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Transfinite Number in Wittgenstein's *Tractatus*

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In his highly perceptive, if underappreciated introduction to Wittgenstein's *Tractatus*, Russell identifies a "lacuna" within Wittgenstein's theory of number, relating specifically to the topic of transfinite number. The goal of this paper is two-fold. The first is to show that Russell's concerns cannot be dismissed on the grounds that they are external to the Tractarian project, deriving, perhaps, from logicist ambitions harbored by Russell but not shared by Wittgenstein. The extensibility of Wittgenstein's theory of number to the case of transfinite cardinalities is, I shall argue, a *desideratum* generated by concerns intrinsic, and internal to Wittgenstein's logical and semantic framework. Second, I aim to show that Wittgenstein's theory of number as espoused in the *Tractatus* is consistent with Russell's assessment, in that Wittgenstein meant to leave open the possibility that transfinite numbers could be generated within his system, but did not show explicitly how to construct them. To that end, I show how one could construct a transfinite number line using ingredients inherent in Wittgenstein's system, and in accordance with his more general theories of number and of operations.

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1. Introduction

In his highly perceptive, if underappreciated introduction to Wittgenstein's *Tractatus Logico-Philosophicus* (TLP), Russell identifies a "lacuna" (TLP xxiii) within Wittgenstein's theory of number, relating specifically to the topic of transfinite number. According to Russell, Wittgenstein's theory "is only capable of dealing with finite numbers", and "(n)o logic can be considered adequate until it has been shown to be capable of dealing with transfinite numbers" (TLP xxiii). While Russell thus identifies transfinite number as an issue upon which Wittgenstein's theory "stands in need of greater technical development" (TLP xxiii), he is also careful to note that, there is nothing "in Mr. Wittgenstein's system to make it impossible for him to fill this lacuna" (TLP xxiii).

This paper has two principal, and interrelated goals. The first is to show that these concerns of Russell's cannot be dismissed on the grounds that they are external to the Tractarian project, deriving, perhaps, from logicist ambitions harbored by Russell but not shared by Wittgenstein. By contrast, the extensibility of Wittgenstein's theory of number to the case of transfinite number is, I shall argue, a *desideratum* generated by concerns internal to Wittgenstein's logical and semantic theory. Second, I aim to show that Wittgenstein's theory of number as espoused in the *Tractatus* is consistent with Russell's assessment, in the sense that Wittgenstein meant to leave open the possibility that transfinite numbers could be generated within his system,[1] by making recourse to his theory of operations, and to the general form of an operation (TLP 6.01). Though he did not show explicitly how this was to be done, he specified two distinct operations, denoted by Σ and $\overline{\xi}$ respectively, the structure of each of which is such that it would be plausible to deploy them in such a construction. The way the Σ operation in particular is introduced at TLP 4.27 and 4.42, strongly suggests that Wittgenstein intended it to have further applications beyond those in association with which it is immediately deployed in that context.

To these ends, I will show how requirements internal to the *Tractatus* require that the number of elementary propositions constitutes an infinite totality, while also therefore requiring that the number of truth-possibilities, and truth-functions of those elementary propositions, must constitute higher, transfinite totalities. The number of elementary propositions must be an infinite total, and the relationship between elementary propositions on the one hand, and their truth-possibilities as well as truth-

[1] At a first blush, the idea that Wittgenstein meant it to be possible to extend his theory of number to the domain of transfinite number might seem incompatible with the voluminous evidence compiled and summarized by Rodych (2000) to the effect that Wittgenstein was strongly critical of transfinite set theory (or TST). Yet, Rodych's own thesis is that Wittgenstein was strongly critical of TST "(f)rom his return to philosophy in 1929 through at least 1949". (2000, 281) The *Tractatus* does not lie within this period of Wittgenstein's philosophical development and thus the evidence contained in Rodych's paper is inconclusive with respect to the issue of whether Wittgenstein was strongly critical of TST when he wrote TLP. The general picture which will emerge over the course of this paper is that in the *Tractatus*, Wittgenstein is critical of some aspects of TST but not others. In particular, though he appears to reject set theory as "superfluous in mathematics" (TLP 6.031), and to dispute the special status of axioms, or "primitive propositions", relative to that of other logical propositions (TLP 6.1271), this does not amount to a wholesale rejection of Cantor's theory of transfinite number. Instead, Wittgenstein hopes to lay the groundwork for a theory of both finite and transfinite number, consistent in many respects with Cantor's, by deploying a theory of iterative operations *in lieu* of an axiomatic theory of classes.

functions on the other, must be one-to-many. Explicating and supporting these points will first involve delving into the mechanics of Wittgenstein's N-operator notation in Section 2. In Section 3, it will then involve reflecting on the size of the domain of quantification, and defending the idea that the number of Tractarian objects must be infinite (though this may only be *shown* but not literally *said* in a logically adequate notation). Building on this, Section 4 will probe the mechanics of Wittgenstein's truth-tabular notation, and show how it implicates and involves transfinite cardinalities. In Section 5 I will then show how, using ingredients already present within Wittgenstein's logical system, one could extend his theory of number from natural, to transfinite numbers. In sum, this could be done by first applying an iterative Ω operation to generate the successive progression of finite, natural numbers, and then taking the limit ordinal of these numbers, ω, as the base of an iterative power operation $\sum'(\overline{\xi})$ which, applied successively, generates the series of transfinite Beth numbers.[2]

In a Tractarian notation, the general form of transfinite number yielded by such a power operation could be symbolized as follows: $[\overline{\omega}, \overline{\xi}, \sum(\overline{\xi})]$, where $\overline{\omega}$ represents all digitally encoded subselections of members of the natural number series, $\overline{\xi}$ represents any arbitrary sub-selection of bases or results of $\sum'(\overline{\xi})$, and $\sum(\overline{\xi})$ represents an operation which calculates summations of binomial coefficients corresponding to numbers of subselections of selections of digital sequences (which in the base case, encode subselections of natural numbers). Applying this operation iteratively, $\sum'(\overline{\xi})$ could be used to calculate the number of subselections of natural numbers ($\sum(\overline{\omega})$), yielding $\beth_1 (= 2^{\omega})$, then calculate the number of subselections of the original subselections, yielding $\beth_2 (= 2^{2^{\omega}})$, and so on.

[2] In Section 5, it will be made clear why I choose to develop the Beth series instead of the Aleph series.

Aside from going some way to address the "lacuna" identified by Russell, an added benefit of this demonstration will be to display the deep, and philosophically illuminating internal relationships between Wittgenstein's treatments of logic, language and mathematics, all of which make essential recourse, as we shall see, to structurally analogous operations, as well as to the general form of an operation, $\Omega'(\overline{\eta})$, identified at TLP 6.01. Indeed, $N'(\overline{\xi})$ and $\sum'(\overline{\xi})$ are each straightforward substitution instances of $\Omega'(\overline{\eta})$.

Russell is not alone in noting the existence of gaps within Wittgenstein's theory of number. (see, e. g., Frascolla 1997, 362, Frascolla 2017, 307, and Ramsey 1931, 17.) However, and as Frascolla (2017, 307) notes, it was "misdirected" of Ramsey to characterize Wittgenstein as providing a "ridiculously narrow view of mathematics" (1931, 17). Such gaps exist in Wittgenstein's presentation not because Wittgenstein offered "the equational fragment of arithmetic as an exhaustive exposition" (Frascolla 2017, 307) of the general form of number, but instead insofar as he intended that fragment to serve "as a model for the interpretation of the remaining parts of mathematics" (2017, 307). In other words, the "equational fragment" provided in the *Tractatus* is not supposed to constitute a robust formal exposition of a theory of number and of the foundations of mathematics, so much as it aims merely to suggest, and work out the beginnings of a novel and alternative approach to number, particularly relative to the logicism of Frege and Russell. (Something analogous is true regarding Wittgenstein's exposition of his N-operator notation.) Indeed, Wittgenstein himself is careful to note, in his preface, that the *Tractatus* should not be construed as a textbook (TLP 4), and that he has "fallen a long way short of what is possible" (TLP 4) in terms of clearly expressing his logical, mathematical, and philosophical thought. Wittgenstein subsequently invites others to "come and do it better" (TLP 4), that is, to clarify and flesh out the details of his system. That said, it will be important to keep in mind as we exposit Wittgenstein's system and attempt to flesh out the relevant details, that ultimately, for many interrelated reasons, the Tractarian system is irreparable and does not work. Nevertheless, it is a highly influential and illuminating system which we can learn from by carefully examining both its insights and deficiencies alike.

2. Operator N and the General Form of a Truth-Function

It is tempting to dismiss Russell's admonition of Wittgenstein, that he fails to sufficiently develop a theory of transfinite number, on the grounds that it emerges from and reflects logicist concerns external to the Tractarian project. Perhaps, it might be argued, Wittgenstein should simply be read as a finitist (see, e. g., Li 2018), who would have no truck with transfinite numbers. If this were in fact the case, then Russell's assessment of Wittgenstein's system as being compatible with the introduction of transfinite numbers would be both odd and incorrect. In Section 4 of this paper, however, it will be shown that contrary to this temptation, transfinite cardinalities are implicit within features internal to Wittgenstein's early philosophy of logic and language, consistent with Russell's assessment. In order to understand how and why, it will be helpful to first explicate some of these features. In this section, more specifically, we will take a closer look at Wittgenstein's N-operator notation and reflect on how Wittgenstein likely intended to deploy it to recover quantification over infinite domains. Doing so will provide crucial background context, for subsequent discussion of the scholarly controversy over the size of the Tractarian domain, and of the reasons to think that Wittgenstein's logical system involves implied transfinite cardinalities. It will also serve to highlight crucial, structural similarities between Wittgenstein's construction of truth-functions, and the construction of transfinite cardinalities which can be undertaken by deploying elements of his theory of operations and truth-functions.

