin{matrix}\mathfrak{D}_3\\ \exists!u\end{matrix} \quad [u\in t]^1}{u\in t}\{\}E_{1,1} \quad \begin{matrix}[\varphi[x/u]]^2\\ \mathfrak{D}_4\\ \psi\end{matrix}}{\psi}\{\}E_{2,2} \quad \trianglerighteq_{EF} \quad \begin{matrix}\mathfrak{D}_2\\ \varphi[x/u]\\ \mathfrak{D}_4\\ \psi\end{matrix}$$

The term Ekman maximum formula is used to refer to the minor premise $u\in t$ of E_2-rule. Subsequently, the following outcome emerges:

Proposition 3.2. *If the set $\mathbb{R}_F$ of reductions of S_F includes an Ekman-type reduction process in generalized form for set-abstraction, $\trianglerighteq_{EF}$, S_F has an open derivation of $\bot$ from $[a=\{x|\neg x\in x\}]$ which generates an infinite reduction sequence and is not normalizable.*

Proof. Employing open derivations Σ_1, Σ_2, and Σ_3 as presented in Proposition 3.1, the minor premise $a\in a$ of the E_2-rule within Σ_3 is an Ekman maximum formula. Through the application of $\triangleright_{EF}$ on Σ_3, the resulting open derivation Σ_5 of $\bot$ is derived from $[a=x|\neg x\in x]$.

$$\dfrac{\dfrac{\begin{matrix}[a=\{x|\neg x\in x\}]^1,[a\in a]^2\\ \Sigma_1\\ \bot\end{matrix}}{\neg a\in a}\neg I_{,2} \quad \begin{matrix}[a=\{x|\neg x\in x\}]^1\\ \Sigma_2\\ a\in a\end{matrix} \quad [\bot]^7}{\bot}\neg E_{,7}$$

Seeing that Σ_5 involves a major premise $\neg a\in a$, the application of $\triangleright_{\neg}$ to Σ_5 produces an identical derivation to Σ_5. Consequently, Σ_5 embarks on an enduring reduction sequence and resists being reduced to a normal form. □

Proposition 3.2 poses significant challenges to Tennant's stance that Russell's paradox lacks genuineness. Should Σ_5 incorporate id est inferences from $a \in a$ to $\neg a \in a$ and from $\neg a \in a$ to $a \in a$, *PCP* then establishes Σ_5 as paradoxical. Consequently, contrary to Tennant's view, the acceptance of $\trianglerighteq_{EF}$ brings Russell's paradox into the realm of genuine paradoxes.[9]

3.2 Is the Liar Pradox Genuine?

Tennant (2016, 2017) claimed that Russell's paradox lacks genuineness due to the normal derivation of $\neg\exists y(y = x|\neg x \in x)$ in S_F relative to $\mathbb{R}_F$. Contrarily, the Liar paradox is considered a genuine paradox with an equivalent outcome to Proposition 2.3. Analogous to the method questioning the genuineness of Russell's paradox, an analysis of the Liar paradox is performed, applying generalized elimination rules in an attempt to present a derivation intimating the Liar paradox is not a genuine paradox.

With the aim of establishing non-genuineness of the Liar, the rules for second-order existential quantification $\exists^2$ is borrowed from Prawitz (1965, Ch.5). Let X^n, Y^n be n-ary predicate variables, and φ^n, ψ^n, σ^n be n-ary predicate formulas. $x_1,...,x_n$ (or $t_1,...,t_n$) is ab-

[9] In Tennant (1982), the standard form of the set-abstraction elimination rule was employed. Tennant (1982, p. 276) stated that the derivation of $\bot$ from the assumption $\exists!a$, where $a = \{x|\neg x \in x\}$ generates an infinite reduction sequence and said, "Russell's [paradox] remains an intrinsically troublesome case of paradox." Later, based on the findings of Proposition 3.1, Tennant (2016, Sec. 3) asserted that Russell's paradox is not genuinely paradoxical, given that the examined derivation does not generate an infinite reduction sequence. He diagnosed that the standard form of the elimination rule for set-abstraction creates an artifact feature of the infinite reduction sequence. Unfortunately, as demonstrated in Σ_5 of Proposition 3.2, the generalized form of the elimination rule either creates an infinite reduction sequence. The crux of the matter lies not in the choice of elimination rules, but rather in the adoption of reduction procedures. Additionally, even with the standard elimination rules in place, a normal derivation of $\neg\exists y(y = \{x|\neg x \in x\})$ can be readily established. (See Appendix A for details.)

