Logics Based on Linear Orders of Contaminating Values

Roberto Ciuni[1], Thomas Macaulay Ferguson[2], and Damian Szmuc[3]

[1] Department FISPPA, Section of Philosophy, University of Padova
roberto.ciuni@unipd.it
[2] Cycorp and Saul Kripke Center, CUNY Graduate Center
tferguson@gradcenter.cuny.edu
[3] IIF-SADAF, CONICET and Department of Philosophy, University of Buenos Aires
szmucdamian@gmail.com

Abstract. A wide family of many-valued logics—for instance, those based on the weak Kleene algebra—include a non-classical truth-value that is "contaminating" in the sense that whenever the value is assigned to a formula φ, any complex formula in which φ appears is assigned that value as well. In such systems, the contaminating value enjoys a wide range of interpretations, suggesting scenarios in which more than one of these interpretations are called for. This calls for an evaluation of systems with *multiple* contaminating values. In this paper, we consider the countably infinite family of multiple-conclusion consequence relations in which classical logic is enriched with one or more contaminating values whose behavior is determined by a linear ordering between them. We consider some motivations and applications for such systems and provide general characterizations for all consequence relations in this family. Finally, we provide sequent calculi for a pair of four-valued logics including two linearly ordered contaminating values before defining two-sided sequent calculi corresponding to each of the infinite family of many-valued logics studied in this paper.

1 Introduction

A broad family of many-valued logics [3, 10, 11, 21, 26] impose a syntactic *filter* on logical consequence, to the effect that:

$$\Gamma \vDash \varphi \quad \text{only if} \quad Var(\varphi) \subseteq Var(\Gamma)$$

where $Var(\Gamma)$ represents the collection of propositional variables in a formula or set of formulae. This filter condition[4] and similar requirements are achieved by including a semantic value that is "contaminating" or "infectious" in the sense that whenever the value is assigned to a formula φ, any complex formula in which φ appears is assigned that value as well.[5] The most famous among the systems that include such a value is the three-valued *weak Kleene logic* $\mathsf{K}_3^{\mathsf{w}}$ by [3, 19], which obeys a *weaker* version of the filter condition above, namely:

$$\Gamma \vDash \varphi \quad \text{only if} \quad \begin{cases} \text{either} & Var(\varphi) \subseteq Var(\Gamma) \\ \text{or} & \Gamma \vDash \psi \quad \text{for all } \psi \end{cases}$$

[4] The condition is called *Proscriptive Principle* by [21], and the logics obeying it are among the systems usually called *containment logics*—we believe the reason for this is clear enough. In [24] the logics obeying the condition above or related ones are called *filter logics*, whence our name for the condition.

[5] This property has been very well-studied, under different names, in relation to particular systems or fragments of some systems. In [1] it is called *predominance of the atheoretical element*, in [7] it is referred to as *principle of contamination*, whereas [17] calls it *principle of component homogeneity*, and [13, 28] calls it *infectiousness*.

The contaminating value of $\mathsf{K}_3^{\mathsf{w}}$ and its sublogics [10, 11, 26] has been proposed as an adequate model for a remarkably diverse range of phenomena, including *linguistic*, *epistemic*, and *computer-theoretical* ones. In particular, Bochvar [3] uses the contaminating value of $\mathsf{K}_3^{\mathsf{w}}$ to reason about class-theoretic antinomies, while Fitting [14] uses it, in his bilattice-based semantics, to capture lack of shared expertise among groups of *experts*. Finally, Avron and others [2] use the contaminating value as a model for *catastrophic errors* encountered by a computer program, in the spirit of [20].

Now, each of the above phenomena may have different sources, or come in different varieties. For instance, the meaninglessness of a sentence can be due to category mistakes [27], Chomsky-style nonsense [6], or ill-formedness [1], and all or some of these traits can be found concurrently in a given set of complex expressions. Also, in computer-program applications, we often have multiple *virtual machines* running within one another (*e.g.*, a Java VM running inside Wine running inside Linux), with each of these possibly facing errors (be they catastrophic or not) or faults of some kind. Also, we can receive non-uniform expert advice because, along with some experts having no take on a given issue, two or more of them propose conflicting replies. The logic $\mathsf{K}_3^{\mathsf{w}}$ [3, 14] and related ones [2, 20] can model only one of these sources in isolation, and they cannot give an adequate insight on their possible interactions. These can be modeled just if many contaminating values are available.

In this paper, we serve this purpose by providing a general many-valued semantics in which classical logic is augmented by a linear order of contaminating values in which some values may be designated and others not. Depending on the range of the contaminating values admitted, many different consequence relations arise.[6] We present *general characterization* results of all such consequence relations in terms of the satisfaction of variable-inclusion properties between sets of premises and (sets of) conclusions—Theorem 1 and Theorem 2. Throughout the paper, we focus on standardly defined *multiple-conclusion* consequence.[7]

In this vein, throughout this paper we will be focusing on logics satisfying the following filter condition:

$$\Gamma \vDash \Delta \quad \text{only if} \quad Var(\Gamma') \subseteq Var(\Delta') \text{ for some non-empty } \Gamma' \subseteq \Gamma \text{ and } \Delta' \subseteq \Delta$$

or some weaker versions of it, later establishing that sometimes particular chains of variable-inclusion conditions are needed for logical consequence in matrices that extend classical logic with more than two contaminating values—Theorem 3 and Theorem 4. We also include proof-theoretical results, by providing decorated complete sequent calculi for a pair of four-valued logics whose non-classical values are contaminating and linearly ordered before defining two-sided sequent calculi for a countably infinite family of such systems.

The paper proceeds as follows. Section 2 introduces the basic notation and definitions that we use throughout the paper. Section 3 introduces the basic semantic machinery of contaminating values, which can be exemplified with the three-valued logics $\mathsf{K}_3^{\mathsf{w}}$ [3, 19] and PWK [18], and the simplest combination of contaminating values, which gives rise to the four-valued logics HYB_1 and HYB_2 [28]. Theorem 1, Theorem 2 and their corollaries are presented in this section. Section 4 deploys a straightforward, general method for the construction of matrices endowed

[6] There is a close connection between some of the many-valued matrices presented in this paper and an algebraic construction known as *Płonka sums of (direct systems of) logical matrices*, initially explored in [22, 23] and recently discussed in [4, 5]. In this paper we do not discuss the relation of our matrices with these constructions, but we hope to make a thorough examination of this topic in future works.

[7] The investigation of single-conclusion consequence relations induced by many-valued semantics counting with a linear order of contaminating values is another, deeply interesting project that we hope to explore in the near future.

with a linear order of (finitely many) contaminating values. Theorems 3–6 are presented in this section. In Section 5, we prove that the infinitely many LOC-matrices built on the matrix for CL induce infinitely many multiple-conclusion consequence relations. Section 6 presents sound and complete sequent calculi for the logics HYB_1 and HYB_2 (Theorem 8 and Theorem 10, respectively) and two-sided sequent calculi for an infinite family of their subsystems (Theorem 11). Finally, Section 7 presents some concluding remarks.

2 Preliminaries

Throughout the paper, we adopt the standard notation and basic definitions from *Abstract Algebraic Logic*, as presented *e.g.* in [15]. One important exception with regard to [15], however, concerns our definition of multiple-conclusion matrix consequence (see below).

Given a similarity type ν and a countably infinitely set X of generators, the absolutely free algebra $\mathbf{Fml}$ over X is called the *formula algebra* of type ν. Fml denotes the universe of $\mathbf{Fml}$. We call *propositional variables*—or *variables*, simply—the members of X, and we denote them by $p, q, r, \ldots$. We call *ν-formulae* the members of Fml, and we denote them by $\phi, \psi, \theta, \ldots$. We use $\varGamma, \varDelta, \varPsi \ldots$ to denote *sets of formulae*.[8] We omit reference to the type ν when this does not create confusion. In this paper, if no particular remark is made, $\mathbf{Fml}$ is assumed to be a formula algebra of type $(1, 2, 2)$, namely, of the type containing the connectives $\neg, \vee, \wedge$.

A *logic* of type ν is a pair $\mathsf{S} = \langle \mathbf{Fml}, \vdash_{\mathsf{S}} \rangle$, where $\mathbf{Fml}$ is a formula algebra of type ν and $\vdash_{\mathsf{S}} \subseteq \mathcal{P}(Fml) \times \mathcal{P}(Fml)$ is a substitution invariant *multiple-conclusion consequence relation*. A *ν-matrix*—or, simply, a *matrix*—is a pair $\mathcal{M} = \langle \mathbf{A}, \mathcal{D} \rangle$ with $\mathbf{A}$ an algebra of type ν with universe A and $\mathcal{D} \subset A$. $\mathcal{D}$ is called the *filter* of $\mathcal{M}$. Informally, we think of the members of A as *truth-values*. Under this informal reading, the members of $\mathcal{D}$ are naturally thought of as *designated values*.[9]

Just to make an example, classical logic CL is defined as $\langle \mathbf{Fml}, \models_{\mathcal{M}_{\mathsf{CL}}} \rangle$, and $\mathcal{M}_{\mathsf{CL}}$ is defined as $\langle \mathbf{B}_2, \{1\} \rangle$, where $\mathbf{B}_2 = \langle \{0, 1\}, \neg, \vee, \wedge \rangle$ is the well-known two-element Boolean algebra of type $(1, 2, 2)$. The elements 0 and 1 of its universe are informally interpreted as 'false' and 'true', respectively, with 1 being the only designated value. In this paper, we will focus especially in matrices that have $\mathcal{M}_{\mathsf{CL}}$ as a submatrix, in the following sense:

Definition 1. *A matrix $\mathcal{M} = \langle \mathbf{A}, \mathcal{D} \rangle$ is a submatrix of a matrix $\mathcal{M}' = \langle \mathbf{A}', \mathcal{D}' \rangle$ ($\mathcal{M} \sqsubseteq \mathcal{M}'$) if and only if $\mathbf{A}$ is a subalgebra of $\mathbf{A}'$ and $\mathcal{D} = \mathcal{D}' \cap A$.*

Logical matrices, in turn, can be seen to give raise to substitution invariant multiple-conclusion consequence relations, as Definition 3 illustrates.

Definition 2. *A valuation is a* homomorphism $v : \mathbf{Fml} \longrightarrow \mathbf{A}$ *from a formula algebra* $\mathbf{Fml}$ *into an algebra* $\mathbf{A}$ *of the same type.*

We denote by $\mathrm{Hom}_{\mathbf{Fml},\mathbf{A}}$ the set of valuations for $\mathbf{Fml}$ defined on $\mathbf{A}$. When $\mathbf{Fml}$ is clear by the context and we wish to focus on the matrix rather than on the algebra, we write $\mathrm{Hom}_{\mathcal{M}}$. For every $\mathcal{M} = \langle \mathbf{A}, \mathcal{D} \rangle$, we let $\mathrm{Hom}_{\mathcal{M}}(\varGamma)$ be the set $\{v \in \mathrm{Hom}_{\mathcal{M}} \mid v[\varGamma] \subseteq \mathcal{D}\}$ of the *models* of $\varGamma$ based on $\mathcal{M}$. In this paper, we focus on *matrix consequence relations*:

[8] Unless specified otherwise, in this paper we consider just *finite* sets of formulae, with the exception, of course, of Fml itself.

[9] Notice that, in using these notions, we do not assume or even try to stress that we do not allow the presence of matrices whose algebraic reduct is the trivial algebra. However, as will become clear shortly, in this paper our interest is in investigating logics induced by matrices having contaminating values which, in turn, extend the two-valued matrix inducing classical logic—i.e. the matrix whose algebraic reduct is the two-element Boolean algebra. We would like to thank an anonymous reviewer for urging us to clarify this issue.

Definition 3. *Given a matrix $\mathcal{M} = \langle \mathbf{A}, \mathcal{D} \rangle$, the relation $\models_{\mathcal{M}} \subseteq \mathcal{P}(Fml) \times \mathcal{P}(Fml)$ defined as follows:*

$$\Gamma \models_{\mathcal{M}} \Delta \;\Leftrightarrow\; \text{for every } v \in \mathrm{Hom}_{\mathcal{M}}, \nu[\Gamma] \subseteq \mathcal{D} \text{ implies } \nu(\psi) \in \mathcal{D} \text{ for some } \psi \in \Delta$$

is a multiple-conclusion matrix consequence *relation.*

We say that Δ is a *tautology* if and only if $\varnothing \models_{\mathcal{M}} \Delta$, and we say that Γ is *unsatisfiable* if and only if $\Gamma \models_{\mathcal{M}} \varnothing$. We write $\phi \models_{\mathcal{M}} \psi$ instead of $\{\phi\} \models_{\mathcal{M}} \{\psi\}$, and $\phi, \psi \models_{\mathcal{M}} \gamma, \delta$ instead of $\{\phi, \psi\} \models_{\mathcal{M}} \{\gamma, \delta\}$. We also use other notation, writing *e.g.* Γ, Δ for $\Gamma \cup \Delta$, or Γ, ϕ for $\Gamma \cup \{\phi\}$. Finally, when $\models_{\mathcal{M}_{\mathsf{S}}}$ is the matrix consequence relation of a logic S, we refer to $\models_{\mathcal{M}_{\mathsf{S}}}$ as to $\mathcal{M}_{\mathsf{S}}$-consequence.[10]

Before closing this section, it is of high importance to notice that the notion of multiple-conclusion consequence that we define here is different from the one defined in [15], which provides all the other basic notation and definitions in the present paper. In particular, Definition 3 above comes with the standard *disjunctive* reading of the right side of $\models_{\mathcal{M}}$, while [15, Definition 1.7] comes with a *conjunctive* reading of it—implying that *all* the formulae in the conclusion-set have to be satisfied. In fact, in [15], the author himself notices that his definition is not standard. In this paper, a particular reason to stick to the standard definition, as we did, is that the disjunctive reading of the right side of $\models_{\mathcal{M}}$ fits the interpretation of two-sided sequents in sequent calculi, and a uniform reading seems more appropriate in view of the results on sequent calculi from Section 6.

3 Basic Contaminating Logics

As we previously advertised, in this paper we are interested in logics with *contaminating* truth-values, that is, in logics induced by single logical matrices containing contaminating truth-values. Thus, in order to proceed to their study and analysis, we will distinguish two classes of such logics and, consequently, of such matrices.

The first class will comprise the basic contaminating logics, *i.e.* those logics induced by matrices complying with the most basic understanding of what a matrix with a contaminating logic is. The second class will comprise the logics equipped with a linear order of contaminating values, *i.e.* those logics induced by matrices having a plurality of linearly ordered contaminating values. In what follows, we begin our journey towards understanding basic contaminating logics by defining what an algebra with a contaminating element looks like.

Definition 4. *An algebra $\mathbf{A}$ of type ν has a contaminating element k if and only if there is a non-empty $A' \subseteq A$, with $A' \neq \{k\}$, such that for every m-ary $g \in \nu$ and every $\{a_1, \dots, a_m\} \subseteq A'$:*

$$\textit{if } k \in \{a_1, \dots, a_m\} \textit{ then } g^{\mathbf{A}}(a_1, \dots, a_m) = k$$

If $A' = A$, we say that k is *absolutely contaminating*; if A does not satisfy Definition 4 relative to k, but some $A' \subset A$ does, we say that k is *partially contaminating*. With the exception of $\mathsf{K}_3^{\mathsf{w}}$ and PWK, defined below, all the logics in this paper include one or more partially contaminating values alongside an absolutely contaminating one. In this regard, if $y \in A'$ and k contaminates the elements of A', we write $\mathcal{C}(y, k)$ for "y is contaminated by k".

Definition 5. *A matrix $\mathcal{M} = \langle \mathbf{A}, \mathcal{D} \rangle$ has a contaminating value k if $\mathbf{A}$ has a contaminating element k. Otherwise, we say $\mathcal{M}$ has no contaminating value.*

[10] For the basic setting defined in this section, see also [15, Chapter 1].

Our first example of a matrix extending the two-valued matrix $\mathcal{M}_{\mathsf{CL}}$ with a contaminating value are the matrices inducing the three-valued logics $\mathsf{K}_3^{\mathsf{w}}$ and PWK by [3] and [18], respectively. These are built using the so-called *weak Kleene algebra* **WK**, an algebra with an absolutely contaminating element introduced in [19]. More precisely, **WK** is the algebra of type $(1, 2, 2)$ whose universe is $\{0, n, 1\}$ and whose operations are given by Table 1.

Table 1.

	$\neg$
1	0
n	n
0	1

$\vee$	1	n	0
1	1	n	1
n	n	n	n
0	1	n	0

$\wedge$	1	n	0
1	1	n	0
n	n	n	n
0	0	n	0

It is clear from Table 1 that value n satisfies Definition 4 relative to $\{0, n, 1\}$. This is, in a way, the *simplest* case of contamination, where a value n contaminates all the values in the universe of the algebra in question.

Moving to the logics themselves, it is interesting to observe some features of the three-valued systems $\mathsf{K}_3^{\mathsf{w}}$ and PWK—for which sound and complete sequent calculi were provided in [9].

Definition 6. *$\mathsf{K}_3^{\mathsf{w}} = \langle \mathbf{Fml}, \models_{\mathcal{M}_{\mathsf{K}_3^{\mathsf{w}}}} \rangle$ and $\mathsf{PWK} = \langle \mathbf{Fml}, \models_{\mathcal{M}_{\mathsf{PWK}}} \rangle$, where:*

$$\mathcal{M}_{\mathsf{K}_3^{\mathsf{w}}} = \langle \mathbf{WK}, \{1\} \rangle \qquad \mathcal{M}_{\mathsf{PWK}} = \langle \mathbf{WK}, \{n, 1\} \rangle$$

$\mathsf{K}_3^{\mathsf{w}}$ lacks any tautology, exactly as its more famous kin K_3 by [19]. By contrast, PWK shares tautologies with classical logic CL, but it fails to validate some classical inference rules (most notably, *Ex Falso Quodlibet* and *Reductio ad Absurdum*), exactly as its more famous kin LP by [25]. The presence of a contaminating value determines further failures. In particular, we have $v(\varphi \vee \psi) = n$ in any valuation v such that $v(\varphi) = 1$ and $v(\psi) = n$, and $v(\varphi \wedge \psi) = n$ in any valuation v such that $v(\varphi) = 0$ and $v(\psi) = n$. Since $\mathcal{D}_{\mathcal{M}_{\mathsf{K}_3}} = \{1\}$ and $\mathcal{D}_{\mathcal{M}_{\mathsf{PWK}}} = \{n, 1\}$, this implies:

$$\varphi \not\models_{\mathcal{M}_{\mathsf{K}_3^{\mathsf{w}}}} \varphi \vee \psi \qquad \textit{Failure of Disjunctive Addition}$$
$$\varphi \wedge \psi \not\models_{\mathcal{M}_{\mathsf{PWK}}} \varphi \qquad \textit{Failure of Conjunctive Simplification}$$

However, notice that the following *local* versions of these properties hold:

$$\varphi \vee \psi \models_{\mathcal{M}_{\mathsf{K}_3^{\mathsf{w}}}} \varphi \vee \neg\varphi \qquad \textit{Local Excluded Middle}$$
$$\varphi \wedge \neg\varphi \models_{\mathcal{M}_{\mathsf{PWK}}} \varphi \wedge \psi \qquad \textit{Local Explosion}$$

Our second example of a matrix extending the two-valued matrix $\mathcal{M}_{\mathsf{CL}}$ with a contaminating value are the matrices inducing the four-valued logics HYB_1 and HYB_2, introduced in [28], themselves sublogics of $\mathsf{K}_3^{\mathsf{w}}$ and PWK. These matrices are built on the algebra **HYB**, which includes two contaminating elements. More precisely, **HYB** is the algebra of type $(1, 2, 2)$ whose universe is $\{0, n_1, n_2, 1\}$ and whose operations are given by Table 2.

It is clear, again, by looking at Table 2, that n_2 satisfies Definition 4 relative to the entire universe $\{0, n_1, n_2, 1\}$. By contrast, n_1 satisfies Definition 4 relative to $\{0, n_1, 1\}$ only. As a consequence, n_2 is *absolutely* contaminating, while n_1 is just *partially* contaminating.

Yet again, let us now turn to two logics induced by logical matrices built using the **HYB** algebra, the systems HYB_1 and HYB_2—for which we will provide sound and complete sequent calculi in Section 6.

Table 2.