At the heart of Wittgenstein's early philosophy of logic and language is what he calls the "general form of a proposition" (TLP 6). Within his symbol for the general form of a proposition ($[\overline{p}, \overline{\xi}$ $N(\overline{\xi})]$), which he also calls "the general form of a truth-function" (TLP 6), Wittgenstein specifies a procedure whereby all possible truth-functions may be built up *via* successive applications of a single, truth-functional operator, specifically N, to selections of elementary propositions. Hence, according to Wittgenstein's symbol for the general form of a truth-function, we start with all selections of elementary propositions ($\overline{p}$) as our base, and from there, generate truth-functions by applying N successively, first to any such selection of those elementary propositions ($\overline{\xi}$), and then to the N-expressed truth-functions of those elementary propositions which result. (To say that a truth-function is "N-expressed" means that it is symbolized using successive iterations of N alone, exclusive of other, equivalent combinations of truth-functional operators such as "∼" (i. e., negation) or "&" (i. e., conjunction)). Notably, N functions similarly to joint negation (e. g., ∼ p & ∼ q) except that it may be applied to more than two propositions at a time. Indeed, N is defined such that it may take an indefinite number of arguments, from 1 to an infinite number of arguments. The bar on top of "*p*" or "ξ" represents an operation which generates selections of whatever arguments it applies to.[3] For instance, within Wittgenstein's symbol for

[3]Russell has engendered some confusion about the bar notation by describing the symbol $\overline{p}$ in his introduction as standing "for all atomic propositions" (TLP, xvii). Yet when Wittgenstein himself describes the bar notation in TLP 5.501 he is careful to note that it indicates not all elementary propositions but rather selections thereof (though 'all' is clearly one such selection among others). Specifically, he says it indicates all "values" which are "terms of the bracketed expression", e. g., "$(\overline{\xi})$ = (P, Q, R)". (TLP 5.501) Wittgenstein calls these "values" because they are the results of a selection operation performed on whatever base they are applied to. The bar indicates this selection operation, and when it appears over ξ it indicates an arbitrary selection from among those indicated by $\overline{p}$, which serves as the base for the N-operation characteristic of the general form of a truth-function. Wittgenstein alludes to Russell's

the general form of a proposition, "*p*" serves as a sentence-letter metavariable which ranges over all elementary propositions, and the bar on top of the "*p*" generates all subselections of those arguments (i.e., elementary propositions). $\overline{\xi}$ represents any arbitrary one of those selections, for instance (p, q, r). (TLP 5.101) Subsequently, the $\overline{\xi}$ operation can be deployed again in succession, to select the results of N-operations, namely N-expressions, for presentation to N. The first term within Wittgenstein's symbol for the general form of a truth-function, $\overline{p}$, thus tells us what the $\overline{\xi}$ operation is being applied to. In other words, it tells us what the base of the selection operation specified therein is. In the case of the general form of a proposition it is elementary propositions, but it need not always be. As we will see in more detail in Section 5, at TLP 6.01 Wittgenstein deploys $\overline{\eta}$ as a metavariable within his symbol for the general form of an operation $\Omega'(\overline{\eta})$, to stand for any arbitrary base of the selection operation represented by $\overline{\xi}$.

In order to express negation using Wittgenstein's $N'(\overline{\xi})$ operation, for instance, we would apply N to a single elementary proposition, like so: N(p). In order to express the conjunction of p and q, we would apply N, successively, first to each of p and q, and then place each of these resulting N-expressions under an additional iteration of N as follows: N(N(p), N(q)). The disjunction of p and q, we would express by applying N first to (p, q), and then, successively, applying another iteration of N to the resulting N-expression, like so: N(N(p, q)). Given that conditional statements are true whenever either their consequents are true, or their antecedents are false, $p \rightarrow q$ is equivalent to $\sim p \vee q$, and that in turn can be expressed *via* successive applications of Wittgenstein's N-operator, as follows: N(N(N(p), q)).

Furthermore, by allowing N to take an infinite selection of arguments, and by allowing those arguments to be each of the infinite number of substitution instances of a propositional function such as fx, for example, we may then use N to express something equivalent to Russellian quantification. As Wittgenstein explains in TLP 5.52, for instance, "if ξ has as its values all the values of a function fx for all values of x, then $N(\overline{\xi}) = \sim (\exists x).fx$". In other words, if we substitute each of an infinite number of individual constants for x in fx, and then place each of the infinite number of elementary propositions which result within the brackets under the scope of the N operator, like so: N(fa, fb, fc, fd, . . . , fω), then we will thereby express something that is equivalent to the negation of $(\exists x)$.fx. To express the equivalent of $(\exists x)$.fx, then, we need simply to apply an additional "N" to the front of this N expression in succession, as follows: N(N(fa, fb, fc, fd, . . . , fω)).

Like $(\exists x)$.fx, {N(N(fa, fb, fc, fd, . . . , fω))} is equivalent to a truth-functional expansion, a disjunction which takes each of an infinite number of substitution instances of the propositional function fx, as disjuncts. The equivalent truth-functional expansion can thus be given in the form of a disjunction as follows: fa $\vee$ fb $\vee$ fc $\vee$ fd $\vee$ . . . $\vee$ fω. Earlier, we saw how to express the conjunction of p and q using the N-operator, and from this it is easy to see how the N-operator may be used to recover universal quantification. For instance, we may express $(\forall x)$.fx by using N to recover something equivalent to the conjunction of each of the substitution instances of fx, like so: N(N(fa), N(fb), N(fc), N(fd), . . . , N(fω)). This N-expression is then equivalent to the following, conjunctive, truth-functional expansion: fa & fb & fc & fd & . . . & fω.

In explicating how to deploy the N operator in order to construct several common truth-functions, for illustrative purposes we have followed Wittgenstein (TLP 5.52, 5.501) in focusing on how N may be used to recover quantification into monadic propositional functions. In TLP 5.501, Wittgenstein briefly indicates one way in which we might expand this same basic method to capture quantification over propositional functions of higher arity. Specifically, the third of the three kinds of description he

misunderstanding in an August 1919 letter where he writes to Russell that "I suppose you don't understand the notation of '$\overline{\xi}$'. It does not mean 'for all values of ξ. . . '" (Wittgenstein 1995, 126) identifies, by which we may select arguments for presentation to the N operator (i. e., stipulate terms to occur within the brackets under the scope of an iteration of N) involves "giving a formal law that governs the construction of the propositions" (TLP 5.501). In that case, what will occur in the brackets under the scope of one or more iterations of N will be "all the terms of a series of forms" (TLP 5.501). Here Wittgenstein seems to have in mind the sorts of formal series described in TLP 4.1252, which involves several distinct objects (a, b, etc.,) being ordered in a series by iterations of a dyadic relation R. In this case, the propositional function "*xRy*" serves as a sort of prototype, or recipe which can be used to gather together substitution instances of a dyadic, as opposed to monadic propositional function, for presentation to the N-operator. With regards to each of the second and third kinds of description identified at TLP 5.501, there would seem to be no barrier to using the same basic procedures, or others like them, to gather together substitution instances of triadic, quadratic, or other propositional functions of even higher arity, in order to use N to recover quantification over such propositional functions, *via* the method of truth-functional expansion exposited above.

In developing our exposition, however, we have also made two additional, and crucial presumptions. The first is that existential and universal quantifications are equivalent to infinite, truth-functional expansions, i. e., disjunctions and conjunctions, respectively. The second is that that the totality of substitution instances of any monadic propositional function is infinite. Defending the idea that Wittgenstein adhered to this second presumption will be the focus of Section 3. In the remainder of Section 2, we will look at the first presumption. It will be recalled that our goal here is not to defend the Tractarian logical system. Instead it is to give a general indication of the lines along which Wittgenstein thought it would be possible to construct a transfinite number line, given other things we know about his system.

One thing we can be fairly confident about with regards to his system is that he did indeed think that existential and universal quantifications were equivalent to infinite, truth-functional expansions, and that these could be expressed with his N-operator notation. Hence in a lecture on 25 Nov 1932, G. E. Moore records Wittgenstein as identifying "a most important mistake in the *Tractatus*" (Stern, Rogers and Citron 2016, 216) which is "muddling up a sum with the limit of a sum" (2016, 217). In this context, he claims that "if <u>all</u> general propositions were identical with logical products or logical sums. . . then any <u>general</u> proposition could be written" (2016, 217) using N. The notes don't contain the letter "N", but it is clear that his N operator is what he means by the expression "$(\widehat{\xi})$[-------T}", which he says represents the "negation of all propositions that are values of $\underline{\xi}$" (2016, 217). Continuing on, he says that in the *Tractatus*, he mistakenly supposed "that (∃x)fx = fa ∨ fb ∨ fc <u>& so on</u> was of laziness, when it wasn't". Moreover, he claims that if the "& so on" were "of laziness" then "it could be replaced by an enumeration" (2016, 217).