breviated as $\overrightarrow{x_n}$ (or $\overrightarrow{t_n}$). $\exists^2 I-$ and $\exists^2 E-$rules are of the following form.

$$\dfrac{\varphi[\psi^n[\overrightarrow{t_n}/\overrightarrow{x_n}]/X^n]}{\exists^2 X^n \varphi}\,\exists^2 I \qquad \dfrac{\exists^2 X^n \varphi \qquad \begin{matrix}[\varphi[Y^n/X^n]]^1\\ \mathfrak{D}\\ \sigma\end{matrix}}{\sigma}\,\exists^2 E_{,1}$$

where $\varphi[\psi^n[\overrightarrow{t_n}/\overrightarrow{x_n}]/X^n]$ is obtained from φ by replacing each occurrence of a subformula $X^n(\overrightarrow{t_n})$ in φ by $\psi^n[\overrightarrow{t_n}/\overrightarrow{x_n}]$; in $\exists^2 E$, Y^n does not occur free in any undischarged assumptions on which σ depends except $\varphi[Y^n/X^n]$, nor does Y^n occur free in σ.[10] The reduction procedure for $\exists^2$ is as below:

$$\dfrac{\dfrac{\varphi[\psi^n[\overrightarrow{t_n}/\overrightarrow{x_n}]/X^n]}{\exists^2 X^n \varphi}\,\exists^2 I \qquad \begin{matrix}[\varphi[Y^n/X^n]]^1\\ \mathfrak{D}\\ \sigma\end{matrix}}{\sigma}\,\exists^2 E_{,1} \quad \rhd_{\exists^2} \quad \begin{matrix}\mathfrak{D}_1\\ \varphi[\psi^n[\overrightarrow{t_n}/\overrightarrow{x_n}]/X^n][Y^n/\psi^n[\overrightarrow{t_n}/\overrightarrow{x_n}]]\\ \mathfrak{D}_2\\ \sigma\end{matrix}$$

Let S_{L^2} be a natural deduction system consisting of rules for $\wedge$, $\rightarrow$, $\neg$, $T(x)$, and $\exists^2$. The set R_{L^2} of reduction procedures for $\wedge$, $\rightarrow$, $\neg$, $T(x)$, and $\exists^2$ are given. $\varphi \leftrightarrow \psi$ is defined as $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$. As a consequence, the liar sentence Φ holds the relation $\Phi \leftrightarrow \neg T(\ulcorner \Phi \urcorner)$. In light of Proposition 3.3, the Liar paradox is not regarded as a genuine paradox.

Proposition 3.3. *S_{L^2} relative to $\mathbb{R}_{L^2}$ has a closed normal derivation of $\neg\exists^2 X(X \leftrightarrow \neg T(\ulcorner X \urcorner))$.*

Proof. Initiation begins with demonstrating the closed derivation Π_4 of $\neg\exists^2 X(X \leftrightarrow \neg T(\ulcorner X \urcorner))$ and verifying that Π_4 is in normal form.

[10]Note that the notion of *subformula* is defined inductively by (1) φ is a subformula of φ, (2) if $\psi \circ \sigma$ is a subformula of φ then so are ψ, σ where $\circ$ is $\vee$ or $\wedge$ or $\rightarrow$, (3) if $\forall x \psi$ or $\exists x \psi$ is a subformula of φ, then so is $\psi[t/x]$.

Claim 1. There is a closed derivation Π_4 of $\neg\exists^2 X(X \leftrightarrow \neg T(\ulcorner X \urcorner))$.

