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Normality Operators and Classical Collapse

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Abstract: In this paper, we extend the expressive power of the logics K_3, LP and FDE with a *normality operator*, which is able to express whether a formula is assigned a classical truth value or not. We then establish *classical recapture* theorems for the resulting logics. Finally, we compare the approach via normality operator with the classical collapse approach devised by Jc Beall.

Keywords: Many-valued Logic, Classical Recapture, Normality Operators, Classical Collapse, Logic of Formal Inconsistency, Logic of Formal Undeterminedness

Introduction

Theories of *classical recapture* (Beall, 2011, 2013; Priest, 1979, 1991) specify at which conditions we can safely draw *classically valid* inferences while having a (subclassical) *many-valued logic* as our reasoning tool of choice. For instance, if we use Strong Kleene logic K_3 (Kleene, 1952), a theory of classical recapture will specify at which conditions we can assert an instance $\phi \vee \neg\phi$ of the *Law of Excluded Middle*—a principle that fails in the logic. Similarly, if we use the Logic of Paradox LP by Priest (1979, 2006), the theory will specify at which conditions we can apply Modus Ponens and infer ψ from $\phi, \phi \supset \psi$, or apply Ex Contradictione Quodlibet and infer ψ from $\phi \wedge \neg\phi$ — again, the logic fails the rules in question. Something along these lines would be done also for the four-valued FDE (Belnap, 1977).

This endeavor is motivated by a philosophical background that is shared by a number of many-valued logicians: use of a many-valued reasoning tool is necessary because we may face a number of 'abnormal phenomena' that allegedly cannot be treated classically (logical paradoxes, partial information, vagueness, denotational failure), but as long as these phenomena are not at stake, classical logic is perfectly in order as it is.

In this paper, we generalize the expressive power of the *Logic of Formal Inconsistency* (Carnielli, Coniglio, & Marcos, 2007; Carnielli, Marcos,

& De Amo, 2000; da Costa, 1974; Marcos, 2005) and the *Logic of Formal Undeterminedness* (Corbalan, 2012) in order to specify at which conditions we can reason classically when deploying some given many-valued reasoning tools. In particular, we present the many-valued logics $\mathsf{K_3}^{\circledast}$, $\mathsf{LP}^{\circledast}$, and $\mathsf{FDE}^{\circledast}$, which increase the expressive power of $\mathsf{K_3}$, LP and FDE, respectively, and we establish classical recapture results for these logics. Finally, we compare our approach with the classical recapture strategy provided by the *classical collapse* approach by (Beall, 2011, 2013).

The paper proceeds as follows. In the remainder of this Introduction, we provide some background on the Logic of Formal Inconsistency (LFI), the Logic of Formal Undeterminedness (LFU), and the normality operator that we use in the paper. In Section 1, we introduce the logics $\mathsf{K_3}$, LP and FDE, which provide the basic many-valued reasoning tools of the paper. In Section 2, we augment the three logics with the normality operator $\circledast$, thus obtaining systems in the LFI and LFU tradition, and we establish our main results: Theorem 1 and Theorem 2. Interestingly, $\mathsf{FDE}^{\circledast}$ requires a slightly different recapture strategy than $\mathsf{K_3}^{\circledast}$ and $\mathsf{LP}^{\circledast}$. Section 3 introduces the approach by (Beall, 2011, 2013), and Section 4 compares our 'recapture via normality' and classical collapse. Finally, Section 5 summarizes the content of the paper and presents some conclusions.

Background. LFI is a family of systems originating in da Costa (1974). Systems in this family control the behavior of inconsistency by internalizing the notion in the object language. This is done by a *consistency operator*—see end of Section 2. LFI includes a huge variety of formalisms, which may receive highly diversified semantical treatments. Here, we follow Carnielli et al. (2000) in focusing on a formalism that has a straightforward truth-functional semantics (see Sections 1 and 2). LFU dualizes da Costa's project and includes systems controlling the behavior of undeterminedness (failure of Excluded Middle). This is done by an *determinedness* operator that, together with negation, internalizes the notion in the object language.

The *normality operator* from this paper generalizes the operators from LFI and LFU. While the consistency (determinedness) operator expresses that a formula ϕ is consistent (determined), the normality operator expresses the stricter notion that a formula ϕ has a classical truth value (or 'is normal'). While the operators of normality and consistency (determinedness) coincide in a paraconsistent (paracomplete) three-valued logic, they are in principle distinct in a four-valued logic that is both paraconsistent and paracomplete, such as logic $\mathsf{FDE}^{\circledast}$ from Section 2.

1 Preliminaries

Given an infinitely denumerable set $\mathcal{P}$ of propositional variables, standard propositional language $\mathcal{L}_1(\mathcal{P})$ is defined by the following Backus-Naur Form (BNF):

$$\Phi ::= p \mid \neg\phi \mid \phi \vee \psi \mid \phi \wedge \psi$$

where $p \in \mathcal{P}$ and $\neg, \vee, \wedge$ are negation, disjunction, conjunction, respectively. As usual, we define $\phi \supset \psi = \neg\phi \vee \psi$. We denote sets of arbitrary formulas by $\Sigma, \Gamma, \Delta, \ldots$, and we omit reference to $\mathcal{P}$ when possible. We interpret the formulas in $\mathcal{L}_1$ via valuation functions:

Definition 1 (Valuations) *We let $\mathcal{V}$ be the class of all functions $\nu : \Phi \longmapsto \{0, \frac{1}{2}, 1\}$ that satisfy the following clauses:*

- $\nu(\neg\phi) = 1 - \nu(\phi)$
- $\nu(\phi \vee \psi) = max(\nu(\phi), \nu(\psi))$
- $\nu(\phi \wedge \psi) = min(\nu(\phi), \nu(\psi))$

We denote by $\mathcal{V}_{\mathsf{CL}}$ the set of valuations $\nu \in \mathcal{V}$ such that $\nu(p) \in \{0, 1\}$ for every $p \in \mathcal{P}$.[1] We define a *logic* S semantically as a pair $\langle \mathcal{L}, \vDash_{\mathsf{S}} \rangle$, where $\mathcal{L}$ is a language and $\vDash_{\mathsf{S}}$ is a relation of *logical consequence*—from now on, we will often talk about S-consequence, depending on the system we are focusing on. For every logic S, we define a set $\mathcal{D}_{\mathsf{S}} \subseteq \mathcal{T}$ of *designated values* of S. We define S-consequence as preservation of designated values in S:

Definition 2 (S-consequence) *For every logic S, S-consequence is a relation $\vDash_{\mathsf{S}} \subseteq 2^{\Phi} \times \Phi$ such that:*

$$\Sigma \vDash_{\mathsf{S}} \psi \iff \nu(\psi) \in \mathcal{D}_{\mathsf{S}} \textit{ if } \nu(\phi) \in \mathcal{D}_{\mathsf{S}} \textit{ for every } \phi \in \Sigma$$

The following is a useful notation: $var(\Sigma)$ is the set of variables p that occur in some $\phi \in \Sigma$. We write $var(\psi)$ instead of $var(\{\psi\})$. We call a *tautology* any formula that follows from the empty set of premises.

1.1 Strong Kleene Logic and the Logic of Paradox

Strong Kleene Logic K_3 (Kleene, 1952) and the Logic of Paradox LP (Priest, 1979, 2006) have found prominent applications in philosophical logic, especially with respect to *logical paradoxes* and *truth theory* (Field, 2008;

[1] We believe the reason for the label is clear: the valid rules and principles of Classical Logic CL are determined by these valuations.

Kripke, 1975; Priest, 1979, 2006).[2] Their interpretation is based on the valuation functions from Definition 1. The difference between the two logics is in their designated values: $\mathcal{D}_{\mathsf{K}_3} = \{1\}$, while $\mathcal{D}_{\mathsf{LP}} = \{\frac{1}{2}, 1\}$.

A straightforward consequence of Definition 1 and $\mathcal{D}_{\mathsf{K}_3} = \{1\}$ is that no formula $\phi \in \Phi_{\mathcal{L}_1}$ is a *tautology* in K_3: we have $\nu(\phi) = \frac{1}{2}$ if $\nu(p) = \frac{1}{2}$ for every $p \in var(\phi)$. *A fortiori*, the *Law of Excluded Middle* fails:

$$\varnothing \not\vDash_{\mathsf{K}_3} \phi \vee \neg\phi \qquad \text{(Failure of LEM)}$$

According to standard terminology, this makes K_3 a *paracomplete* logic. Another consequence of K_3 having no tautology is failure of the *Law of Identity* (LI) $\phi \supset \phi$. Definition 1 and $\mathcal{D}_{\mathsf{LP}} = \{\frac{1}{2}, 1\}$ imply that *Ex Contradictione Quodlibet* fails:

$$\phi \wedge \neg\phi \not\vDash_{\mathsf{LP}} \psi \qquad \text{(Failure of ECQ)}$$

Any $v \in \mathcal{V}$ such that $\nu(p) = \frac{1}{2}$ and $\nu(q) = \mathbf{f}$ provides a countermodel. Also, notice that every formula $\phi \in \Phi_{\mathcal{L}_1}$ is *satisfiable* in LP: $\nu(\phi) = \frac{1}{2}$ whenever $\nu(p) = \frac{1}{2}$ for every $p \in var(\phi)$. Finally, all tautologies from Classical Logic CL are LP-tautologies, and vice versa. We refer the reader to Priest (1979) for this.

More in general, presence of the third value implies departure from *classical* consequence $\vDash_{\mathsf{CL}}$, to the effect that some classically valid inferences fail in K_3 and LP. The following observation details some validities and the most notable failures of K_3 and LP:[3]

Observation 1 K_3*-consequence and* LP*-consequence satisfy:*

$1a$	$\psi \not\vDash_{\mathsf{K}_3} \phi$ *for ϕ a classical tautology*	$1b$	$\psi \vDash_{\mathsf{LP}} \phi$ *for ϕ a classical tautology*
$2a$	$\phi, \neg\phi \vDash_{\mathsf{K}_3} \psi$	$2b$	$\phi, \neg\phi \not\vDash_{\mathsf{LP}} \psi$
$3a$	$\phi \supset (\psi \wedge \neg\psi) \vDash_{\mathsf{K}_3} \neg\phi$	$3b$	$\phi \supset (\psi \wedge \neg\psi) \not\vDash_{\mathsf{LP}} \neg\phi$
$4a$	$\phi, \phi \supset \psi \vDash_{\mathsf{K}_3} \psi$	$4b$	$\phi, \phi \supset \psi \not\vDash_{\mathsf{LP}} \psi$
$5a$	$\neg\psi, \phi \supset \psi \vDash_{\mathsf{K}_3} \neg\phi$	$5b$	$\neg\psi, \phi \supset \psi \not\vDash_{\mathsf{LP}} \neg\phi$
$6a$	$\phi \supset \psi, \psi \supset \zeta \vDash_{\mathsf{K}_3} \phi \supset \zeta$	$6b$	$\phi \supset \psi, \psi \supset \zeta \not\vDash_{\mathsf{LP}} \phi \supset \zeta$

[2] Other applications include *partial functions* (Kleene, 1938, 1952), *partial information* (Abdallah, 1995), *logic programs* (Fitting, 1985) (K_3), and *vagueness* (Priest, 2013; Ripley, 2013; Shapiro, 2006) (K_3 and LP).

[3] We refer the reader to (Beall 2011, 2013; Priest 1979, 2006) for these failures and validities.

As is well known, departure from classical reasoning is the key of K_3 and LP's success in approaching the logical paradoxes. However, this success comes at a cost: LP fails Modus Ponens (MP)—failure 4b above—and K_3 fails LI, which are crucial in our understanding of a conditional. This suffices to explain why we may want to reason classically when we are sure that no abnormal phenomenon is around. In Sections 2 and Sections 3, we approach two different ways to recapture *classical reasoning* in K_3 and LP.

Remark 1 (Reading of the third value) Third value $\frac{1}{2}$ has two natural informal readings in K_3 and LP, respectively. Failure of LEM in K_3 suggests that $\frac{1}{2}$ is read as 'neither true nor false', or 'undetermined', or 'undefined'. Failure of ECQ in LP suggests that $\frac{1}{2}$ is read as 'both true and false', or 'overdetermined', or 'inconsistent'. These readings will help in what follows.