The picture Wittgenstein is sketching of the nature of the relationship between N and quantification here, is as follows: N could be used to express infinite, truth-functional expansions corresponding to existential and universal quantification, by operating on enumerable lists of elementary propositions. (If the expansions in question were not infinite, then it could not have been a "mistake" to think that their terms could be enumerated.) The thought here seems to be that, the propositional function fx describes a selection of propositions, specifically, substitution instances of that propositional function, which may then be listed under one or more iterations of N, as outlined above, in order to recover expressions involving quantification. In the simplest instance, described at TLP 5.52, listing them under one iteration of N recovers the negation of an existential quantification, for example. Immediately following this remark, Wittgenstein goes on in TLP 5.521 to explicate the way in which logical products and sums provide a bridge between quantification, and the N-notation. There he says that quantification is "embedded" in the notions of logical product and sum, but that there are certain complications, or difficulties, with introducing generality in association with them. In order to obviate these difficulties, we have to treat quantified expressions as indicating logical prototypes (e. g., "fx" in (∃x).fx), in which variables and constants are clearly disambiguated, since constants are prominently displayed therein as those symbols which are not bound by quantifiers (TLP 5.522). Using these prototypes, we can then gather together lists of substitution instances for presentation to the N-operator, and thereby use N to express something equivalent to existential and universal quantifications. The complications identified by Wittgenstein, here, may be of precisely the sort identified by Fogelin (1982, 1987) and Soames (1983), that emerge when N is applied to expressions containing certain sorts of combinations, or iterations, of quantifiers. Such difficulties would then explain why Wittgenstein insists that we must "dissociate" generality from truth-functions, meaning that N does not apply to anything with quantifiers in it. Instead it will apply to lists of elementary propositions which we use logical prototypes contained in quantified expressions to generate.

While it would obviously be impracticable to write these whole lists down, it must nevertheless be possible, according to Wittgenstein, in principle. At the very least, it would have to be the case that these lists could be treated as if they were enumerable. On this view, the second and third, "recipe" or "prototype" methods of stipulating arguments for presentation to N identified at TLP 5.501, could thus, in principle if not in practice, be replaced by the first, i. e., direct enumeration. Hence, referring back to the *Tractatus*, Wittgenstein explains in the *Big Typescript* that

> My understanding of the general proposition was that (∃x).fx is a logical sum, and that although its terms weren't enumerated there, they could be enumerated (from the dictionary and the grammar of language) (Wittgenstein 2005, 249).

Importantly, Wittgenstein never thought of the N-operator notation as something which it would be practically convenient to use. Its significance within the Tractarian system is philosophical, not practical (see Connelly 2017). It shows that, provided we can treat it as if operating on enumerable lists of elementary propositions, there is a single, truth-functional operator, N, which can in principle be used to recover the expressive capacity of any other possible notation. N thus yields the general form of a truth-function, and that, in turn, is the general form of a proposition (TLP 6).

Later, however, Wittgenstein came to realize that the idea of a general propositional form is a mistake, the nature of which we can see by, *inter alia*, looking more closely at the cases in which the expressive power of N breaks down. A vast literature has grown around this issue of the expressive completeness of N, where and how exactly it fails.[4] In the portions of Moore's notes quoted above, Wittgenstein is telling us how. It is not that N cannot express mixed, multiply general quantifications because it cannot operate on different variables within its scope in different ways. (compare Fogelin 1987, 79, Soames 1983, 575–76, Landini 2007, 136–37.) Wittgenstein means to obviate problems like this by "dissociating" (TLP 5.521) quantifiers from N. N thus does not operate on anything containing variables, or internal, quantificational structure (Connelly 2017). Instead, though such lists may be *generated* with the help of propositional functions containing variables (in which case they serve as a sort of "recipe" for selecting, or "prototype" of, the propositions comprising the list), N ultimately instead operates upon lists of elementary propositions

[4]See, e. g., Fogelin (1982, 1987), Geach (1981, 1982), Soames (1983), McGray (2006), Rogers and Wehmeier (2012), and Connelly (2017).

(Connelly 2017, 4). Because elementary propositions are logically independent, they may be assigned semantically atomic sentences letters which lack any internal structure (Connelly 2017, 20). It is these semantically atomic sentence letters which occur in the brackets under the scope of N, just as Wittgenstein indicates in his symbol for the general form of a truth function at TLP 6. The operation this symbol specifies, however, does ultimately break down, and it does so in the infinite case because in that case the lists upon which N must operate are not enumerable, even in principle, and therefore cannot convincingly be treated as if they are.

In the portion of Moore's notes from the 1932 lecture quoted above, Wittgenstein is telling us that in the *Tractatus*, he mistakenly presumed that the terms of an infinite, truth-functional expansion could be treated as if they were enumerable. That is why he describes his mistake as akin to "muddling up a sum with the limit of a sum". In Section 4 when we examine Wittgenstein's $\sum$ notation and consider its associations with the binomial theorem and Pascal's triangle, we will explore some additional possible reasons why he might have thought this. In an ordinary, arithmetical sum which has a final term, in any case, you can enumerate all of the terms of the sum. For instance, in $1 + 1 = 2$ the final term of the sum is "1". When you are dealing with a sum which approaches a limit, by contrast, you cannot enumerate all of the terms it contains because they are infinite, and so endless, in number. So, for example, as in the case of $1 + 1$, 2 is also the sum of $1 + \frac{1}{2} + \frac{1}{4} + \ldots =$. Both sums "terminate" in the number 2 in the sense that 2 is the "answer" or solution to both sums. But from this it does not follow that, like $1 + 1$, this second sum has a final term. On the contrary, in the second case 2 is the "answer" in the sense that it is the limit of a sum with an endless number of terms. In his *Tractatus*, Wittgenstein seems to have wanted to be able to treat truth-functional expansions as if they could be infinite in length while still having a "limit" in the sense of an end, or terminus. This explains why, when Ramsey (1931, 74) describes Wittgenstein's analysis of universal quantification (e. g., "For all x, x is red") in terms of conjunction, he provides as his illustration of that analysis a terminal, truth-functional expansion (i. e., "a is red and b is red and ... and z is red".) As Wittgenstein later realized, however, this notion involved a mistake analogous to conflating these two distinct sorts of arithmetical sums.

A somewhat less charitable way of characterizing Wittgenstein's mistake is that he has made the "elementary blunder" (compare Soames 1983, 578) of muddling up infinite with finite lists. Reading the author of the *Tractatus* more charitably, however, what has happened is that he has adopted a conception of infinity according to which infinite lists can be treated as terminal, and though counterintuitive, such a conception is required in order for language to have determinate sense. If infinite, truth-functional expansions do not terminate, or cannot be treated as if they do, then you cannot express quantification over infinite domains within the scope of the general form of a truth-function. From the extensionalist perspective of the author of the *Tractatus*, what it means to be a proposition *is* to be expressible *via* iterations of the general form of a truth-function. So, from that perspective we have to bite the bullet and accept that infinite truth-functional expansions can be treated as if they are terminal. A counter-intuitive conception of infinity is, from this perspective, the cost of making sense of classical logic and of its relationship to language.[5]

[5]Based upon an interpolated notecard from a 1946 lecture of Quine's, Burgess (2008, 74–77) considers the prospects of a proposal for handling difficulties in the expression of prohibitively lengthy truth-functional expansions "by admitting "and so on" as an irreducible part of the language" (2008, 74). Could, as an anonymous referee has suggested, Wittgenstein similarly deploy an "and so on" operator in association with his N-operator in order to obviate the need to appeal to this counter-intuitive conception of infinity, so as to address concerns regarding the recovery of quantification over infinite domains? Conceivably. Though intriguing, such a suggestion seems anachronistic at a first blush, however, given that it appears to be a brainchild of Quine's from the

In remarks to Desmond Lee sometime in 1930-31, Wittgenstein thus identifies an associated error on which his extensionalism depends. Elaborating on the Tractarian notion that "there are elementary propositions, each describing an atomic fact, into which all propositions can be analysed" (King and Lee 1980, 120), he claims that this is an erroneous notion which "arises from two sources" (1980, 120). The second has to do with colour exclusion and the problem it poses for the logical independence of elementary propositions. Further discussion of this issue would take us two far afield of our present purposes.[6] The first, which is directly relevant to our present concerns, and to the issue of whether N may be used to express infinite, truth-functional expansions, is the error of "(t)reating infinity as a number" (1980, 120). In other words, the error involved here is thinking that you could treat ω as if it were a huge number at the end of an infinitely long list of natural numbers. If you could, then it is easy to see how N could be used, as outlined above, to express the infinite, truth-functional expansions corresponding to existential and universal quantifications. In the illustrative case of the monadic propositional function fx, you would just list all of its substitution instances within the scope of an N expression of the appropriate structure. For instance, in order to express (∃x).fx, you would simply list each of the infinite number of substitution instances of fx produced by substituting a name for each of the infinite number of objects under a single iteration of N. You would then add another N on the left end, like so: N(N(fa, fb, fc, fd, . . . , fω)). The ". . . " here corresponds to the interpretation of "& so on" Wittgenstein is talking about in Moore's notes, quoted above. I have used the limit ordinal "ω" as the name for the object referred to within the last substitution instance, in order to covey the idea that it is not only the last but also the infinitieth substitution instance on the list. Though Wittgenstein does not himself use this notation, it is a helpful way of capturing what he must have meant, given the textual evidence provided above.