The proof begins with producing an open derivation Π_1 of $\bot$ sourced from $[T(\ulcorner Y \urcorner)]$ and $[Y \leftrightarrow \neg T(\ulcorner Y \urcorner)]$

$$\dfrac{\dfrac{[Y \leftrightarrow \neg T(\ulcorner Y \urcorner)]^7}{(Y \to \neg T(\ulcorner Y \urcorner)) \wedge (\neg T(\ulcorner Y \urcorner) \to Y)}\,def \qquad \dfrac{[Y \to \neg T(\ulcorner Y \urcorner)]^5 \quad \dfrac{[T(\ulcorner Y \urcorner)]^1 \quad [Y]^2}{Y}\,TE_{,2} \quad \dfrac{[\neg T(\ulcorner Y \urcorner)]^3 \quad [T(\ulcorner Y \urcorner)]^1 \quad [\bot]^4}{\bot}\,\neg E_{,4}}{\bot}\to E_{,3}}{\bot}\wedge E_{,5}$$

Then, an open derivation Π_2 of Y from $[Y \leftrightarrow \neg T(\ulcorner Y \urcorner)]$ is achieved.

$$\dfrac{\dfrac{[Y \leftrightarrow \neg T(\ulcorner Y \urcorner)]^7}{(Y \to \neg T(\ulcorner Y \urcorner)) \wedge (\neg T(\ulcorner Y \urcorner) \to Y)}\,def \qquad \dfrac{[\neg T(\ulcorner Y \urcorner \to Y)]^5 \quad \dfrac{\begin{array}{c}[T(\ulcorner Y \urcorner)]^1\\ \Pi_1\\ \bot\end{array}}{\neg T(\ulcorner Y \urcorner)}\,\neg I_{,1} \quad [Y]^6}{Y}\to E_{,6}}{Y}\wedge E_{,5}$$

In the presence of Π_1 and Π_2, an open derivation Π_3 of $\bot$ from $[Y \leftrightarrow \neg T(\ulcorner Y \urcorner)]$ is established.

$$\dfrac{\dfrac{[Y \leftrightarrow \neg T(\ulcorner Y \urcorner)]^7}{(Y \to \neg T(\ulcorner Y \urcorner)) \wedge (\neg T(\ulcorner Y \urcorner) \to Y)}\,def \qquad \dfrac{[Y \to \neg T(\ulcorner Y \urcorner)]^6 \quad \begin{array}{c}[Y \leftrightarrow \neg T(\ulcorner Y \urcorner)]^7\\ \Pi_2\\ Y\end{array} \quad \dfrac{[\neg T(\ulcorner Y \urcorner)]^8 \quad \dfrac{\begin{array}{c}[Y \leftrightarrow \neg T(\ulcorner Y \urcorner)]^7\\ \Pi_2\\ Y\end{array}}{T(\ulcorner Y \urcorner)}\,TI \quad [\bot]^9}{\bot}\,\neg E_{,9}}{\bot}\to E_{,8}}{\bot}\wedge E_{,6}$$

Finally, the closed derivation Π_4 of $\neg\exists^2 X(X \leftrightarrow \neg T(\ulcorner X \urcorner))$ is given.

$$\dfrac{\dfrac{\left[\exists^2 X(X \leftrightarrow \neg T(\ulcorner X \urcorner))\right]^{10} \quad \begin{array}{c}[Y \leftrightarrow \neg T(\ulcorner Y \urcorner)]^7\\ \Pi_3\\ \bot\end{array}}{\bot}\,\exists^2 E_{,7}}{\neg\exists^2 X(X \leftrightarrow \neg T(\ulcorner X \urcorner))}\,\neg I_{,10}$$

Claim 2. Π_4 is in normal form.

Considering that every major premise in Π_1, Π_2, Π_3, and Π_4 consists of assumptions and no reduction procedure in $\mathbb{R}_{L^2}$ is applicable, it follows that Π_4 is in normal form. $\square$

As demonstrated by Proposition 3.3, no formula X within S_{L^2} upholds the relation $X \leftrightarrow \neg T(\ulcorner X \urcorner)$. Considering that the liar sentence Φ forms the relation $\Phi \leftrightarrow \neg T(\ulcorner \Phi \urcorner)$, the conclusion drawn from Proposition 3.3 is that no derivation of the Liar paradox in conformity with *PCP* is attainable. In light of Tennant's viewpoint that Russell's paradox is not a genuine paradox due to the normal derivation of the formula excluding the set of all sets not members of themselves, the normal derivation of Π_4 of $\neg\exists^2 X(X \leftrightarrow \neg T(\ulcorner X \urcorner))$ lends credence to the notion that the Liar paradox is similarly not genuine.