1.2 First-degree entailment

The logic FDE has been first introduced by Anderson and Belnap (1962), and it has been later generalized to a 'useful four-valued logic' by Belnap (1977). In FDE, formulas from $\mathcal{L}_1$ are interpreted by adjusting Definition 1 to a poset $\{0, \mathbf{n}, \mathbf{b}, 1\}$ of truth values,[4] whose weak partial order $\leq$ is defined as follows:[5]

- $0 < \mathbf{n} < 1$
- $0 < \mathbf{b} < 1$
- $\mathbf{n} \nleq \mathbf{b}$ and $\mathbf{b} \nleq \mathbf{n}$

Definition 3 (Valuations, 2) *We let $\mathcal{U}$ be the class of all functions $u : \Phi \longmapsto \{0, \mathbf{n}, \mathbf{b}, 1\}$ that satisfy the following clauses:*

- $u(\neg\phi) = \begin{cases} 1 - u(\phi) & \text{if} \quad u(\phi) \in \{0,1\} \\ x & \text{if} \quad u(\phi) = x \quad \text{for } x \notin \{0,1\} \end{cases}$
- $u(\phi \vee \psi) = glb(u(\phi), u(\psi))$
- $u(\phi \wedge \psi) = lub(u(\phi), u(\psi))$

[4]The usual notation for the truth values of FDE is $\mathbf{f}, \mathbf{n}, \mathbf{b}, \mathbf{t}$. However, $\mathbf{f}$ and $\mathbf{t}$ behave exactly as 0 and 1 in K_3 and LP, and we keep the numerical notation here, for the sake of uniformity.

[5]A weak partial order is any *reflexive* and *transitive relation* $\mathcal{R}$ on a domain D that obeys $\forall x, y \in D : \mathcal{R}(x, y)$ and $\mathcal{R}(y, x) \Rightarrow x = y$.

Besides, we have $\mathcal{D}_{\mathsf{FDE}} = \{\mathbf{b}, 1\}$. From the specification of the order and the definition of $\mathcal{D}_{\mathsf{FDE}}$, it is easy to see that, if we restrict valuations in $\mathcal{U}$ to $\{0, \mathbf{b}, 1\}$, we obtain LP. Dually, if we restrict valuations in $\mathcal{U}$ to $\{0, \mathbf{n}, 1\}$, we obtain K_3. This suffices to understand that FDE is a sublogic of both K_3 and LP. The following failure guarantees that FDE is a *proper* sublogic of the two formalisms:

$$\phi \wedge \neg\phi \nvDash_{\mathsf{FDE}} \psi \vee \neg\psi \qquad \text{(Failure of Confusion)}$$

2 Recapture via normality

In this section, we propose to recapture classical reasoning by improving the expressive power of K_3 and LP by devices that tell apart the situations where a formula ϕ has a classical truth value ('is normal') from the situations where ϕ has some non-classical value ('is abnormal'). In particular, the logics $\mathsf{K_3}^{\circledast}$, $\mathsf{LP}^{\circledast}$ and $\mathsf{FDE}^{\circledast}$ that we introduce in this section are obtained by extending K_3, LP and FDE, respectively, with a normality operator $\circledast$. The following is a semantic definition of a normality operator:

Definition 4 (Normality operator) *Given a language $\mathcal{L}$, a set $\mathcal{T}$ of truth values including 0 and 1, and valuation functions $v : \Phi_{\mathcal{L}} \longmapsto \mathcal{T}$, a unary connective k is a normality operator iff, for every $\phi \in \Phi_{\mathcal{L}}$:*

$$v(k\phi) = 1 \Leftrightarrow v(\phi) \in \{0, 1\} \textit{ and } v(k\phi) = 0 \Leftrightarrow v(\phi) \notin \{0, 1\}$$

The logic $\mathsf{LP}^{\circledast}$ is a LFI along the tradition of da Costa (1974), Carnielli et al. (2007), and Marcos (2005). The logic $\mathsf{K_3}^{\circledast}$ is a LFU along the lines of Corbalan (2012). We come back to the connections between normality operators and the two families of logics at the end of this section.

2.1 Normality operator

Given an infinitely denumerable set $\mathcal{P}$ of propositional variables, the language $\mathcal{L}_2(\mathcal{P})$ is defined by the following BNF:

$$\Phi ::= p \mid \neg\phi \mid \phi \vee \psi \mid \phi \wedge \psi \mid \circledast\phi$$

where $p \in \mathcal{P}$ and $\circledast$ is a *normality* operator, with $\circledast\phi$ reading 'ϕ has a classical truth value'. We generalize the definition of valuation functions from Section 1:

Definition 5 (Valuations, 3) *We let $\mathcal{V}^+$ be the class of all functions $\nu : \Phi_{\mathcal{L}_2} \longmapsto \{0, \frac{1}{2}, 1\}$ that satisfy the clauses from Definition 1 together with:*

$$\nu(\circledast\phi) = \begin{cases} 1 & \text{if} \quad \nu(\phi) \in \{0,1\} \\ 0 & \text{if} \quad \nu(\phi) \notin \{0,1\} \end{cases}$$

We define $\mathsf{K_3}^{\circledast} = \langle \mathcal{L}_2, \vDash_{\mathsf{K_3}^{\circledast}} \rangle$ as the *extension* of $\mathsf{K_3}$ with $\circledast$. $\mathsf{K_3}^{\circledast}$-consequence $\vDash_{\mathsf{K_3}^{\circledast}}$ is defined according to Definition 2 by assuming $\mathcal{D}_{\mathsf{K_3}^{\circledast}} = \mathcal{D}_{\mathsf{K_3}} = \{1\}$. We define $\mathsf{LP}^{\circledast} = \langle \mathcal{L}_2, \vDash_{\mathsf{LP}^{\circledast}} \rangle$ as the *extension* of LP with $\circledast$. $\mathsf{LP}^{\circledast}$-consequence $\vDash_{\mathsf{LP}^{\circledast}}$ is defined according to Definition 2 by assuming $\mathcal{D}_{\mathsf{LP}^{\circledast}} = \mathcal{D}_{\mathsf{LP}} = \{\frac{1}{2}, 1\}$.