We will see in more detail in Section 4 that, from within the perspective of Wittgenstein's Tractarian logical system, adding substitution instances corresponding to other monadic predicates and logical forms to the list will not increase the size of the infinite totality it comprises. In other words, just as the even numbers, as well as the odd numbers, constitute infinite totalities which are equivalent in size to the natural numbers, the number of substitution instances of an infinite number of monadic predicates, and of an infinite number of logical forms, will each be the same size as the infinite totality of substitution instances of any monadic predicate. These facts will be important when we explore the ways in which Wittgenstein's system *can* be seen to require higher, transfinite numbers. For the time being they are also significant because they suggest that, from Wittgen-

40's and TLP was published in the 20's. In any case, there appears to be little evidence Wittgenstein considered such a proposal, which is perhaps due to its incompatibility with other integral aspects of his philosophy of logic, such as his claim that *all* propositions, including those which lack prohibitively lengthy truth-functional expansions, have a common form which can be expressed in a single, uniform variable (which contains $N(\overline{\xi})$ as its sole primitive operator, to the exclusion of "and so on" along with any other operations). What he did do, instead, was to build the very concept of "and so on" *into* his concept of successive applications of an operation. (TLP 5.2523) And this approach gives rise to precisely the concerns that are raised by Wittgenstein in Moore's notes on his lecture dealing with the N-operator: 'and so on' is ambiguous with respect to whether or not it can be replaced by an enumeration of its terms. It is not clear that a distinct "and so on" operator fares any better in this respect, though perhaps the ambiguity could be resolved by appeal to two distinct operators. This latter suggestion, however, strays implausibly far from Wittgenstein's intent, in multiplying the number of primitive operators far beyond the "one and only general primitive sign" (TLP 5.472) he says is involved in the general form of a proposition. Thus, while there may be something to this proposal as a means of tackling truth-functional expansions, it is not Wittgenstein's and does not appear compatible with his philosophy of logic. It thus cannot be appealed to as a plausible alternative reading of Wittgenstein's N-operator notation which obviates the need to appeal to the counter-intuitive conception of infinity outlined above.

[6]See Ramsey (1923, 473) for further discussion of the nature of the problem which "colour-exclusion" poses for Wittgenstein's early logical and semantic system.

stein's Tractarian perspective, it should be possible to express any meaningful truth-function by applying N successively to an infinite number of elementary propositions, since this will involve no more or less steps than were involved in applying N successively so as to recover quantification into monadic predicates. Wittgenstein's Tractarian conception of infinite lists is one that combines two agreeable, if ultimately incompatible properties: its terms are enumerable in principle, but you can always add more items to the list without increasing its size. The incompatibility of these two properties reflects the "muddle" between ordinary sums and limits of sums, finite lists and infinite lists, alluded to above in Moore's notes. At the time of authoring the *Tractatus*, Wittgenstein saw this "muddle" as a counterintuitive conception of infinity that he was willing to accept, in order to preserve other things, such as extensionalism, that he was committed to. In some sense, Wittgenstein's conception is not especially less intuitive than the Cantorian, and Russellian conception of the actual infinite from which it is derived and by which it is inspired. The latter, for example, also entails the counterintuitive consequence that the set of even numbers has the same cardinality as that of the natural numbers, even though the latter includes some numbers, the odds, that the former does not. Looked at from this perspective, Wittgenstein's conception of an infinite totality is hardly the most counter intuitive view defended by an early analytic philosopher in an attempt to make classical logic work. Ramified types, objective falsehoods, and subsistent but non-existent entities come to mind. Russell's own experiences in this endeavor, and his appreciation of the difficulties inherent in it, explains why "(a)s one with a long experience in the difficulties of logic", he goes on to praise Wittgenstein for having "constructed a theory of logic which is not at any point obviously wrong" (TLP, xxv).

3. Fixed and Infinite: the Domain of Quantification in Wittgenstein's *Tractatus*

In any case, turning now to the second of the two presumptions of our exposition of the general form of a truth-function, undertaken above in Section 2, in this section we will see that the number of substitution instances of any monadic propositional function must be infinite, since the number of Tractarian objects and so the Tractarian domain of quantification must be infinite. Within the context of attempts to explicate Wittgenstein's N-operator notation, and to assess the expressive completeness as well as consistency of the Tractarian logical system, by contrast, a number of commentators have detected an alleged ambivalence within Wittgenstein's characterization of the size of the domain of quantification. In particular, Wittgenstein is claimed to be in some way indecisive regarding, uncertain about, or wishes to leave as an open, empirical question, the issue of whether the domain of quantification is to be construed as finite or infinite. Soames, for instance, claims that for Wittgenstein "the number of actual objects, and hence names, is not a matter of logic. One cannot specify the number of names in an ideal symbolism, nor can one assume that it must be finite" (1983, 574). Likewise, according to Rogers and Wehmeier, Wittgenstein wishes to leave the size of the domain as an open question, "entertaining the possibility that it might be infinite", but ultimately concluding that it "is not a question that can be decided by logic alone" (2012, 539). According to James McGray, finally, the number of objects in the Tractarian domain is "perhaps" (2006, 148) or "possibly" (2006, 152) infinite, though in any case "unknown, and perhaps even unknowable" (2006, 168). In this section of the paper, I will show that readings of these sorts are unstable in the sense that, given other things we know him to be committed to, Wittgenstein cannot be thought to be ambivalent concerning the size of the domain of quantification without thereby also, and contrarily being thought to take a definite stand on it. Interpretive charity thus demands that we not attribute to Wittgenstein the idea that the question of domain size might be an open one, since the alternative is to view him as being embroiled in a quagmire of confusions and self-contradictions.

While Wittgenstein *can* conceivably leave as open the issue of whether the *empirical universe* contains a finite or infinite number of complexes, it is important to note that, within the logical perspective of the *Tractatus*, variables of quantification do not, ultimately, range over complexes, but instead over metaphysically simple objects. Ordinary names for complexes are merely disguised descriptions. They are shorthand for complete descriptions of complexes (TLP 2.0201) *via* conjunctions of elementary propositions which depict the several atomic facts that make the complex up. While quantifiers do not occur within such elementary propositions, they can be used, similarly to ordinary names, as shorthand for much lengthier truth-functional expansions of elementary propositions (conjunctions or disjunctions). And in that case, these quantifiers range over the simple constituents, that is objects, which make up the atomic facts portrayed by elementary propositions.

Tractarian simple objects are, however, logical as opposed to empirical objects. They constitute both the scaffolding or "unalterable form" (TLP 2.023) of logical space, as well as the underlying substance of the world (TLP 2.0211) which is common to all possible worlds (TLP 2.024), including those comprised of an infinite number of facts. For elementary propositions to have sense at all (TLP 2.0211), the names they contain must stand for timeless, indecomposable, simple objects. Atomic facts are combinations of such objects, (TLP 2.01) and complexes are made up of existing atomic facts. (TLP 2, 2.021) The *existence* of facts and complexes thus presupposes the *subsistence* of objects.

If, then, it is even *possible* for an infinite number of empirical complexes (i. e., contingent particulars) to *exist* in logical space (which it must be if Wittgenstein wants to leave it as an open question whether they do), an infinite domain of simple, logical objects must inevitably *subsist* in order to secure and manifest that possibility. If, on Wittgenstein's view, it is even *possible* that logical space contains an infinite number of *simple* objects, moreover (which it must be if Wittgenstein wants to leave it as an open question whether it does) it must contain an infinite domain of simple objects to secure and manifest that possibility. While we have good reason to think that Tractarian quantifiers ultimately range over simple objects, a domain of infinite size is implied, regardless of which way we interpret them. Logical space after all, is a space of *possibility*. If, on Wittgenstein's view, it is even *possible* that the cardinality of Tractarian objects is infinite (and if Wittgenstein wants to leave that as an open question then it must be on his view), it must be *permissible* for quantifiers to range over infinite domains: "Whatever is possible in logic is always permitted" (TLP 5.473). Reading Wittgenstein charitably, then, he does not regard an infinite domain of quantification as an open possibility, so much as a modal reality.

Concerns about the charity of readings which attribute ambivalence on this issue arise, because the idea that Wittgenstein might think of it as an open possibility that the domain is infinite, but also an open question whether it is permissible for quantifiers to range over infinite domains, is in conflict with several remarks he makes regarding the nature of the relationship between logic and its application. For example, at TLP 5.551 Wittgenstein identifies as his "fundamental principle" the idea that

> whenever a question can be decided by logic it must be possible to decide it without further ado. (And if we get into a position where we have to look at the world for an answer to such a problem, that shows we are on a completely wrong track) (TLP 5.551).

