Exact Verification in Modal Languages

Michael Cohen

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Abstract

The basic modal language into which we embed intuitionistic logic cannot express the difference between exact and inexact verification of a sentence given a state. I describe few ways to expand the language of basic modal logic that allow us to express this difference. The specific expansion needed depends on the intermediate logic we are interested in.

Kripke semantics is a successful tool for studying various aspects of propositional intuitionistic logic (henceforth **Int**). Very roughly, with Kripke semantics we can think of sentences of **Int** as true or false relative to states of information that expand monotonically over time. A sentence φ is true in state s iff s, and all its extensions, contain enough information to verify φ. This informal idea can be made precise using the semantic tools developed in modal logic, as Kripke has shown.

The Kripke semantics of **Int** are contrasted with the older B-H-K semantics, which state how a sentence φ of **Int** can be constructed. In other words, the semantics tells us how to construct complex sentences from simpler ones. The prime example is the clause for implication: very roughly again, an implication $\alpha \to \beta$ is true under the B-H-K semantics iff there is a way of transforming a construction of α into a construction of β.

The B-H-K semantics seems very different from the Kripke semantics. In particular, as noted in (Fine 2014), the B-H-K semantics are *exact*, in the sense that each clause of the semantics tells you what exactly is needed for constructing a sentence. At no point can we add to a construction c irrelevant content. On the other hand, the Kripke semantics are *inexact*, in the sense that if state s' is a proper extension of state s, then if s verifies φ, then so does s'. Since s' is a proper extension of s, it might very well contain information which is irrelevant to the verification of φ. The standard Kripke semantics is blind to this difference between s and s'.

Kit Fine has developed an exact semantics for **Int**, that combines elements from both the Kripke semantics and the B-H-K semantics (Fine 2014). Fine's semantics provides an exact state that verifies φ, or makes it true (i.e. the exact *truth-maker* of φ). Unlike the B-H-K semantics, Fine's exact states are not to be understood as constructions of proofs. Instead, they can be thought of as facts which are part of a reality. And unlike the Kripke semantics, Fine's system distinguishes between states that exactly verify a statement, and states that inexactly verify a statement.

In this paper, I explore a different, more conservative way to distinguish between exact and inexact states that verify sentences of **Int**. The question is how can we express the property of a state verifying φ s.t. any proper part of it does not verify φ. Various ways to extend the language of the modal logic **S4** will be tested in order to express this idea.

I stat by presenting the Kripke semantics and the truth maker semantics, along with some results which will be relevant later on. I then move on to show how exact states are defined in Boolean modal logic. Thereafter I discus some consequences and limitations of this approach.

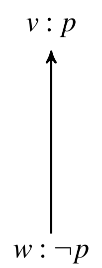
1 Kripke Semantics and Truth-maker Semantics for **Int**

1.1 Kripke Semantics

An S4 Kripke frame is a pair (W,R), where W is a non empty set of worlds (or states), and R is a transitive reflexive relation on W. A Kripke model (W,R,V) is a Kripke frame together with a valuation function V that assigns each sentence letter a subset of W.

The language $\mathscr{L}_{PL}$ of propositional logic is generated from sentence letters, the unary operators $\neg$, and the binary $\vee, \wedge, \rightarrow$. The language $\mathscr{L}_{ML}$ of basic modal logic extends $\mathscr{L}_{PL}$ with the unary operator $\Box$. Sentences of modal logic are evaluated w.r.t pairs (M,w) of Kripke models and worlds. In particular:
– $M,w \models \Box\varphi$ iff $M,v \models \varphi$ for all wRv,
while the sentential connectives are interpreted as usual. A sentence φ is valid iff it is true on all worlds of all models of a class of Kripke frames. The set of all sentences valid on the class of S4 frames is the modal logic **S4**.

Denote with $(\varphi)^t$ the function from $\mathscr{L}_{PL}$ to $\mathscr{L}_{ML}$ that inserts a $\Box$ in front of every sub-formula of φ. It is well known that $(\varphi)^t$ is a validity of **S4** iff φ is a theorem of **Int** (see e.g., Mints 2012). In other words, we can faithfully embed **Int** into **S4** via the $(\varphi)^t$ translation. For example, note that $(p \vee \neg p)^t$ is $\Box(\Box p \vee \Box\neg\Box p)$, and that the latter is not a theorem of **S4**, as evident by the following counter (Kripke) model (reflexive arrows are omitted throughout):



Note that $M,w \not\models \Box(\Box p \vee \Box\neg\Box p)$, as in w, both $\Box p$ and $\Box\neg\Box p$ are false. A sentence φ is verified in a state w iff $\Box\varphi$ is true at w. Roughly speaking, we interpret the arrows as denoting the extension relation between states. Thus, in the above example, v extends w, in the sense that if something is verified in w, then it will be verified in v. I will call the converse of the extension relation in this work a *part relation*. More accurately, recall that the *cluster* of w (denoted $[w]$) in an S4 model is a set of states v s.t. wRv and vRw. If wRv and v is not in the cluster of w, we say that v is a *proper extension* of w and conversely that w is a *proper part* of v. It it thus better to think of clusters as the verifiers of sentences.