Given the clause from Definition 5 and $\mathcal{D}_{\mathsf{K_3}^{\circledast}} = \{1\}$, we have $\nu(\circledast\phi) \in \mathcal{D}_{\mathsf{K_3}^{\circledast}}$ iff $\nu(\phi \vee \neg\phi) = 1$: in the logic, $\circledast\phi$ states that ϕ is *determined*—it verifies its corresponding instance of LEM. In turn, this equates with stating that ϕ has a classical value. Dually, given the clause from Definition 5 and $\mathcal{D}_{\mathsf{LP}^{\circledast}} = \{\frac{1}{2}, 1\}$, we have $\nu(\circledast\phi) \in \mathcal{D}_{\mathsf{LP}^{\circledast}}$ iff $\nu(\phi \wedge \neg\phi) = 0$: in the logic, $\circledast\phi$ states that ϕ is *consistent*—it does not satisfy the corresponding contradiction. Again, this means that ϕ has a classical value. The following observation details some validities for the two logics:

Observation 2 $\mathsf{K_3}^{\circledast}$*-consequence and* $\mathsf{LP}^{\circledast}$*-consequence satisfy:*

$1a.$	$\varnothing \vDash_{\mathsf{S}^{\circledast}} \circledast \circledast \phi$	$1b.$	$\varnothing \vDash_{\mathsf{S}^{\circledast}} \circledast \neg \circledast \phi$
$2a.$	$\circledast\phi \vDash_{\mathsf{K_3}^{\circledast}} \phi \vee \neg\phi$	$2b.$	$\phi \vee \neg\phi \vDash_{\mathsf{K_3}^{\circledast}} \circledast\phi$
$3a.$	$\neg \circledast \phi \vDash_{\mathsf{LP}^{\circledast}} \phi \wedge \neg\phi$	$3b.$	$\phi \wedge \neg\phi \vDash_{\mathsf{LP}^{\circledast}} \neg \circledast \phi$
$4a.$	$\neg \circledast \phi \vDash_{\mathsf{S}^{\circledast}} \neg \circledast \neg\phi$	$4b.$	$\neg \circledast \neg\phi \vDash_{\mathsf{S}^{\circledast}} \neg \circledast \phi$
$5a.$	$\circledast(\phi \wedge \psi) \vDash_{\mathsf{S}^{\circledast}} \circledast\phi \wedge \circledast\psi$	$5b.$	$\circledast\phi \wedge \circledast\psi \vDash_{\mathsf{S}^{\circledast}} \circledast(\phi \wedge \psi)$
$6a.$	$\circledast(\phi \vee \psi) \vDash_{\mathsf{S}^{\circledast}} \circledast\phi \vee \circledast\psi$	$6b.$	$\circledast\phi \wedge \circledast\psi \vDash_{\mathsf{S}^{\circledast}} \circledast(\phi \vee \psi)$

where $\mathsf{S}^{\circledast} \in \{\mathsf{K_3}^{\circledast}, \mathsf{LP}^{\circledast}\}$

In particular, 1a and 1b states that, in $\mathsf{K_3}^{\circledast}$ ($\mathsf{LP}^{\circledast}$), talk about determinedness (consistency) and undeterminedness (inconsistency) are themselves determined (consistent). 2a and 2b state the above equivalence between 'ϕ has a classical truth value' and 'ϕ is determined' in $\mathsf{K_3}^{\circledast}$. Dually, 3a and 3b state the equivalence between 'ϕ does not have a classical truth value' and 'ϕ is inconsistent' in $\mathsf{LP}^{\circledast}$. The remaining items detail how $\circledast$ interacts with the other connectives.

2.2 Normality operators and classical recapture

We need some preliminaries before we establish our classical recapture theorems from $\mathsf{K_3}^{\circledast}$, $\mathsf{LP}^{\circledast}$ and $\mathsf{FDE}^{\circledast}$. First, we need an auxiliary notion:

Definition 6 (Normal counterpart) *Given a set $\Sigma \subseteq \Phi_{\mathcal{L}_1}$, we say that the set $\Sigma^{\circledast} = \{\circledast\phi \in \Phi_{\mathcal{L}_2} \mid \phi \in \Sigma\}$ is the* normal counterpart of Σ.

Second, the following observation will prove useful in the next theorem:

Observation 3 *For every $\nu \in \mathcal{V}$, there exists a $\nu' \in \mathcal{V}_{\mathsf{CL}}$ such that (1) $\nu'(p) = \nu(p)$ if $\nu(p) \in \{0, 1\}$, and (2) $\nu'(p) = 1$ if $\nu(p) = \frac{1}{2}$. By the clauses on ν by Definition 1, it follows that:*

1. $\nu'(\phi) = 0$ if $\nu(\phi) = 0$
2. $\nu'(\phi) = 1$ if $\nu(\phi) = 1$

Now we can establish our *classical recapture theorem*:[6]

Theorem 1 (Recapture via normality) *If $\Sigma, \psi \subseteq \Phi_{\mathcal{L}_1}$, then:*

$$\Sigma \vDash_{\mathsf{CL}} \psi \;\Leftrightarrow\; \begin{cases} \Sigma, \circledast\psi \vDash_{\mathsf{K_3}^{\circledast}} \psi \\ \Sigma, \Sigma^{\circledast} \vDash_{\mathsf{LP}^{\circledast}} \psi \end{cases}$$

Proof. We start with $\mathsf{K_3}^{\circledast}$: ($\Rightarrow$) Assume $\Sigma \vDash_{\mathsf{CL}} \psi$. Now suppose that $\Sigma, \circledast\psi \nvDash_{\mathsf{S}^+} \psi$. This implies that there is a $\nu \in \mathsf{V}$ such that $\nu[\Sigma] = \{1\}$ and $\nu(\psi) = 0$, which in turn contradicts the initial assumption.
($\Leftarrow$) Suppose $\Sigma, \circledast\psi \vDash_{\mathsf{K_3}^{\circledast}} \psi$. Since $\Sigma, \psi \subseteq \Phi_{\mathcal{L}_1}$, this equates with the fact that, for every $\nu \in \mathcal{V}$, $\nu(\psi) = 1$ if $\nu[\Sigma] = \{1\}$ and $\nu(\psi) \neq \frac{1}{2}$. Thus, the supposition that there is a classical valuation $\nu \in \mathcal{V}_{\mathsf{CL}}$ such that $\nu(\psi) = 0$ and $\nu[\Sigma] = \{1\}$ would contradict the initial hypothesis. As as consequence, we have $\nu(\psi) = 1$ for every $\nu \in \mathcal{V}_{\mathsf{CL}}$ such that $\nu[\Sigma] = \{1\}$; but this implies $\Sigma \vDash_{\mathsf{CL}} \psi$.