	$\neg$
1	0
n_1	n_1
n_2	n_2
0	1

$\vee$	1	n_1	n_2	0
1	1	n_1	n_2	1
n_1	n_1	n_1	n_2	n_1
n_2	n_2	n_2	n_2	n_2
0	1	n_1	n_2	0

$\wedge$	1	n_1	n_2	0
1	1	n_1	n_2	0
n_1	n_1	n_1	n_2	n_1
n_2	n_2	n_2	n_2	n_2
0	0	n_1	n_2	0

Definition 7. $\mathsf{HYB}_1 = \langle \mathbf{Fml}, \models_{\mathcal{M}_{\mathsf{HYB}_1}} \rangle$ *and* $\mathsf{HYB}_2 = \langle \mathbf{Fml}, \models_{\mathcal{M}_{\mathsf{HYB}_2}} \rangle$*, where:*

$$\mathcal{M}_{\mathsf{HYB}_1} = \langle \mathbf{HYB}, \{n_1, 1\} \rangle \qquad \mathcal{M}_{\mathsf{HYB}_2} = \langle \mathbf{HYB}, \{n_2, 1\} \rangle$$

Each of HYB_1 and HYB_2 shares all the failures of $\mathsf{K}_3^{\mathsf{w}}$ and PWK. Additionally, the following distinguish the two logics HYB_1 and HYB_2 from $\mathsf{K}_3^{\mathsf{w}}$ and PWK:

$$\varphi \vee \psi \models_{\mathcal{M}_{\mathsf{HYB}_1}} \varphi \vee \neg\varphi \qquad \varphi \wedge \neg\varphi \not\models_{\mathcal{M}_{\mathsf{HYB}_1}} \varphi \wedge \psi$$

$$\varphi \vee \psi \not\models_{\mathcal{M}_{\mathsf{HYB}_2}} \varphi \vee \neg\varphi \qquad \varphi \wedge \neg\varphi \models_{\mathcal{M}_{\mathsf{HYB}_2}} \varphi \wedge \psi$$

As for Local Excluded Middle, any valuation v such that $v(\psi) = v(\varphi \vee \psi) = n_2$ and $v(\varphi) = n_1$ is such that $v(\varphi \vee \psi) \in \mathcal{D}_{\mathcal{M}_{\mathsf{HYB}_2}}$ and $v(\varphi \vee \neg\varphi) \notin \mathcal{D}_{\mathcal{M}_{\mathsf{HYB}_2}}$. Also, for every valuation v such that $v(\varphi \vee \psi) \in \{n_1, 1\}$, we have $v(\varphi \vee \neg\varphi) \in \{n_1, 1\}$. Since $\mathcal{D}_{\mathcal{M}_{\mathsf{HYB}_1}} = \{n_1, 1\}$, the rule has no countermodel in $\mathcal{M}_{\mathsf{HYB}_1}$. As for Local Explosion, any valuation v where $v(\varphi \wedge \neg\varphi) = n_1$ and $v(\psi) = n_2$ provides a countermodel to the rule in HYB_1; for every valuation v where $v(\varphi) = v(\varphi \wedge \neg\varphi) = n_2$, we have $v(\varphi \wedge \psi) = n_2$ by contamination. Since $\mathcal{D}_{\mathcal{M}_{\mathsf{HYB}_2}} = \{n_2, 1\}$, the rule has no countermodel in $\mathcal{M}_{\mathsf{HYB}_2}$.

After analyzing these examples of basic contaminating logics, it is interesting to consider the rather general idea of obtaining an extension of a given matrix $\mathcal{M}$ by adjoining it an absolutely contaminating value n—to later study the logic induced by this single matrix. In order to do this, we make precise what extending a given matrix $\mathcal{M}$ with an absolutely contaminating value would amount to.

Definition 8. *Given an algebra* $\mathbf{A}$ *of type* ν*, let* $\mathbf{A}[n]$ *be the algebra of the same type that results from adjoining to* $\mathbf{A}$ *an absolutely contaminating element* n *such that* $n \notin A$*,* i.e. $\mathbf{A}[n]$ *is such that for every* m*-ary* $g \in \nu$ *and every* $\{a_1, \ldots, a_m\} \subseteq A \cup \{n\}$*:*

$$g^{\mathbf{A}[n]}(a_1, \ldots, a_m) = \begin{cases} n & \textit{if } n \in \{a_1, \ldots, a_m\} \\ g^{\mathbf{A}}(a_1, \ldots, a_m) & \textit{otherwise} \end{cases}$$

Definition 9. *Given a matrix* $\mathcal{M} = \langle \mathbf{A}, \mathcal{D} \rangle$*, let* $\mathcal{M}[n] = \langle \mathbf{A}[n], \mathcal{D} \cup \mathcal{D}' \rangle$*, where* $\mathcal{D}' \subseteq \{n\}$ *be the matrix with a contaminating value that results from adjoining an absolutely contaminating value* n *to* $\mathcal{M}$*.*

With the help of these definitions, we are now in a position to study the case of some basic logics induced by single logical matrices which have a contaminating value. To this extent, the following two theorems establish that, for every matrix $\mathcal{M}[n]$ extending a matrix $\mathcal{M}$ with a contaminating value, the corresponding $\mathcal{M}[n]$-consequence can be characterized on the ground of $\mathcal{M}$-consequence alone, together with certain filter conditions.

Theorem 1. *Given a matrix* $\mathcal{M} = \langle \mathbf{A}, \mathcal{D} \rangle$*, let* $\mathcal{M}[b] = \langle \mathbf{A}[b], \mathcal{D} \rangle$ *be the matrix with a contaminating value that results from adjoining a non-designated absolutely contaminating value* b *to* $\mathcal{M}$*. Then,* $\mathcal{M}[b]$*-consequence can be characterized as follows:*

$$\Gamma \models_{\mathcal{M}_{[b]}} \Delta \;\Leftrightarrow\; Var(\Delta') \subseteq Var(\Gamma) \textit{ for some } \Delta' \subseteq \Delta \textit{ s.t. } \Gamma \models_{\mathcal{M}} \Delta'$$

Proof. For left-to-right, we prove this by contraposition. Assume it is not the case that $Var(\Delta') \subseteq Var(\Gamma)$ for some $\Delta' \subseteq \Delta$ such that $\Gamma \vDash_{\mathcal{M}} \Delta'$.

If $\Gamma \vDash_{\mathcal{M}} \varnothing$, then for every $\mathcal{M}$ valuation v we have that $v(\Gamma) \nsubseteq \mathcal{D}$—i.e. there are no $\mathcal{M}$ valuations under which each formula in Γ is designated. If this is the case, the filter condition is trivially satisfied by letting Δ' be $\varnothing$. Furthermore, given the set of designated values of $\mathcal{M}[b]$ is the same than those of $\mathcal{M}$, in this case we would also have that $\Gamma \vDash_{\mathcal{M}[b]} \varnothing$.

If $\Gamma \nvDash_{\mathcal{M}} \varnothing$, *i.e.* if Γ is satisfiable in $\mathcal{M}$, we reason as follows. Suppose that for every $\Delta' \subseteq \Delta$, either $Var(\Delta') \nsubseteq Var(\Gamma)$ or $\Gamma \nvDash_{\mathcal{M}} \Delta'$. We construct an $\mathcal{M}[b]$ valuation witnessing that $\Gamma \nvDash_{\mathcal{M}[b]} \Delta$.

Now, by the condition assumed on Δ we can split Δ into two sets, the set $\Delta^{\diamond} = \{\psi \in \Delta \mid Var(\psi) \nsubseteq Var(\Gamma)\}$ and its complement $\Delta^{\bullet} = \Delta \smallsetminus \Delta^{\diamond}$. Importantly, because $Var(\Delta^{\bullet}) \subseteq Var(\Gamma)$, our supposition entails that $\Gamma \nvDash_{\mathcal{M}} \Delta^{\bullet}$. Also, because Γ is by hypothesis satisfiable in $\mathcal{M}$ and because $\Gamma \nvDash_{\mathcal{M}} \Delta^{\bullet}$, there exists an $\mathcal{M}$ valuation v such that $v[\Gamma] \subseteq \mathcal{D}$ and $v[\Delta^{\bullet}] \cap \mathcal{D} = \varnothing$.

Now, from this valuation v, we define an $\mathcal{M}[b]$ valuation $v^{\star}$ by the following scheme:

$$v^{\star}(p) = \begin{cases} v(p) \text{ if } p \in Var(\Gamma) \\ b \text{ otherwise} \end{cases}$$

Recall that $\mathcal{D}_{\mathcal{M}[b]} = \mathcal{D}_{\mathcal{M}}$, by definition. Then, because $v^{\star}$ agrees with v with respect to the propositional variables appearing in Γ, $v^{\star}(\Gamma) \subseteq \mathcal{D}_{\mathcal{M}[b]}$. Moreover, because $Var(\Delta^{\bullet}) \subseteq Var(\Gamma)$, for each $\psi \in \Delta^{\bullet}$, $v^{\star}(\psi) \notin \mathcal{D}_{\mathcal{M}[b]}$. If $\Delta^{\diamond} = \varnothing$, this suffices to have a countermodel witnessing that $\Gamma \nvDash_{\mathcal{M}[b]} \Delta$, since $Var(\Delta) = Var(\Delta^{\bullet})$ and $Var(\Delta^{\bullet}) \subseteq Var(\Delta)$. If $\Delta^{\diamond} \neq \varnothing$, then by construction every $\psi \in \Delta^{\diamond}$ contains a propositional variable p such that $v^{\star}(p) = b$. Whence, for each $\psi \in \Delta^{\diamond}$, $v^{\star}(\psi) = b \notin \mathcal{D}_{\mathcal{M}[b]}$. Because $\Delta = \Delta^{\bullet} \cup \Delta^{\diamond}$, $v^{\star}$ provides a countermodel witnessing that $\Gamma \nvDash_{\mathcal{M}[b]} \Delta$.

As for right-to-left, assume there exists a $\Delta' \subseteq \Delta$ such that $Var(\Delta') \subseteq Var(\Gamma)$ and $\Gamma \vDash_{\mathcal{M}} \Delta'$. Hence, any $\mathcal{M}[b]$ valuation v for which $v(\Gamma) \subseteq \mathcal{D}$ is—when restricted to $Var(\Gamma)$—essentially an $\mathcal{M}$ valuation. This implies $\mathrm{Hom}_{\mathcal{M}[b]}(\Gamma) \subseteq \mathrm{Hom}_{\mathcal{M}[b]}(\Delta')$ if and only if $\mathrm{Hom}_{\mathcal{M}}(\Gamma) \subseteq \mathrm{Hom}_{\mathcal{M}}(\Delta')$ if $\Delta' = \varnothing$. Otherwise, the valuation maps each $\psi \in \Delta'$ to a designated value. As v was selected arbitrarily, this reasoning extends to *any* $\mathcal{M}[b]$ valuation, whence $\Gamma \vDash_{\mathcal{M}[b]} \Delta'$ and *a fortiori* $\Gamma \vDash_{\mathcal{M}[b]} \Delta$. ■

Theorem 2. *Given a matrix $\mathcal{M} = \langle \mathbf{A}, \mathcal{D} \rangle$, let $\mathcal{M}[h] = \langle \mathbf{A}[h], \mathcal{D} \cup \{h\} \rangle$ be the matrix with a contaminating value that results from adjoining a designated absolutely contaminating value h to $\mathcal{M}$. Then, $\mathcal{M}[h]$-consequence can be characterized as follows:*

$$\Gamma \vDash_{\mathcal{M}[h]} \Delta \;\Leftrightarrow\; Var(\Gamma') \subseteq Var(\Delta) \textit{ for some } \Gamma' \subseteq \Gamma \textit{ s.t. } \Gamma' \vDash_{\mathcal{M}} \Delta$$

Proof. For left-to-right, we again prove this by contraposition. Assume that for every $\Gamma' \subseteq \Gamma$, either $Var(\Gamma') \nsubseteq Var(\Delta)$ or $\Gamma' \nvDash_{\mathcal{M}} \Delta$. As before, we may split Γ into two sets: $\Gamma^{\bullet} = \{\psi \in \Gamma \mid Var(\psi) \nsubseteq Var(\Delta)\}$ and $\Gamma^{\diamond} = \Gamma \smallsetminus \Gamma^{\bullet}$.

By construction, $Var(\Gamma^{\diamond}) \subseteq Var(\Delta)$, whence $\Gamma^{\diamond} \nvDash_{\mathcal{M}} \Delta$, in which case we fix an $\mathcal{M}$ valuation v witnessing the failure of this inference. From v, we again define an $\mathcal{M}[h]$ valuation v^{*}:

$$v^{*}(p) = \begin{cases} v(p) \text{ if } p \in Var(\Delta) \\ h \text{ otherwise} \end{cases} .$$

Recall that $\mathcal{D}_{\mathcal{M}[h]} = \mathcal{D}_{\mathcal{M}} \cup \{h\}$, by definition. Because v^{*} restricted to the propositional variables of Δ—and *a fortiori* to the propositional variables of $\Gamma^{\diamond}$—is coextensional with v, we know that $v^{*}(\Delta) \cap \mathcal{D}_{\mathcal{M}[h]} = \varnothing$ while $v^{*}(\Gamma^{\diamond}) \subseteq \mathcal{D}_{\mathcal{M}[h]}$. If $\Gamma^{\bullet} = \varnothing$, this suffices to have a countermodel witnessing that $\Gamma \nvDash_{\mathcal{M}[h]} \Delta$, since $Var(\Gamma) = Var(\Gamma^{\diamond})$ and $Var(\Gamma^{\diamond}) \subseteq Var(\Delta)$. If $\Gamma^{\bullet} \neq \varnothing$, since h contaminates all other values, by construction we have that $v^{*}(\Gamma^{\bullet}) = \{h\} \subseteq \mathcal{D}_{\mathcal{M}[h]}$. Hence, v^{*} maps every formula of $\Gamma = \Gamma^{\diamond} \cup \Gamma^{\bullet}$ to a designated value yet fails to map any $\psi \in \Delta$ to a designated value, *i.e.* v^{*} witnesses that $\Gamma \nvDash_{\mathcal{M}[h]} \Delta$.

For right-to-left, we assume that there is a $\Gamma' \subseteq \Gamma$ such that $Var(\Gamma') \subseteq Var(\Delta)$ for which $\Gamma' \vDash_{\mathcal{M}} \Delta$. If $\Gamma' = \varnothing$, since $A \setminus \mathcal{D} = (A \cup \{h\}) \setminus \mathcal{D}_{\mathcal{M}[h]}$, and h contaminates every other value, then every $\mathcal{M}[h]$ valuation that is a countermodel for every $\psi \in \Delta$ is—when restricted to the variables in Δ—an $\mathcal{M}$ valuation that is a countermodel to every $\psi \in \Delta$.

If $\Gamma' \neq \varnothing$, then for any $\mathcal{M}[h]$ valuation v such that $v(\Gamma') \subseteq \mathcal{D}_{\mathcal{M}[h]}$, if $h \in v(\Delta)$ then there is a $\psi \in \Delta$ such that $v(\psi) = h$. Otherwise—if $h \notin v(\Delta)$—then because all propositional variables appearing in Γ' appear in Δ, also $h \notin v(\Gamma')$. Hence, v restricted to the propositional variables appearing in Δ is essentially an $\mathcal{M}$ valuation, and the fact that $\Gamma' \vDash_{\mathcal{M}} \Delta$ ensures that $v(\psi) \in \mathcal{D}$ for some $\psi \in \Delta$. Hence, in either case we conclude that $\Gamma' \vDash_{\mathcal{M}[h]} \Delta$ and *a fortiori* that $\Gamma \vDash_{\mathcal{M}[h]} \Delta$. ■

Interestingly, these two theorems have immediate corollaries concerning our previous examples of logics induced by single matrices which extend the two-valued matrix for classical logic $\mathcal{M}_{\mathsf{CL}}$ with contaminating values. As is easy to observe, $\mathcal{M}_{\mathsf{K}_3^{\mathsf{w}}}$ is the matrix $\mathcal{M}_{\mathsf{CL}}[b]$ obtained by extending $\mathcal{M}_{\mathsf{CL}}$ with a non-designated contaminating value b, while $\mathcal{M}_{\mathsf{PWK}}$ is the matrix $\mathcal{M}_{\mathsf{CL}}[h]$ obtained by extending $\mathcal{M}_{\mathsf{CL}}$ with a designated contaminating value h. Thus, from Theorem 1 and Theorem 2 we obtain the next result.

Corollary 1. *$\mathcal{M}_{\mathsf{K}_3^{\mathsf{w}}}$- and $\mathcal{M}_{\mathsf{PWK}}$-consequence can be characterized as follows:*

$$\Gamma \vDash_{\mathcal{M}_{\mathsf{K}_3^{\mathsf{w}}}} \Delta \;\Leftrightarrow Var(\Delta') \subseteq Var(\Gamma) \textit{ for some } \Delta' \subseteq \Delta \textit{ s.t. } \Gamma \vDash_{\mathcal{M}_{\mathsf{CL}}} \Delta'$$

$$\Gamma \vDash_{\mathcal{M}_{\mathsf{PWK}}} \Delta \Leftrightarrow Var(\Gamma') \subseteq Var(\Delta) \textit{ for some } \Gamma' \subseteq \Gamma \textit{ s.t. } \Gamma' \vDash_{\mathcal{M}_{\mathsf{CL}}} \Delta$$

Corollary 1 improves the characterization results by [7, 12]. Additionally, it offers a different look at the above failures and validities in $\mathsf{K}_3^{\mathsf{w}}$ and PWK—especially concerning the lack of tautologies and the failure of Disjunctive Addition in $\mathsf{K}_3^{\mathsf{w}}$, as well as the identity between classical tautologies and PWK-tautologies and the failure of Conjunctive Simplification in PWK.

Moreover, the generality of Theorem 1 and Theorem 2 allows us to observe that $\mathcal{M}_{\mathsf{HYB}_1}$ is the matrix $\mathcal{M}_{\mathsf{PWK}}[b]$, whereas $\mathcal{M}_{\mathsf{HYB}_2}$ is the matrix $\mathcal{M}_{\mathsf{K}_3^{\mathsf{w}}}[h]$. Furthermore, this allows us to establish that $\mathcal{M}_{\mathsf{HYB}_1}$ is the matrix $\mathcal{M}_{\mathsf{CL}}[hb]$ obtained by extending $\mathcal{M}_{\mathsf{CL}}$ first with a designated contaminating value h, and then with a non-designated contaminating value b. On the other hand, $\mathcal{M}_{\mathsf{HYB}_2}$ is the matrix $\mathcal{M}_{\mathsf{CL}}[bh]$ obtained by inverting h and b in the extension procedure. Thus, from Corollary 1, Theorem 1 and Theorem 2 we obtain the next result.

Corollary 2. *$\mathcal{M}_{\mathsf{HYB}_1}$-consequence and $\mathcal{M}_{\mathsf{HYB}_2}$-consequence can be characterized as follows:*

$$\begin{aligned}\Gamma \vDash_{\mathcal{M}_{\mathsf{HYB}_1}} \Delta \Leftrightarrow\; & Var(\Gamma') \subseteq Var(\Delta') \subseteq Var(\Gamma) \\ & \textit{for some } \Gamma' \subseteq \Gamma, \Delta' \subseteq \Delta \textit{ s.t. } \Gamma' \vDash_{\mathcal{M}_{\mathsf{CL}}} \Delta'\end{aligned}$$

$$\begin{aligned}\Gamma \vDash_{\mathcal{M}_{\mathsf{HYB}_2}} \Delta \Leftrightarrow\; & Var(\Delta') \subseteq Var(\Gamma') \subseteq Var(\Delta) \\ & \textit{for some } \Gamma' \subseteq \Gamma, \Delta' \subseteq \Delta \textit{ s.t. } \Gamma' \vDash_{\mathcal{M}_{\mathsf{CL}}} \Delta'\end{aligned}$$

Corollary 2 improves the characterization results suggested by [28], and gives a different perspective on the above failures and validities in HYB_1 and HYB_2—especially concerning the failure or validity of Local Excluded Middle and Local Explosion, respectively.

4 Contaminating Logics with a Linear Ordered of Contaminating Values

In computer programs, two prominent kinds of errors may cause a system to permanently halt. On the level of *software*, we can have errors in code (such as an attempt to assign a value to an undeclared variable), which in turn may cause a process to halt. On the level of *hardware*, we can have physical errors that are caused, for instance, when an environment attempts to retrieve a value from a physical address that is corrupt. As noticed by [2], we may want to distinguish between the two kinds of errors when modeling the behavior of a program that is encountering a fault in some of its procedures.

An application of $\mathsf{K}_3^{\mathsf{w}}$ to errors at the level of code has been provided by [12]. More precisely, [12] represents code errors in the language C++ by means of the value n from $\mathcal{M}_{\mathsf{K}_3^{\mathsf{w}}}$. In C++, *undeclared* variables are not treated as variables, and an expression in which they appear will not be computed, exactly as an ill-formed string of symbols.[11]

Note that these two types of errors—errors in code and errors in the physical constitution of hardware—enjoy the type of linear ordering that has been central to this paper. The triggering of the syntactic error at the local level—that is, within the virtual machine—may cause the *environment within which the executable was run* to halt prematurely. But that this occurs within the scope of a virtual machine insulates the operating system from such local errors. On the other hand, if the *operating system* attempts to retrieve a value *on behalf of* the virtual machine from a bad address, the error that causes the operating system to fail will bring down the virtual machine alongside it.