We can avoid the detour through the modal language and use Kripke semantics directly to interpret **Int**, by imposing a hereditary condition on the V function. Since this work will employ tools of modal logic, I will keep using the S4 detour.

I assume familiarity with some basic notions and results of standard modal logic, such as the finite model property and the invariance of modal logic under bisimulation (see Blackburn et al. 200).

A (transitive) Kripke frame is rooted iff there is a world r s.t. rRv for all v in the frame. A finite Kripke frame is a frame with a finite set of worlds. **S4** has the finite frame property, meaning that every non-theorem of **S4** has a counter model based on finite frame (i.e., no sentence can force the model to be infinite).

The claim that standard Kripke semantics is blind to the difference between exact and inexact verifies can now be made precise. Consider the following two S4 models:

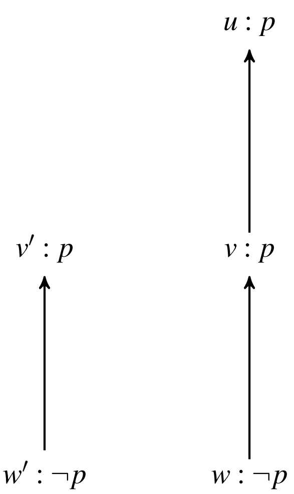


Figure 1: Two models, M_1 (left) and M_2 (right), which are bisimilar.

Note that state u in the right model (M_2) verifies p, but that u is a proper extension of state v which also verifies p. u is a proper extension of v and v is a proper part of u. Thus, we would like to say that u is an inexact verifier of p because it has a proper part v that verifies p. On the other hand, the only proper part of v, state w, does not verify p, so we would like to say that v is an exact verifier of p. Hence, v exactly verifies p while u inexactly verifies it. Talking so, we retain the intuition that exact verifies of φ are such that every proper part of them does not verify φ:

exact verifies contain *exactly* what is needed for verifying a given sentence.

Note that M_1 and M_2 are bisimilar, with the bisimulation relation being $w'Zw$, $v'Zv$, $v'Zu$. Now suppose that there is a sentence φ of the basic modal language that can distinguish exact states form inexact states, s.t. it is (say) false in the inexact state (M_2, u) but true in the exact state (M_2, v). By the bisimulation between M_1 and M_2, φ will have to be both true and false in (M_1, v'), as bisimilar states satisfy the same modal formulas and φ is true in v' and false in u. Thus, there is no such sentence φ, and so the basic modal language is indifferent to the difference between v and u. Basic modal logic cannot define exact verification.

1.2 Truth-maker Semantics for **Int**

Fine advances a semantics for **Int** that can distinguish between exact and inexact states, which is based on truth makers.

A pair $(S, \sqsubseteq)$ is a partial order iff $\sqsubseteq$ is a relation on S which is transitive, reflexive and anti-symmetric. A partial order is a complete semi lattice iff every subset of S has a least upper bound. $\sqcup T$ denotes the least upper bound of T and $\sqcap T$ denote the greatest lower bound. $\sqcup T$ is infomally understood as the fusion of all the states in T. $s \rightsquigarrow t$ denotes the state $\sqcap\{u : t \sqsubseteq s \sqcup u\}$ and is called the *residuation* of s and t. Very informally, the residuation $s \rightsquigarrow t$ describes the state the we need to add to s to get t.

An exact frame (E-frame) is a partial order $(S, \sqsubseteq)$ which is a complete semi lattice and satisfies the following condition:

residuation For all $s, t \in S$: $t \sqsubseteq s \sqcup (s \rightsquigarrow t)$

An E-model (M^E) is a an E-frame with a valuation relation V. Since Fine takes $\bot$ to be a sentential atom, we understand V to be a relation holding between states and sentential atoms. The relation V satisfies the strict falsum condition, that states that if $sV\bot$ then sVp, for any atomic p.

A state s is

contradictory iff $sV\bot$.

inconsistent iff there is a t s.t. $t \sqsubseteq s$ and t is contradictory, and

consistent otherwise.

compatible with t iff $s \sqcup t$ is consistent, and

incompatible with t otherwise.

Note that there are no counterparts to these conditions in basic modal logic. There are no worlds that get $\bot$ as an assignment.