In other words, either the issue of domain size is undecidable, or it *is* decidable "without further ado". It is not instead the case, that the question of whether the domain is finite or infinite is a mystery about which we might have to speculate, or investigate empirically, and which could conceivably turn out either way. Hence, Wittgenstein insists: "there can *never* be surprises in logic" (TLP 6.1251). And again:

> It is clear that logic must not clash with its application. . . logic has to be in contact with its application. Therefore, logic and its application must not overlap (TLP 5.557).

The idea that a domain of infinite size might be logically possible, and yet it might nevertheless not be permissible to quantify over it, would be precisely the kind of case that Wittgenstein means to rule out here when he says that logic cannot "overlap" with its application. The illegitimate proposal, here, would be that the infinite domain of objects belongs to one category, that of the logically possible, and that quantification over infinite domains belongs to the other category, that of logically permissible applications, and quantification over infinite domains might, or might not be in the intersection. But that idea, Wittgenstein claims, is bogus. There is no such "overlap" between what's logically possible and syntactically permissible. If a domain of infinite size is logically possible, then, by virtue of the internal relatedness of the two formal concepts, it is also syntactically permissible to quantify over that domain. The very idea that Wittgenstein is ambivalent on the size of the domain implies that it is possibly infinite and therefore, by Tractarian principles, that it is permissible to quantify over the infinite domain and thus that the domain is infinite.

To be clear, the argument here is not that "Wittgenstein does not know whether the domain is finite or infinite, thus it must be possible for it to be infinite, and therefore it is infinite". That would involve a fallacy associated with moving from an epistemic premise about Wittgenstein's beliefs to a modal conclusion about logical space. Moreover, Wittgenstein's considered beliefs are not ambivalent on this issue at all, as we shall see in more detail momentarily. Instead, my argument, aimed at a would be Wittgenstein interpreter, is that "If *you* suppose that Wittgenstein regards it as an open possibility that logical space contains an infinite number of objects, then unless you want to attribute inconsistent views to him, *you* must also suppose him to regard it as logically permissible to quantify over all of them". The argument, that is, moves from one interpretive claim about what Wittgenstein regards as a possibility, to a second claim about what a charitable interpreter must therefore hold him to regard as permissible. The argument is not given *from* the perspective of Wittgenstein and does not concern what he does or does not know. It is instead offered *to* an interpreter who is constrained to offer a charitable, and thus consistent reading of Wittgenstein's views.

Another conceivable objection to my argument would be to insist that, on the supposition that Wittgenstein wishes to leave it an open question whether the domain is infinite, he must also wish to leave it as an open question whether the domain is finite. On that supposition, there must therefore be a possible world in which the domain is finite. But then, since my argument presupposes constancy of domain across possible worlds, does it not imply, absurdly, that a fixed, *finite* domain of Tractarian objects must be a "modal reality" just as much as a fixed, and infinite domain of objects is?

Again, however, this objection glosses over the integral distinction between empirical and logical possibility, and fails to note a crucial, and related asymmetry between the finite and the infinite case. Wittgenstein can regard it as an open, empirical possibility that the universe contains only a finite number of existing complexes. But he cannot charitably be thought to regard it as an open possibility that it contains only a finite number of logical objects. This is because, an infinite domain of logical objects can accommodate quantification over both an infinite number of either empirical complexes, or logical objects, as well as a finite number of either empirical complexes, or logical objects. Given an infinite domain of objects, for instance, one can easily quantify over a finite subset of them by applying a restriction on the domain. On the other hand, a fixed, finite domain of logical objects can account for quantification only over a finite, but not an infinite number of either empirical complexes, or logical objects. If it turns out that the domain is finite in the logical sense, then it cannot be an open question whether it is infinite in either the empirical or the logical sense. Thus, the supposition that Wittgenstein wishes to leave as an open question the issue of whether the domain of quantification is finite or infinite, is consistent only with a fixed, infinite, domain of logical objects, not a fixed, finite domain of logical objects. In other words, if Wittgenstein wants to leave it as an open question whether the domain is infinite, then this would require him, ultimately, to deny that it is an open question since one of the supposedly open possibilities, the finite case, is incompatible with the initial assumption. Wittgenstein therefore cannot leave it as an open possibility that the domain of Tractarian objects is finite, and a charitable interpreter cannot credibly interpret him that way.

Reading Wittgenstein as ambivalent on the question of whether the domain is finite, or infinite, is thus deeply problematic, and uncharitable given other things we know him to be committed to. But can any positive, textual evidence be provided for the claim that he regards the domain as fixed, and infinite? Much of the textual evidence for the claim that Wittgenstein regards the domain of objects as fixed across possible worlds can be found in the 1's and 2's of TLP. So, for example, at TLP 1.13 he writes that "The facts in logical space are the world". This remark anticipates a distinction developed in greater detail later in the text, and which we discussed earlier, between subsistence and existence. Logical space *subsists* independently of the world, which is all that *exists*. As Wittgenstein remarks in TLP 1 and then in TLP 2, "The world is all that is the case ... what is the case—a fact—is the existence of states of affairs". Logical space, as opposed to the world, is thus not made of up facts but rather of objects. As Wittgenstein insists at TLP 2.021, and 2.024, "Objects make up the substance of the world", and "Substance is what subsists independently of what is the case". Tractarian objects constitute not only the substance of the world, but also its "unalterable form" (TLP 2.023). The world thus has a logical form, constituted by objects, which subsist in all possible worlds regardless of the actual facts that exist in any of them. Indeed, it is obvious, Wittgenstein insists, "that an imagined world, however different it may be from the real one, must have *something*—a form—in common with it" (TLP 2.022). So the domain of objects is fixed across possible worlds in the sense that they provide both the substance, and the form of the logical space in which various possibilities may manifest as actual. All possible worlds contain the same objects; what differs from world to world is the way these objects are combined to produce facts which, as we have seen, consist in the existence of states of affairs. As Wittgenstein explains at TLP 2.0271–2.0272, "Objects are what is unalterable and subsistent; their configuration is what is changing and unstable. The configuration of objects produces states of affairs".

But where is the textual evidence that the domain of objects, which is fixed across possible worlds, is also infinite in number? Well, at TLP 2.0131 Wittgenstein writes, for example, that "(a) spatial object must be situated in infinite space". This provides *prima facie* evidence that Tractarian logical space is infinite, on the assumption that geometrical space is included in, or coextensive with logical space. Evidence that Wittgenstein thinks the space of geometry is coextensive with logical space comes at TLP 3.032–3.0321, where he writes that

> It is as impossible to represent in language anything that "contradicts logic" as it is in geometry to represent by its coordinates a figure that contradicts the laws of space ... Though a state of affairs that would contravene the laws of physics can be represented by us spatially, one that would contravene the laws of geometry cannot (TLP 3.032–3.0321).

Moreover, at TLP 3.41–3.411 Wittgenstein claims that propositional signs provide the "logical coordinates" (TLP 3.41) of a "logical place" (TLP 3.411), and that these "agree" (TLP 3.411) with the coordinates of a "geometrical place" (TLP 3.411) in which existence is possible. In both passages, again, we find Wittgenstein appealing to a distinction between logical space and the world, but in this case the emphasis is placed on the fact that the former (logical space) is described by geometry while the latter (the world) is described by physics. Physical laws are contingent, but the logical and geometrical space the phenomena they describe exist in has certain *a priori* logical features, including that of being infinite.

Wittgenstein clearly regards not only logical space as infinite, but also, and correlatively, regards it as an open possibility that the empirical universe, i.e., the world, is infinite. Thus, he claims at TLP 4.2211 that

> Even if the world is infinitely complex, so that every fact consists of infinitely many states of affairs and every state of affairs is composed of infinitely many objects, there would still have to be objects and states of affairs. (TLP 4.2211)

Wittgenstein's main point here appears to be that, regardless of how complex the world is, the logical space it exists in must be characterized by certain basic *a priori* features, including the subsistence of objects, and (possibly, at least) the existence of states of affairs. Indirectly, though, in reflecting on the possibility of an infinite empirical universe, Wittgenstein has suggested that the logical space in which that possibility may become manifest must itself be infinite, and so that there must be an infinite number of objects to compose it. This interpretation is confirmed by Wittgenstein's characterization of logical space at TLP 4.463 as an "infinite whole" which is left open by a tautologous assertion.

So, we can be sure both that Wittgenstein regards it as an open possibility that the empirical universe is infinite, and that the logical space which contains the world, whether that world is infinite or not, is infinite. But what about the possibility that there are only a finite number of objects, and an infinite volume of logical space between, or around them? Couldn't logical space be infinite without the domain of Tractarian objects being infinite? Implicitly, this seems to be ruled out in so far as logical space is constituted by objects. As we have seen, objects make up the unalterable form of logical space which subsists in any actual, possible, or imaginable world. In any case, however, Wittgenstein explicitly rules this combination of views out at TLP 5.511, where, referring back to TLP 5.51, he describes his N-operator notation as part of an "all embracing" logical calculus, which serves as a "great mirror" in which is reflected the "infinitely fine network" that is the world. Here Wittgenstein is thinking of the world of physics, and its *a priori* logical features as described by geometry. But obviously, if the world so conceived is actually an infinitely fine network, it must be possible for it to be. Logical space, in other words, like the geometrical space with which it is co-extensive, is infinitely divisible. It must therefore contain an infinite number of objects into which logical, and geometrical space may be divided.