As for $\mathsf{LP}^{\circledast}$: ($\Rightarrow$) Assume $\Sigma \vDash_{\mathsf{CL}} \psi$. If $\mathcal{V}_{\mathsf{CL}}(\Sigma) = \varnothing$, then by Observation 3.2, $\{\nu \in \mathcal{V} \mid \nu[\Sigma] = 1\} = \varnothing$. This in turn implies $\Sigma, \Sigma^{\circledast} \vDash_{\mathsf{LP}^{\circledast}} \psi$. If $\mathcal{V}_{\mathsf{CL}}(\Sigma) \neq \varnothing$, then $\{\nu \in \mathcal{V} \mid \nu[\Sigma] = 1\} \neq \varnothing$. Now suppose that $\nu[\Sigma] = \{1\}$ and $\nu(\psi) = 0$ for some $\nu \in \mathcal{V}$. Take a valuation $\nu' \in \mathcal{V}_{\mathsf{CL}}$ such that: (1) $\nu'(p) \in \{0, 1\}$ for every $p \in \mathcal{P}$, (2) $\nu'(p) = 1$ if $\nu(p) \in \{\frac{1}{2}, 1\}$, (3) $\nu'[\Sigma] = \{1\}$. By Observation 3.1, $\nu'(\psi) = 0$. Since ν' is classical, this

[6]In what follows, we abuse notation a bit and write $\Sigma, \psi \subseteq \Phi$ instead of $\Sigma \cup \{\psi\} \subseteq \Phi$.

contradicts the initial assumption. Thus, $\nu(\psi) = 1$ for every $\nu \in \mathcal{V}_{\mathsf{LP}}$ such that $\nu[\Sigma] = \{1\}$. This in turn implies that $\Sigma, \Sigma^{\circledast} \vDash_{\mathsf{LP}^{\circledast}} \psi$.
($\Leftarrow$) Suppose $\Sigma, \Sigma^{\circledast} \vDash_{\mathsf{LP}^{\circledast}} \psi$. Since $\Sigma, \psi \subseteq \Phi_{\mathcal{L}_1}$, this equates with the fact that $\nu(\psi) \in \{\frac{1}{2}, 1\}$ if $\nu[\Sigma] = \{1\}$. Since $\mathcal{V}_{\mathsf{CL}}(\Sigma) \subseteq \{\nu \in \mathcal{V} \mid \nu[\Sigma] = \{1\}\}$, this implies that $\nu(\psi) = 1$ for every $\nu \in \mathcal{V}_{\mathsf{CL}}(\Sigma)$. But this implies $\Sigma \vDash_{\mathsf{CL}} \psi$. $\square$

Let us see how *recapture via normality* recovers the classical inferences or laws that fail in K_3 and LP—see Section 1:

$$(\phi \wedge \neg\phi), \circledast(\phi \wedge \neg\phi) \vDash_{\mathsf{LP}^{\circledast}} \psi \qquad \text{(Recapture of ECQ)}$$

Similarly, LEM and LI can be recaptured in ${\mathsf{K}_3}^{\circledast}$:

$$\circledast\phi \vDash_{{\mathsf{K}_3}^{\circledast}} \phi \vee \neg\phi \qquad \text{(Recapture of LEM)}$$
$$\circledast\phi \vDash_{{\mathsf{K}_3}^{\circledast}} \phi \supset \phi \qquad \text{(Recapture of LI)}$$

More in general, all classical tautologies can be recaptured in ${\mathsf{K}_3}^{\circledast}$ as follows: $\circledast\phi \vDash_{{\mathsf{K}_3}^{\circledast}} \phi$ for ϕ a classical tautology.

Observation 4 ${\mathsf{K}_3}^{\circledast}$*-consequence and* $\mathsf{LP}^{\circledast}$*-consequence satisfy:*

$$\psi, \circledast\phi \vDash_{{\mathsf{K}_3}^{\circledast}} \phi \qquad \textit{for } \phi \textit{ a classical tautology}$$
$$\phi \supset (\psi \wedge \neg\psi), \circledast(\phi \supset (\psi \wedge \neg\psi)) \vDash_{\mathsf{LP}^{\circledast}} \neg\phi$$
$$\phi, \phi \supset \psi, \{\phi, \phi \supset \psi\}^{\circledast} \vDash_{\mathsf{LP}^{\circledast}} \psi$$
$$\neg\psi, \phi \supset \psi, \{\neg\psi, \phi \supset \psi\}^{\circledast} \vDash_{\mathsf{LP}^{\circledast}} \neg\phi$$
$$\phi \supset \psi, \psi \supset \zeta, \{\phi \supset \psi, \psi \supset \zeta\}^{\circledast} \vDash_{\mathsf{LP}^{\circledast}} \phi \supset \zeta$$

2.3 Recapture via normality in $\mathsf{FDE}^{\circledast}$

In order to develop a 'recapture via normality' strategy for $\mathsf{FDE}^{\circledast}$, we consider once again language $\mathcal{L}_2$, which we interpret via the following valuation functions:

Definition 7 (Valuations, 4) *We let $\mathcal{U}^{+}$ be the class of all functions $u : \Phi_{\mathcal{L}_2} \longmapsto \{0, \mathbf{n}, \mathbf{b}, 1\}$ that satisfy the clauses from Definition 3 together with:*

$$\nu(\circledast\phi) = \begin{cases} 1 & \text{if} \quad \nu(\phi) \in \{0, 1\} \\ 0 & \text{if} \quad \nu(\phi) \notin \{0, 1\} \end{cases}$$