Now, there is a distinction between the semantic features of an "error" value's being *contaminating* and with its being *designated*. In the former case, the semantic features are forced upon us by the scenario itself. In the latter case, however, whether or not a value should be taken to be designated is a *pragmatic* decision, determined by an end user's interest. For example, Halldén, in [18], allowed that some formulae should be valid even if there are occasions in which they are meaningless. Halldén defended this by arguing that the *validity* of a formula should be judged solely on the basis of its *meaningful* instances. In a similar vein, an end user may similarly be concerned with the stability of the code itself and *not* in the stability of the physical memory. The parallel with Halldén's treatment of the contaminating value in PWK, then, suggests that one might justifiably consider this global error to be designated.

In the case of a large ontology with an integrated theorem prover, for example, one might wish for certain theorems to be derivable, in spite of the potential for hardware errors. In this case, practical concerns make lead the ontology's developers to *discount* this type of situation from consideration when judging validity, just as Halldén *elects* to discount meaninglessness. Furthermore, when one is testing code, some tiers of errors are important to acknowledge while others are not. Simply put, whether one's code leads to a software error is part of a developer's concern; the fact that a particular piece of hardware upon which the software runs crashes due to faulty RAM is not. If we follow Halldén in taking practical concerns to determine whether a particular semantic category is designated or not, then we clearly encounter scenarios in which some contaminating values ought to be designated while others should not.

In the present era—in which development is increasingly virtualized—the line between software errors and hardware errors rapidly blurs. One might develop in a language run in a virtual machine hosted in a Docker container running on a server. Given the prevalence of these types of linearly nested development environments, one might as well be interested in situations where we have an arbitrarily deep cascade of situations featuring aspects that deserve to be modelled

[11] A syntactic object p is treated as a variable—or, is a *declared* variable—if the interpreter is informed that p is to be used in this manner.

by contaminating values, some of which we may choose to be designated and some of which we may choose not to be.

Thus, in the next sections we extend our previous considerations to build appropriate semantic tools to model such settings. We do this by appealing to the idea of a linear order of contaminating values, such that the greater contaminating values contaminate the smaller ones and, of course, the non-contaminating values.

4.1 Formal Definitions

The extension procedure mentioned in the previous section allows to generate an infinity of matrices with contaminating values extending the two-valued matrix $\mathcal{M}_{\mathsf{CL}}$ that induces classical logic—and, in general, extending any given matrix $\mathcal{M}$. In particular, we focus particularly on the case of those matrices that have a *linear order* of contaminating values. To this extent, we begin this section by defining what an algebra with a linear order of contaminating values amounts to.

Definition 10. *An algebra* $\mathbf{A}$ *of type* ν *has a linear order of contaminating elements* $n_1 \dots n_k$ *(with* $1, \dots, k \in \omega$*) if and only if each* $n_j \in \{n_1, \dots, n_k\}$ *is an absolutely contaminating element in the subalgebra* $\mathbf{A}[n_1 \dots n_j]$ *of the same type, whose universe is* $A \setminus \{n_i \mid i > j \geq 1\}$.

Definition 11. *A matrix* $\mathcal{M} = \langle \mathbf{A}, \mathcal{D} \rangle$ *has a linear order of contaminating values* $n_1 \dots n_k$ *if* $\mathbf{A}$ *has a linear order of contaminating elements* $n_1 \dots n_k$*. If this is the case, we say that* $\mathcal{M}$ *is a LOC-matrix.*

It is indeed easy to check that, given an algebra with a set of contaminating elements $\{n_1, \dots, n_k\}$ complying with the definition above, then the following holds for every $n_i, n_j, n_m \in \{n_1, \dots, n_k\}$:

1. If $\mathcal{C}(n_i, n_j)$ and $\mathcal{C}(n_j, n_i)$, then $n_i = n_j$
2. If $\mathcal{C}(n_i, n_j)$ and $\mathcal{C}(n_j, n_m)$, then $\mathcal{C}(n_i, n_m)$
3. $\mathcal{C}(n_i, n_j)$ or $\mathcal{C}(n_j, n_i)$

The properties above correspond, respectively, to the *antisymmetry*, *transitivity* and *totality* of the relation $\mathcal{C}$, whence by definition $\mathcal{C}$ turns to be a linear order on $\{n_1, \dots, n_k\}$. Given this, we believe that talk of a linear order of contaminating elements of an algebra or—alternatively—of a *linear order of contaminating values* of a matrix, is justified.

Notice that for some LOC-matrices, the linear order of the contaminating values can be described in terms of some independent orderings induced by the underlying algebra.[12] In particular, given the matrix $\mathcal{M}_{\mathsf{CL}}$, consider the LOC-matrix $\mathcal{M}_{\mathsf{CL}}[n_1, \dots, n_k]$ whose underlying algebra is $\mathbf{B}_2[n_1, \dots, n_k] = \langle \{0, 1, n_1, \dots, n_k\}, \neg, \vee, \wedge \rangle$, where $\mathbf{B}_2$ is the previously referred two-element Boolean algebra. Then, we can define:

$$a \leq_\vee c \Leftrightarrow a \vee c = c \qquad\qquad a \leq_\wedge c \Leftrightarrow a \wedge c = a$$

In this regard, it is easy to see that both $\leq_\vee$ and $\leq_\wedge$ linearly order $\{0, 1, n_1, \dots, n_k\}$. Indeed, we have $0 <_\vee 1 <_\vee n_1 <_\vee \dots <_\vee n_{k-1} <_\vee n_k$ and $n_k <_\wedge n_{k-1} <_\vee \dots n_1 <_\wedge 0 <_\wedge 1$. Furthermore, we can observe that with the help of $\leq_\vee$ and $\leq_\wedge$ it is possible to provide an alternative definition of the contaminating relation, in the following terms:

$$\mathcal{C}(a, c) \Leftrightarrow a \leq_\vee c \text{ and } c \leq_\wedge a$$

[12] We thank an anonymous reviewer for noticing this fact.

This alternative definition allows us to interpret "a is contaminated by c" as "a is lesser than c according to order $\leq_\vee$ and greater than c according to order $\leq_\wedge$".

This highlights an interesting connection between LOC-matrices and a family of algebraic structures known as *involutive bisemilattices*. These are algebras $\mathbf{A} = \langle A, \vee, \wedge, \neg \rangle$ such that (i) $\langle A, \vee \rangle$ and $\langle A, \wedge \rangle$ are *semilattices*, and (ii) $\neg\neg a = a$, $a \wedge c = \neg(\neg a \vee \neg c)$, $a \wedge (\neg a \vee c) = a \wedge c$.[13] More concretely, given a LOC-matrix $\mathcal{M}[n_1, \ldots, n_k]$ whose underlying algebra $\mathbf{A}[n_1, \ldots, n_k]$ is an involutive bisemilattice $\langle A \cup \{n_1, \ldots, n_k\}, \vee, \wedge, \neg \rangle$, as is the case with any LOC-matrix extending $\mathcal{M}_{\mathsf{CL}}$, then the contamination order of $\mathcal{M}[n_1, \ldots, n_k]$ can be described as in the previous paragraph—using $\leq_\vee$ and $\leq_\wedge$. Furthermore, involutive bisemilattices can be represented in terms of *Płonka sums of (direct systems of) algebras* (cf. [22, 23]). This is, in fact, of special interest for us given that some—but perhaps not all—LOC-matrices whose algebra reduct is an involutive bisemilattice can be represented in terms of *Płonka sums of (direct systems of) logical matrices*, which are themselves based on Płonka sums of their underlying algebras (cf. [4, 5, 22, 23]).

Having made these remarks, we now focus on the analysis of the extensions of $\mathcal{M}_{\mathsf{CL}}$—and, in general, of any given matrix $\mathcal{M}$—obtained by adjoining it a linear order of contaminating values $n_1 \ldots n_k$, to later study the logic induced by this single matrix. In order to do this, in what follows we make precise what extending a given matrix $\mathcal{M}$ with such a linear order of contaminating values amounts to.

Definition 12. *Given an algebra $\mathbf{A}$ of type ν, let $\mathbf{A}[n_1 \ldots n_k]$ be the algebra of the same type that results from adjoining to $\mathbf{A}$ a linear order of contaminating elements $n_1 \ldots n_k$ such that $A \cap \{n_1 \ldots n_k\} = \emptyset$,* i.e. *$\mathbf{A}[n_1 \ldots n_k]$ is such that each $n_j \in \{n_1, \ldots, n_k\}$ is an absolutely contaminating element in the algebra $\mathbf{A}[n_1 \ldots n_j]$, whose universe is $A \cup \{n_i \mid 1 \leq i \leq j\}$.*

Alternatively, $\mathbf{A}[n_1 \ldots n_k]$ can be seen as the result of adjoining an absolutely contaminating value to the algebra $\mathbf{A}[n_1 \ldots n_{k-1}]$. Whence:

$$\mathbf{A}[n_1 \ldots n_k] = \mathbf{A}[n_1 \ldots n_{k-1}][n_k] = \mathbf{A}[n_1 \ldots n_{k-2}][n_{k-1}][n_k] = \cdots = \mathbf{A}[n_1] \ldots [n_k]$$

Definition 13. *Given a matrix $\mathcal{M} = \langle \mathbf{A}, \mathcal{D} \rangle$, let $\mathcal{M}[n_1 \ldots n_k] = \langle \mathbf{A}[n_1 \ldots n_k], \mathcal{D} \cup D' \rangle$, where $D' \subseteq \{n_1, \ldots, n_k\}$ be the LOC-matrix that results from adjoining a linear order of contaminating values $n_1 \ldots n_k$ to $\mathcal{M}$.*

We reprise the convention from the previous section and use b to denote a *non-designated* contaminating value and h to denote a *designated* contaminating value. In this vein, we can think of any LOC-matrix $\mathcal{M}[n_1 \ldots n_k]$ as a matrix having alternations of the value b and the value h, *i.e.* by replacing each undesignated contaminating value in $n_1 \ldots n_k$ for b, and each designated contaminating value in $n_1 \ldots n_k$ for h. Thus, for instance the LOC-matrix $\mathcal{M}[n_1, n_2, n_3]$ where n_1 and n_2 are undesignated would become the matrix $\mathcal{M}[bbh]$, whereas the the LOC-matrix $\mathcal{M}[n_1, n_2, n_3, n_4]$ where n_2 and n_4 are designated would become the matrix $\mathcal{M}[bhbh]$.

In fact, to be precise enough, in these cases we should differentiate each instance of a non-designated and a designated contaminating value by enumerating each of these in parallel and consecutively. That is, for instance, by referring to the LOC-matrix $\mathcal{M}[bbh]$ in more precise terms as the matrix $\mathcal{M}[b_1 b_2 h_1]$, and similarly by referring to the LOC-matrix $\mathcal{M}[bhbh]$ as the matrix $\mathcal{M}[b_1 h_1 b_2 h_2]$. For the sake of simplicity, however, we will try to keep the simpler notation referring *e.g.* to $\mathcal{M}[h_1 b_1 b_2 b_3 h_2 b_4]$ as $\mathcal{M}[hbbbhb]$, and so on and so forth, hoping that the reader bears in mind the ultimate meaning of this nomenclature.

[13] We refer the reader to [4, 23] for a detailed treatment of involutive bisemilattices.

Finally, with regard to LOC-matrices we will say that $\mathcal{M}[\dots b]$ has a contaminating undesignated value "on top" of its linear order of contaminating values, while the $\mathcal{M}[\dots h]$ has a contaminating designated value "on top" of its linear order of contaminating values.

4.2 Characterization Results

In this section, for any given LOC-matrix $\mathcal{M}_{\mathsf{CL}}[n_1 \dots n_k]$ extending $\mathcal{M}_{\mathsf{CL}}$ we provide a characterization result for the notion of $\mathcal{M}_{\mathsf{CL}}[n_1 \dots n_k]$-consequence. It should be remarked, nevertheless, that our characterization results have full generality and do not depend on the fact that $\mathcal{M}_{\mathsf{CL}}$ is the matrix that gets extended with a linear order of contaminating values—the results will hold without loss of generality for any given matrix $\mathcal{M}$. Moreover, these results will be of particular interest when we discuss the completeness results for the sequent calculi associated to these systems.

For the purpose of proving our characterization results, let us begin by noticing that for each LOC-matrix with a linear order of contaminating values we can consider a *simplified* linear order of such contaminating values. To do this, we replace every b-block (i.e. every consecutive block of contaminating undesignated values of any length) and every h-block (i.e. every consecutive block of contaminating designated values of any length) with a single appearance of a non-designated, or a designated contaminating value—respectively. In this regard, the following result about LOC-matrices and simplified LOC-matrices is a corollary of Theorem 1 and Theorem 2:

Corollary 3. *Given a matrix $\mathcal{M} = \langle \mathbf{A}, \mathcal{D} \rangle$, let $\mathcal{M}[n]$ be the extension of $\mathcal{M}$ with a contaminating value n, and let $\mathcal{M}[nn']$ be the extension of $\mathcal{M}[n]$ with a contaminating value n', such that either $\{n, n'\} \subseteq \mathcal{D}_{\mathcal{M}[nn']}$ or $\{n, n'\} \cap \mathcal{D}_{\mathcal{M}[nn']} = \varnothing$. Then, $\mathcal{M}[nn']$-consequence can be characterized as follows:*

$$\Gamma \vDash_{\mathcal{M}[nn']} \Delta \Leftrightarrow \Gamma \vDash_{\mathcal{M}[n]} \Delta$$

Thus, Corollary 3 tells us that instead of working with a given LOC-matrix we can work with the corresponding simplified LOC-matrix, without loss of generality.

Let us notice that this does not mean that one can mix *designated* values and *non-designated* ones, inducing the same logic, but rather than one will induce the same logic by collapsing blocks of designated contaminating values and blocks of non-designated contaminating values, into single appearances thereof. To illustrate this, the reader is encouraged to straightforwardly check that what holds of, *e.g.*, a matrix $\mathcal{M}[bhb]$ will hold without loss of generality *e.g.* of the matrices $\mathcal{M}[bbbhhhhb]$ and $\mathcal{M}[bhbbb]$.

Furthermore, given our previous equivalence result concerning LOC-matrices and simplified LOC-matrices, let us refer to the cardinality m of the simplified linear order of contaminating values of a given LOC-matrix $\mathcal{M}$, as its number of *alternations*. We will, correspondingly, state m as $2n + 1$ if it is odd, and as $2n$ if it is even.

In Section 5, we will se that the infinitely many LOC-matrices based on $\mathcal{M}_{\mathsf{CL}}$ induce infinitely many distinct multiple-conclusion relations.

Before moving on, we prove one further logical property that will be useful in the sequel.

Lemma 1. *Let $\mathcal{M}[hb \dots hb]$ be a classical matrix $\mathcal{M}$ endowed with a linear order $hb \dots hb$ of contaminating values, and let $\mathcal{M}[bh \dots bh]$ be the matrix resulting from $\mathcal{M}[hb \dots hb]$ by replacing each h with a b and vice versa. The consequence relations $\vDash_{\mathcal{M}[hb \dots hb]}$ and $\vDash_{\mathcal{M}[bh \dots bh]}$ are* dual*, that is:*

$$\Gamma \vDash_{\mathcal{M}[hb \dots hb]} \Delta \Leftrightarrow \Delta^{\neg} \vDash_{\mathcal{M}[bh \dots bh]} \Gamma^{\neg}$$

where, for every $\Gamma \subseteq Fml$, $\Gamma^{\neg} = \{\neg\varphi \in Fml \mid \varphi \in \Gamma\}$.

Proof. Take a matrix $\mathcal{M}_{[n_1,n_2,\ldots,n_k]}$ where $\mathcal{M}$ is the matrix of classical logic, $[n_1, n_2, \ldots, n_k]$ is a sequence of contaminating values, and (i) for every n_i, n_i is designated if and only if n_{i+1} is non-designated, (ii) n_1 is designated, and n_k is non-designated. Take now matrix $\mathcal{M}'_{[n_1,n_2,\ldots,n_k]}$, which is like $\mathcal{M}_{[n_1,n_2,\ldots,n_k]}$ except that (ii) is replaced by (ii'): n_1 is non-designated, and n_k is designated. We prove that

$$\Gamma \vDash_{\mathcal{M}[n_1,n_2,\ldots,n_k]} \Delta \;\Leftrightarrow\; \Delta^{\neg} \vDash_{\mathcal{M}'[n_1,n_2,\ldots,n_k]} \Gamma^{\neg}$$

Suppose that $\Gamma \vDash_{\mathcal{M}[n_1,n_2,\ldots,n_k]} \Delta$. This means that, for all valuations $v \in \mathrm{Hom}_{\mathcal{M}[n_1,n_2,\ldots,n_k]}$, if $v(\psi) = \{0, n_i\}$ for every $\psi \in \Delta$ and some non-designated n_i, then $v(\varphi) = \{0, n_i\}$ for some $\varphi \in \Gamma$. We have $\mathrm{Hom}_{\mathcal{M}[n_1,n_2,\ldots,n_k]} = \mathrm{Hom}_{\mathcal{M}'[n_1,n_2,\ldots,n_k]}$ by construction of the two matrices, whence the above transfers to $\mathcal{M}'[n_1, n_2, \ldots, n_k]$. From this and the fact that a contaminating value will be designated in the matrix $\mathcal{M}'[n_1, n_2, \ldots, n_k]$ if and only if it is non-designated in $\mathcal{M}[n_1, n_2, \ldots, n_k]$, we have that, for every $v \in \mathrm{Hom}_{\mathcal{M}'[n_1,n_2,\ldots,n_k]}$, if $v(\neg\psi) = \{1, n_i\}$ for every $\psi \in \Delta$ and $\mathcal{M}'[n_1, n_2, \ldots, n_k]$-designated n_i, then $v(\neg\varphi) = \{1, n_i\}$ for some $\varphi \in \Gamma$. As a consequence we have that $\Delta^{\neg} \vDash_{\mathcal{M}'[n_1,n_2,\ldots,n_k]} \Gamma^{\neg}$. The other direction of the equivalence is proved with the same procedure. Given the definitions of $\mathcal{M}[n_1, n_2, \ldots, n_k]$ and $\mathcal{M}'[n_1, n_2, \ldots, n_k]$ and our convention on hs and bs, it is clear that the former is a matrix $\mathcal{M}[hb \ldots hb]$ where $[bh \ldots hb]$ has cardinality k, and the latter is a matrix $\mathcal{M}[bh \ldots bh]$ where $[bh \ldots bh]$ has cardinality k. This proves Lemma 1. ■

Having proven Lemma 1, let us move to the main results of this section.

Definition 14. *Given a non-empty $\Gamma \subseteq Fml$, we say that $\Gamma_0, \ldots, \Gamma_n \in \mathcal{P}(\Gamma)$ is a* decreasing chain *of subsets of Γ if and only if $\Gamma_0 \supseteq \Gamma_1 \supseteq \cdots \supseteq \Gamma_n$.*

Theorem 3. *Given a matrix $\mathcal{M}$, let $\mathcal{M}[\ldots b]$ be a LOC-matrix extending $\mathcal{M}$ with a linear order of contaminating values that has an odd number of alternations $2n+1$ (where $n \geq 1$), and a non-designated contaminating value b "on top". Then, $\mathcal{M}[\ldots b]$-consequence can be characterized as follows:*

$$\begin{aligned}\Gamma \vDash_{\mathcal{M}[\ldots b]} \Delta \Longleftrightarrow\; & Var(\Delta_n) \subseteq Var(\Gamma_{n-1}) \subseteq Var(\Delta_{n-1}) \subseteq \cdots \subseteq Var(\Gamma_0) \subseteq Var(\Delta_0) \subseteq Var(\Gamma) \\ & \textit{for some } \Gamma_0, \ldots, \Gamma_{n-1} \in \mathcal{P}(\Gamma) \textit{ and } \Delta_0, \ldots, \Delta_n \in \mathcal{P}(\Delta) \textit{ s.t. } \Gamma_{n-1} \vDash_{\mathcal{M}} \Delta_n\end{aligned}$$

where $\Gamma_0, \ldots, \Gamma_{n-1}$ and $\Delta_0, \ldots, \Delta_n$ are decreasing chains.

Proof. We prove this claim by induction on the number of alternations.

Base case: $n = 1$. In such a case, we have $2(1) + 1 = 3$ alternations, *i.e.* we can assume without loss of generality that we are dealing with the simplified LOC-matrix $\mathcal{M}[bhb]$.