This feature of logical space, that it contains an infinite number of simple, logical objects, is not "unknown" (McGray 2006, 168) so much as it is "unsayable". Hence Wittgenstein writes (in TLP 4.1272) not that it is "impossible to know" the number of objects, but only that it is "senseless to speak" of that number (compare Rogers and Wehmeier 2012, 539–40). Russell lends support to this interpretation when he writes in his introduction that "the totality of possible values of x which might seem to be involved in the totality of propositions of the form fx is not admitted by Mr. Wittgenstein among the things that can be spoken of" (TLP, xxiv). With regards to Russell's "axiom of infinity", moreover, Wittgenstein explains that what it "is intended to say would express itself in a language through the existence of infinitely many names with different meanings" (TLP 5.535). Again, however, if Wittgenstein thinks such a language is logically permissible, then interpreting him charitably requires he hold that an infinite domain of simple objects subsist in order to ensure that possibility. (That which is logically/syntactically permitted is also logically possible.) Hence, the axiom of infinity is neither false nor dubitable, on Wittgenstein's view, so much as it simply tries to *say* what may only be *shown* in an infinitary logical notation. Contrast this with what Wittgenstein has to say about Russell's axiom of reducibility (TLP 6.1232), which he criticizes on the grounds that it is a contingent, rather than a logical proposition.

Finally, what of Wittgenstein's remark at TLP 5.55 that "we are unable to give the number of names with different meanings"? Does this not show that since the number of names with different meanings is unspecifiable, the number of objects is likewise unspecifiable? And, thus, that the number of names and so objects is not infinite, as I have suggested, but rather open and indeterminate as has been suggested by Soames, McGray, and others? On the contrary, if the number of names cannot be specified and so is open and indeterminate, it must be possible for the number of names to be infinite. For it to be possible that there be an infinite number of names with different meanings, there must be an infinite domain of objects to ensure that possibility. It is notable that Wittgenstein does not mention the word "object" in TLP 5.55, and nor does he say that we cannot give the number of objects. What he says is that we cannot give the number of different names and this is because, thanks to the subsistence of an infinite domain of objects, there is an endless, open possibility of producing new names. We thus cannot give their specific number, and nor can we give the specific composition of all possible elementary propositions. This is because, given the infinite domain of objects it will always be possible to produce endlessly new forms.

4. Implied Transfinite Cardinalities

Since we know that in Wittgenstein's system, there are an infinite number of objects, names of each of which may be substituted into any monadic propositional function, the totality of possible substitution instances of any monadic propositional function must therefore constitute an infinite sub-selection of the infinite totality of all possible substitution instances of all monadic propositional functions. The totality of all possible substitution instances of *all* monadic propositional functions must also constitute an infinite sub-selection of the infinite totality of all possible substitution instances of all propositional functions of whatever arity. None of these totalities can be finite, since they each involve an infinite number of possible substitution instances. But some of these selections nevertheless contain members which other selections do not. Thus, just as the even numbers constitute an infinitely large sub-selection of the natural numbers while having the same cardinality as the natural numbers, and prime numbers constitute an infinitely large sub-selection of the natural numbers while having the same cardinality as both the natural and the even numbers (see Steinhart 2009, 156–58, Bostock 2012, 25–26, and Russell 1919, 80), the infinite totality of all possible substitution instances of monadic propositional functions has the same cardinality as both the infinite totality of possible substitution instances of any monadic propositional function, as well as the infinite totality of all possible substitution instances of all propositional functions of whatever arity. Moreover, if you add each of these infinite totalities together, the result is an infinite totality which is the same size as each of the three individually, specifically, $\aleph_0$. This reflects the feature of transfinite arithmetic that, if we add the ordinal number ω of each of these totalities together, we get a totality which has the same cardinality, $\aleph_0$, as we would get by adding 1 to ω, 147 to ω, or if we simply left it alone (see Steinhart 2009, 168, Bostock 2012, 25–26). Addition does not increase the size of any transfinite ordinal any more than multiplication or exponentiation does.

Given what was said in Section 3, we know that the domain of Tractarian objects must be infinite. But why should the cardinality of that domain be $\aleph_0$ and not, say, $\aleph_1$? We know that the number of Tractarian objects cannot be $\aleph_1$ because $\aleph_1$ is the number of points on a continuous line, or in a continuous space. A continuous line or space is "dense" in that it is divisible without limit. However, Tractarian logical space cannot be divisible without limit because the Tractarian objects which compose its scaffolding are indivisible. We saw above in Section 3 that Tractarian logical space must be infinitely divisible, but since it is made up of indivisible objects, it cannot be continuous. This implies that the analysis of actual and possible facts in logical space will terminate at a level on which logical space, though infinitely divided, cannot be further divided. On the plausible assumption that Tractarian objects are akin to geometrical, space time points,[7] the infinity of points (objects) which make up logical space must therefore be denumerable. I will say a bit more on Wittgenstein's views about the continuum, and about Cantor's continuum hypothesis in Section 5, but for now the important point is that if the transfinite cardinality of Tractarian objects must be one or the other, it must be that of the natural as opposed to the real numbers. The cardinality of the set of real numbers is equal to that of the continuum, and while the continuum is "dense" in the sense of being divisible without limit, configurations of objects within Tractarian logical space are not.

In the case of all elementary propositions of the form fx, each substitution instance of a monadic propositional function at the elementary level can thus be paired off with a natural number, in a one-to-one relation. No matter how many substitution instances we add to this list, provided it is an infinite list, it will remain the same size, and it will be countable *via* a list of natural numbers which has 1 at the beginning and the limit ordinal ω at the end. If we wanted to construct a truth-table listing all possible truth-functions of all elementary propositions of the form fx, we could start by listing each of the infinite totality of elementary propositions of this form in columns on the left hand side of our table like so:

fa	fb	fc	...	fω
T	T	T	...	T
T	T	T	...	F
T	T	T	...	T
T	T	T	...	F
.	.	.	...	.
.	.	.	...	.
.	.	.	...	.
F	F	F	...	F

We could then pair each column up with one of the natural numbers 1- ω as follows:

1	2	3	...	ω
fa	fb	fc	...	fω
T	T	T	...	T
T	T	T	...	F
T	T	T	...	T
T	T	T	...	F
.	.	.	...	.
.	.	.	...	.
.	.	.	...	.
F	F	F	...	F

The cardinality of this list would then be $\aleph_0$. Theoretically, we could then list all other elementary propositions, of whatever logical form, or arity, along the top of the left-hand side of our truth-table, without increasing the size of the total number of elementary propositions listed. In this sense, assigning propositions to the cells at the top of columns on the left hand side of our table would involve a process very similar to that which characterized the thought experiment of the "Hilbert Hotel" (see Steinhart 2009, 158–59), which is a fully occupied hotel with an

[7] See Bizarro (2010), Connelly (2015) and Eisenthal (2018) for more thorough explication, discussion and defense of this idea, and of related "Hertzian" themes and influences inherent in Wittgenstein's early philosophy.

infinite number of rooms, but new busses show up with new infinite totalities of guests and so we make room for them by emptying rooms corresponding to infinitely large subselections of the rooms (e. g., we might empty all even numbered rooms, or all prime numbered rooms).

At a first blush, the procedure I am attempting to explicate here might be dismissed out of hand as a blatant violation of Cantor's theorem. Cantor (1895/1995) proved that you cannot list all of the natural numbers and nor, *a fortiori*, can you list all of their subsets (see Bostock 2012, 20–22). Does not the idea of a truth-table of the sort I am describing here presuppose the (incoherent) notion that one might actually write down a series equal in length to the natural numbers, and to a set of all subsets of the natural numbers? Well, yes. Assuming a reduction of quantified propositions to truth-functions of elementary propositions of the sort sketched in Section 2, moreover, it is also contrary to Church's theorem as well, since if that reduction went through, and we could write down all truth-functions of elementary propositions in a table, we would have a decision procedure for predicate logic. That, however, is just what Church (1936) showed to be impossible. But in a sense, that is the whole point of what I am arguing in this section of the paper: Wittgenstein's system involves commitment to transfinite cardinalities which he, in turn, problematically construes as applying to in principle enumerable, but nevertheless transfinite totalities. Again, Wittgenstein wants to be able to adhere to a conception of the actual infinite which combines the two agreeable, but mutually incompatible properties of being endless but limited, where actually infinite lists which are *limited* may be treated as if they were *terminal* (just as we might erroneously treat the *limit* of an endless sum as if it were the *terminus* of the sum).[8] That is the only way he saw it as possible for logic to come out as a decision procedure. And from his perspective, writing long before Church proved otherwise, it was obvious that it was one: "One can calculate whether a proposition belongs to logic, by calculating the logical properties of the symbol" (TLP 6.126).