The above definition qualifies $\mathsf{FDE}^{\circledast}$ as the extension of FDE with normality operator $\circledast$. Notice that FDE does not obey Observation 3. In order to see this, take a valuation $u \in \mathcal{U}$ such that $u(p) = \mathbf{n}$, and $u(q) = \mathbf{b}$. Since $glb(\mathbf{n}, \mathbf{b}) = 1$, we have $u((p \wedge \neg p) \vee (q \wedge \neg q)) = 1$. This in turn implies that the recapture recipe from Thoerem 1 does not apply to FDE. Take again the example above, and suppose that, additionally, $u(r) = 0$. Since $u((p \wedge \neg p) \vee (q \wedge \neg q)) = 1$, this suffices to falsify
$(p \wedge \neg p) \vee (q \wedge \neg q), \circledast((p \wedge \neg p) \vee (q \wedge \neg q)), \circledast r \nvDash_{\mathsf{FDE}^{\circledast}} r$. However,
$(p \wedge \neg p) \vee (q \wedge \neg q) \vDash_{\mathsf{CL}} r$. We deploy a *stronger* recapture strategy, that requires *all* variables from premises and conclusion to have a classical value:

Theorem 2 (Recapture via normality in $\mathsf{FDE}^{\circledast}$) *If* $\Sigma, \psi \subseteq \Phi_{\mathcal{L}_1}$, *then:*

$$\Sigma \vDash_{\mathsf{CL}} \psi \ \Leftrightarrow\ \Sigma, (var(\Sigma))^{\circledast}, (var(\psi))^{\circledast} \vDash_{\mathsf{FDE}^{\circledast}} \psi$$

Proof. ($\Rightarrow$) Assume $\Sigma \vDash_{\mathsf{CL}} \psi$. For every valuation $u \in \mathcal{U}$ such that (1) $u[\Sigma] = \{1\}$ and (2) $u(p) \in \{0, 1\}$ for every $p \in var(\Sigma) \cup var(\psi)$, there is a corresponding classical valuation $u' \in \mathcal{U}$. This implies that $u(\psi) = 1$. Otherwise, we would have $u'(\psi) = 0$, which just contradicts the initial hypothesis. Since $\Sigma, \psi \subseteq \Phi_{\mathcal{L}_1}$, this implies $\Sigma, (var(\Sigma))^{\circledast}, (var(\psi))^{\circledast} \vDash_{\mathsf{S}^{\circledast}} \psi$.
($\Leftarrow$) Assume $\Sigma, (var(\Sigma))^{\circledast}, (var(\psi))^{\circledast} \vDash_{\mathsf{FDE}^{\circledast}} \psi$. Since $\Sigma, \psi \subseteq \Phi_{\mathcal{L}_1}$, this implies that $u(\psi) = \{1\}$ for every $u \in \mathcal{U}$ such that (1) $u[\Sigma] = \{1\}$, and (2) $u(p) \in \{0, 1\}$ for every $p \in var(\Sigma) \cup var(\psi)$. Every such valuation u can be turned into a corresponding classical valuation $u' \in \mathcal{U}$ where, by construction, $u'[\Sigma] = \{1\}$ and $u'(\psi) = 1$. Since these exhaust the classical models of Σ, we can conclude that $\Sigma \vDash_{\mathsf{CL}} \psi$. □

Notice, however, that the rule of Confusion $\phi \wedge \neg\phi \vDash \psi \vee \neg\psi$ can be recaptured by just imposing that the formulas in the premises and conclusion have a classical truth value. Indeed, it is easy to check that:

$$\phi \wedge \neg\phi, \{\phi \wedge \neg\phi\}^{\circledast}, \circledast\psi \vDash_{\mathsf{FDE}^{\circledast}} \psi \vee \neg\psi \qquad \text{(Recapture of Confusion)}$$

2.4 Normality operators and logics of formal inconsistency and undeterminedness

The normality operator is a generalization of the consistency and determinedness operators from LFI and LFU, respectively. These families of systems are defined as follows:

Definition 8 (Logic of Formal Inconsistency, Carnielli et al., 2007) *A logic* S *is a* Logic of Formal Inconsistency *if and only if it satisfies the following, for some connective k:*

- $\phi, \neg\phi \nvDash_{\mathsf{S}} \psi$ *(Failure of ECQ)*
- $\phi, \neg\phi, k\phi \vDash_{\mathsf{S}} \psi$ *(Principle of Gentle Explosion)*

Definition 9 (Logic of Formal Undeterminedness, Corbalan, 2012) *A logic* S *is a* Logic of Formal Undeterminedness *if and only if it satisfies the following, for some connective k:*

- $\varnothing \nvDash_{\mathsf{S}} \phi \vee \neg\phi$ *(Failure of LEM)*
- $\varnothing \vDash_{\mathsf{S}} \phi \vee \neg\phi \vee k\phi$ *(Principle of Gentle Implosion)*

Any operator k obeying the criteria from Definition 8 works as a *consistency operator*: the Principle of Gentle Explosion (PGE) implies that $k\phi$ will be designated if and only if ϕ does not verify a contradiction.[7] By contrast, if k obeys the criteria from Definition 9, it will work as a *determinedness operator*: the Principle of Gentle Implosion (PGI) implies that $k\phi$ will be designated if and only if ϕ verifies the corresponding instance of LEM.[8]

This suffices to understand that the normality operator $\circledast$ collapses on a consistency operator in $\mathsf{LP}^{\circledast}$ and any logic that is paraconsistent but not paracomplete. Dually, $\circledast$ collapses on a determinedness operator in $\mathsf{K_3}^{\circledast}$ and any logic that is paracomplete but not paraconsistent.