By Theorem 1 we are guaranteed to infer that $\Gamma \vDash_{\mathcal{M}[bhb]} \Delta$ is equivalent to there being a $\Delta_0 \in \mathcal{P}(\Delta)$ such that $\Gamma \vDash_{\mathcal{M}[bh]} \Delta_0$ and, of course, $\Delta_0 \subseteq \Delta$, and more importantly $Var(\Delta_0) \subseteq Var(\Gamma)$. In addition, by Theorem 2 the fact that $\Gamma \vDash_{\mathcal{M}[bh]} \Delta_0$ is guaranteed to be equivalent to there being a $\Gamma_0 \in \mathcal{P}(\Gamma)$ such that $\Gamma_0 \vDash_{\mathcal{M}[b]} \Delta_0$ and, of course, $\Gamma_0 \subseteq \Gamma$, and more importantly $Var(\Gamma_0) \subseteq Var(\Delta_0)$. Finally, again by Theorem 1 the fact that $\Gamma_0 \vDash_{\mathcal{M}[b]} \Delta_0$ is guaranteed to be equivalent to there being a $\Delta_1 \in \mathcal{P}(\Delta_0)$ such that $\Gamma_0 \vDash_{\mathcal{M}} \Delta_1$ and, of course, $\Delta_1 \subseteq \Delta_0$, and more importantly $Var(\Delta_1) \subseteq Var(\Gamma_0)$.

All these facts together guarantee the equivalence of $\Gamma \vDash_{\mathcal{M}[bhb]} \Delta$ with there being sets $\Gamma_0 \in \mathcal{P}(\Gamma)$ and $\Delta_0, \Delta_1 \in \mathcal{P}(\Delta)$ such that $\Gamma_0 \vDash_{\mathcal{M}} \Delta_1$, where $\Delta_1 \subseteq \Delta_0$, and $Var(\Delta_1) \subseteq Var(\Gamma_0) \subseteq Var(\Delta_0) \subseteq Var(\Gamma)$.

Inductive step: $n > 1$. We assume that $\mathcal{M}[\ldots b]$ is a simplified LOC-matrix with $2(n-1) + 1$ alternations, and a non-designated value on top. Given this, we consider the LOC-matrix

$\mathcal{M}[\dots bhb]$, *i.e.* a simplified LOC-matrix with $2n+1$ alternations, and a non-designated value on top.

By Theorem 1 we are guaranteed to infer that $\Gamma \vDash_{\mathcal{M}[\dots bhb]} \Delta$ is equivalent to there being a $\Delta_0 \in \mathcal{P}(\Delta)$ such that $\Gamma \vDash_{\mathcal{M}[\dots bh]} \Delta_0$, and more importantly $Var(\Delta_0) \subseteq Var(\Gamma)$. Moreover, by Theorem 2 that $\Gamma \vDash_{\mathcal{M}[\dots bh]} \Delta_0$ implies that there is a $\Gamma_0 \in \mathcal{P}(\Gamma)$ such that $\Gamma_0 \vDash_{\mathcal{M}[\dots b]} \Delta_0$, for which $Var(\Gamma_0) \subseteq Var(\Delta_0)$. Furthermore, by the Inductive Hypothesis, that $\Gamma_0 \vDash_{\mathcal{M}[\dots b]} \Delta_0$ is equivalent to there being $\Gamma_1, \dots, \Gamma_{n-1} \in \mathcal{P}(\Gamma_0)$ and $\Delta_1, \dots, \Delta_n \in \mathcal{P}(\Delta_0)$ such that $\Gamma_{n-1} \vDash_{\mathcal{M}} \Delta_n$, where $\Gamma_1, \Gamma_2, \dots, \Gamma_{n-1}$ and $\Delta_1, \Delta_2, \dots, \Delta_n$ are decreasing chains, and more importantly $Var(\Delta_n) \subseteq Var(\Gamma_{n-1}) \subseteq Var(\Delta_{n-1}) \subseteq \dots \subseteq Var(\Gamma_1) \subseteq Var(\Delta_1) \subseteq Var(\Gamma_0)$.

Finally, all these facts together imply our desired result, *i.e.* that $\Gamma \vDash_{\mathcal{M}[\dots bhb]} \Delta$ is equivalent to there being $\Gamma_0, \dots, \Gamma_{n-1} \in \mathcal{P}(\Gamma)$ and $\Delta_0, \dots, \Delta_n \in \mathcal{P}(\Delta)$ such that $\Gamma_{n-1} \vDash_{\mathcal{M}} \Delta_n$, where $\Gamma_0, \Gamma_1, \dots, \Gamma_{n-1}$ and $\Delta_0, \Delta_1, \dots, \Delta_n$ are decreasing chains, and more importantly $Var(\Delta_n) \subseteq Var(\Gamma_{n-1}) \subseteq Var(\Delta_{n-1}) \subseteq \dots \subseteq Var(\Gamma_0) \subseteq Var(\Delta_0) \subseteq Var(\Gamma)$. ■

Theorem 4. *Given a matrix $\mathcal{M}$, let $\mathcal{M}[\dots h]$ be a LOC-matrix extending $\mathcal{M}$ with a linear order of contaminating values that has an odd number of alternations $2n+1$ (where $n \geq 1$), and a designated contaminating value h "on top". Then, $\mathcal{M}[\dots h]$-consequence can be characterized as follows:*

$$\begin{aligned}\Gamma \vDash_{\mathcal{M}[\dots h]} \Delta \iff & Var(\Gamma_n) \subseteq Var(\Delta_{n-1}) \subseteq Var(\Gamma_{n-1}) \subseteq \dots \subseteq Var(\Delta_0) \subseteq Var(\Gamma_0) \subseteq Var(\Delta) \\ & \textit{for some } \Gamma_0, \dots, \Gamma_n \in \mathcal{P}(\Gamma) \textit{ and } \Delta_0, \dots, \Delta_{n-1} \in \mathcal{P}(\Delta) \textit{ s.t. } \Gamma_n \vDash_{\mathcal{M}} \Delta_{n-1}\end{aligned}$$

where $\Gamma_0, \dots, \Gamma_n$ and $\Delta_0, \dots, \Delta_{n-1}$ are decreasing chains.

Proof. Similar to the proof of Theorem 3. ■

Theorem 5. *Given a matrix $\mathcal{M}$, let $\mathcal{M}[\dots b]$ be a LOC-matrix extending $\mathcal{M}$ with a linear order of contaminating values that has an even number of alternations $2n$ ($n \geq 1$), and a non-designated contaminating value b "on top". Then, $\mathcal{M}[\dots b]$-consequence can be characterized as follows:*

$$\begin{aligned}\Gamma \vDash_{\mathcal{M}[\dots b]} \Delta \iff & Var(\Gamma_{n-1}) \subseteq Var(\Delta_{n-1}) \subseteq Var(\Gamma_{n-2}) \subseteq \dots \subseteq Var(\Gamma_0) \subseteq Var(\Delta_0) \subseteq Var(\Gamma) \\ & \textit{for some } \Gamma_0, \dots, \Gamma_n \in \mathcal{P}(\Gamma) \textit{ and } \Delta_0, \dots, \Delta_n \in \mathcal{P}(\Delta) \textit{ s.t. } \Gamma_{n-1} \vDash_{\mathcal{M}} \Delta_{n-1}\end{aligned}$$

where $\Gamma_0, \dots, \Gamma_{n-1}$ and $\Delta_0, \dots, \Delta_{n-1}$ are decreasing chains

Proof. Similar to the proof of Theorem 6. ■

Theorem 6. *Given a matrix $\mathcal{M}$, let $\mathcal{M}[\dots h]$ be a LOC-matrix extending $\mathcal{M}$ with a linear order of contaminating values that has an even number of alternations $2n$ (where $n \geq 1$), and a designated contaminating value h "on top". Then, $\mathcal{M}[\dots h]$-consequence can be characterized as follows:*

$$\begin{aligned}\Gamma \vDash_{\mathcal{M}[\dots h]} \Delta \iff & Var(\Delta_{n-1}) \subseteq Var(\Gamma_{n-1}) \subseteq Var(\Delta_{n-2}) \subseteq \dots \subseteq Var(\Delta_0) \subseteq Var(\Gamma_0) \subseteq Var(\Delta) \\ & \textit{for some } \Gamma_0, \dots, \Gamma_n \in \mathcal{P}(\Gamma) \textit{ and } \Delta_0, \dots, \Delta_n \in \mathcal{P}(\Delta) \textit{ s.t. } \Gamma_{n-1} \vDash_{\mathcal{M}} \Delta_{n-1}\end{aligned}$$

where $\Gamma_0, \dots, \Gamma_{n-1}$ and $\Delta_0, \dots, \Delta_{n-1}$ are decreasing chains.

Proof. We prove this claim by induction on the number of alternations.

<u>Base case:</u> $n = 1$. In such a case, we have $2(1) = 2$ alternations, *i.e.* we can assume without loss of generality that we are dealing with the simplified LOC-matrix $\mathcal{M}[bh]$.

By Theorem 2 we are guaranteed to infer that $\Gamma \vDash_{\mathcal{M}[bh]} \Delta$ is equivalent to there being a $\Gamma_0 \in \mathcal{P}(\Gamma)$ such that $\Gamma_0 \vDash_{\mathcal{M}[b]} \Delta$, and more importantly $Var(\Gamma_0) \subseteq Var(\Delta)$. In addition, by

Theorem 1 the fact that $\mathit{\Gamma}_0 \vDash_{\mathcal{M}[b]} \mathit{\Delta}$ is guaranteed to be equivalent to there being a $\mathit{\Delta}_0 \in \mathcal{P}(\mathit{\Delta})$ such that $\mathit{\Gamma}_0 \vDash_{\mathcal{M}[b]} \mathit{\Delta}_0$, and more importantly $Var(\mathit{\Delta}_0) \subseteq Var(\mathit{\Gamma}_0)$.

All these facts together guarantee the equivalence of $\mathit{\Gamma} \vDash_{\mathcal{M}[bh]} \mathit{\Delta}$ with there being sets $\mathit{\Gamma}_0 \in \mathcal{P}(\mathit{\Gamma})$ and $\mathit{\Delta}_0 \in \mathcal{P}(\mathit{\Delta})$ such that $\mathit{\Gamma}_0 \vDash_{\mathcal{M}} \mathit{\Delta}_0$, where $Var(\mathit{\Delta}_0) \subseteq Var(\mathit{\Gamma}_0) \subseteq Var(\mathit{\Delta})$.

Inductive step: $n > 1$. We assume that $\mathcal{M}[\ldots h]$ is a simplified LOC-matrix with $2(n-1)$ alternations, and a designated value on top. Given this, we consider the LOC-matrix $\mathcal{M}[\ldots hbh]$, *i.e.* a simplified LOC-matrix with $2n$ alternations, and an designated value on top.

By Theorem 2 we are guaranteed to infer that $\mathit{\Gamma} \vDash_{\mathcal{M}[\ldots hbh]} \mathit{\Delta}$ is equivalent to there being a $\mathit{\Gamma}_0 \in \mathcal{P}(\mathit{\Gamma})$ such that $\mathit{\Gamma}_0 \vDash_{\mathcal{M}[\ldots hb]} \mathit{\Delta}$, and more importantly $Var(\mathit{\Gamma}_0) \subseteq Var(\mathit{\Delta})$. Moreover, by Theorem 1 that $\mathit{\Gamma}_0 \vDash_{\mathcal{M}[\ldots hb]} \mathit{\Delta}$ implies that there is a $\mathit{\Delta}_0 \in \mathcal{P}(\mathit{\Delta})$ such that $\mathit{\Gamma}_0 \vDash_{\mathcal{M}[\ldots h]} \mathit{\Delta}_0$, and $Var(\mathit{\Delta}_0) \subseteq Var(\mathit{\Gamma}_0)$. Furthermore, by the Inductive Hypothesis, that $\mathit{\Gamma}_0 \vDash_{\mathcal{M}[\ldots h]} \mathit{\Delta}_0$ is equivalent to there being $\mathit{\Gamma}_1, \ldots, \mathit{\Gamma}_{n-1} \in \mathcal{P}(\mathit{\Gamma}_0)$ and $\mathit{\Delta}_1, \ldots, \mathit{\Delta}_{n-2} \in \mathcal{P}(\mathit{\Delta}_0)$ such that $\mathit{\Gamma}_{n-1} \vDash_{\mathcal{M}} \mathit{\Delta}_{n-1}$, where $\mathit{\Gamma}_1, \mathit{\Gamma}_2, \ldots, \mathit{\Gamma}_{n-1}$ and $\mathit{\Delta}_1, \mathit{\Delta}_2, \ldots, \mathit{\Delta}_{n-1}$ are decreasing chains, and more importantly $Var(\mathit{\Delta}_{n-1}) \subseteq Var(\mathit{\Gamma}_{n-1}) \subseteq Var(\mathit{\Delta}_{n-2}) \subseteq \cdots \subseteq Var(\mathit{\Delta}_1) \subseteq Var(\mathit{\Gamma}_1) \subseteq Var(\mathit{\Delta}_0)$.

Finally, all these facts together imply our desired result, *i.e.* that $\mathit{\Gamma} \vDash_{\mathcal{M}[\ldots hbh]} \mathit{\Delta}$ is equivalent to there being $\mathit{\Gamma}_0, \ldots, \mathit{\Gamma}_{n-1} \in \mathcal{P}(\mathit{\Gamma})$ and $\mathit{\Delta}_0, \ldots, \mathit{\Delta}_{n-1} \in \mathcal{P}(\mathit{\Delta})$ such that $\mathit{\Gamma}_{n-1} \vDash_{\mathcal{M}} \mathit{\Delta}_{n-1}$, where $\mathit{\Gamma}_0, \mathit{\Gamma}_1, \ldots, \mathit{\Gamma}_{n-1}$ and $\mathit{\Delta}_0, \mathit{\Delta}_1, \ldots, \mathit{\Delta}_{n-1}$ are decreasing chains, and more importantly $Var(\mathit{\Delta}_{n-1}) \subseteq Var(\mathit{\Gamma}_{n-1}) \subseteq Var(\mathit{\Delta}_{n-2}) \subseteq \cdots \subseteq Var(\mathit{\Delta}_0) \subseteq Var(\mathit{\Gamma}_0) \subseteq Var(\mathit{\Delta})$. ■

5 Infinitely Many Multiple-Conclusion Consequence Relations

Section 3 makes it clear that $\mathsf{K}_3^{\mathsf{w}}$, PWK, HYB_1, and HYB_2 are *distinct* logics. Thus, we know that the infinitely many LOC-matrices that are definable from $\mathcal{M}_{\mathsf{CL}}$ induce *at least* four multiple-conclusion consequence relations. In this section, we prove that such matrices actually induce *infinitely many multiple-conclusion consequence relations.* This just follows from Proposition 3 below. Additionally, we provide further results, which contribute to have a clear insight on the relations among the multiple-conclusion consequence relations that are induced by the infinitely many LOC-matrices $\mathcal{M}_{\mathsf{CL}}[n_1, \ldots, n_k]$ that have k alternations for $k \geq 2$.

First, we consider the case where the number of alternations k in a matrix is $k = 2n$ for $n \geq 1$. This case will suffice to show that there are infinitely many multiple-conclusion relations based of LOC-matrices. Then we go to the case where $k = 2n+1$ for $n \geq 1$. This case will help us understand the relations between the infinitely many multiple-conclusion consequence relations in terms of inclusion and distinctness. In what follows, we will often mention the following:

Observation 1 *Let $\mathcal{M}[n_1, \ldots, n_k]$ and $\mathcal{M}[n_1, \ldots, n_m]$ be LOC-matrices, with $m \geq k$ and k, m alternations, respectively. Then, $\langle \mathbf{Fml}, \vDash_{\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_m]} \rangle$ is a* sublogic *of $\langle \mathbf{Fml}, \vDash_{\mathcal{M}[n_1,\ldots,n_k]} \rangle$. That is:*

$$\textit{If } \mathit{\Gamma} \vDash_{\mathcal{M}[n_1,\ldots,n_m]} \psi, \textit{ then } \mathit{\Gamma} \vDash_{\mathcal{M}[n_1,\ldots,n_k]} \psi$$

Proof. Suppose that $n_i \in \mathcal{D}_{\mathcal{M}[n_1,\ldots,n_m]}$ iff $n_i \in \mathcal{D}_{\mathcal{M}[n_1,\ldots,n_k]}$. Then, every $v \in \mathcal{M}[n_1, \ldots, n_k]$ is such that $v \in \mathcal{M}[n_1, \ldots, n_k]$. Hence, if $\mathit{\Gamma} \vDash_{\mathcal{M}[n_1,\ldots,n_m]} \psi$, then $\mathit{\Gamma} \vDash_{\mathcal{M}[n_1,\ldots,n_k]} \psi$. Suppose that $n_i \in \mathcal{D}_{\mathcal{M}[n_1,\ldots,n_m]}$ iff $n_i \notin \mathcal{D}_{\mathcal{M}[n_1,\ldots,n_k]}$. Take the set $\mathcal{G}_{k,m}$ of the k most contaminating values in $\mathcal{M}[n_1, \ldots, n_m]$. It is easy to see that, $(*)$ for every $n_j \in \mathcal{G}_{k,m}$, $n_j \in \mathcal{D}_{\mathcal{M}[n_1,\ldots,n_m]}$ iff $n_{j-(m-k)} \in \mathcal{D}_{\mathcal{M}[n_1,\ldots,n_k]}$, and $n_j \notin \mathcal{D}_{\mathcal{M}[n_1,\ldots,n_m]}$ otherwise. Define now a function $f : A_{\mathcal{M}[n_1,\ldots,n_k]} \to \{0,1\} \cup \mathcal{G}_{k,m}$ such that $f(0) = 0$, $f(1) = 1$, and $f(n_i) = f(n_{i+(k-m)})$, and that. For every valuation $v \in \mathcal{M}[n_1, \ldots, n_k]$, we can build a valuation $v' \in \mathcal{M}[n_1, \ldots, n_m]$ such that (i) $v'(p) \in \{0,1\} \cup \mathcal{G}_k$, and (ii) $v'(p) = v(p)$ if $v(p) \in \{0,1\}$, and (iii) $v'(p) = f(v(p))$. Given the definition of f and

(∗), we have that $v'(\phi) \in \mathcal{D}_{\mathcal{M}[n_1,\dots,n_m]} \cap \mathcal{G}_{k,m}$ iff $v(\phi) \in \mathcal{D}_{\mathcal{M}[n_1,\dots,n_k]}$, and $v'(\phi) \in (A \setminus \mathcal{D}_{\mathcal{M}[n_1,\dots,n_m]}) \cap \mathcal{G}_{k,m}$ iff $v(\phi) \notin \mathcal{D}_{\mathcal{M}[n_1,\dots,n_k]}$. Hence, every $v \in \mathrm{Hom}_{\mathcal{M}[n_1,\dots,n_k]}$ can be redefined as a special valuation $v \in \mathrm{Hom}_{\mathcal{M}[n_1,\dots,n_m]}$. As a consequence, Hence, if $\Gamma \vDash_{\mathcal{M}[n_1,\dots,n_m]} \psi$, then $\Gamma \vDash_{\mathcal{M}[n_1,\dots,n_k]} \psi$. Since the two cases discussed exhaust all the possible cases, we have the statement proven.