The above semantic structure allows Fine to define exact verification as a relation between a pair (M^E, s) and a φ of **Int** in the following recursive fashion:
- $s \Vdash p$ iff sVp

- $s \Vdash \bot$ iff $sV\bot$
- $s \Vdash \varphi \vee \psi$ iff $s \Vdash \varphi$ or $s \Vdash \psi$
- $s \Vdash \varphi \wedge \psi$ iff $t \Vdash \varphi$ and $u \Vdash \psi$ and $s = u \sqcup t$.
- $s \Vdash \varphi \rightarrow \psi$ iff there is a function f s.t. for any $t \Vdash \varphi$, $f(t) \Vdash \psi$ and $s = \sqcup\{t \Vdash \varphi : t \rightsquigarrow f(t)\}$.

The negation clause $\neg\varphi$ is given by the clause for $\varphi \rightarrow \bot$. Note that from $s \Vdash p$ (s exactly verifies p) and $s \sqsubset s'$ (s is a proper part of s'), we cannot conclude that $s' \Vdash p$ (the extension of s verifies p). We denote inexact verification with $s' | \rhd \varphi$, which holds iff $s \sqsubseteq s'$ and $s \Vdash \varphi$.

All theorems of **Int** are inexactly verified on every (M^E, s) pair. It follows that all theorems of **Int** are exactly verified by the null state. Example: $s \Vdash p \rightarrow p$ iff there is a function that takes states that verify p to states that verify p. This is the identity function. Thus, $s = \sqcup\{t \rightsquigarrow t\}$ and so $s = \emptyset$ (which will denote the null element, whose existence follows from Fine's frame E conditions.) See (Fine 2014) for a more comprehensive presentation of the truth maker semantics for **Int**.

1.3 Intermediate logics and their respective companions

Intermediate logics are logics that extend **Int** with an additional set of axioms Γ. There are uncountable many intermediate logics between **Int** and classical logic. A few intermediate logics which will be discussed later are:
KC= **Int** + $\neg p \vee \neg\neg p$ (Jankov logic)
LC= **Int** + $(p \rightarrow q) \vee (q \rightarrow p)$ (Dummett logic)
KP= **Int** + $\neg p \rightarrow (q \vee r) \rightarrow (\neg p \rightarrow q) \vee (\neg p \rightarrow r)$ (Kreisel Putnam logic)

The relation between the S4 $()^t$ translation and intermediate logics is well studied. In particular, we know that the same as **Int** can is embedded in **S4** using $()^t$, the following relations hold:
- **KC** is embedded in **S4.2**
- **LC** is embedded in **S4.3**
- **KP** is embedded in **S4** + $\Diamond(p_1 \wedge \Box q) \wedge \Diamond(p_2 \wedge \Box q) \rightarrow \Diamond(\Diamond p_1 \wedge \Diamond p_2 \wedge \Box q)$[1]

where:
S4.2= **S4** + $\Diamond\Box p \rightarrow \Box\Diamond p$,
S4.3= **S4** + $\Box(\Box p \rightarrow p) \vee \Box(\Box q \rightarrow q)$.

Many intermediate logics can be give a Kripke semantics with first order frame properties, which correspond to modal systems between **S4** and **S5**. It is also known that **S4** is not the only logic that embeds **Int** under the $()^t$ translation. **S4.1** and **GRz** are modal logics that also embed **Int** with the $()^t$ translation (see Chagrov and Zakharyashchev 1992).

Fine has a few remarks about the connection between his truth maker semantics and intermediate logics. In particular, he notes that the class of E-models without any contradictory states validate **KC**, and that the class of models with only one contradictory state satisfy **KP**.

[1]See (Holliday 2107) for the modal Kreisel Putnam axiom.

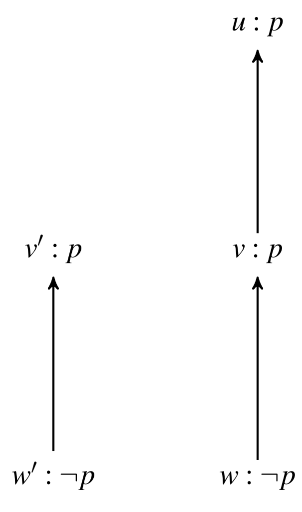
2 Exact verification in Kripke semantics for intermediate logics

We have seen in the last section that while Fine's E-models can distinguish between exact verification and inexact verification, the basic modal logic, which is based on Kripke semantics, cannot make this distinction. In this section, the focus will be on the following question: what do we need to add to the basic language of modal logic in order to express the difference between exact and inexact verification?

I start with investigating this question for the case of the intermediate logic **LC**. I will assume for now that all Kripke frames which I discus are finitely rooted; this assumption will be reconsidered later. Since we know that we can embed **LC** in the modal logic **S4.3**, the question is what extensions of **S4.3** can express exact verification.