However, $\mathsf{FDE}^{\circledast}$ makes it clear that the normality operator is more general than its kins from LFI and LFU. Definition 4 implies that the normality operator satisfies the criteria from *both* Definition 8 *and* Definition 9 in a logic that is both paraconsistent and paracomplete, like $\mathsf{FDE}^{\circledast}$. By contrast, if we extend FDE with an operator $\circ$ such that $u(\circ\phi) = 1 \Leftrightarrow u(\phi) \neq \mathbf{b}$ and $u(\circ\phi) = 0 \Leftrightarrow u(\phi) = \mathbf{b}$, then we will have a *consistency* operator—that is, an operator satisfying the conditions from Definition 8. Dually, if we extend FDE with an operator $\star$ such that $u(\star\phi) = 1 \Leftrightarrow u(\phi) \neq \mathbf{n}$ and $u(\star\phi) = 0 \Leftrightarrow u(\phi) = \mathbf{n}$, then we will have a *determinedness* operator—that is, an operator satisfying the conditions from Definition 8. Neither of them, however, satisfy Definition 4.

Remark 2 $\mathsf{K_3}^{\circledast}$ and $\mathsf{LP}^{\circledast}$ have already appeared in the literature under different names. In particular, $\mathsf{K_3}^{\circledast}$ has been first discussed by Gupta and Belnap (1993). As for $\mathsf{LP}^{\circledast}$, this is equivalent with the logic LFI_1 by Carnielli et

[7] Notice that what we call recapture of ECQ is equivalent to PGE.
[8] What we call recapture of LEM is equivalent to PGI.

al. (2000). In particular, the latter extends $\mathcal{L}_2$ with a strong negation connective $\sim$ and a detachable conditional $\rightarrow$,[9] and it can be obtained from $\mathsf{LP}^{\circledast}$ by defining $\sim\phi = \circledast\neg\phi$ and $\phi \rightarrow \psi = \sim\phi \vee \psi$. On the other hand, $\mathsf{LP}^{\circledast}$ obtains from LFI_1 since, in that very logic, the consistency operator is definable: $\circledast\phi = \sim\phi \vee \sim\sim\phi$.

3 Classical collapse

Classical collapse is an approach to classical recapture that has been developed by Jc Beall (2011, 2013). This approach is cast against a twofold background: (1) a philosophical view that sees classical logic as our default reasoning tool—a view known as *default classicality*—and (2) a distinction between *logical principles*, which codify a more or less fine-grained reasoning tool, and *extra-logical principles*. The technical results in classical collapse give formal expression to the first background, while the philosophical interpretation of Beall's approach rely on the second background. We start with the formal results.

Default classicality and classical collapse. *Default classicality* is the view that '*classical logic is the default logic, and the weaker logic kicks into when necessary*' (Beall, 2011, p. 326). This view prompts the familiar question—*'How can we* recapture *classical reasoning in our* weaker *logic?'* Beall's reply to it is *classical collapse*.

In order to deploy classical collapse, we need to upgrade standard single-conclusion consequence to multiple-conclusion consequence:

Definition 10 *For every logic* S, S^+*-consequence is a relation* $\vDash_{\mathsf{S}^+} \subseteq 2^{\Phi} \times 2^{\Phi}$ *such that:*

$$\Sigma \vDash_{\mathsf{S}^+} \Delta \;\Leftrightarrow\; \mathcal{V}_{\mathsf{S}}(\Sigma) \subseteq \textstyle\bigcup_{\psi\in\Delta} \mathcal{V}_{\mathsf{S}}(\psi)$$

with the proviso that, if $\Sigma \vDash_{\mathsf{S}^+} \Delta$, then Σ and Δ have *finitely many elements*. As usual, if $\Delta = \{\psi\}$, we will write $\Sigma \vDash_{\mathsf{S}^+} \psi$ instead of $\Sigma \vDash_{\mathsf{S}^+} \{\psi\}$.[10] Beside, we need the following:

[9] Remember that, in a paraconsistent and not paracomplete logic, $\circledast$ turns to be a *consistency* operator. We follow standard terminology and say that a conditional is detachable if it obeys MP.

[10] Beall (2011, 2013) does not impose the finiteness requirement in its definition of LP^+ and similar multiple-conclusion reasoning tools. However, the restriction is standard, and so we will follow it here.

Definition 11 (Auxiliary definitions)

$$
\begin{array}{lcl}
\iota(\Sigma) & = & \{p \wedge \neg p \in \Phi_{\mathcal{L}} \mid p \in var(\Sigma)\} \\
\kappa(\Sigma) & = & \{p \vee \neg p \in \Phi_{\mathcal{L}} \mid p \in var(\Sigma)\}
\end{array}
$$

$\iota(\Sigma)$ is the set of the *contradictions* that can be formed out of the variables from Σ. Thus, if $var(\Sigma) = \{p, q, r\}$, then $\iota(\Sigma) = \{p \wedge \neg p, q \wedge \neg q, r \wedge \neg r\}$. $\kappa(\Sigma)$ is the set of the *instances of LEM* that can be formed from variables from Σ. Again, if $var(\Sigma) = \{p, q, r\}$, then $\kappa(\Sigma) = \{p \vee \neg p, q \vee \neg q, r \vee \neg r\}$. We write $\iota(\psi)$ and $\kappa(\psi)$ instead of $\iota(\{\psi\})$ and $\kappa(\{\psi\})$. Beall (2011,2013) proves the following recapture result for $\mathsf{K_3}^+$ and LP^+:

Proposition 1 (Beall, 2013, Theorem 4.2 and Beall, 2011, Theorem 3.7)

$$
\Sigma \vDash_{\mathsf{CL}} \Delta \iff \begin{cases} \Sigma, \kappa(\Delta) \vDash_{\mathsf{K_3}^+} \Delta \\ \Sigma \vDash_{\mathsf{LP}^+} \Delta, \iota(\Sigma) \\ \Sigma, \kappa(\Delta) \vDash_{\mathsf{FDE}^+} \Delta, \iota(\Sigma) \end{cases}
$$

The proposition states that, when it comes to $\mathsf{K_3}^+$, we can draw the classical conclusions of a given premise-set Σ if *all* variables in the *conclusion verify LEM*. Dually, when it comes to LP^+, from a given premise-set we can conclude *either* the classical conclusions of the premise-set, *or* that 'there is something inconsistent in the premise-set.' As for FDE^+, it combines the conditions for the other two logics.

Default classicality as an extra-logical principle. So far so good. But of course, when it comes to LP-reasoning (and its multiple-conclusion version), classical collapse cannot tell us, case by case, *whether we are in an abnormal situation, or in a perfectly classical one*: no formula from $\Phi_{\mathcal{L}}$ can express that 'ϕ has a classical value'. Thus, there is no way for LP (and LP^+) to express *that we are in a normal situation*.