5.1 The case where $k = 2n$ for $n \geq 1$

We need some preliminary constructions first. Given a natural number $k \geq 2$ such that $k = 2n$ or $k = 2n + 1$ for some natural number n, we consider formulas of the form $p_1 \wedge (p_1 \vee \cdots \vee p_i)$ for $1 \leq i \leq k$, and we set the following abbreviations:

$$\zeta_1 = p_1 \wedge (p_1 \wedge p_2) \qquad \theta_1 = \begin{cases} p_1 \wedge (p_1 \vee p_2 \vee p_3) & \textit{if } 3 < k \textit{ for } k = 2n \\ & \textit{or } 3 \leq k \textit{ for } k = 2n+1 \\ \text{undefined} & \textit{otherwise.} \end{cases}$$

$$\zeta_2 = \begin{cases} p_1 \wedge (p_1 \vee \cdots \vee p_4) & \textit{if } 4 \leq k \textit{ for } k = 2n \\ & \textit{or } 4 < k \textit{ for } k = 2n+1 \\ \text{undefined} & \textit{otherwise.} \end{cases} \qquad \theta_2 = \begin{cases} p_1 \wedge (p_1 \vee \cdots \vee p_5) & \textit{if } 5 < k \textit{ for } k = 2n \\ & \textit{or } 5 \leq k \textit{ for } k = 2n+1 \\ \text{undefined} & \textit{otherwise.} \end{cases}$$

$$\vdots \qquad\qquad \vdots$$

$$\zeta_{n-1} = \begin{cases} p_1 \wedge (p_1 \vee \cdots \vee p_{k-2}) & \textit{if } k = 2n \\ p_1 \wedge (p_1 \vee \cdots \vee p_{k-3}) & \textit{if } k = 2n+1 \\ \text{undefined} & \textit{if } n = 1 \end{cases} \qquad \theta_{n-1} = \begin{cases} p_1 \wedge (p_1 \vee \cdots \vee p_{k-1}) & \textit{if } k = 2n \\ p_1 \wedge (p_1 \vee \cdots \vee p_{k-2}) & \textit{if } k = 2n+1 \\ \text{undefined} & \textit{if } n = 1 \end{cases}$$

$$\zeta_n = \begin{cases} p_1 \wedge (p_1 \vee \cdots \vee p_k) & \textit{if } k = 2n \\ p_1 \wedge (p_1 \vee \cdots \vee p_{k-1}) & \textit{if } k = 2n+1 \end{cases} \qquad \theta_n = \begin{cases} \text{undefined} & \textit{if } k = 2n \\ p_1 \wedge (p_1 \vee \cdots \vee p_k) & \textit{if } k = 2n+1 \end{cases}$$

For the time being, we focus on cases where $k = 2n$ for some natural $n \geq 1$, since this is relevant for Proposition 2 and Proposition 3 below. The following are two particular examples of the construction, with $k = 4$, and $k = 6$:

$k = 4, n = 2$

$\zeta_1 = p_1 \wedge (p_1 \wedge p_2)$
$\theta_1 = p_1 \wedge (p_1 \vee p_2 \vee p_3)$
$\zeta_2 = \zeta_n = p_1 \wedge (p_1 \vee \cdots \vee p_4)$
θ_j undefined for every $j \geq 2$
ζ_m undefined for every $m > 2$

$k = 6, n = 3$

$\zeta_1 = p_1 \wedge (p_1 \wedge p_2)$
$\theta_1 = p_1 \wedge (p_1 \vee p_2 \vee p_3)$
$\zeta_2 = \zeta_{n-1} = p_1 \wedge (p_1 \vee \cdots \vee p_4)$
$\theta_2 = \theta_{n-1} = p_1 \wedge (p_1 \vee \cdots \vee p_5)$
$\zeta_3 = \zeta_n = p_1 \wedge (p_1 \vee \cdots \vee p_6)$
θ_j undefined for every $j \geq 3$
ζ_m undefined for every $j > 3$

Notice that, for every $k = 2n$ with $n \geq 1$, the sequence $\zeta_1, \dots \theta_{n-1}, \zeta_n$ has length $k - 1$.

Proposition 1. *Let $\mathcal{M}_{\mathsf{CL}}[n_1, \dots, n_k]$ be a LOC-matrix with $k = 2n$ alternations for some natural $n \geq 1$. Then:*

If $n_k \notin \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}}[n_1, \dots, n_k]$, $\quad (\star) \quad \zeta_1, \dots, \zeta_n \vDash_{\mathcal{M}_{\mathsf{CL}}[n_1,\dots,n_k]} p_1 \vee \neg p_1, \theta_1, \dots, \theta_{n-1}$
If $n_k \in \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}}[n_1, \dots, n_k]$, $\quad (\star\star) \quad p_1 \wedge \neg p_1, \theta_1, \dots, \theta_{n-1} \vDash_{\mathcal{M}_{\mathsf{CL}}[n_1,\dots,n_k]} \zeta_1, \dots, \zeta_n$

Proof. We first prove that, if $n_k \notin \mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_k]$, then $(\star)$ is the case. Consider the following construction:

$\mathit{\Gamma}_{n-1} = \varnothing$
$\mathit{\Delta}_{n-1} = \{p_1 \vee \neg p_1\}$
$\mathit{\Gamma}_{n-2} = \{\zeta_1\} = \{p_1 \wedge (p_1 \vee p_2)\}$ defined and relevant only if $n \geq 2$.
$\mathit{\Delta}_{n-2} = \mathit{\Delta}_{n-2} = \{\theta_1\} = \{p_1 \wedge (p_1 \vee p_2 \vee p_3)\}$ defined and relevant only if $n \geq 2$.

$\vdots$

$\mathit{\Delta}_0 = \{\theta_{n-1}\} = \{p_1 \wedge (p_1 \vee \cdots \vee p_{k-1})\}$
$\mathit{\Gamma} = \bigcup_{0\leq i\leq n-1} \mathit{\Gamma}_i \ \cup \{\zeta_n\}$

Clearly, $\mathit{\Gamma}_0,\ldots,\mathit{\Gamma}_{n-1}$ and $\mathit{\Delta}_0,\ldots,\mathit{\Delta}_{n-1}$ are decreasing chains such that:

1. $\mathit{\Gamma}_{n-1} \vDash_{\mathcal{M}_{\mathsf{CL}}} \mathit{\Delta}_{n-1}$;
2. $Var(\mathit{\Gamma}_{n-1}) \subseteq Var(\mathit{\Delta}_{n-1}) \subseteq Var(\mathit{\Gamma}_{n-2}) \subseteq \cdots \subseteq Var(\mathit{\Gamma}_0) \subseteq Var(\mathit{\Delta}_0) \subseteq Var(\mathit{\Gamma})$.

From 1–2 and Theorem 5, $(\star)$ follows.

The proof that $(\star\star)$ is the case if $n_k \in \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_k]}$ goes along the very same lines, with the relevant construction being: $\mathit{\Delta}_{n-1} = \varnothing$, $\mathit{\Gamma}_{n-1} = \{p_1 \wedge \neg p_1\}$, $\mathit{\Delta}_{n-2} = \{\zeta_1\} = \{p_1 \wedge (p_1 \vee p_2)\}$, $\mathit{\Gamma}_{n-2} = \{\theta_1\} = \{p_1 \wedge (p_1 \vee p_2 \vee p_3)\}$, $\ldots$, $\mathit{\Gamma}_0 = \{\theta_{n-1}\} = \{p_1 \wedge (p_1 \vee \cdots \vee p_{k-1}\}$, $\mathit{\Delta} = \bigcup_{0\leq i\leq n-1} \mathit{\Delta}_i \cup \{\zeta_n\}$—with each $\mathit{\Delta}_{n-i}$ and $\mathit{\Gamma}_{n-i}$ being defined and relevant only if $n \geq i$. It is easy to check that the construction satisfies conditions (1) $\mathit{\Gamma}_{n-1} \vDash_{\mathcal{M}_{\mathsf{CL}}} \mathit{\Delta}_{n-1}$; and (2) $Var(\mathit{\Delta}_{n-1}) \subseteq Var(\mathit{\Gamma}_{n-1}) \subseteq Var(\mathit{\Delta}_{n-2}) \subseteq \cdots \subseteq Var(\mathit{\Delta}_0) \subseteq Var(\mathit{\Gamma}_0) \subseteq Var(\mathit{\Delta})$ that Theorem 6 sets for every $\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_k]$ that is relevant for the proposition. ■

We distinguish infinitely many instances of $(\star)$ and $(\star\star)$, depending on the value of $k = 2n$. For every $k = 2n$, we call $(\star k)$ and $(\star\star k)$ its particular instances of $(\star)$ and $(\star\star)$, respectively. We list some examples here:

$(\star 2)$ $\quad p_1 \wedge (p_1 \vee p_2) \vDash_{\mathcal{M}_{\mathsf{CL}}[hb]} p_1 \vee \neg p_1$ $\quad (k=2)$
$(\star 4)$ $\quad p_1 \wedge (p_1 \vee p_2), p_1 \wedge (p_1 \vee \cdots \vee p_4) \vDash_{\mathcal{M}_{\mathsf{CL}}[hbhb]} p_1 \vee \neg p_1, p_1 \wedge (p_1 \vee p_2 \vee p_3)$ $\quad (k=4)$
$(\star 6)$ $\quad p_1 \wedge (p_1 \vee p_2), p_1 \wedge (p_1 \vee \cdots \vee p_4), p_1 \wedge (p_1 \vee \cdots \vee p_6) \vDash_{\mathcal{M}_{\mathsf{CL}}[hbhbhb]}$
$\quad p_1 \vee \neg p_1, p_1 \wedge (p_1 \vee p_2 \vee p_3), p_1 \wedge (p_1 \vee \cdots \vee p_5)$ $\quad (k=6)$

$(\star\star 2)$ $\quad p_1 \wedge \neg p_1 \vDash_{\mathcal{M}_{\mathsf{CL}}[bh]} p_1 \wedge (p_1 \vee p_2)$ $\quad (k=2)$
$(\star\star 4)$ $\quad p_1 \wedge \neg p_1, p_1 \wedge (p_1 \vee p_2 \vee p_3) \vDash_{\mathcal{M}_{\mathsf{CL}}[bhbh]} p_1 \wedge (p_1 \vee p_2), p_1 \wedge (p_1 \vee \cdots \vee p_4)$ $\quad (k=4)$
$(\star\star 6)$ $\quad p_1 \wedge \neg p_1, p_1 \wedge (p_1 \vee p_2 \vee p_3), p_1 \wedge (p_1 \vee \cdots \vee p_5) \vDash_{\mathcal{M}_{\mathsf{CL}}[bhbhbh]}$
$\quad p_1 \wedge (p_1 \vee p_2), p_1 \wedge (p_1 \vee \cdots \vee p_4), p_1 \wedge (p_1 \vee \cdots \vee p_6)$ $\quad (k=6)$

Proposition 2. *Let $\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_k]$ be a LOC-matrix with k alternations, $k = 2n$ for some $n \geq 1$, and let $\mathcal{M}^*_{\mathsf{CL}}[n_1,\ldots,n_k]$ be the LOC-matrix such that $n_i \in \mathcal{D}_{\mathcal{M}^*_{\mathsf{CL}}[n_1,\ldots,n_k]}$ iff $n_i \notin \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_k]}$ for $1 \leq i \leq k$. Then*

If $\quad n_k \notin \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}}[n_1,\ldots,n_k]$, $\quad \zeta_1,\ldots,\zeta_n \nvDash_{\mathcal{M}^*_{\mathsf{CL}}[n_1,\ldots,n_k]} p_1 \vee \neg p_1, \theta_1,\ldots,\theta_{n-1}$
If $\quad n_k \in \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}}[n_1,\ldots,n_k]$, $\quad p_1 \wedge \neg p_1, \theta_1,\ldots,\theta_{n-1} \nvDash_{\mathcal{M}^*_{\mathsf{CL}}[n_1,\ldots,n_k]} \zeta_1,\ldots,\zeta_n$

Proof. We first prove that, if $n_k \notin \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_k]}$, then $(\star k)$ is not valid w.r.t. $\mathcal{M}^*_{\mathsf{CL}}[n_1,\ldots,n_k]$. For every $\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_k]$, if $k = 2n$ for some $n \geq 1$ and $n_k \notin \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_k]}$, then $n_i \in \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_k]}$ if i is odd, and $n_i \notin \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_k]}$ if i is even. Given the constraint imposed on $\mathcal{D}_{\mathcal{M}^*_{\mathsf{CL}}[n_1,\ldots,n_k]}$, we have that $n_i \in \mathcal{D}_{\mathcal{M}^*_{\mathsf{CL}}[n_1,\ldots,n_k]}$ if i is even, and $n_i \notin \mathcal{D}_{\mathcal{M}^*_{\mathsf{CL}}[n_1,\ldots,n_k]}$ if i is odd. Take now variables $p_1, \ldots, p_k$ and a valuation $v \in \mathrm{Hom}_{\mathcal{M}^*_{\mathsf{CL}}[n_1,\ldots,n_k]}$ such that $v(p_i) = n_i$. This implies that (i) $v(p_i) \in \mathcal{D}_{\mathcal{M}^*_{\mathsf{CL}}[n_1,\ldots,n_k]}$ if i is even, and $v(p_i) \notin \mathcal{D}_{\mathcal{M}^*_{\mathsf{CL}}[n_1,\ldots,n_k]}$ if i is odd. Notice that, from the constraints imposed on v, we have (ii) for every $j, i \in \{1, k\}$, if $j > i$, then $v(p_j)$ is more contaminating than $v(p_i)$. By construction of ζ_is and θ_is, we have that (iii) $Var(\zeta_i) = Var(\theta_{i-1}) \cup \{p_{2i}\}$, and (iv) $2i > j$ for every $p_j \in Var(\theta_{i-1})$. (iii), (i), and $v(p_i) = n_i$ together imply $v(p_{2i}) = n_{2i}$, and hence $v(p_{2i}) \in \mathcal{D}_{\mathcal{M}^*_{\mathsf{CL}}[n_1,\ldots,n_k]}$. (iv) and (ii) together imply that $v(\zeta_i) = v(p_{2i})$. Hence, we have $v(\zeta_i) \in \mathcal{D}_{\mathcal{M}^*_{\mathsf{CL}}[n_1,\ldots,n_k]}$. Since choice of i is arbitrary, we have $v(\zeta_j) \in \mathcal{D}_{\mathcal{M}^*_{\mathsf{CL}}[n_1,\ldots,n_k]}$ for every $j \in \{1, \ldots, n\}$. By construction of of ζ_is and θ_is, we have that (v) $Var(\theta_i) = Var(\zeta_i) \cup \{p_{2i+1}\}$, and (vi) $2i+1 > j$ for every $p_j \in Var(\zeta_i)$. (v), (i), and $v(p_i) = n_i$ together imply $v(p_{2i+1}) = n_{2i+1}$, and hence $v(p_{2i+1}) \notin \mathcal{D}_{\mathcal{M}^*_{\mathsf{CL}}[n_1,\ldots,n_k]}$. (vi) and (ii) together imply that $v(\theta_i) = v(p_{2i+1})$. Hence, we have $v(\theta_i) \notin \mathcal{D}_{\mathcal{M}^*_{\mathsf{CL}}[n_1,\ldots,n_k]}$. Since choice of i is arbitrary, we have $v(\theta_j) \notin \mathcal{D}_{\mathcal{M}^*_{\mathsf{CL}}[n_1,\ldots,n_k]}$ for every $j \in \{1, \ldots, n\}$. Additionally, since $v(p_1) \notin \mathcal{D}_{\mathcal{M}^*_{\mathsf{CL}}[n_1,\ldots,n_k]}$, we have $v(p_1 \vee \neg p_1)$. Together with the fact that $v(\zeta_j) \in \mathcal{D}_{\mathcal{M}^*_{\mathsf{CL}}[n_1,\ldots,n_k]}$ for every $j \in \{1, \ldots, n\}$, this implies that v satisfies all the premises from $(\star k)$, while dissatisfying all the conclusions.

We now prove that, if $n_k \in \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_k]}$, then $(\star\star k)$ is not valid w.r.t. $\mathcal{M}^*_{\mathsf{CL}}[n_1, \ldots, n_k]$. for every $\mathcal{M}_{\mathsf{CL}}[n_1, \ldots, n_k]$, if $k = 2n$ for some $n \geq 1$ and $n_k \in \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_k]}$, then $n_i \in \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_k]}$ if i is even, and $n_i \notin \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_k]}$ if i is odd. Given the constraint above on $\mathcal{D}_{\mathcal{M}^*_{\mathsf{CL}}[n_1,\ldots,n_k]}$, we have that $n_i \in \mathcal{D}_{\mathcal{M}^*_{\mathsf{CL}}[n_1,\ldots,n_k]}$ if i is odd, and $n_i \notin \mathcal{D}_{\mathcal{M}^*_{\mathsf{CL}}[n_1,\ldots,n_k]}$ if i is even. Take now variables $p_1, \ldots, p_k$ and a valuation $v \in \mathrm{Hom}_{\mathcal{M}^*_{\mathsf{CL}}[n_1,\ldots,n_k]}$ such that $v(p_i) = n_i$. This implies that (i) $v(p_i) \in \mathcal{D}_{\mathcal{M}^*_{\mathsf{CL}}[n_1,\ldots,n_k]}$ if i is odd, and $v(p_i) \notin \mathcal{D}_{\mathcal{M}^*_{\mathsf{CL}}[n_1,\ldots,n_k]}$ if i is even, to the effect that v provides a counterexample to $(\star\star k)$. ■

Just to get a concrete example of this: take $\mathcal{M}_{\mathsf{CL}}[hb]$, with $\mathcal{M}^*_{\mathsf{CL}}[hb] = \mathcal{M}_{\mathsf{CL}}[bh]$. We have $p_1 \wedge (p_1 \vee p_2) \not\models_{\mathcal{M}_{\mathsf{CL}}[bh]} p_1 \vee \neg p_1$, and any valuation $v \in \mathrm{Hom}_{\mathcal{M}_{\mathsf{CL}}[bh]}$ provides a counterexample if $v(p_1) = n_1 = b$, and $v(p_2) = n_2 = h$. In a similar way, $p_1 \wedge \neg p_1 \not\models_{\mathcal{M}_{\mathsf{CL}}[hb]} p_1 \wedge (p_1 \vee p_2)$. Any valuation $v \in \mathrm{Hom}_{\mathcal{M}_{\mathsf{CL}}[bh]}$ provides a counterexample if $v(p_1) = n_1 = h$, and $v(p_2) = n_2 = b$.

Proposition 3. *Let $\mathcal{M}_{\mathsf{CL}}[n_1, \ldots, n_k]$ be a LOC-matrix with k alternations, $k = 2n$ for some $n \geq 1$, and $n_k \notin \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_k]}$. Then, for every number of alternations $m > k$, we have:*

1. $\zeta_1, \ldots, \zeta_n \not\models_{\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_m]} p_1 \vee \neg p_1, \theta_1, \ldots, \theta_{n-1}$ *if* $n_k \notin \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_k]}$
2. $p_1 \wedge \neg p_1, \theta_1, \ldots, \theta_{n-1} \not\models_{\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_m]} \zeta_1, \ldots, \zeta_n$ *if* $n_k \in \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_k]}$

Proof. We proof that 1–2 hold if $m = k + 1$ in the relevant cases, by building suitable countermodels to $(\star k)$ and $(\star\star k)$. We then generalize the result to every natural $m > k$. We have two cases:

Case 1: $n_k \notin \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_k]}$. We distinguish two subcases:

Case 1a: $n_i \in \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_m]}$ iff $n_i \in \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_k]}$ for $1 \leq i \leq k$. This implies that, if $\mathcal{M}_{\mathsf{CL}}[n_1, \ldots, n_k]$ is, say, $\mathcal{M}_{\mathsf{CL}}[hbhb]$, then $\mathcal{M}_{\mathsf{CL}}[n_1, \ldots, n_m]$ is $\mathcal{M}_{\mathsf{CL}}[hbhbh]$. For every $\mathcal{M}_{\mathsf{CL}}[n_1, \ldots, n_k]$, if $k = 2n$ for some $n \geq 1$ and $n_k \notin \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_k]}$, then $n_i \in \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_k]}$ if i is odd, and $n_i \notin \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_k]}$ if i is even. Given the constraint imposed on $\mathcal{D}_{\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_m]}$ by this case, the same applies to $\mathcal{M}_{\mathsf{CL}}[n_1, \ldots, n_m]$. Take now variables $p_1, \ldots, p_k$ and a valuation $v \in \mathrm{Hom}_{\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_m]}$ such that $v(p_i) = n_{i+1}$. This implies that (i) $v(p_i) \in \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_m]}$ if i is even, and $v(p_i) \notin \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_m]}$ if i is odd. Notice that (ii)–(vi) from the proof of Proposition 2 also apply here. From this and $v(p_i) = n_{i+1}$, we have $v(p_{2i}) = n_{2i+1}$, $v(p_{2i+1}) = n_{2i+2}$, and hence $v(\zeta_i) \in \mathcal{D}_{\mathcal{M}^*_{\mathsf{CL}}[n_1,\ldots,n_m]}$ and $v(\theta_i) \notin \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_m]}$. Since choice of i is arbitrary, we have $v(\zeta_j) \in \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_m]}$ and $v(\theta_j) \notin \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_m]}$ for every $j \in \{1, \ldots, n\}$. Since, additionally, we have $v(p_1 \vee \neg p_1)$ by construction, we have that v satisfies all the premises from $(\star k)$, while dissatisfying all the conclusions. This proves the statement for this case.

Case 1b: $n_i \in \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_m]}$ iff $n_i \notin \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_k]}$ for $1 \leq i \leq k$—which implies that, if $\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_k]$ is, say, $\mathcal{M}_{\mathsf{CL}}[hbhb]$, then $\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_m]$ is $\mathcal{M}_{\mathsf{CL}}[bhbhb]$. This case follows from Proposition 2 and Observation 1.