4 Which Paradox is Genuine?

An examination has been conducted on two derivations pertaining to the Liar paradox, notably Proposition 2.3 and 3.3. Additionally, two derivations associated with Russell's paradox have been scrutinized, as found in Proposition 3.1 and 3.2. As pointed out, Tennant (2016) argued that Proposition 3.1 serves as evidence that Russell's paradox is not a genuine paradox, whereas the Liar paradox retains its genuine status on the basis of Proposition 2.3. Drawing on a similar method to the one disputing genuineness of Russell's paradox, Proposition 3.3 indicates that the Liar paradox is not a genuine paradox in S_{L^2}. Analogously, Proposition 3.2 has the potential to endorse the belief that Russell's paradox is indeed a genuine paradox. Furthermore, if a derivation Δ_3 of Russell's paradox within a natural deduction system for native set theory in Appendix B, as initially introduced by Prawitz (1965, Appendix B), is selected, this could serve as the basis for asserting Russell's paradox as a genuine paradox.These occurrences necessitate addressing the inquiry, "Which paradox is genuine?"

The completeness conjecture for genuine paradoxes is bifurcated into two facets: for any derivation $\mathfrak{D}$ in a natural deduction, (a) if $\mathfrak{D}$ formalizes a *genuine paradox*, then $\mathfrak{D}$ is paradoxical, and (b) if $\mathfrak{D}$ is paradoxical, then $\mathfrak{D}$ formalizes a *genuine paradox*. Interestingly, Tennant (2016) harnessed (a) to argue against the genuineness of Russell's paradox, whereas he drew on (b) to corroborate the genuineness of the Liar paradox. Tennant's dual arguments pertaining to genuine and non-genuine paradoxes can be concisely depicted in the following manner.

The Argument for the Non-Genuineness of Russell's Paradox

Premise 1. (By Proposition 3.1) No derivation of Russell's paradox in S_F is *paradoxical*.

Premise 2. (By (a)) and the premise 1) No derivation of Russell's paradox in S_F formalizes a genuine paradox.

Conclusion. Russell's paradox (depicted in S_F) is not a genuine paradox.

The Argument for the Genuineness of the Liar Paradox

Premise 1. (By Proposition 2.3) The derivation $\mathfrak{D}_3$ of the Liar paradox in S_L is *paradoxical*.

Premise 2. (By (b) and the premise 2) $\mathfrak{D}_3$ in S_L formalizes a genuine paradox.

Conclusion. The Liar paradox (formulated in S_L) is a genuine paradox.

Suppose there exists solely one derivation encapsulating Russell's (or the Liar) paradox; in such a scenario, Tennant's independent application of conditions (a) and (b) would pose no issues. Nonetheless, an alternate derivation pertaining to Russell's and the Liar paradox is present. Employing the outcome of Proposition 3.3 in conjunction with condition (a) allows for the assertion of the Liar paradox's non-genuineness. Furthermore, utilizing the result of Proposition 3.2 or Prawitz's derivation Δ_3 of Russell's paradox within a native set theory system (refer to Appendix B) enables claims of Russell's paradox being genuine, based on condition (b). Due to the divergent outcomes concerning Russell's and the Liar paradox, determining the genuinely paradoxical status remains a challenge.

The completeness conjecture's predicament concerning genuine paradoxes arises due to the insufficiently elucidated concept of a 'genuine paradox' and the numerous possibilities for formalizing paradoxes. Supposing an understanding of genuine paradox is achieved and the determination has been made that the Liar paradox is genuine, while Russell's is not, then any derivation formalizing the Liar paradox should, according to (a), be paradoxical. Proposition 2.3 supports relation (a) and Proposition 3.3 conflicts with the supposition that the Liar paradox is genuine. Should the supposition be maintained, there may be faults in the formalization methodology of the Liar paradox found within Proposition 3.3.

Correspondingly, by utilizing (b) alongside Russell's paradox, Proposition 3.1 substantiates the stance that Russell's paradox fails to qualify as genuine, and Proposition 3.2 opposes this stance. Identifying the inaccuracies in the proof of Proposition 3.2 may be achieved, for instance, by examining the implementation of Ekman-type reduction. Thus, if the types of genuine paradoxes are determined, methods to formalize paradoxes, such as rule applications and reduction processes, can be adequately explicated.