It turns out the that extension needed for **S4.3** is relatively simple. The frame condition of the logic **S4.3** is the following first order formula: $\forall x, y, z(xRy \wedge xRz \rightarrow (yRz \vee zRy))$ (together with reflexivity and transitivity). Let $\blacksquare$ denote the *complement* operator of the standard $\Box$ operator. This modality has been studied in various places, under various names; see Goranko (1989, 1990) Holliday (2017) and Humberstone (1987). While $\Box$ is evaluated with the R relation, $\blacksquare$ is evaluated with the complement of the R relation. The semantic clause is thus given by:
- $M, w \models \blacksquare\varphi$ iff for all v s.t. $w \not R v$, $M, v \models \varphi$.
In other words, $\blacksquare\varphi$ is true in w iff φ is true in all the worlds *inaccessible* from w.

To see how $\blacksquare$ adds the needed expressive power, the models of Figure 1 (presented again below) will suffice.



Note that $M_2, v \models \Box p \wedge \blacksquare\Diamond\neg p$ (where M_2 is the right model), since the only inaccessible world from v is w, and w sees a world in which p is false, namely itself. On the other hand, $M_2, u \not\models \blacksquare\Diamond\neg p$, since world v is inaccessible from u, but $v \models \Box p$. Hence, the formula $\Box p \wedge \blacksquare\Diamond\neg p$ can distinguish between worlds u and v.

It follows from this observation that ■ strictly increases the expressive power of the basic modal logic, as worlds v and u are bi-similar to v' (clearly, only w.r.t the R relation).

Consider again the translation $()^t$ from the propositional language to the modal language. Recall that we say that a state w verifies φ if $(\varphi)^t$ is true in w. I propose the following explication of exact verification:

> a state w *exactly verifies* the sentence φ of **LC** iff $(\varphi)^t \wedge \blacksquare\neg\Box(\varphi)^t$ is true in w.

The left conjunct is immediate: if w exactly verifies a formula, it should also inexactly verify it. Thus, $(\varphi)^t$ should be true at w. The right disjunct says that all the worlds inaccessible from w are *not* verifiers of φ. Under the strong frame conditions of **S4.3**, together with the assumption that the frame is finite, $M,w \models (\varphi)^t \wedge \blacksquare\neg\Box(\varphi)^t$ expresses the fact that w is the 'lowest' world in which φ is satisfied, i.e., no extension of w is an exact verifier of φ. This is made precise in the following proposition.

Proposition 1. *Let (M,w) be a S4.3 model. If $M,w \models (\varphi)^t \wedge \blacksquare\neg\Box(\varphi)^t$ then for every u which is a proper extension of w, $M,w \not\models \blacksquare\neg\Box(\varphi)^t$*

Proof. u is a proper extension of w if wRu and $u \not R w$. Suppose that $M,u \models \blacksquare\neg\Box(\varphi)^t$. Since $u \not R w$, it follows that $M,w \models \neg\Box(\varphi)^t$, contradicting the assumption that $M,w \models (\varphi)^t$.

□

Instead of talking about $(\varphi)^t \wedge \blacksquare\neg\Box(\varphi)^t$ as the sentence that characterizes exact verification for φ, we can just talk about $\Box\varphi \wedge \blacksquare\neg\Box\varphi$, since we know that the outermost operator of $(\varphi)^t$ is $\Box$ (by the behavior of the translation). In other words, $(\varphi)^t \wedge \blacksquare\neg\Box(\varphi)^t$ is equivalent to $\Box(\varphi)^t \wedge \blacksquare\neg\Box(\varphi)^t$, so we just drop the $()^t$ notation when we talk about $\Box\varphi \wedge \blacksquare\neg\Box\varphi$.

We can show that on finite rooted **S4.3** models the fact that φ is verified by a state implies that there exists a state which exactly verifies φ, and that the cluster which exactly verifying φ is unique. In other words, the existence of a verifier implies the existences of a unique exact verifier in the logic **LC**. Start with existence:

Proposition 2. *Let M be a (finite rooted)* S4.3 *model. Then $M,w \models \Box\varphi$ implies $M,u \models \Box\varphi \wedge \blacksquare\Diamond\neg\varphi$ for some u in M.*