Case 2: $n_k \in \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_k]}$. We distinguish two subcases:

Case 2a: $n_i \in \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_m]}$ iff $n_i \in \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_k]}$ for $1 \leq i \leq k$. This implies that, if $\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_k]$ is, say, $\mathcal{M}_{\mathsf{CL}}[bhbh]$, then $\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_m]$ is $\mathcal{M}_{\mathsf{CL}}[bhbhb]$. For every $\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_k]$, if $k = 2n$ for some $n \geq 1$ and $n_k \in \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_k]}$, then $n_i \in \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_k]}$ if i is even, and $n_i \notin \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_k]}$ if i is odd. Given the constraint above on $\mathcal{D}_{\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_m]}$, the same applies to $\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_m]$. Take now variables $p_1,\ldots,p_k$ and a valuation $v \in \mathrm{Hom}_{\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_m]}$ such that $v(p_i) = n_{i+1}$. The same construction from Case 1a provides a counterexample. This proves the statement for this subcase.

Case 2b: $n_i \in \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_m]}$ iff $n_i \notin \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_k]}$ for $1 \leq i \leq k$—which implies that, if $\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_k]$ is, say, $\mathcal{M}_{\mathsf{CL}}[hbhb]$, then $\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_m]$ is $\mathcal{M}_{\mathsf{CL}}[bhbhb]$. This case follows from Proposition 2 and Observation 1.

The cases above prove the statement for $m = k + 1$. From this and Observation 1, the statement holds for every $m > k$. ∎

Propositions 1–3 together prove that each multiple-conclusion consequence relation induced by a LOC-matrix with k alternations for $k = 2n$ (for some $n \geq 1$) is *distinct* from every multiple-conclusion consequence relation induced by a LOC-matrix with $m > k$ alternations. This in turn implies that there are infinitely many multiple-conclusion consequence relations based on LOC-matrices.

5.2 The case where $k = 2n + 1$ for $n \geq 1$

We go now to the case where $k = 2n+1$ for some natural $n \geq 1$. Remember that, in this case, $\zeta_2 = p_1 \wedge (p_1 \vee \cdots \vee p_4)$ if $4 < k$, and undefined otherwise, $\theta_2 = p_1 \wedge (p_1 \vee \cdots \vee p_5)$ if $5 \leq k$, and undefined otherwise, and so on. Also, $\zeta_n = p_1 \wedge (p_1 \vee \cdots \vee p_{k-1})$ and $\theta_n = p_1 \wedge (p_1 \vee \cdots \vee p_k)$. The following are two particular examples of the construction, with $k = 3$, and $k = 5$:

$k = 3, n = 1$	$k = 5, n = 2$
$\zeta_1 = p_1 \wedge (p_1 \wedge p_2)$	$\zeta_1 = p_1 \wedge (p_1 \wedge p_2)$
$\theta_1 = p_1 \wedge (p_1 \vee p_2 \vee p_3)$	$\theta_1 = p_1 \wedge (p_1 \vee p_2 \vee p_3)$
ζ_j undefined for every $j \geq 2$	$\zeta_2 = \zeta_n = p_1 \wedge (p_1 \vee \cdots \vee p_4)$
θ_m undefined for every $j \geq 2$	$\theta_2 = \theta_n = p_1 \wedge (p_1 \vee \cdots \vee p_5)$
	ζ_j undefined for every $j \geq 3$
	θ_m undefined for every $j \geq 3$

Proposition 4. *Let $\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_k]$ be a LOC-matrix with $k = 2n+1$ alternations for some natural $n \geq 1$. Then:*

If $n_k \notin \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}}[n_1,\ldots,n_k]$,	$(\circ)$	$p_1 \wedge \neg p_1, \theta_1,\ldots,\theta_n \vDash_{\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_k]} \zeta_1,\ldots,\zeta_n$
If $n_k \in \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}}[n_1,\ldots,n_k]$,	$(\circ\circ)$	$\zeta_1,\ldots,\zeta_n \vDash_{\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_k]} p_1 \vee \neg p_1, \theta_1,\ldots,\theta_n$

Proof. We first prove that, if $n_k \notin \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_k]}$, $(\circ)$ is the case. Consider the following construction: $\Delta_n = \varnothing$, $\Gamma_{n-1} = \{p_1 \wedge \neg p_1\}$, $\Delta_{n-1} = \{\zeta_1\} = \{p_1 \wedge (p_1 \vee p_2)\}$, $\Gamma_{n-2} = \{\theta_1\} =$

$\{p_1 \wedge (p_1 \vee p_2 \vee p_3)\}, \ldots, \Delta_0 = \{\zeta_n\} = \{p_1 \wedge (p_1 \vee \cdots \vee p_{k-1})\}$, $\Gamma = \bigcup_{0 \leq i \leq (k/2)-1} \Gamma_i \cup \{\theta_n\}$— with each Δ_{n-i} and Γ_{n-i} being defined and relevant only if $n \geq i$. The construction satisfies the conditions that Theorem 3 sets for every $\mathcal{M}_{\mathsf{CL}}[n_1, \ldots, n_k]$ that is relevant for the proposition.

The proof that ($\circ\circ$) is the case for $n_k \in \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_k]}$ goes along the very same lines, with the relevant construction being: $\Gamma_n = \varnothing$, $\Delta_{n-1} = \{p_1 \vee \neg p_1\}$, $\Gamma_{n-1} = \{\zeta_1\} = \{p_1 \wedge (p_1 \vee p_2)\}$, $\Delta_{n-2} = \{\theta_1\} = \{p_1 \wedge (p_1 \vee p_2 \vee p_3)\}, \ldots, \Gamma_0 = \{\zeta_n\} = \{p_1 \wedge (p_1 \vee \cdots \vee p_{k-1})\}$, $\Delta = \bigcup_{0 \leq i \leq (k/2)-1} \Delta_i \cup \{\theta_n\}$— with each Δ_{n-i} and Γ_{n-i} being defined and relevant only if $n \geq i$. It is easy to check that the construction satisfies the conditions set by Theorem 4, which is the relevant theorem here. ■

We distinguish infinitely many instances of ($\circ$) and ($\circ\circ$), depending on the value of $k = 2n + 1$, and we follow the notational convention that we set when dealing with instances of ($\star$) and ($\star\star$). We list a pair of examples here:

$$(\circ 3) \quad p_1 \wedge \neg p_1, p_1 \wedge (p_1 \vee p_2 \vee p_3), \vDash_{\mathcal{M}_{\mathsf{CL}}[bhb]} p_1 \wedge (p_1 \vee p_2) \qquad (k = 3)$$
$$(\circ 5) \quad p_1 \wedge \neg p_1, p_1 \wedge (p_1 \vee p_2 \vee p_3), p_1 \wedge (p_1 \vee \cdots \vee p_5) \vDash_{\mathcal{M}_{\mathsf{CL}}[bhbhb]} \vDash_{\mathcal{M}_{\mathsf{CL}}[bhbhb]} p_1 \wedge (p_1 \vee p_2), p_1 \wedge (p_1 \vee \cdots \vee p_4) \qquad (k = 5)$$

$$(\circ \circ 3) \quad p_1 \wedge (p_1 \vee p_2) \vDash_{\mathcal{M}_{\mathsf{CL}}[hbh]} p_1 \vee \neg p_1, p_1 \wedge (p_1 \vee p_2 \vee p_3) \qquad (k = 3)$$
$$(\circ \circ 5) \quad p_1 \wedge (p_1 \vee p_2), p_1 \wedge (p_1 \vee \cdots \vee p_4) \vDash_{\mathcal{M}_{\mathsf{CL}}[hbhbh]} \vDash_{\mathcal{M}_{\mathsf{CL}}[hbhbh]} p_1 \vee \neg p_1, p_1 \wedge (p_1 \vee p_2 \vee p_3), p_1 \wedge (p_1 \vee \cdots \vee p_5) \qquad (k = 5)$$

Proposition 5. *Let $\mathcal{M}_{\mathsf{CL}}[n_1, \ldots, n_k]$ be a LOC-matrix with k alternations, $k = 2n + 1$ for some $n \geq 1$, and let $\mathcal{M}^*_{\mathsf{CL}}[n_1, \ldots, n_k]$ be the LOC-matrix such that $n_i \in \mathcal{D}_{\mathcal{M}^*_{\mathsf{CL}}[n_1,\ldots,n_k]}$ iff $n_i \notin \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_k]}$ for $1 \leq i \leq k$. Then:*

If $n_k \notin \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}}[n_1, \ldots, n_k]$, $\quad p_1 \wedge \neg p_1, \theta_1, \ldots, \theta_n \nvDash_{\mathcal{M}^*_{\mathsf{CL}}[n_1,\ldots,n_k]} \zeta_1, \ldots, \zeta_n$
If $n_k \in \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}}[n_1, \ldots, n_k]$, $\quad \zeta_1, \ldots, \zeta_n \nvDash_{\mathcal{M}^*_{\mathsf{CL}}[n_1,\ldots,n_k]} p_1 \vee \neg p_1, \theta_1, \ldots, \theta_n$

Proof. Suppose that $n_k \notin \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_k]}$. For every $\mathcal{M}_{\mathsf{CL}}[n_1, \ldots, n_k]$, if $k = 2n + 1$ for some $n \geq 1$ and $n_k \notin \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_k]}$, then $n_i \in \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_k]}$ if i is even, and $n_i \notin \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_k]}$ if i is odd. Given the constraint imposed on $\mathcal{D}_{\mathcal{M}^*_{\mathsf{CL}}[n_1,\ldots,n_k]}$, we have that $n_i \in \mathcal{D}_{\mathcal{M}^*_{\mathsf{CL}}[n_1,\ldots,n_k]}$ if i is odd, and $n_i \notin \mathcal{D}_{\mathcal{M}^*_{\mathsf{CL}}[n_1,\ldots,n_k]}$ if i is even. Take now variables $p_1, \ldots, p_k$. Any valuation $v \in \mathrm{Hom}_{\mathcal{M}^*_{\mathsf{CL}}[n_1,\ldots,n_k]}$ such that $v(p_i) = n_i$ provides a counterexample to ($\circ k$).

Suppose that $n_k \in \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_k]}$. Again, for every $\mathcal{M}_{\mathsf{CL}}[n_1, \ldots, n_k]$, if $k = 2n + 1$ for some $n \geq 1$ and $n_k \in \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_k]}$, then $n_i \in \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_k]}$ if i is odd, and $n_i \notin \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_k]}$ if i is even. Given the constraint above on $\mathcal{D}_{\mathcal{M}^*_{\mathsf{CL}}[n_1,\ldots,n_k]}$, we have that $n_i \in \mathcal{D}_{\mathcal{M}^*_{\mathsf{CL}}[n_1,\ldots,n_k]}$ if i is odd, and $n_i \notin \mathcal{D}_{\mathcal{M}^*_{\mathsf{CL}}[n_1,\ldots,n_k]}$ if i is even. Take now variables $p_1, \ldots, p_k$. Again, any valuation $v \in \mathrm{Hom}_{\mathcal{M}^*_{\mathsf{CL}}[n_1,\ldots,n_k]}$ such that $v(p_i) = n_i$ provides a counterexample to ($\star\star k$). ■

Just to get a concrete example of this: take $\mathcal{M}_{\mathsf{CL}}[bhb]$, with $\mathcal{M}^*_{\mathsf{CL}}[bhb] = \mathcal{M}_{\mathsf{CL}}[hbh]$. We have $p_1 \wedge \neg p_1, p_1 \wedge (p_1 \vee p_2 \vee p_3) \nvDash_{\mathcal{M}_{\mathsf{CL}}[hbh]} p_1 \wedge (p_1 \vee p_2)$, and any valuation $v \in \mathrm{Hom}_{\mathcal{M}_{\mathsf{CL}}[bh]}$ provides a counterexample if $v(p_1) = n_1 = h_1$, $v(p_2) = n_2 = b_1$, $v(p_3) = n_3 = h_2$. In a similar way, $p_1 \wedge (p_1 \vee p_2) \nvDash_{\mathcal{M}_{\mathsf{CL}}[bhb]} p_1 \vee \neg p_1, p_1 \wedge (p_1 \vee p_2 \vee p_3)$. Any valuation $v \in \mathrm{Hom}_{\mathcal{M}_{\mathsf{CL}}[bhb]}$ provides a counterexample if $v(p_1) = n_1 = b_1$, $v(p_2) = n_2 = h_1$, and $v(p_3) = n_3 = h_2$.

Proposition 6. *Let $\mathcal{M}_{\mathsf{CL}}[n_1, \ldots, n_k]$ be a LOC-matrix with k alternations, $k = 2n+1$ for some $n \geq 1$. Then, for every number of alternations $m > k$:*

1. $p_1 \wedge \neg p_1, \theta_1, \ldots, \theta_n \nvDash_{\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_m]} \zeta_1, \ldots, \zeta_n \quad$ *if* $n_k \notin \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_k]}$
2. $\zeta_1, \ldots, \zeta_n \nvDash_{\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_m]} p_1 \vee \neg p_1, \theta_1, \ldots, \theta_n \quad$ *if* $n_k \in \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}[n_1,\ldots,n_k]}$

Proof. Again, we first prove the statement for $m = k+1$. As for Proposition 3, we have two cases:

Case 1: $n_k \notin \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}[n_1,\dots,n_k]}$. We distinguish two subcases:

Case 1a: $n_i \in \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}}[n_1,\dots,n_m]$ iff $n_i \in \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}}[n_1,\dots,n_k]$ for $1 \leq i \leq k$. This goes exactly as Case 1a from Proposition 3, to the effect that $(\circ k)$ does not hold w.r.t $\mathcal{M}_{\mathsf{CL}}[n_1,\dots,n_m]$-consequence for $m = k+1$, if the latter meets the conditions of the present Case 1a.

Case 1b: $n_i \in \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}}[n_1,\dots,n_m]$ iff $n_i \in \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}}[n_1,\dots,n_k]$ for $1 \leq i \leq k$. This goes exactly as Case 1b from Proposition 3, to the effect that $(\circ k)$ does not hold w.r.t $\mathcal{M}_{\mathsf{CL}}[n_1,\dots,n_m]$-consequence for $m = k+1$, if the latter meets the conditions of the present Case 1b.

Case 2: $n_k \in \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}[n_1,\dots,n_k]}$. We distinguish two subcases:

Case 2a: $n_i \in \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}}[n_1,\dots,n_m]$ iff $n_i \in \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}}[n_1,\dots,n_k]$ for $1 \leq i \leq k$. This goes exactly as Case 2a from Proposition 3, to the effect that $(\circ \circ k)$ does not hold w.r.t $\mathcal{M}_{\mathsf{CL}}[n_1,\dots,n_m]$-consequence for $m = k+1$, if the latter meets the conditions of the present Case 2a.

Case 2b: $n_i \in \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}}[n_1,\dots,n_m]$ iff $n_i \in \mathcal{D}_{\mathcal{M}_{\mathsf{CL}}}[n_1,\dots,n_k]$ for $1 \leq i \leq k$. This goes exactly as Case 2a from Proposition 3, to the effect that $(\circ \circ k)$ does not hold w.r.t $\mathcal{M}_{\mathsf{CL}}[n_1,\dots,n_m]$-consequence for $m = k+1$, if the latter meets the conditions of the present Case 2b.

The case above prove the statement for $m = k+1$. From this and Observation 1, we have that that statement holds for every $m > k$.

Propositions 4–6 together prove that each multiple-conclusion consequence relation induced by a LOC-matrix with k alternations for $k = 2n+1$ (for some $n \geq 1$) is *distinct* from every multiple-conclusion consequence relation induced by a LOC-matrix with $m > k$ alternations. Together with Section 3 and Propositions 1–3, this determines the relations illustrated by Figure 1:

$$
\begin{tikzcd}
 & \models_{\mathcal{M}_{\mathsf{CL}}[h]} \arrow[r] \arrow[ddr] & \models_{\mathcal{M}_{\mathsf{CL}}[h_1 b_1]} \arrow[r] & \cdots \arrow[r] & \models_{\mathcal{M}_{\mathsf{CL}}[h_1 b_1 \dots h_i]} \arrow[r] \arrow[ddr] & \models_{\mathcal{M}_{\mathsf{CL}}[h_1 b_1 \dots h_i b_i]} \arrow[r] & \cdots \\
\models_{\mathcal{M}_{\mathsf{CL}}} \arrow[ur] \arrow[dr] & & & & & & \\
 & \models_{\mathcal{M}_{\mathsf{CL}}[b]} \arrow[r] \arrow[uur] & \models_{\mathcal{M}_{\mathsf{CL}}[b_1 h_1]} \arrow[r] & \cdots \arrow[r] & \models_{\mathcal{M}_{\mathsf{CL}}[b_1 h_1 \dots b_i]} \arrow[r] \arrow[uur] & \models_{\mathcal{M}_{\mathsf{CL}}[b_1 h_1 \dots b_i h_i]} \arrow[r] & \cdots
\end{tikzcd}
$$

Fig. 1. Diagram of the infinitely many multiple-conclusion consequence relations induced by LOC-matrices based on $\mathcal{M}_{\mathsf{CL}}$

6 Proof Theory for Contaminating Logics

In this section, we present sequent calculi for the logics HYB_1 and HYB_2, thus extending similar results from [9] for $\mathsf{K}_3^{\mathsf{w}}$ and PWK. More precisely, we provide sound and complete calculi of *annotated* sequents for the two four-valued logics from Section 3. An *annotated* sequent is an object of the form $\Gamma, [\![\Gamma']\!] \Rightarrow \Delta, [\![\Delta']\!]$ where $\Gamma, \Gamma', \Delta, \Delta'$ are sets of formulae of the language.

In annotated sequent calculi, additional rules are provided in order to capture the interaction among formulae within squared brackets, outside square brackets, and the interaction of formulae within square brackets and formulae outside the brackets.

As in [9], each of our calculi places *restrictions* on several rules—more precisely, the rules need some variable inclusion condition to be satisfied in order to be applicable. We will detail the corresponding provisos when needed.

One further peculiarity of the calculi that follow should be acknowledged and discussed. Our calculi for HYB_1 and HYB_2 are *decorated* insofar as we employ a bracketing device in each of the antecedent and succedent to track variable-inclusion properties. On the surface, one might interpret this as an instance of a four-sided sequent calculus. If this were the case, it would be disappointing for several reasons. On the one hand, many-sided sequents are far less intuitive and natural than two-sided sequents (or one-sided sequents, for that matter). On the other, there exist tools such as `MUltseq` (described, *e.g.*, in [16]) that can construct sound and complete many-sided sequent calculi for any finitely-valued logic.

We do not believe that this is a reasonable concern, however. Whereas the standard reading of a many-sided sequent is one in which each "side" plays the role of a distinct truth-value, which might be considered an inauthentic smuggling of semantics into the proof theory, it is not clear that a similar alignment exists in our calculi for HYB_1 and HYB_2. The motivation for our bracketing device is not semantic, but rather, syntactic in nature, which seems to offend our own proof theoretic sensibilities far less. In any case, should the reader remain unconvinced, the general method for authentically *two-sided* sequent calculi that will be presented later—in Section 6.2—count the consequence relations for HYB_1 and HYB_2 as special cases.

6.1 Sequent Calculi for HYB_1 and HYB_2

Both systems include the following three rules, where for every $\Gamma \subseteq Fml$, Γ^* is any modification of Γ by permuting elements, absorbing redundancies, or duplicating formulae:

$$\frac{}{\varnothing, [\![p]\!] \Rightarrow \varnothing, [\![p]\!]}\ [\text{Axiom}]$$

$$\frac{\Gamma, [\![\Xi]\!] \Rightarrow \Delta, [\![\Theta]\!]}{\Gamma^*, [\![\Xi^*]\!] \Rightarrow \Delta^*, [\![\Theta^*]\!]}\ [\text{Structural}]$$

$$\frac{\Gamma, [\![\Gamma']\!] \Rightarrow \Delta, [\![\Delta']\!]}{\Gamma, \Xi, [\![\Gamma']\!] \Rightarrow \Delta, \Theta, [\![\Delta']\!]}\ [\text{Weak}]$$

[Axiom] secures the validity of those classical axioms in which a propositional variable is within the scope of a square bracket in each sequent. [Structural] grants standard structural rules, but Weakening, *within any of the four slots.* [Weak] differs from the Weakening for non-annotated calculus in that we can only allow Weakening *outside* the scope of the bracket. The following "push" rules below meet the need to shift formulae from outside the scope of a square bracket to within its scope. It is with these rules that variable-inclusion restrictions come into play:

$$\frac{\Gamma, \varphi, [\![\Gamma']\!] \Rightarrow \Delta, [\![\Delta']\!]}{\Gamma, [\![\Gamma', \varphi]\!] \Rightarrow \Delta, [\![\Delta']\!]}\ [\text{PushL}] \qquad \frac{\Gamma, [\![\Gamma']\!] \Rightarrow \Delta, \psi, [\![\Delta']\!]}{\Gamma, [\![\Gamma']\!] \Rightarrow \Delta, [\![\Delta', \psi]\!]}\ [\text{PushR}]$$

In the HYB_1 calculus, [PushL] requires the restriction $Var(\varphi) \subseteq Var(\Delta')$ and [PushR] requires $Var(\psi) \subseteq Var(\Gamma \cup \Gamma')$. In the HYB_2 calculus, the two rules require $Var(\varphi) \subseteq Var(\Delta \cup \Delta')$ and $Var(\psi) \subseteq Var(\Gamma')$, respectively.