In sum, LP and LP^+ leave us with a *choice* between *classical reasoning* and the *weaker* reasoning tool that is crafted for abnormal situations. In order to make a choice, we need to appeal to *extra-logical* principles (Beall, 2011, p. 331)—principles of rationality, pragmatic principles, epistemic principles, and so on. *Default classicality* can be read as a principle of this sort, stating: *as a first go, reject the inconsistent options* Beall (2011, p. 332)—or, more in general, the *abnormal* options. However, if we face an abnormal case, we switch to the appropriate *weaker* reasoning tool (Beall, 2011, p. 332).[11]

[11] This combination of logical and extra-logical principles explains how Beall can insist on

4 Comparing the two approaches

Proposition 1, Theorem 1, and Theorem 2 together imply that *recapture by normality* and *classical collapse* are equivalent:

Corollary 1 *If $\Sigma, \psi \subseteq \Phi_{\mathcal{L}}$, we have:*

$$\begin{array}{rcl} \Sigma^{\circledast}, \Sigma \vDash_{\mathsf{LP}^{\circledast}} \bigvee_{\psi \in \Delta} \psi & \Leftrightarrow & \Sigma \vDash_{\mathsf{LP}^{+}} \Delta, \iota(\Sigma) \\ \Sigma, \Delta^{\circledast} \vDash_{\mathsf{K_3}^{\circledast}} \bigvee_{\psi \in \Delta} \psi & \Leftrightarrow & \Sigma, \kappa(\Delta) \vDash_{\mathsf{K_3}^{+}} \Delta \\ \Sigma, (var(\Sigma))^{\circledast}, (var(\Delta))^{\circledast} \vDash_{\mathsf{FDE}^{\circledast}} \bigvee_{\psi \in \Delta} \psi & \Leftrightarrow & \Sigma, \kappa(\Delta) \vDash_{\mathsf{FDE}^{+}} \Delta, \iota(\Sigma) \end{array}$$

This is expected: all the methods of classical recapture do exactly the same thing—namely, they establish some kind of *equivalence* with classical reasoning. From a logical point of view, then, the two methods are on a par. However, the two approaches can be compared on a number of extra-logical features.

Semantically closed languages. One virtue of classical collapse over recapture via normality is that Beall's approach can be applied to a semantically closed extension of $\mathsf{K_3}^+$, LP^+ and FDE^+, while recapture via normality cannot apply to semantically closed versions of $\mathsf{K_3}^{\circledast}$, $\mathsf{LP}^{\circledast}$, $\mathsf{FDE}^{\circledast}$. A semantically closed language $\mathcal{L}$ is a language that can express its own concept of truth. This is done by expressions of the form $Tr(\underline{\phi})$, where Tr is a *truth predicate* and $\underline{\phi}$ is the *name* of formula ϕ. Gupta and Belnap (1993) proved that a semantically closed extension of $\mathsf{K_3}^{\circledast}$ is *trivial*, and Barrio, Pailos, and Szmuc (2016) proved a similar result for $\mathsf{LP}^{\circledast}$. By contrast, the semantically closed extensions of $\mathsf{K_3}^+$ and LP^+ are not trivial, exactly as those of $\mathsf{K_3}$ and LP.

Informational reading of the many-valued setting. Truth theory and logical paradoxes aren't the only applications of many-valued logic. A consistent track of research applies $\mathsf{K_3}$ and other paracomplete logics to the issue of reasoning with *partial information* (see especially Abdallah, 1995; D'Agostino, 2014; D'Agostino, Finger, & Gabbay, 2013). On the paraconsistent camp, the informational focus is receiving growing attention: Mares (2002); Priest (2001); Restall and Slaney (1995) present theories of *belief revision* that accommodate the presence of inconsistent information.

default reasoning without dropping monotonicity: the logic is monotonic, our choices are defeasible (Beall, 2011, p. 332).

If we adopt an informational reading of many-valued reasoning, recapture via normality has some virtues over classical collapse. Indeed, classical collapse for $\mathsf{K_3}^+$ requires the information that *all* variables in the conclusion-set Δ have a classical truth value. From an informational point of view, this is more demanding than the condition provided by Theorem 1 for $\mathsf{K_3}^{\circledast}$, where we just need the conclusion to have a classical value. [12]

As for the paraconsistent case, the classical collapse strategy turns not to be much informative. Indeed, it states what we know from the start: *either* we can reason classically, *or* there is some inconsistency around. This is not much of a limit *if* classical collapse is complemented by the philosophy of choice in reasoning that is endorsed by Beall (2011, 2013)—indeed, in this case we can opt for the extra-logical principle of *default classicality* and choose one of the two options available. However, the relevance of classical collapse for LP^+ seems to be somewhat smaller out of this philosophical background. By contrast, recapture via normality for $\mathsf{LP}^{\circledast}$ is able to secure that we are in a consistent (classical, normal) situation, thus providing more information than its classical collapse kin. Also, the relevance of this feature does not depend, at least apparently, on a given philosophical background, and it applies to different philosophies of reasoning that one may want to couple with the formal techniques that we have presented in Section 2.

5 Conclusions

In this paper, we have introduced a *normality operator*, which is a linguistic device allowing to distinguish 'normal situations' (where a formula ϕ has a classical value) from 'abnormal situations' (where a formula ϕ has a non-classical value). We have applied the operator in order to build systems of Logic of Formal Inconsistency and Undeterminedness from the many-valued logics $\mathsf{K_3}$, LP and FDE, and we have established classical recapture theorems for the resulting logics $\mathsf{K_3}^{\circledast}, \mathsf{LP}^{\circledast}, \mathsf{FDE}^{\circledast}$ (Theorem 1 and Theorem 2). Finally, we have compared *recapture via normality* with another approach to classical recapture, namely *classical collapse* by Beall (2011, 2013).

[12] Notice that, in $\mathsf{K_3}$, some formula can have a classical value even when some of its variables have a non classical one. For instance, $\nu(p \vee q) = 1$ if $\nu(p) = \frac{1}{2}$ and $\nu(q) = 1$.

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