Proof. Assume otherwise for reductio. Assume $M,w \models \Box\varphi$. By the reductio assumption, $M \models \neg(\Box\varphi \wedge \blacksquare\Diamond\neg\varphi)$, i.e. $M \models \Diamond\neg\varphi \vee \blacklozenge\Box\varphi$. Note that for the root world r, $M,r \models \Diamond\neg\varphi$, since there is no x s.t. $rR^c x$ (thus $\blacklozenge\Box\varphi$ must be false at r). By the assumed disjunction, $w \models \blacklozenge\Box\varphi$. Thus, there is a u_1 s.t. $w \not R u_1$ and $u_1 \models \Box\varphi$. Note that u_1 is not w (since wRw by assumption), nor r (since $M,r \models \Diamond\neg\varphi$). By the assumed disjunction, $u_1 \models \blacklozenge\Box\varphi$, so there is a u_2 s.t. $u_2 \models \Box\varphi$ and $u_1 \not R u_2$. Note that u_2 is not r (since $M,r \models \Diamond\neg\varphi$) and not u_1 (since $u_1 R u_1$). By the frame conditions of $S4.3$, since $w \not R u_1$ it follows that $u_1 R w$, thus u_2 is not w. Conclusion: u_2 is a new world. This reasoning can repeat again for any u_n, contradicting the assumption that M is finite. □

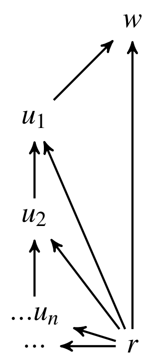


Figure 2: Proof of Proposition 2, depicted: an infinite descending chain $u_1, u_2, \ldots$

Similar to the way that the theorems of an intermediate logic are exactly verified on all null states in Fine's system, we have that the theorems of the intermediate **LC** are exactly verifier on all roots (of finite models). If r is a root of a finite rooted model then r satisfies $(\varphi)^t \wedge \blacksquare\Diamond\neg(\varphi)^t$, for any φ which is a theorem of **LC**.

Proposition 3. *If φ is a theorem of LC, then $(\varphi)^t \wedge \blacksquare\Diamond\neg(\varphi)^t$ is true on all the roots of finite $S4.3$ models.*

Proof. Consider an arbitrary finitely rooted $S4.3$ model. By the correctness of the $()^t$ translation, we have that $M, r \models (\varphi)^t$. Under the assumption that r is the root, there is no x s.t. $rR^c x$, so $M, r \models \blacksquare\psi$ holds for any ψ. □

In **LC**, the uniqueness of the exact verifier is relative to clusters. Clearly, if w exactly verifies φ and v is part of the w cluster, then v also exactly verifies φ. Uniqueness is shown in the following proposition.

Proposition 4. *On finite rooted S4.3 models, if $M, w \models \Box\varphi \wedge \blacksquare\Diamond\neg\varphi$ then there is no other cluster $[v]$ different from $[w]$ s.t. $M, v \models \Box\varphi \wedge \blacksquare\Diamond\neg\varphi$.*

Proof. Assume $M, w \models \Box\varphi \wedge \blacksquare\Diamond\neg\varphi$. Assume for reductio that $M, v \models \Box\varphi \wedge \blacksquare\Diamond\neg\varphi$. Since $[w] \neq [v]$, either $w \not R v$ or $v \not R w$. WLOG, assume $w \not R v$. Since $w \models \blacksquare\Diamond\neg\varphi$, it follows that $M, v \models \Diamond\neg\varphi$ – contradiction. □

The argument that guarantees the existence of an exact verifier for each inexact verifier (in the proof of proposition 2) fails on (possibly non-rooted) infinite models. In particular, if $M, w \models (\varphi)^t$ and there is an infinite chain of worlds s.t. $u_0 R w$ (but not the converse), and $u_1 R u_0$ (but not the converse) ... and so on, then there is no guarantee that φ has an exact verifier.

The resources of basic modal logic, however, allow us to block this possibility. The modal logic **Grz** is obtained by adding to **S4** the axiom $\Box(\Box(p \to \Box p) \to p) \to p$. **Grz** is known to be valid on a Kripke frame F iff F does not contain an infinite chain of distinct elements (together being transitive and reflexive) (van-Benthem et al. 2016). It follows that **Grz** frames are finite frames (a frame property which is not first order definable). It is also known that the $()^t$ translation faithfully embeds **Int** in **Grz**, although **Grz** is stronger than **S4** (Chagrov and Zakharyashchev 1992). It is thus possible to replace **S4** in the above results with **Grz** and drop the requirement that then frame must be finite.

The requirement that the frame must be rooted cannot be immediately dropped, however. For consider the finite and non-rooted model of Figure 2.

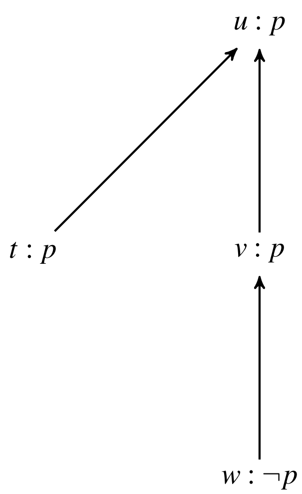


Figure 3: A non-rooted finite Kripke frame

We would like to say that both world v and world t are exact verifies of p (both of them do no have proper parts that verify p). However, $M, v \not\models \Box p \wedge \blacksquare\Diamond\neg p$ since t is inaccessible from v but $t \models \Box p$.