On the other hand, by identifying a singular way to formalize paradoxes using natural deduction, discerning the genuineness of a paradox can be achieved. Assume the most appropriate method for formalizing Russell's paradox incorporates the natural deduction system S_F for free logic of sets, and that applying Ekman-type reduction is improper. Even if Σ_3 from Proposition 3.2 fulfills *PCP* and generates an infinite reduction sequence, it may be surmised that Σ_3 is an unsuitable formalization for Russell's paradox and that the employment of Ekman-type reduction is impermissible. According to (a), the classification of Russell's paradox as genuine can be resolved. Moreover, should the ideal method for formalizing the Liar paradox involve applying rules and reduction procedures in S_L relative to $\mathbb{R}_L$, the determination of the Liar paradox's genuineness can be made through (b). Such a unification of the formalization process yields definitive answers regarding the genuineness of paradoxes.

In sum, to assess the completeness conjecture pertaining to genuine paradoxes, there are two aspects requiring thorough elucidation. The first aspect entails the identification of the class of genuine paradoxes. The second aspect involves the unique formalization method of paradoxes, which encompasses the application of rules and reduction procedures.

5 Conclusion

The philosophical inquiry at hand has delved into two derivations of the Liar paradox and another two concerning Russell's paradox. Tennant's stance, which maintains that the Liar paradox is a genuine paradox while Russell's paradox is not, is contested by the findings in Sections 3 and 4. These sections present an opposing conclusion, asserting that it is Russell's paradox that embodies genuine paradoxicality, while the Liar paradox does not reach this threshold, as corroborated by Proposition 3.2 and Proposition 3.3.

The analysis conducted by Tennant (2016, 2017) was limited to the consideration of $\mathfrak{D}_3$ from Proposition 2.3 for the genuineness of the Liar and Σ_4 of Proposition 3.1 for the non-genuineness of Russell's. There has been no persuasive case made to dismiss the idea that Π_4 of Proposition 3.3 bolsters the viewpoint that the Liar paradox does not qualify as a genuine paradox.

Tennant (2016, 2017) only considered $\mathfrak{D}_3$ of Proposition 2.3 for the genuineness of the Liar and Σ_4 of Proposition 3.1 for the non-genuineness of Russell's. He has not yet had a good reason to repudiate that Π_4 of Proposition 3.3 supports the view that the Liar paradox is not a genuine paradox. In a similar vein, when Prawitz's derivation Δ_3 of Proposition 5.2 is considered, it serves as a basis for asserting that Russell's paradox is a genuine paradox. Given that Tennant has not discussed grounds for genuine paradoxes from a proof-theoretic viewpoint, a comprehensive evaluation of the completeness conjecture for genuine paradoxes necessitates clarification regarding which paradoxes are genuine and which formalization method is appropriate for them. Addressing either of these matters could potentially validate the conjecture or, at a minimum, contribute to a philosophically significant understanding of proof-theoretic aspects of paradoxicality.[11]

[11] The anonymous reviewer suggests that the results of the present work establish "there is no right or wrong judgement of paradoxicality *per se*," given that such judgements are contingent upon one's choice of fundamental principles. While conferring considerable weight to this view, it is asserted that the infinite reduction sequence harbors deeper philosophical import, and that Tennant's completeness conjecture may be framed within his intuitionistic relevant system, also known as core logic. Recognition is offered to the anonymous reviewer for the valuable commentary rendered.

Appendix A. The Proof of $\neg\exists y(y = \{x|\neg x \in x\})$ in a Natural Deduction System for Free Logic of Sets with Standard Forms of Rules

A natural deduction system $S_{F'}$ for the free logic of sets has the following rules for a set-forming operator and the rule of denotation (RD) with $\neg-$ and $\exists-$rules stated in standard form. $S_{F'}$ has a set $\mathbb{R}_{F'}$ of reduction procedures for $\neg$, $\exists$, and $\{\}$.

$$\cfrac{\begin{matrix}[\varphi[a/x]]^1, [\exists!a]^1 \\ \mathfrak{D}_1 \\ a \in t\end{matrix} \quad \begin{matrix} \\ \mathfrak{D}_2 \\ \exists!t\end{matrix} \quad \begin{matrix}[a \in t]^1 \\ \mathfrak{D}_3 \\ \varphi[x/a]\end{matrix}}{t = \{x|\varphi(x)\}} \{\,\}I_{,1}$$

where a does not occur in $t = \{x|\varphi(x)\}$ nor in any undischarged assumptions of the subderivations other than those of the form of rules displayed. Furthermore, a rule of denotation, RD, is present in $S_{F'}$, characterized by a form akin to the one in S_F.