On Wittgenstein's view, the propositions of logic are all, essentially, tautologies. Any valid argument can be put in the form of a "corresponding material conditional" (Bergmann, Moor and Nelson 2014, 100–1) which would be a logical proposition and thus a tautology. And if it were a tautology, then it would have to be possible in principle to symbolically calculate and thereby prove that it was. Perhaps fallible, finite creatures such as human beings could not conceivably write out the relevant truth-table, which proves quantificational validity, in its entirety. But from Wittgenstein's Tractarian perspective, it does not follow that the table is in principle impossible to write out, check, or comprehend. A sufficiently powerful and long-lived God, supercomputer, or temporally un-situated and so timeless "metaphysical subject" (TLP 5.641) might readily do so. Logic is not contingent upon human psychology, and thus neither is the information encoded in Tractarian truth-tables.

From the perspective of the author of the *Tractatus*, moreover, the information contained in such a table, though infinite, is limited in a way which allows us to treat it as enumerable. Thus, checking such a truth-table would not amount to completing an infinite task in a finite amount of time, but rather to completing an infinite but limited task in an infinite, but limited amount of time. Human beings can use Russellian quantifiers and other short-

[8] To conceive of or intuit the conception of endless, but limited totalities of the sort I am attributing to Wittgenstein, the reader may find it helpful to think of M. C. Escher's "Circle Limit" artworks based on the Poincaré disk model of the hyperbolic plane (Dunham 2010). The diagrams in these artworks have an outer limit, but there is always an infinite distance between the center of the diagram and the limit, regardless of where the center is stipulated to be. If an imaginary subject located within the picture tried to travel from the center to the outer limit of the circle, no matter how far they travelled from their original location they would never reach the limit and would always remain located in the center of the space. So in one sense the space is endless, but in another it comprises a bounded totality.

hand techniques to encode this information. But the information has to be truth-functionally determinate in the first place in order for it to be possible to encode it. From Wittgenstein's Tractarian perspective, if the truth-functional expansions corresponding to quantifiers do not terminate, then they do not convey any truth-functionally determinate information and nothing does, or does not follow from them. If you cannot write down the truth-conditions of a sentence in a table, then it is not an intelligible sentence with sense. If something is not an intelligible sentence with sense then it cannot be entailed by anything else.

At this point, it is important to recall, however, what was said in Section 2. First, we are not claiming that the Tractarian system actually works. We are merely trying to indicate how one might attempt to construct a transfinite number line within that that system given what we know about it, and to show why it would require such a construction to be possible. Second, recall that Wittgenstein identified muddling up finite and infinite lists as "a most important mistake" (Stern, Rogers and Citron 2016, 216) in the *Tractatus*. If we are going to faithfully represent Wittgenstein's system, we are going to have to provisionally make that same mistake here too.

In any case, undertaking this allocation of cells along the top of the columns on the left-hand side of our truth-table may require a bit more ingenuity than is involved in rearranging guests in rooms where each is assigned a natural number, but, granting the (mistaken) idea that such lists are possible, there is no reason why it could not be done in principle. In principle, within the Tractarian perspective, any ordered, infinite list of elementary propositions we might come up with could be paired up one-to-one with the infinite series of natural numbers that begins with 1 and ends with the limit ordinal ω. This means that ω is the limit of the natural number series, though it is not the immediate successor of any natural number in the series. It thus limits the natural number series though it has no unique, immediate predecessor and is not itself a finite natural number. In Wittgenstein's system, just as the "totality of objects" limits empirical reality without being finite, ω limits the natural numbers without being finite.

By an argument straightforwardly analogous to Cantorian diagonalization, however, we could then show that the number of truth-possibilities, and truth-functions of those elementary propositions must each have larger, transfinite cardinalities. The truth-possibilities of any list of elementary propositions cannot be paired off one-to-one with those elementary propositions, because no matter how many elementary propositions we have, they will always stand in a one-to-many relationship to their truth-possibilities. This one-to-many relationship is defined by the operation 2^n, where n = the number of elementary propositions. Something similar is true of the possible truth-functions of any number of elementary propositions. This number we may arrive at by raising 2 to the power of m, where m= the number of truth-possibilities. Since $m = 2^n$, the number of truth-functions of a selection of elementary propositions is simply the exponent of an operation which raises 2 to an exponent. First 2 is raised to the power of n, and then 2 is raised to the power of m, where m is the result of raising 2 to the power of n.

When Wittgenstein describes these one-to-many relations of relative cardinality first at TLP 4.27–28 and then at TLP 4.42, it is notable that he expresses them as summations, yielding the number of possible combinations of truth-values, and thus truth-possibilities and truth-functions, of n elementary propositions.[9] In each of TLP 4.27 and TLP 4.42, Wittgenstein means to reference the binomial theorem (Brauldi 2009, 130), and to use it to calculate the values K_n and L_n which are, respectively, the number of truth-possibilities and truth-functions of n elementary propositions. At TLP 4.27, the value K_n is defined by the following summation:

[9]Credit is due to my student Jeremiah Cashore for his assistance in working out important philosophical and mathematical details in this section dealing with Wittgenstein's $\sum$ notation, and with helping to create the required mathematical expressions.

$$K_n = \sum_{v=0}^{n} \binom{n}{v}$$

What this says is that, in order to obtain the number of truth-possibilities of n elementary propositions we may sum up the binominal coefficients (Brauldi 2009, 127) that are solutions to the equation $K_n = \frac{n!}{v!(n-v)!}$, where v ranges from v=0 (lower limit) to v=n (upper limit). The notation $\binom{n}{v}$ which occurs within the formula above is read "n choose v". By summing up the binomial coefficients which are solutions to the above equation we arrive at the number of possible ways of choosing v positive facts from n states of affairs; v is the number of positive facts within a given combination of n negative and/or positive states of affairs (or atomic facts). Because such combinations are isomorphic to the truth-possibilities of elementary propositions, the above summation also yields the number of truth-possibilities of n elementary propositions. v corresponds to the number of "T's" within a selection of n truth-values that makes up either a truth-possibility or a truth-function of elementary propositions (e. g., for (T, F, T, T) v=3). Expanding this summation yields

$$K_n = \left(\frac{n!}{v!(n-v)!}\right)_{v=0} + \left(\frac{n!}{v!(n-v)!}\right)_{v=1} + \left(\frac{n!}{v!(n-v)!}\right)_{v=2} + \ldots + \left(\frac{n!}{v!(n-v)!}\right)_{v=n}$$

Building on the value K_n, L_n is then the number of truth-functions of n elementary propositions, and is defined by the following summation given at TLP 4.42:

$$L_n = \sum_{k=0}^{K_n} \binom{K_n}{k}$$

What this summation says is that, in order to obtain the number of truth-functions of n elementary propositions, we may sum up the binomial coefficients which are solutions to the equation $L_n = \frac{K_n!}{k!(K_n-k)!}$ where k ranges from k=0 (lower limit) to $k = K_n$ (upper limit). The expression $\binom{K_n}{k}$ is read "K_n choose k" where k is the number of cases on which a proposition agrees with the truth-possibilities of the elementary propositions of which it is a truth-function, and K_n is the number of such truth-possibilities. At TLP 5.101, Wittgenstein lists the 16 different ways (truth-functions) that propositions can agree or disagree with the truth possibilities of n elementary propositions, where n=2. Expanding the above summation yields

$$L_n = \left(\frac{K_n!}{k!(K_n-k)!}\right)_{k=0} + \left(\frac{K_n!}{k!(K_n-k)!}\right)_{k=1} + \left(\frac{K_n!}{k!(K_n-k)!}\right)_{k=2} + \ldots + \left(\frac{K_n!}{k!(K_n-k)!}\right)_{k=K_n}$$

Each of these two summations can be illustrated in the following table, which corresponds to the infinite array known as Pascal's triangle (compare Brauldi 2009, 127-28):

n/K_n \ v/k	0	1	2	3	4	5	…	Σ
0	1							1
1	1	1						2
2	1	2	1					4
3	1	3	3	1				8
4	1	4	6	4	1			16
5	1	5	10	10	5	1		32
⋮	⋮	⋮	⋮	⋮	⋮	…	⋱	⋮

When calculating K_n, we take any value for n listed vertically in the column furthest to the left, and we pair it with each of the values for v listed horizontally along the top row, from v=0 to v=n. Taking each of these pairs as values for n and v within the binomial theorem yields a series of coefficients which are listed in the row beside any given n for each v. Summing them up gives us K_n in the column furthest to the right on the same row as our chosen value for n. If n=2, for example then $K_n = 4$.

When calculating L_n, in turn, we take a value of those listed vertically in the column furthest to the left as a value for K_n, and pair it up with each of the values listed along the top row for k, from k=0 to $k = K_n$. Applying the binomial theorem to each pair we produce a series of coefficients listed alongside K_n with one for each k. Summing these coefficients yields L_n, which is listed in the column furthest to the right along the same row as our chosen value for K_n. For example, if $K_n = 4$ then $L_n = 16$. Note that $\sum$ always yields a power of 2. This means that while 5 is a possible value for n it is not a possible value for K_n.