Such non-rooted counter models can be avoided in the extended modal language that we use. Observe that if r is a root of a frame, then $\blacksquare\bot$ is true at r, regardless of the valuation. This is because if r is the root, then rRw for all w, and so there is no t s.t. $r \not R t$. From the perspective of the complement modality, the root is a dead end, a world that does not see any world. With this observation, we can see that rooted frames are definable with the complement modality:

Proposition 5. *Let F be a reflexive transitive frame. Then:*
$\blacksquare\neg\blacksquare\bot \to \blacksquare\bot$ *is valid on F iff F has a root.*

Proof. Left to right: Assume that $\blacksquare\neg\blacksquare\bot \to \blacksquare\bot$ is valid on the frame F. Assume for reductio that F is not rooted, i.e. for any world x in F there is a world y s.t. $x \not R y$. Pick an arbitrary w. By assumption $w \models \blacksquare\neg\blacksquare\bot \to \blacksquare\bot$. The consequent

implies that w is a root, so it must be false, and hence the antecedent is false as well. So $w \models \neg\blacksquare\neg\blacksquare\bot$. Hence, there is a world v inaccessible from w s.t. $v \models \blacksquare\bot$. This implies that that v is the root – contradiction.
Right to left: Assume F has a root r. Assume for reductio that there is a w s.t. $w \models \blacksquare\blacklozenge\top \wedge \blacklozenge\top$. By the right conjunct, we know that w is not a root r. However, we know that rRw. If wRr then by transitivity w is also a root, a possibility we have just ruled out. So $w \not R r$ is the case. Thus, by the left conjunct, $r \models \blacklozenge\top$, a contradiction again.

$\square$

For more on the extension of the basic modal logic with the $\blacksquare$ operator, including a completeness result for **S4.3** with $\blacksquare$, see (Goranko, 1990).

The definition of exact verification as $\Box\varphi \wedge \blacksquare\Diamond\neg\varphi$ does not work on frames that impose weaker conditions than that of $S4.3$. For instance, consider the model in Figure 4. In that model, both t and v should be considered as exact verifies of p, as their only proper part, world w, does not verify p. However, note that $\Box p \wedge \blacksquare\Diamond\neg p$ is false at v, since t is inaccessible from v, i.e. $v \not R t$, but $v \models \Box p$. For similar reasons, $\Box p \wedge \blacksquare\Diamond\neg p$ is false at t. The model in Figure 4 is thus an example of a finitely rooted $S4$ model which verifies p but dose not have a state that exactly verifies p, according to the above definition.

The frame of Figure 4 satisfies the condition that if xRy and xRz, then there is a w s.t. yRw and zRw, a frame condition that correspond to the logic **S4.2**. As mentioned earlier, this modal logic embeds the intermediate propositional logic **KC**. It follows that we currently do not have a way of expressing exact verification in **KC** and lower logics. The next subsection is devoted to such logics.

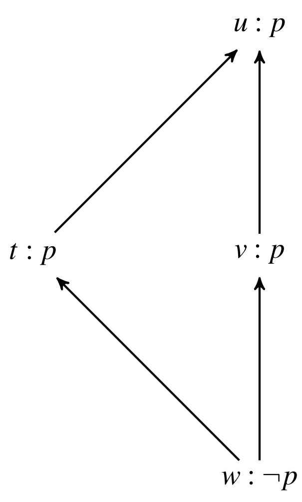


Figure 4: An S4.2 frame

2.1 Logics below **LC**

In order to define exact verification in logics weaker than **LC**, it should be possible to distinguish between states that are proper parts of a state w and states that are just inaccessible from w. On finitely rooted S4.3 models, the two categories coincide, but on weaker frames they do not.

What we would like is to construct a proper part relation, s.t., for instance in Figure 4, t will bear this relation to w but not to v. A natural candidate is the converse relation R^-, s.t. wR^-v iff vRw. Of course, since our basic R relation is reflexive, the R^- relation will be reflexive as well, in contrast with the proper part relation, which should be irreflexive.

The proper part relation lies in the intersection of the complement and converse relation: for v to be a proper part of w, v needs to be related to w with R, and w should not be related to v: vRw & $w \not R v$. Under this intersection relation, world v of Figure 4 is related to world w, while not related to t.