$$\cfrac{t = \{x|\varphi(x)\} \quad \begin{matrix}\mathfrak{D}_4 \\ \varphi[u/x]\end{matrix} \quad \begin{matrix}\mathfrak{D}_5 \\ \exists!u\end{matrix}}{u \in t} \{\,\}E_1 \qquad\qquad \cfrac{t = \{x|\varphi(x)\} \quad \begin{matrix}\mathfrak{D}_6 \\ u \in t\end{matrix}}{\varphi[u/t]} \{\,\}E_2$$

where φ is atomic. To prove that a closed normal derivation of $\neg\exists y(y = \{x|\neg x \in x\})$ is obtainable, let $\neg x \in x$ stand for φ within the $\{\}E_1-$ and $\{\}E_2-$ rules, and assign the parameter a to both term t and u. Consequently, the subsequent rules emerge as instances of the $\{\}E_1-$ and $\{\}E_2-$ rules.

$$\frac{a=\{x|\neg x\in x\} \quad \overset{\mathfrak{D}_4}{\neg a\in a} \quad \overset{\mathfrak{D}_5}{\exists! a}}{a\in a}\{\ \}E_1 \qquad\qquad \frac{a=\{x|\neg x\in x\} \quad \overset{\mathfrak{D}_6}{a\in a}}{\neg a\in a}\{\ \}E_2$$

The followings are reduction procedures for $\{\}$ in $\mathbb{R}_{F'}$.

$$\frac{\dfrac{\begin{matrix}[\varphi[a/x]]^1,[\exists!a]^1\\ \mathfrak{D}_1\\ a\in t\end{matrix} \quad \begin{matrix}\mathfrak{D}_2\\ \exists!t\end{matrix} \quad \begin{matrix}[a\in t]^1\\ \mathfrak{D}_3\\ \varphi[a/x]\end{matrix}}{t=\{x|\varphi(x)\}}\{\}I_{,1} \quad \begin{matrix}\mathfrak{D}_4\\ \varphi[u/x]\end{matrix} \quad \begin{matrix}\mathfrak{D}_5\\ \exists!u\end{matrix}}{u\in t}\{\}E_1 \quad \rhd_{\{\}_{1'}} \quad \begin{matrix}\mathfrak{D}_4 & \mathfrak{D}_5\\ \varphi[u/x] & \exists!u\\ \mathfrak{D}_1 & \\ u\in t & \end{matrix}$$

$$\frac{\dfrac{\begin{matrix}[\varphi[a/x]]^1,[\exists!a]^1\\ \mathfrak{D}_1\\ a\in t\end{matrix} \quad \begin{matrix}\mathfrak{D}_2\\ \exists!t\end{matrix} \quad \begin{matrix}[a\in t]^1\\ \mathfrak{D}_3\\ \varphi[a/x]\end{matrix}}{t=\{x|\varphi(x)\}}\{\}I_{,1} \quad \begin{matrix}\mathfrak{D}_6\\ u\in t\end{matrix}}{\varphi[u/t]}\{\}E_2 \quad \rhd_{\{\}_{2'}} \quad \begin{matrix}\mathfrak{D}_6\\ u\in t\\ \mathfrak{D}_3\\ \varphi[u/t]\end{matrix}$$

Then, the closed normal derivation of $\neg\exists y(y=\{x|\neg x\in x\})$ is established.

Proposition 5.1. *There is a normal derivation of* $\neg\exists y(y=\{x|\neg x\in x\})$ *in* $S_{F'}$ *relative to* $\mathbb{R}_{F'}$.

Proof. Starting with a closed derivation Σ_4' of $\neg\exists y(y=\{x|\neg x\in x\})$, it shall be demonstrated that Σ_3' is in normal form.

Claim 1. there is a closed derivation Σ_4' of $\neg\exists y(y=\{x|\neg x\in x\})$.