Negation rules come with a pair of right rules and a pair of left rules, since we need to distinguish the case where we are introducing the sign within the scope of a square bracket from that where we are introducing the sign without such a scope:

$$\frac{\Gamma, [\![\Gamma', \varphi]\!] \Rightarrow \Delta, [\![\Delta']\!]}{\Gamma, [\![\Gamma']\!] \Rightarrow \Delta, [\![\Delta', \neg\varphi]\!]}\ [\neg \mathrm{R}_1] \qquad \frac{\Gamma, \varphi, [\![\Gamma']\!] \Rightarrow \Delta, [\![\Delta']\!]}{\Gamma, [\![\Gamma']\!] \Rightarrow \Delta, \neg\varphi, [\![\Delta']\!]}\ [\neg \mathrm{R}_2]$$

In the HYB_1 calculus, $[\neg \mathrm{R}_1]$ and $[\neg \mathrm{R}_2]$ require $Var(\varphi) \subseteq Var(\Gamma \cup \Gamma')$; in the HYB_2 calculus, $[\neg R_1]$ requires that $Var(\varphi) \subseteq Var(\Gamma')$, and $[\neg R_1]$ has no proviso. As for the left rules:

$$\frac{\Gamma, [\![\Gamma']\!] \Rightarrow \Delta, [\![\Delta', \psi]\!]}{\Gamma, [\![\Gamma', \neg\psi]\!] \Rightarrow \Delta, [\![\Delta']\!]}\ [\neg \mathrm{L}_1] \qquad \frac{\Gamma, [\![\Gamma']\!] \Rightarrow \Delta, \psi, [\![\Delta']\!]}{\Gamma, \neg\psi, [\![\Gamma']\!] \Rightarrow \Delta, [\![\Delta']\!]}\ [\neg \mathrm{L}_2]$$

where $[\neg \mathrm{L}_1]$ requires that $Var(\psi) \subseteq Var(\Delta')$ and $[\neg \mathrm{L}_2]$ has no proviso. Additionally, we consider a couple of rules for conjunction:

$$\frac{\Gamma, [\![\Gamma', \varphi, \psi]\!] \Rightarrow \Delta, [\![\Delta']\!]}{\Gamma, [\![\Gamma', \varphi \wedge \psi]\!] \Rightarrow \Delta, [\![\Delta']\!]}\ [\wedge \mathrm{L}_1] \qquad \frac{\Gamma, \varphi, \psi, [\![\Gamma']\!] \Rightarrow \Delta, [\![\Delta']\!]}{\Gamma, \varphi \wedge \psi, [\![\Gamma']\!] \Rightarrow \Delta, [\![\Delta']\!]}\ [\wedge \mathrm{L}_2]$$

Rules $[\wedge \mathrm{L}_1]$ and $[\wedge \mathrm{L}_2]$ require no provisos in either HYB_1 or HYB_2. However, the following mixed rule requires a variable-inclusion restriction:

$$\frac{\Gamma, \varphi, [\![\Gamma', \psi]\!] \Rightarrow \Delta, [\![\Delta']\!]}{\Gamma, [\![\Gamma', \varphi \wedge \psi]\!] \Rightarrow \Delta, [\![\Delta']\!]}\ [\wedge \mathrm{L}^*]$$

In HYB_1, the rule is admissible provided that $Var(\varphi) \subseteq Var(\Delta')$, while in HYB_2, $Var(\varphi) \subseteq Var(\Delta \cup \Delta')$ is required. For the right rules, we consider the case in which both conjuncts are outside of the scope of $[\![-]\!]$ and the case in which both are within its scope. Note, again, that we can appeal to [PushR] in order to cover mixed cases.

$$\frac{\Gamma, [\![\Gamma']\!] \Rightarrow \Delta, [\![\Delta', \varphi]\!] \qquad \Gamma, [\![\Gamma']\!] \Rightarrow \Delta, [\![\Delta', \psi]\!]}{\Gamma, [\![\Gamma']\!] \Rightarrow \Delta, [\![\Delta', \varphi \wedge \psi]\!]}\ [\wedge \mathrm{R}_1]$$

$$\frac{\Gamma, [\![\Gamma']\!] \Rightarrow \Delta, \varphi, [\![\Delta']\!] \qquad \Gamma, [\![\Gamma']\!] \Rightarrow \Delta, \psi, [\![\Delta']\!]}{\Gamma, [\![\Gamma']\!] \Rightarrow \Delta, \varphi \wedge \psi, [\![\Delta']\!]}\ [\wedge \mathrm{R}_2]$$

Again, neither $[\wedge \mathrm{R}_1]$ nor $[\wedge \mathrm{R}_2]$ requires a proviso in the two logics, but one could define an admissible rule that requires that $Var(\varphi) \subseteq Var(\Gamma \cup \Gamma')$ in HYB_1 and $Var(\varphi) \subseteq Var(\Gamma')$ in HYB_2:

$$\frac{\Gamma, [\![\Gamma']\!] \Rightarrow \Delta, \varphi, [\![\Delta']\!] \qquad \Gamma, [\![\Gamma']\!] \Rightarrow \Delta, [\![\Delta', \psi]\!]}{\Gamma, [\![\Gamma']\!] \Rightarrow \Delta, [\![\Delta', \varphi \wedge \psi]\!]}\ [\wedge \mathrm{R}^*]$$

Finally, we consider also the rules for disjunction:

$$\frac{\Gamma, [\![\Gamma', \varphi]\!] \Rightarrow \Delta, [\![\Delta']\!] \qquad \Gamma, [\![\Gamma', \psi]\!] \Rightarrow \Delta, [\![\Delta']\!]}{\Gamma, [\![\Gamma', \varphi \vee \psi]\!] \Rightarrow \Delta, [\![\Delta']\!]}\ [\vee \mathrm{L}_1]$$

$$\frac{\Gamma, \varphi, [\![\Gamma']\!] \Rightarrow \Delta, [\![\Delta']\!] \qquad \Gamma, \psi, [\![\Gamma']\!] \Rightarrow \Delta, [\![\Delta']\!]}{\Gamma, \varphi \vee \psi, [\![\Gamma']\!] \Rightarrow \Delta, [\![\Delta']\!]}\ [\vee \mathrm{L}_2]$$

Neither $[\vee \mathrm{L}_1]$ nor $[\vee \mathrm{L}_2]$ require provisos. Again, for the right rules, we consider the case in which both disjuncts are outside of the scope of $[\![-]\!]$ and the case in which both are within its scope. Note, again, that we can appeal to [PushR] in order to cover mixed cases.

$$\frac{\Gamma, [\![\Gamma']\!] \Rightarrow \Delta, [\![\Delta', \varphi, \psi]\!]}{\Gamma, [\![\Gamma']\!] \Rightarrow \Delta, [\![\Delta', \varphi \vee \psi]\!]}\ [\vee \mathrm{R}_1] \qquad \frac{\Gamma, [\![\Gamma']\!] \Rightarrow \Delta, \varphi, \psi, [\![\Delta']\!]}{\Gamma, [\![\Gamma']\!] \Rightarrow \Delta, \varphi \vee \psi, [\![\Delta']\!]}\ [\vee \mathrm{R}_2]$$

Now we state *soundness* and *completeness* of HYB_1 and HYB_2 with respect to $\mathcal{M}_{\mathsf{CL}}[hb]$ and $\mathcal{M}_{\mathsf{CL}}[bh]$, respectively.

Theorem 7 (Soundness of HYB_1). *If $\Gamma, [\![\Gamma']\!] \Rightarrow \Delta, [\![\Delta']\!]$ is provable in HYB_1, then $\Gamma \cup \Gamma' \vDash_{\mathcal{M}_{\mathsf{CL}}[hb]} \Delta \cup \Delta'$.*

Proof. Any initial sequent $\varnothing, [\![p]\!] \Rightarrow \varnothing, [\![p]\!]$ has the form $\Gamma, [\![\Gamma']\!] \Rightarrow \Delta, [\![\Delta']\!]$ in which Γ and Δ are empty and $\Gamma' = \Delta' = \{p\}$. In this case, the sequent enjoys the property that:[14]

1. $Var(\Gamma') \subseteq Var(\Delta') \subseteq Var(\Gamma \cup \Gamma')$
2. $\Gamma' \subseteq \Gamma \cup \Gamma'$ and $\Delta' \subseteq \Delta \cup \Delta'$
3. The sequent $\Gamma' \Rightarrow \Delta'$ is derivable in **LK**

It can be easily checked that that this property is preserved under each of the foregoing rules. The case of the Exchange and Contraction rules, and Weakening (outside the scope of the square brackets) can be noted to preserve this property, since they correspond to properties that are valid in every Tarskian logic and HYB_1 is a Tarskian logic, as every matrix logic is—see [29]. We notice that this property is preserved by the other rules as follows. Moreover, this can also be checked to apply straightforwardly to the "push" rules and the operational rules (in- and outside the square brackets). Hence, any derivable sequent enjoys the above tripartite property.

Now, we know that $\Xi \vDash_{\mathcal{M}_{\mathsf{CL}}[hb]} \Theta$ if and only if there exists a $\Xi' \subseteq \Xi$ and a $\Theta' \subseteq \Theta$ such that $Var(\Xi') \subseteq Var(\Theta') \subseteq Var(\Xi)$ and $\Xi' \vDash_{\mathcal{M}_{\mathsf{CL}}} \Theta'$. Because of soundness of **LK** (a presentation of which is described in [9]), the above tripartite property entails validity in $\mathcal{M}_{\mathsf{CL}}[hb]$. Soundness of HYB_2 with respect to $\mathcal{M}_{\mathsf{CL}}[bh]$ is proved by similar reasoning. ■

In the sequel, when we refer to the two-sided sequent calculus for PWK (and similarly for $\mathsf{K}_3^{\mathsf{w}}$), we will be talking about the calculi designed by Coniglio and Corbalán, presented in [9] as a fragment of Gentzen's sequent calculus for classical logic—indeed, as a fragment where some of the operational rules were restricted with *variable inclusion* requirements.

Given these, the following will help prove the completeness of HYB_1 with respect to $\mathcal{M}_{\mathsf{CL}}[hb]$.

Lemma 2. *If $\Gamma \vDash_{\mathcal{M}_{\mathsf{CL}}[hb]} \Delta$ such that $\Gamma' \subseteq \Gamma$, $\Delta' \subseteq \Delta$, $Var(\Gamma') \subseteq Var(\Delta') \subseteq Var(\Gamma)$ and $\Gamma' \vDash_{\mathcal{M}_{\mathsf{CL}}} \Delta'$, then $\Gamma' \Rightarrow \Delta'$ is provable in the calculus for PWK.*

Proof. Assume $\Gamma \vDash_{\mathcal{M}_{\mathsf{CL}}[hb]} \Delta$. Then by Corollary 2 for $\mathcal{M}_{\mathsf{CL}}[hb]$, we know that there are $\Gamma' \subseteq \Gamma$, $\Delta' \subseteq \Delta$, with $Var(\Gamma') \subseteq Var(\Delta') \subseteq Var(\Gamma)$ and $\Gamma' \vDash_{\mathcal{M}_{\mathsf{CL}}} \Delta'$. By completeness of **LK**, this implies that $\Gamma' \Rightarrow \Delta'$ is provable in **LK**. We also know that $Var(\Gamma') \subseteq Var(\Delta')$. Hence, by [9, Lemma 21], these two observations jointly imply that $\Gamma' \Rightarrow \Delta'$ is provable in the sequent calculus for PWK. ■

Definition 15. *In the HYB_1 calculus, a PWK rule that applies only to formulae within brackets is a "bracketed rule".*

Theorem 8 (Completeness of HYB_1). *If $\Gamma \vDash_{\mathcal{M}_{\mathsf{CL}}[hb]} \Delta$ such that $\Gamma' \subseteq \Gamma$, $\Delta' \subseteq \Delta$, $Var(\Gamma') \subseteq Var(\Delta') \subseteq Var(\Gamma)$ and $\Gamma' \vDash_{\mathcal{M}_{\mathsf{CL}}} \Delta'$, then $\Gamma', [\![\Gamma'']\!] \Rightarrow \Delta', [\![\Delta'']\!]$ is provable in HYB_1, where $\Gamma = \Gamma' \cup \Gamma''$ and $\Delta = \Delta' \cup \Delta''$.*

[14] As usual, this label denotes the standard sequent calculus for classical logic CL.

Proof. Assume that $\Gamma \vDash_{\mathcal{M}_{\mathsf{CL}}[hb]} \Delta$. Then, by Lemma 2, there is a PWK proof of $\Gamma' \Rightarrow \Delta'$. Call this proof, *i.e.* a rooted binary tree, Π. We can design an algorithm to transform a PWK proof of this sequent into an HYB_1 proof of $\Gamma, [\![\Gamma']\!] \Rightarrow \Delta, [\![\Delta']\!]$.

First, replace every node $\Xi \Rightarrow \Theta$ of Π by a node $\varnothing, [\![\Xi]\!] \Rightarrow \varnothing, [\![\Theta]\!]$. Then, place below each leaf, or axiom node, one instance of [Weak], such that from an axiom $\varnothing, [\![p]\!] \Rightarrow \varnothing, [\![p]\!]$ we infer in one step the sequent $\Gamma, [\![p]\!] \Rightarrow \Delta, [\![p]\!]$. After that, for each non-axiom node place Γ to the left of the square brackets in the antecedent and Δ to the left of the square brackets in the succedent. In the resulting proof, each PWK rule is applied within the scope of the square brackets. Moreover, we can check that every application of a PWK rule corresponds to a "bracketed rule" in HYB_1 that respects the corresponding provisos.

Actually, since Weakening is not fully admissible within the scope of square brackets, something must be said about this case. Suppose in an H proof of $\Gamma' \Rightarrow \Gamma'$ there is an *ineliminable* application of Weakening that allows to go from a node $\Xi \Rightarrow \Theta$ to a node $\Xi, \Xi' \Rightarrow \Theta, \Theta'$—whence we can legitimately call Ξ' and Θ' the active (sets of) formulae in this step. Then the current algorithm can be further specified by saying that if Π is a proof which has no ineliminable application of Weakening, then we proceed as previously stated. However, if Π *has* an ineliminable application of Weakening, then we enlarge every node (outside the square brackets) with Γ and Ξ', and Δ and Θ', in their respective sides. Finally, when the Π requires the corresponding application of Weakening, we mimic this in HYB_1 applying the [PushL] and [PushR] rules to Ξ' and Θ', as needed.

This renders a rooted binary tree Π^* with $\Gamma, [\![\Gamma']\!] \Rightarrow \Delta, [\![\Delta']\!]$ as its terminal sequent. We then proceed to apply the rules [PushL], [PushR] followed by elimination of duplicate formulae in Γ' and Δ'. We end up with a HYB_1 proof ending with $\Gamma'', [\![\Gamma']\!] \Rightarrow \Delta'', [\![\Delta']\!]$, for which $\Gamma'' \cup \Gamma' = \Gamma$ and $\Delta'' \cup \Delta' = \Delta$ and $Var(\Gamma') \subseteq Var(\Delta') \subseteq Var(\Gamma'' \cup \Gamma') = \Gamma$. ■

By similar means, we arrive at the corresponding results for HYB_2.

Theorem 9 (Soundness of HYB_2). *If $\Gamma, [\![\Gamma']\!] \Rightarrow \Delta, [\![\Delta']\!]$ is provable in HYB_2, then $\Gamma \cup \Gamma' \vDash_{\mathcal{M}_{\mathsf{CL}}[bh]} \Delta \cup \Delta'$.*

Theorem 10 (Completeness of HYB_2). *If $\Gamma \vDash_{\mathcal{M}_{\mathsf{CL}}[bh]} \Delta$ such that $\Gamma' \subseteq \Gamma$, $\Delta' \subseteq \Delta$, $Var(\Delta') \subseteq Var(\Gamma') \subseteq Var(\Delta)$ and $\Gamma' \vDash_{\mathcal{M}_{\mathsf{CL}}} \Delta'$, then $\Gamma', [\![\Gamma'']\!] \Rightarrow \Delta', [\![\Delta'']\!]$ is provable in HYB_2, where $\Gamma = \Gamma' \cup \Gamma''$ and $\Delta = \Delta' \cup \Delta''$.*

Proof. By Theorem 8 and Lemma 1. ■

Finally, the above calculi suggest that they may be adapted to the cases of matrices with three or more alternations by allowing some sort of nesting of brackets $[\![-]\!]$.

6.2 Sequent Calculi for the General Case

Now, the foregoing calculi seem to follow from non-trivial modifications to the Coniglio-Corbalán methods in which we have added a device that essentially tracks variable inclusions. It is clearly attractive to be able to provide a schematic method to give a sound and complete sequent calculus for *each* of the infinitely many consequence relations discussed in this paper. However, in the process of generalizing these sequent calculi to provide proof theories for each of our $\mathcal{M}[\sigma]$ systems, we are presented with a challenge.

For one, we have the option of trying to give a straightforward generalization of the calculi for HYB_1 and HYB_2 by nesting instances of the $[\![-]\!]$ device within one another and adding provisos and modifications to operational rules to preserve the structure of appropriate variable inclusion properties. Such an approach, however, is on its face perilous, as it would lead to an exponential blow-up in the number of rules. For example, if we have a calculus with the $[\![-]\!]$ device nested to a depth of (say) eighteen, it looks as though an appropriate suite of $[\vee L]$ rules on formulae φ and ψ would need independent special cases for occasions in which φ appears at depth m and ψ appears at depth n for all $m, n < 18$. While in principle such provisos could be described schematically, the resulting blowup in number of operational rules would drastically inhibit the utility of the resulting calculi.

A second approach would be to treat each system with an appropriate many-sided sequent calculus. There are two apparent problems with this approach. On the one hand, it seems as though the foregoing concern about explosion in the number of rules might apply to this case, so that in an m sided sequent calculus, we would need 2^m many distinct cases of a disjunction rule. On the other hand, tools such as `MUltseq` are capable of producing such calculi already and the importance of such a general scheme would be thereby severely diminished.

The third approach would be to make a straightforward (and shameless) appeal to our semantic characterizations by describing a way to take a classically provable sequent $\Gamma \Rightarrow \Delta$ and iterate a carefully controlled succession of application of Weakening on alternative sides to construct a sequent enjoying the appropriate variable inclusion properties. This approach risks the loss of some of the novelty found in the foregoing calculi HYB_1 and HYB_2 but retains a novelty of a different sort. Furthermore, for any of the matrix logics endowed with a linear order of contaminating values described in this work, this approach would permit us to describe a succinct and natural way to determine an appropriate sequent calculus. Such an approach would also have the benefit of being immediately recognizable as generating authentically two-sided sequent calculi.

Among the options, the third seems to fare the best, so we present a general description of appropriate sequent calculi that readily applies to any of the matrices discussed in this paper. If we look closely at the form of Theorems 1 and 2, a rough roadmap to appropriate sequent calculi can be inferred. In the case of (*e.g.*) $\mathcal{M}_{\mathsf{CL}}[hbh]$, we might follow something like the following algorithm, where [WeakL] and [WeakR] are left and right Weakening, respectively:

1. Give a classical proof in **LK** for a sequent $\Gamma \Rightarrow \Delta$
2. Apply arbitrary applications of [WeakR] to yield a sequent $\Gamma \Rightarrow \Delta'$
3. Apply [WeakL] to yield a $\Gamma' \Rightarrow \Delta'$ where $Var(\Delta') \subseteq Var(\Gamma')$
4. Apply [WeakR] to yield a $\Gamma' \Rightarrow \Delta''$ where $Var(\Gamma') \subseteq Var(\Delta'')$

A cursory reappraisal of our semantic characterization of $\mathcal{M}_{\mathsf{CL}}[hbh]$ suffices to reveal that $\Gamma' \Rightarrow \Delta''$ is provable by this algorithm if and only if $\Gamma' \vDash_{\mathcal{M}_{\mathsf{CL}}[hbh]} \Delta''$ is valid. In the general case, all that is necessary is that we *track* the steps at which we may apply [WeakL] and [WeakR] and at what stage a proof can be said to have terminated. To gain the ability to track these steps, we choose to make the novel decision to *label the sequent separator itself* by an index ranging over the natural numbers.