In turns out that this kind of reasoning in modal logic has been extensively studied under the name *Boolean modal logic* (BML), in which boolean operations are applied on a given set of atomic relations $R_1, R_2 ... R_n$ (see Gargov and Passay 1987). Assume a poly-modal language, in which the box operators are denoted as $[R_1]...[R_n]$. Each such operator $[R_i]$ is evaluated, as usual, with a relation R_i in a Kripke frame. In boolean modal logic, if $[R_1]$ and $[R_2]$ are box operators of the poly-modal language, then the following box operators are well defined:
$[R_1 \cap R_2]$, which is evaluated w.r.t the composite relation $R_1 \cap R_2$,
$[R_1 \cup R_2]$, which is evaluated w.r.t the composite relation $R_1 \cup R_2$,
$[R_1^c]$, which is evaluated w.r.t the relation $W^2 - R$.
Diamond versions of the modalities are defined in a similar fashion.

Note that the $\blacksquare$ *complement* relation from the earlier section is just $[R^c]$ of BML, so the logical language of the earlier section can be thought of as a fragment of the latter language.

To investigate exact verification in **Int** (embbeded in **S4**), we will assume two atomic relations: a relflexive transitive R relation, and its converse relation R^- (note that the latter relation is not definable from R with boolean operators). For syntactic continuity with earlier parts of this work, assume that $[R] = \Box$, $[R^c] = \blacksquare$, and let $[R^-]$ be $\Box^-$. Moreover, we abuse notation and write $R = \Box$, $R^c = \blacksquare$ when the reference to the semantic relation itself is clear from context.

With the expressive power of BML, we can define exact verification in **S4** as $\Box\varphi \wedge [\Box^- \cap \blacksquare]\Diamond\neg\varphi$. Note, for instance, that in the model in Figure 4, $\Box p \wedge [\Box^- \cap \blacksquare]\Diamond\neg p$ is true in worlds v and t. The extension of the $[\Box^- \cap \blacksquare]$ relation is $< u,v >, < u,w >, < u,t >, < t,w >, < v,w >$. Thus, $M, v \models [\Box^- \cap \blacksquare]\Diamond\neg p$, since $< v,w >$ is the case under the $[\Box^- \cap \blacksquare]$ relation and $M, w \models \Diamond\neg p$ is the case.

It can be shown that inexact verification implies exact verification, understood as $\Box\varphi \wedge [\Box^- \cap \blacksquare]\Diamond\neg\varphi$, in finitely rooted S4 Kripke models.

Proposition 6. *Let M be a (finite rooted) S4 model. Then $M, w \models \Box\varphi$ implies $M, u \models \Box\varphi \wedge [\Box^- \cap \blacksquare]\Diamond\neg\varphi$ for some u in M.*

Proof. Assume otherwise that $M,w \models \Box\varphi$ and that $M \models \Diamond\neg\varphi \vee \langle R^- \cap R^c\rangle\Box\varphi$. Since in the root r the relation R^c is empty, the right disjunct cannot be true, and so $M,r \models \Diamond\neg\varphi$. Since in w $\Box\varphi$ is the case, the left disjunct of the model validity cannot be true and so $M,w \models \langle R^- \cap R^c\rangle\Box\varphi$. This implies that there is a v s.t. $w \not R v$ and vRw, and $M,v \models \Box\varphi$. Thus, v is distinct from the root and from w. It follows similarly that that there is a v_1 s.t. $v \not R v_1$ and $v_1 R v$, and $M,v_1 \models \Box\varphi$. Thus v_1 is distinct from the root r, v and w. For assume $v_1 = w$. Then, since vRw, vRv_1, contradicting $v \not R v_1$. This can be repeated for any v_n, contradicting the assumption that M is finite. □

Thus, in order to see whether a formula φ of **Int** is exactly verified in a given state w of a Kripke model M, one needs to check if $M,w \models (\varphi)^t \wedge [\Box^- \cap \blacksquare]\neg(\varphi)^t$ is the case.

The existence of exact verification in S4 Kripke models does not imply uniqueness (up to clusters). As can be seen in Figure 4, both state t and state v are exact verifies of p, but they do not occupy the same cluster.

As in the case of **LC** and $S4.3$, one can avoid the restriction of the above results to finitely rooted $S4$ models by embedding **Int** into the modal logic **Grz**, and adding the axiom that guarantees the existence of a root (Proposition 5).

3 Fine's third problem and positive formulas

In his paper, Fine emphasizes an interesting difference between the Kripke and BHK semantics of **Int**. In the BHK semantics of **Int**, constructions are intrinsic in the sense that they do not refer to anything outside the construction in order to establish what it establishes. The Kripke semantics is different. For in order for a state to verify the sentence $\neg p$, it must be the case that no extension of that state verifies p. Form the modal perspective, $\Box\neg\Box p$ must be the case. Such condition, however, relies on states beyond the state of evaluation. In other words, the intrinsic content of the state is not sufficient on order to determine whether a state verifies a statement. This is Fine's third problem in his project of Exactification:

> This feature of the Kripke semantics raises a problem for the project of Exactification. For exact verification is most naturally regarded as an internal matter; we must somehow be able to see it as intrinsic to the content of a given state that it exactly verifies what it does. (Fine 2014: p.554)

This subsection is devoted to this issue in Kripke semantics.