Initially, an open derivation Σ_1' of $\bot$ is derived from the assumptions $[a=\{x|\neg x\in x\}]$ and $[a\in a]$.

$$\frac{\dfrac{[a=\{x|\neg x\in x\}]^1 \quad [a\in a]^2}{\neg a\in a}\{\}E_2 \qquad [a\in a]^2}{\bot}\neg E$$

With the derivation Σ_1', an open derivation Σ_2' of $a \in a$ from the assumtion $[a = \{x|\neg x \in x\}]$ is achieved.

$$\dfrac{[a=\{x|\neg x\in x\}]^1 \qquad \dfrac{\begin{matrix}[a=\{x|\neg x\in x\}]^1,[a\in a]^3\\ \Sigma_1'\\ a\in a\end{matrix}}{\neg a\in a}\,\neg I_{,3} \qquad \dfrac{[a=\{x|\neg x\in x\}]^1}{\exists! a}\,RD}{a\in a}\,\{\}E_1$$

With the possession of derivations Σ_1' and Σ_2', an open derivation Σ_3' of $\bot$ from the assumption $[a = \{x|\neg x \in x\}]$ is established.

$$\dfrac{\dfrac{[a=\{x|\neg x\in x\}]^1 \qquad \dfrac{[a=\{x|\neg x\in x\}]^1 \qquad \dfrac{\begin{matrix}[a=\{x|\neg x\in x\}]^1,[a\in a]^2\\ \Sigma_1'\\ a\in a\end{matrix}}{\neg a\in a}\,\neg I_{,2} \qquad \dfrac{[a=\{x|\neg x\in x\}]^1}{\exists! a}\,RD}{a\in a}\,\{\}E_1}{\neg a\in a}\,\{\}E_2 \qquad \begin{matrix}[a=\{x|\neg x\in x\}]^1\\ \Sigma_3'\\ a\in a\end{matrix}}{\bot}\,\neg E$$

Fianlly, a closed derivation Σ_4' of $\neg\exists y(y = \{x|\neg x \in x\})$ is established.

$$\dfrac{\dfrac{[\exists y(y=\{x|\neg x\in x\})]^4 \qquad \begin{matrix}[a=\{x|\neg x\in x\}]^1\\ \Sigma_3'\\ \bot\end{matrix}}{\bot}\,\exists E_{,1}}{\neg\exists y(y=\{x|\neg x\in x\})}\,\neg I_{,4}$$

Claim 2. Σ_4' is in normal form.

Given the lack of a maximum formula in Σ_4' and the fact that no applicable reduction process exists within $\mathbb{R}_{F'}$, it follows that Σ_4' constitutes a normal derivation. □

Appendix B. Prawitz's Derivation of Russell's Paradox in a Natural Deduction System for Naive Set Theory

Let us consider the natural deduction system S_N for naive set theory, constructed in a manner analogous to Prawitz (1965, Appendix B). S_N encompasses the rules for $\wedge$, $\rightarrow$, $\neg$, and the following additional rules:

$$\frac{\begin{array}{c}\mathfrak{D}\\ \varphi[t/x]\end{array}}{t \in \{x|\varphi(x)\}} \in I \qquad \frac{t \in \{x|\varphi(x)\}}{\varphi[t/x]} \in E$$

$\in -$rules have the following standard reduction process.

$$\dfrac{\dfrac{\begin{array}{c}\mathfrak{D}\\ \varphi[t/x]\end{array}}{t \in \{x|\varphi(x)\}} \in I}{\varphi[t/x]} \in E \qquad \rhd_{\in} \qquad \begin{array}{c}\mathfrak{D}\\ \varphi[t/x]\end{array}$$

Introduce a parameter a in the following manner: $\{x|\neg x \in x\}$. An application of the $\in I-$rule to $\neg a \in a$ yields the result $a \in a$, while an application of the $\in E-$rule to $a \in a$ gives rise to $\neg a \in a$. Prawitz (1965, p. 95) explores the inability to transform a derivation of $\perp$ originating from Russell's paradox into a normal derivation due to the generation of an infinite reduction sequence.

Proposition 5.2. *Let us define a parameter a as $\{x|\neg x \in x\}$. Then, there is a closed derivation of $\perp$ in S_N which generates an infinite reduction sequence and so is not normalizable.*

Proof. Two claims justify the result.

Claim 1. there exists a closed derivation Δ_3 of $\perp$.