Let us first describe the raw materials from which we will define the appropriate calculi. The core of each system will be an indexed variant of a classical sequent calculus. Take a standard two-sided sequent calculus for classical propositional logic—for convenience, let us fix the sequent calculus **LK** described in [9]—and annotate the turnstiles in each rule with a subscript "0". Call this system $\mathbf{LK}_0$. By the definitions to follow, it will turn out that derivability in $\mathbf{LK}_0$ (*i.e.*, derivability of a classically provable sequent $\Gamma \Rightarrow_0 \Delta$) will correspond to the system $\mathcal{M}_{\mathsf{CL}}[\Lambda]$, *i.e.*, classical logic enriched with an *empty* linear order of contaminating values.

To $\mathbf{LK}_0$ we add the structural rules of Contraction and Permutation at every stage in a proof (*i.e.*, for every sequent separator $\Rightarrow_i$), a fact that is codified by the schematic rule where $i \in \omega$:

$$\frac{\Gamma \Rightarrow_i \Delta}{\Gamma' \Rightarrow_i \Delta'}\ [\text{Structural}_i]$$

where Γ' and Δ' are the result of applying instances of Contraction, Exchange, or Duplication to Γ and Δ, respectively.

Our earlier example demonstrated a need to alternate between stages at which [WeakL] is appropriate and stages at which [WeakR] is appropriate. To permit Weakening only in appropriate positions at appropriate times, we *stratify* Weakening with the schematic rules for each $i \in \omega$:

$$\frac{\Gamma \Rightarrow_i \Delta}{\Gamma, \Gamma' \Rightarrow_i \Delta}\,[\text{WeakL}_i] \qquad\qquad \frac{\Gamma \Rightarrow_i \Delta}{\Gamma \Rightarrow_i \Delta, \Delta'}\,[\text{WeakR}_i]$$

Finally, we track the iterations by *ascension* rules in which we lift the operations on a sequent $\Gamma \Rightarrow_i \Delta$ to $\Gamma \Rightarrow_{i+1} \Delta$ when certain variable-inclusion provisos are met. These schematic rules are presented below, where $i \in \omega$:

$$\frac{\Gamma \Rightarrow_i \Delta}{\Gamma \Rightarrow_{i+1} \Delta}\,[\text{AscensionL}_i] \qquad\qquad \frac{\Gamma \Rightarrow_i \Delta}{\Gamma \Rightarrow_{i+1} \Delta}\,[\text{AscensionR}_i]$$

These rules have the provisos that in order to apply [AscensionL$_i$], it must be established that $Var(\Delta) \subseteq Var(\Gamma)$ while correct applications of [AscensionR$_i$] require that $Var(\Gamma) \subseteq Var(\Delta)$.

In order to more conveniently define our general suite of sequent calculi, we define four types of collections of Weakening and Ascension rules:

Definition 16. $\mathbf{LOdd}_m = \{[\mathit{WeakL}_{2i+1}], [\mathit{AscensionL}_{2i+1}] \mid 2i+1 \lneq m\}$.

Definition 17. $\mathbf{ROdd}_m = \{[\mathit{WeakR}_{2i+1}], [\mathit{AscensionR}_{2i+1}] \mid 2i+1 \lneq m\}$.

Definition 18. $\mathbf{LEven}_m = \{[\mathit{WeakL}_{2i}], [\mathit{AscensionL}_{2i}] \mid 2i \lneq m\}$.

Definition 19. $\mathbf{REven}_m = \{[\mathit{WeakR}_{2i}], [\mathit{AscensionR}_{2i}] \mid 2i \lneq m\}$.

These collections permit us to perspicuously define sequent calculi for every one of the infinitely many consequence relations described in this paper. For each string σ of alternating instances of h and b, we define a calculus $\mathbf{LK}[\sigma]$. These systems are given a bipartite definition, broken apart on the basis of the initial element of σ. Where a string begins with h, we define $\mathbf{LK}[\sigma]$ in the following terms, where $\oplus$ indicates enriching a sequent calculus with additional rules.

Definition 20. *For a string hb... of length n, the calculus* $\mathbf{LK}[hb...]$ *is the following:*

$$\mathbf{LK}_0 \oplus [\text{Structural}_i] \oplus \mathbf{REven}_n \oplus \mathbf{LOdd}_n$$

We say that a sequent $\Gamma \Rightarrow \Delta$ *is provable in* $\mathbf{LK}[hb...]$ *if the labeled sequent* $\Gamma \Rightarrow_n \Delta$ *is provable.*

When σ, on the other hand, counts b as its initial element, we give a dual definition for $\mathbf{LK}[\sigma]$.

Definition 21. *For a string bh... of length n, the calculus* $\mathbf{LK}[bh...]$ *is the following:*

$$\mathbf{LK}_0 \oplus [\text{Structural}_i] \oplus \mathbf{ROdd}_n \oplus \mathbf{LEven}_n$$

Again, we say that a sequent $\Gamma \Rightarrow \Delta$ *is provable in* $\mathbf{LK}[hb...]$ *if the labeled sequent* $\Gamma \Rightarrow_n \Delta$ *is provable.*

Now, we may proceed to observe that these sequent calculi are indeed appropriate for our matrices by a general soundness and completeness proof:

Theorem 11. *Let σ be a string of alternating instances of h and b. Then:*

$$\Gamma \vDash_{\mathcal{M}_{\mathsf{CL}}[\sigma]} \Delta \textit{ if and only if } \Gamma \Rightarrow \Delta \textit{ is provable in } \mathbf{LK}[\sigma]$$

Proof. We prove this by induction on complexity of σ for the two cases in which the terminal element of σ is either h or b.

The basis step is when $\sigma = \Lambda$, *i.e.*, the empty string. Then derivability of $\Gamma \Rightarrow_0 \Delta$ corresponds to derivability in **LK**. As induction hypothesis, then, suppose that for the cases for which σ is of length m have been covered. Then we cover two cases to establish the result for σ of length $m + 1$.

In the case in which the terminal element of σ is h, let σ' be the string σ without its terminal element. Then we arrive at $\mathbf{LK}[\sigma]$ by adding the rules [WeakR$_m$] and [AscensionR$_m$] to the calculus $\mathbf{LK}[\sigma']$. Then we know that $\Gamma \Rightarrow_m \Delta$ is derivable in $\mathbf{LK}[\sigma]$ if and only if $Var(\Gamma) \subseteq Var(\Delta)$ and for a $\Delta' \subseteq \Delta$, $\Gamma \Rightarrow_{m-1} \Delta'$ is derivable in $\mathbf{LK}[\sigma']$. By induction hypothesis, this holds if and only if $\Gamma \vDash_{\mathcal{M}_{\mathsf{CL}}[\sigma']} \Delta'$ is valid in $\mathcal{M}_{\mathsf{CL}}[\sigma']$. But by Theorem 2, this is equivalent to the existence of a $\Delta \supseteq \Delta'$ for which $\Gamma \vDash_{\mathcal{M}_{\mathsf{CL}}[\sigma' h]} \Delta$ is valid, and $\mathcal{M}_{\mathsf{CL}}[\sigma' h]$ is just $\mathcal{M}_{\mathsf{CL}}[\sigma]$.

The case in which the terminal element of σ is b is carried out in an identical fashion by dualizing each of the foregoing steps and appealing to Theorem 1 rather than Theorem 2. ■.

With Theorem 11 in hand, we have provided a recursively defined and countably infinite suite of authentically two-sided sequent calculi that correspond to any case in which two-valued classical logic is supplemented with a linear order of contaminating values.

6.3 Cut Admissibility in the Calculi LK[σ]

There are many proof-theoretic properties that are worth investigating in the case of the calculi $\mathbf{LK}[\sigma]$ and we are unable to examine them all. The plight of the rule of [Cut], however, has been identified by a referee as one particularly worthy of investigation and we will consider this question before closing this section.

The rule [Cut], of course, in the case of **LK** is the following:

$$\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \varphi, \Sigma \Rightarrow \Xi}{\Gamma, \Sigma \Rightarrow \Delta, \Xi}\,[\text{Cut}]$$

For sequent calculi in which [Cut] is included as a rule, one is frequently interested in whether a system enjoys *cut elimination*, that is, whether any sequent provable with [Cut] can be proven without the rule. The systems $\mathbf{LK}[\sigma]$ do not include [Cut], however, so the question we will investigate is whether these systems enjoy *cut admissibility*, that is, whether [Cut] can be *emulated* in the calculi.

We note that it is not on its face obvious that any of the systems $\mathbf{LK}[\sigma]$ (where $\sigma \neq \Lambda$) enjoys cut admissibility. Where $\Gamma \Rightarrow \Delta, \varphi$ and $\varphi, \Sigma \Rightarrow \Xi$ are provable there exist relatively delicate, back-and-forth-type variable-inclusion properties between Γ and $\Delta \cup \{\varphi\}$ on the one hand and $\Sigma \cup \{\varphi\} \Rightarrow \Xi$ on the other. The roles that the cut formula φ plays in these back-and-forth containments might differ between the two cases and, moreover, in either of these cases, φ may be critical in the satisfaction appropriate variable-inclusion properties right. That φ is eliminable—or that its role may be taken over by some other formula—is not an obvious fact.

Happily, each of these systems enjoys cut admissilibity and, indeed, enjoys a stronger property in which [Cut] is admissible for each sequent seperator $\Rightarrow_i$. We will say that one of our systems $\mathbf{LK}[\sigma]$ enjoys *full* cut admissibility if [Cut] can be emulated for every indexed sequent separator $\Rightarrow_i$. To put this more precisely:

Definition 22. *We say that a calculus* $\mathbf{LK}[\sigma]$ *enjoys* full cut admissibility *if for every index* i*, whenever the sequents* $\Gamma \Rightarrow_i \Delta, \varphi$ *and* $\varphi, \Sigma \Rightarrow_i \Xi$ *are provable in* $\mathbf{LK}[\sigma]$*, then there exists an* $\mathbf{LK}[\sigma]$ *proof of the sequent* $\Gamma, \Sigma \Rightarrow_i \Delta, \Xi$*.*

Note that this is a stronger claim than mere cut admissibility, as $\Gamma \Rightarrow \Delta$ holds in a system $\mathbf{LK}[\sigma]$ when $\Gamma \Rightarrow_i \Delta$ is derivable for the maximum index i. Hence, a proof of full cut admissibility has cut admissibility *simpliciter* as a corollary.

With respect to cut admissibility, we close Section 6 with the observation that all the sequent calculi $\mathbf{LK}[\sigma]$ enjoy full cut admissibility.

Theorem 12. *For all strings σ comprising alternating instances of h and b, the system $\mathbf{LK}[\sigma]$ enjoys full cut admissibility.*

Proof. For an arbitrary σ, we prove this by induction on the subscript of the sequent separator $\Rightarrow_i$. For the basis step in which $i = 0$, we note simply that [Cut] is admissible in $\mathbf{LK}$ (*i.e.*, $\mathbf{LK}_0$), whence we conclude that the special instance

$$\frac{\Gamma \Rightarrow_0 \Delta, \varphi \qquad \varphi, \Sigma \Rightarrow_0 \Xi}{\Gamma, \Sigma \Rightarrow_0 \Delta, \Xi}$$

is admissible.

For the induction step, suppose that for all $j < i$, the corresponding instance of [Cut] for $\Rightarrow_j$ is admissible. Furthermore, suppose that we have $\mathbf{LK}[\sigma]$ proofs Π and Π' of $\Gamma \Rightarrow_i \Delta, \varphi$ and $\varphi, \Sigma \Rightarrow_i \Xi$, respectively. Now, these two sequents are derived from one of two methods, depending on the choice of $\mathbf{LK}[\sigma]$ and value of i:

- There exist $\mathbf{LK}[\sigma]$-provable sequents $\Gamma \Rightarrow_{i-1} \Delta'$ and $\varphi, \Sigma \Rightarrow_{i-1} \Xi'$ such that $\Delta' \subseteq \Delta \cup \{\varphi\}$, $\Xi' \subseteq \Xi$, $Var(\Delta') \subseteq Var(\Gamma)$, and $Var(\Xi') \subseteq Var(\Sigma \cup \{\varphi\})$. Furthermore, the rule [AscensionL$_{i-1}$] is applied to each of these sequents, (possibly) followed by applications of [WeakR$_i$].
- There exist $\mathbf{LK}[\sigma]$-provable sequents $\Gamma' \Rightarrow_{i-1} \Delta, \varphi$ and $\Sigma' \Rightarrow_{i-1} \Xi$ such that $\Gamma' \subseteq \Gamma$, $\Sigma' \subseteq \Sigma \cup \{\varphi\}$, $Var(\Gamma') \subseteq Var(\Delta \cup \{\varphi\})$, and $Var(\Sigma') \subseteq Var(\Xi)$. Furthermore, the rule [AscensionR$_{i-1}$] is applied to each of these sequents, (possibly) followed by applications of [WeakL$_i$].

• First, consider the former case, for which there are two subcases, one in which $\varphi \notin \Delta'$ and another in which $\varphi \in \Delta'$. In the former subcase, we can construct the end sequent easily. By hypothesis, Π contains as a subproof an $\mathbf{LK}[\sigma]$ proof Π'' of the sequent $\Gamma \Rightarrow_{i-1} \Delta'$. Our assumptions about $\mathbf{LK}[\sigma]$ include the fact that [AscensionL$_{i-1}$] is applied to this sequent. Hence, the iterative construction described in Definition 20 means that either [WeakL$_{i-1}$] (if $i \neq 1$) or Weakening *simpliciter* (if $i = 1$) is a valid rule of $\mathbf{LK}[\sigma]$, whence we may derive $\Gamma, \Sigma \Rightarrow_{i-1} \Delta'$. Because $Var(\Delta') \subseteq Var(\Gamma)$, it holds also that $Var(\Delta') \subseteq Var(\Gamma \cup \Sigma)$, whence we may apply [AscensionL$_{i-1}$] to yield $\Gamma, \Sigma \Rightarrow_i \Delta'$. But by hypothesis, we may also apply [WeakR$_i$], whence we can modify Π'' to construct an $\mathbf{LK}[\sigma]$ proof of the sequent $\Gamma, \Sigma \Rightarrow_i \Delta, \Xi$.

On the other hand, if $\varphi \in \Delta'$, then Π has as a subproof an $\mathbf{LK}[\sigma]$ proof of $\Gamma \Rightarrow_{i-1} \Delta' \smallsetminus \{\varphi\}, \varphi$ and Π' contains a subproof of $\varphi, \Sigma \Rightarrow_{i-1} \Xi'$. By induction hypothesis, the instance of [Cut] for $i-1$ is admissible, whence we are guaranteed that there exists an $\mathbf{LK}[\sigma]$ proof Π'' of the sequent $\Gamma, \Sigma \Rightarrow_{i-1} \Delta' \smallsetminus \{\varphi\}, \Xi'$. We know that $Var(\Delta') \subseteq Var(\Gamma)$ and $Var(\Xi') \subseteq Var(\Sigma \cup \{\varphi\})$, entailing that $Var(\Delta' \cup \Xi') \subseteq Var(\Gamma \cup \Sigma \cup \{\varphi\})$. Because $\varphi \in \Delta'$, the variables in φ appear in Γ, and $Var(\Gamma \cup \Sigma \cup \{\varphi\}) = Var(\Gamma \cup \Sigma)$, so we may rewrite this as the fact that $Var(\Delta' \cup \Xi') \subseteq Var(\Gamma \cup \Sigma)$, licensing us to apply [AscensionL$_{i-1}$] to extend Π'' to a proof of $\Gamma, \Sigma \Rightarrow_i \Delta' \smallsetminus \{\varphi\}, \Xi'$. Finally, a single application of [WeakR$_i$] is sufficient to convert Π'' to an $\mathbf{LK}[\sigma]$ proof of $\Gamma, \Sigma \Rightarrow_i \Delta, \Xi$, as desired.

• The second case is largely dual to the first, and we break up subcases in which $\varphi \notin \Sigma'$ and $\varphi \in \Sigma'$. If $\varphi \notin \Sigma'$, then, as before, we have an $\mathbf{LK}[\sigma]$ proof Π of the sequent $\Sigma' \Rightarrow_{i-1} \Xi$.

Because [AscensionR$_{i-1}$] is a rule of $\mathbf{LK}[\sigma]$, by Definition 20, either [WeakR$_{i-1}$] or Weakening without qualification are valid as well, from which we may turn Π into a proof of the sequent $\Sigma' \Rightarrow_{i-1} \Delta, \Xi$. Again, the hypothesis tells us that $Var(\Sigma') \subseteq Var(\Delta \cup \Xi)$, on which basis we may apply [AscensionR$_{i-1}$] to get a proof of $\Sigma' \Rightarrow_i \Delta, \Xi$, and a further application of [WeakL$_i$] converts Π into a $\mathbf{LK}[\sigma]$ proof of $\Gamma, \Sigma \Rightarrow_i \Delta, \Xi$.

When $\varphi \in \Sigma'$, then our proofs Π and Π' have $\mathbf{LK}[\sigma]$ subproofs of the sequents $\Gamma' \Rightarrow_{i-1} \Delta, \varphi$ and $\Sigma' \smallsetminus \{\varphi\}, \varphi \Rightarrow_{i-1} \Xi$, respectively. By hypothesis, [Cut] holds for $\Rightarrow_{i-1}$, whence we are guaranteed that there exists an $\mathbf{LK}[\sigma]$ proof of the sequent $\Gamma', \Sigma' \smallsetminus \{\varphi\} \Rightarrow_{i-1} \Delta, \Xi$. In this case, $Var(\Gamma') \subseteq Var(\Delta \cup \{\varphi\})$, and $Var(\Sigma') \subseteq Var(\Xi)$; again, because $\varphi \in \Sigma'$, $Var(\varphi) \subseteq Var(\Xi)$, the set $Var(\Delta \cup \Xi \cup \{\varphi\})$ may be simplified to $Var(\Delta \cup \Xi)$. Putting this together, then, we conclude that $Var(\Gamma' \cup (\Sigma' \smallsetminus \{\varphi\})) \subseteq Var(\Delta \cup \Xi)$. This satisfies the proviso required to apply [AscensionR$_{i-1}$] to yield a proof of $\Gamma', \Sigma' \smallsetminus \{\varphi\} \Rightarrow_i \Delta, \Xi$. To this proof, we may apply [WeakL$_i$] to yield an $\mathbf{LK}[\sigma]$ proof of $\Gamma, \Sigma \Rightarrow_i \Delta, \Xi$, as we had needed. ■.

We plan to revisit these calculi and variants of them in future work.

7 Concluding Remarks

In this paper, we have identified a countably infinite family of subsystems of classical logic among which are weak Kleene logic and its paraconsistent dual. We have provided characterizations of each of the corresponding consequence relations and provided for each a sound and complete two-sided sequent calculus. These results are exceedingly general and cover a host of very natural many-valued matrices that have both historical and practical relevance. As any logician familiar with Hilbert's Grand Hotel knows, however, the mere fact that one has proven an infinite number of results does not entail that the work has been completed. With this in mind, we end the paper by describing several avenues in which this work can be directed.

One project that springs to mind is an investigation into the *utility* of these systems. As we have suggested, the current state of the art in applied computer science frequently encounters programs running in a cascade virtual machines nested in one another. This fact suggests room for applicability of our results to this field, but much of this hinges on the matter of interpreting a contaminating value as *designated*. We plan to devote future work to an investigation into the matter of designation (or not) of these truth-values.

One formal matter that is entwined with the question of how to best provide a generalization of the calculi HYB_1 and HYB_2 is the matter of *proof complexity*. One way to look at the trade-off between the calculi that we have described and the method of many-sided sequent calculi is that our presentation has limited the number of additional rules at the cost of a possibly exponential increase of the search space. On its face, verifying that $\Gamma \Rightarrow \Delta$ is provable seems to require a back-and-forth procedure grabbing subsets of Γ and Δ with appropriate variable-inclusion properties until landing on $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ for which we can confirm that $\Gamma' \Rightarrow \Delta'$ is classically provable. This seems to indicate a worst-case complexity of verifying provability of a sequent as being in $\mathsf{EXPTIME}$, but we set aside the investigation into proof complexity for future work.

Finally, in a more theoretical vein, although we have out of convenience interpreted the matrix $\mathcal{M}_{\mathsf{CL}}$ as classical logic, our results make clear that many of the characterization results apply *mutatis mutandis* to *any* many-valued logic. (Indeed, the general method for constructing two-sided sequent calculi ought to carry over in many cases as well.) Investigating the landscape of logics with linearly ordered contaminating values in more generality would lead to studying appropriate subsystems of a broad field of many-valued logics. Families of systems like the four-valued logic of first-degree entailment and its cousins or fuzzy logics suggest that it would be interesting to study how contaminating values interact with other logical properties, such as relevance, non-determinism, fuzziness, and so forth (some initial steps with respect to investigating contaminating values in relevance logics can be found in [8]).

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