One way of cashing out the idea that the verification of an **Int** formula at a given state is dependent on further extrinsic states is by asking whether the corresponding modal $(\varphi)^t$ is *preserved under submodels*. A modal formula φ is preserved under submodels iff whenever φ is true, it remains true when we consider a submodel of the original Kripke model. For example, a sentence letter (in the context of modal logic) is preserved under Kripke models, as the valuation of a sentence letter only depends on the valuation of a given world. On the other hand, the formula $\Diamond\neg p \wedge \Diamond p$ does not preserve under submodels. For no matter what model we are given, such formula is false on the single world submodel of that model.

Call a modal formula *universal* iff it is constructed from literals, $\wedge$, $\vee$ and $\Box$. The following connection between these concepts is known:

Proposition 7. *A formula is preserved under submodels (of all relational models) iff it is equivalent (in **K**) to a universal formula (*Holliday and Icard 2010).

This result gives us a precise formulation of Fine's worry. The intuitionistic conditional and negation, as opposed to the conjunction and disjunction, are sensitive to extensions of states, because the conditional and negation are the connectives that allow us (via the $()^t$ translation) to introduce non-universal modal formulas.

There seems to be no way to avoid this issue in the process of Exactification I have considered earlier. For I expand the basic language of modal logic with extra operators, but the problem resides in the basic language of modal logic, as the result concerning universal formulas shows. This can seen as a limitation of my conservative approach, in which I try to expand the basic modal language in order to express exact verification.

4 Conclusions

There are other ways of expressing exact verification of intuitionistic statements translated into a modal language. One approach, which I have yet to mention, is the based on the logic **GL**. **GL** is obtained by adding that axiom $\Box(\Box p \to p) \to \Box p$ to the normal modal logic **K**. **GL** is, in a precise sense that will not be discussed here, the modal logic of formal provability in PA (Boolos 1995).

Consider the following translation from $\mathscr{L}_{ML}$ to $\mathscr{L}_{ML}$:

$(p)^* = p$
$(\neg\varphi)^* = \neg(\varphi)^*$
$(\varphi \wedge \psi)^* = (\varphi)^* \wedge (\psi)^*$
$(\varphi \to \psi)^* = (\varphi)^* \to (\psi)^*$
$(\Box\varphi)^* = \Box(\varphi)^* \wedge (\varphi)^*$

It is known that $\mathbf{Grz} \vdash \varphi$ iff $\mathbf{GL} \vdash (\varphi)^*$ (see Boolos 1995: p. 156). It follows from this result that **Int** can be embedded in **GL**: for recall that if φ is a theorem of **Int** then $(\varphi)^t$ is a theorem of **Grz** (as well as **S4**). Thus, if φ is a theorem of **Int**, $((\varphi)^t)^*$ is a theorem of **GL**.

The Kripke frames of **GL** are finite, transitive and irreflexive. This suggests that exact verification of **Int** can be expressed in **GL** by *only* supplementing it with converse modality $\Box^-$. Such $\Box^-$ relation can be read as a proper part relation between states, as it is irreflexive. I am not familiar with work on such temporal **GL**, so am not pursuing it here.

A possible different approach to exactification might use the tools of hybrid logic. Expressing the uniqueness of certain states becomes a straightforward task in such logic (see the appendix of (Fine 2014) for a different use of hybrid logic in the context of exactification).

The aim here was to test the expressive power one needs to add to modal logics in order to express certain properties on graphs. It did not directly addresses the four problems that motivate Fine in (Fine 2014). The main issue is that the investigations conducted here do not illuminate the connections between the Kripke semantics and the B-H-K semantics for **Int**, unlike Fine's project. In addition, in (Fine 2014), the project of exactification is conducted at the level of the semantics itself, in the metalanguage, so to say. Here the aim is to be able to talk of exactification in the modal object language.

The hope (which is not achieved here) is to be able to tackle questions of decidability and complexity of the satisfiability and model checking problems. Complexity results for **S4** tell us how hard is it to check whether a state inexactly verifies a sentence φ, which amounts for the model checking problem given a pointed model (M,w) and a modal formula $(\varphi)^t$. How *harder* (in terms of complexity theory) is it to check exact verification versus inexact verification? Since the complexity of (many extensions of) modal logics is a well studied area, it seems natural to explore the connection between extensions of the basic modal logic and the property of exact verification to answer such questions.

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