To begin with, below and to the left, an open derivation Δ_1 of $\perp$

arises from $[a \in a]$. Coupled with the derivation Δ_1, a closed derivation Δ_2 of $a \in a$ is obtained below and to the right.

$$\dfrac{\dfrac{\dfrac{[a \in a]^1}{a \in \{x|\neg x \in x\}}\,def}{\neg a \in a}\in E \qquad [a \in a]^1}{\bot}\neg E \qquad\qquad \dfrac{\dfrac{\dfrac{\begin{matrix}[a \in a]^1\\ \Delta_1\\ \bot\end{matrix}}{\neg a \in a}\neg I_{,1}}{a \in \{x|\neg x \in x\}}\in I}{a \in a}\,def$$

Then, a closed derivation Δ_3 of $\bot$ is achieved.

$$\dfrac{\dfrac{\begin{matrix}[a \in a]^1\\ \Delta_1\\ \bot\end{matrix}}{\neg a \in a}\neg I_{,1} \qquad \begin{matrix}\Delta_2\\ a \in a\end{matrix}}{\bot}\neg E$$

Claim 2. Δ_3 generates an infinite reduction sequence and is not normalizable.

Δ_3 has a maximum formula $\neg a \in a$ in the last $\neg E-$rule and, by applying $\triangleright_{\neg}-$reduction, it reduces to the derivation Δ_4 below.

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}[a \in a]^1\\ \Delta_1\\ \bot\end{matrix}}{\neg a \in a}\neg I_{,1}}{a \in \{x|\neg x \in x\}}\in I}{\neg a \in a}\in E \qquad \begin{matrix}\Delta_2\\ a \in a\end{matrix}}{\bot}\neg E$$

Within Δ_4, a maximum formula $a \in \{x|\neg x \in x\}$ exists, which equates to $a \in a$ by definition, also apparent in the $\in E-$rule. Applying the

$\in$ −reduction results in a derivation identical to Δ_3, where the process initially began. Thus, the applications of reduction procedures to Δ_3 and Δ_4 generate an infinite reduction sequence, establishing that Δ_3 is not a normalizable derivation. $\square$

An unending oscillation between $\neg-$ and $\in$ −reductions characterizes the reduction process of Δ_3. Since it constantly results in maximum formulas, such as $a \in x|\neg x \in x$ and $\neg a \in a$, the process cannot eliminate all maximum formulas. This stands in contrast to Tennant's belief that Russell's paradox does not constitute a genuine paradox; conversely, Δ_3 adheres to *PCP* and appears to indicate that Russell's paradox embodies a genuine paradox.

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역설에 대한 증명론적 기준에서 진정한 역설이란 무엇인가?

최 승 락

닐 테넌트는 자연연역에서 형식화된 역설에 관한 증명론적 기준을 최초로 제안했다. 그에 따르면, 자연연역에서 형식화된 역설은 특정 형태의 바꿔쓰기(id est) 추론을 사용하며 받아들일 수 없는 결론을 도출하는 것인데 이것이 무한한 환원열을 양산하는 특징을 지닌다. 그는 자연연역에서 진정한 역설을 형식화하는 모든 도출이 이 기준을 충족한다고 여겼으며 거짓말쟁이 역설은 진정한 역설이지만 러셀의 역설은 그렇지 않다고 주장했다.

본 논문은 테넌트의 진정한 역설에 대한 가설을 자세히 살펴보고, 이 가설을 검증하기 위해서는 두 가지 문제 중 적어도 하나가 해결되어야 함을 제안한다. 첫 번째 문제는 비형식적인 의미에서 진정한 역설이 무엇을 의미하는지에 대한 철학적 합의가 필요하다는 것이다. 두 번째는 자연연역에서 역설을 형식하기 위한 유일한 방식이 제시되어야 한다는 것이다. 이 두 가지 문제 중 하나가 해결된다는 것은, 테넌트의 진정한 역설에 대한 가설이 검증될 수 있거나 최소한 역설에 관한 증명론적 특징을 설명하는 데 있어 철학적 중요성이 있다는데 의의가 있다 할 것이다.

주요어: 거짓말쟁이 역설, 러셀의 역설, 진정한 역설, 역설에 관한 증명론적 기준, 니일 테넌트.