In any case, if Wittgenstein's point is just to help us calculate how many rows we need on our truth-table, and how many truth-functions we can generate out of a selection of elementary propositions, then why not just say that the number of truth-possibilities can be found *via* the formula 2^n, and that the number of truth-functions of those elementary propositions may be found *via* the formula 2^m where $m = 2^n$ (compare Schroeder 2006, 63–65, Black 1964, 215)? Over and above helping us to make these determinations, by framing these calculations of summations of binomial coefficients, Wittgenstein means to draw attention to the possibility of using the binomial theorem to calculate the cardinalities of selections of digital sequences or combinations. He thereby means, in part, to draw attention to the fact that the operation specified in his $\sum$ notation is structurally analagous to the diagonal procedure that Cantor (1891) uses to prove the existence of transfinite cardinalities, and may thus be used to calculate them. Cantor's procedure (1891, 920–21) works by encoding sequences of linear coordinates $a_{\mu,\nu}$ (e. g., $a_{1,\,1}$, $a_{1,\,2}$, $a_{1,\nu}$, . . .) using one of two characters m or w. Within this digital code, there will be one infinite sequence containing all m's (i. e., $E^{\mathrm{I}} = (m, m, m, m, \ldots\ )$), another containing all w's (i. e., $E^{\mathrm{II}} = (w, w, w, w, \ldots\ )$), and the remainder will consist of some mixture of m's and w's (e. g., $E^{\mathrm{III}} = (m, w, m, w, \ldots\ )$). As noted by Ferreirós (2007, 287–88), this procedure encodes the set of all subsets of coordinates within the linear continuum, since each infinite sequence of m's, w's, or a mixture of each, encodes a sub-set of such coordinates. Arbitrarily, we can think of m as indicating that the corresponding coordinate does occur in the sequence, and w as indicating that the corresponding coordinate does not occur. The infinite sequence E^{I} then encodes the set of all coordinates, E^{II} encodes the empty set, and E^{III} encodes a sub-set which contains every other coordinate but not those in between. Cantor's diagonal argument then shows that the cardinality of the set of all subsets of coordinates is larger than the set of coordinates itself. In this and in other cases, the power set of a set, which consists of all of a set's subsets, has a larger cardinality than the set itself. More specifically, if the cardinality of the set is n, then the cardinality of its power set will be 2^n. If n is a transfinite number, then the cardinality of the power set is going to be a transfinite number which is larger than n, and more specifically equal to 2^n.

Obviously, nothing hinges on whether we use the characters "m" and "w" for our digital code, "0" and "1", or "T" and "F". By using the $\sum$ notation to calculate the cardinalities of selections of sequences of T's and F's, Wittgenstein means to highlight the fact that there are more subselections of any such sequence than there are members of the sequence. More specifically, there are 2^n more subselections of any such sequence than there are members of the sequence. Thus, for n states of affairs there will be 2^n ways that such states of affairs can consist of positive and/or negative facts. These correspond to the number of truth-possibilities of n elementary propositions, since each elementary proposition can either be T, or F. If there are $2^n (= K^n)$ truth-possibilities of elementary propositions, then there will be $2^{2^n} (= L^n)$ ways that a proposition can agree or disagree with those truth-possibilities, and thus 2^{2^n} truth-functions of those elementary propositions. If n is a transfinite number, then K_n will be a transfinite number which is larger than n by a power of 2^n, and L_n will be a transfinite number which is larger by a power of 2^{2^n}.

In his 1913 "Notes on Logic", Wittgenstein deploys a notation in which "a" and "b" are used in place of "T" and "F", and building on this notation will help us to draw out the way in which binomial expansions encode the information contained in Tractarian truth-tables. The binomial expansion for $(a + b)^n$ encodes the truth-possibilities for n elementary propositions, while the binomial expansion for $(a + b)^{K_n}$ encodes the truth-functions of n elementary propositions. If n=2, for example, we calculate the binomial expansion for $(a + b)^2$ to reveal the truth-possibilities for 2 elementary propositions, while we calculate the binomial expansion for $(a + b)^4$ to reveal the truth-functions of 2 elementary propositions. The binomial expansion for $(a + b)^2$ is $(a^2 + 2ab + b^2)$, in which the first term a^2 corresponds to the one truth-possibility on which both elementary propositions are true, while the second term corresponds to the 2 truth-possibilities on which one elementary proposition is true and the other is false, while the third term corresponds to the one case on which both elementary propositions are false. The binomial expansion for $(a + b)^4$ is $(a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4)$, in which the first term corresponds to the one truth-function of two elementary propositions on which all truth-arguments agree with its four truth-possibilities, the second term corresponds to the four cases on which three truth-arguments agree with its truth-possibilities and one does not, the third term corresponds to the six cases on which two truth-arguments agree and two do not agree, the fourth term corresponds to the four cases one which one truth-argument agrees and three do not, and finally, the fifth term corresponds to the one case on which all four truth-arguments disagree with the truth-possibilities of the elementary propositions. Binomial expansions will work in this way to encode the truth-possibilities and truth-functions of n elementary propositions for any, arbitrarily large n. Summing the coefficients of these binomial expansions will always yield the number of truth-possibilities (K_n) or truth-functions (L_n) for n elementary propositions. In accordance with Pascal's triangle these numbers will always be powers of 2^n.

One reason why Wittgenstein might have been led to "muddle up" the idea of a sum with that of the limit of a sum is that regardless of how long they are, binomial expansions always end with a final term, whose coefficient is 1. Though Pascal's triangle is an infinite array, it is also an array which is limited by the number "1" on each end of every row. Thinking in terms of truth-tables, each "1" at the end of a row on Pascal's triangle corresponds either to the case of tautology or contradiction. No matter how large n is, there is always only going to be one case on which all n are true, and only one case on which all n are false. (Analogously, in Cantor's digital code there will only be one case in which all coordinates are contained in the set $(m, m, m, m, \ldots)$ and only one case in which no coordinates are contained in the set $(w, w, w, w, \ldots)$). This may go some way to explaining why Wittgenstein characterizes logical space as an "infinite whole" at TLP 4.463, but describes the world as a "limited whole" at TLP 6.45.

The main point, for our present purposes, is simply that n, K_n, and L_n stand to one another in iteratively defined, one-to-many relations. If you have two elementary propositions (n), then you have four possible ordered pairs of truth-possibilities (K_n), and sixteen possible truth-functions (L_n). Moreover, if n is an infinite totality, then K_n, and L_n must each have cardinalities that are transfinite. The transfinite cardinality of K_n will be arrived at by applying a power operation to n, and the (even larger) transfinite cardinality of L_n will be arrived at by applying the same power operation to K_n. More specifically, K_n will be equal to 2^n, and L_n will be equal to 2^{K_n}.

5. Transfinite Number and the General Form of a Number

In Section 2, we saw that Wittgenstein sought to generate all truth-functions of elementary propositions *via* successive applications of the general form of a truth-function. The general form of a truth-function, as we saw, was also identified as the general form of a proposition at TLP 6. Just a few remarks later at TLP

6.02, Wittgenstein goes on to highlight an inner connection or internal relation between the general form of a truth-function and the general form of a number. According to Wittgenstein's analysis, the natural numbers are to be defined as exponents of an operation that shares the same, basic, iterative structure as N. Just as N generates all truth-functions of elementary propositions, by taking elementary propositions as bases and yielding N-expressed truth-functions as results of successive applications of N, Wittgenstein's ω operation, defined at 6.02, takes 0 as a base and yields the progression of all natural numbers as exponents, or iterations of that operation.[10]

Notably, however, Wittgenstein insists at TLP 6.022 that "the concept of number is simply what is common to all numbers, the general form of number". At 6.03, moreover, he singles out the general form of integers, or whole numbers, as an *instance* of the general form of number. This is significant for us, since it suggests that Wittgenstein's characterization of whole numbers as exponents of an operation may be extensible to other sorts of numbers, such as transfinite numbers. Since we saw above that Wittgenstein's N-operator, and truth-table notations implied that infinite, and transfinite cardinalities belong to totalities such as those of elementary propositions, truth-possibilities, and truth-functions, such an extension would both support, and cohere with Wittgenstein's overall logical and semantic program.

Following von Neumann, for instance, we might define the first transfinite number, the limit ordinal ω, not as a *member* of the natural number series (say the last member), but instead *as the series* of natural numbers in its totality, the series generated by applying the Ω operation first to 0 and then in turn to its successors *ad infinitum* (see Ferreirós 2007, 371–74, Steinhart 2009, 151–52, 163). In Section 2 we saw that within the Tractarian system, these two distinct conceptions of ω are conflated, and that

[10]For a much more detailed exposition of this construction, see Frascolla (1994, esp. 1–22), Frascolla (1997).

Wittgenstein later identified this as a "most important mistake". Yet granting Wittgenstein's mistaken, and ambiguous conception of ω for the sake of exposition, there is no reason it could not be subject to an approach analogous to that developed by von Neumann. Like 0 in the case of the natural number series, ω may then serve as the base for an operation which, applied iteratively, generates the series of transfinite numbers known as the Beth numbers (see Steinhart 2009, 175–76). In this case, the relevant operation will not simply be a successor operation but will be a power operation. Much as the number of truth-possibilities and truth-functions were given as exponents of an operation on a number of elementary propositions n, successive Beth numbers $\beth_1 - \beth_\omega$ may be obtained by iteratively applying a Cantorian power operation.

In the case of finite numbers, we saw that by raising 2 to the power of n, where n is the number of elementary propositions, we obtained the number of truth-possibilities of those elementary propositions, and that by raising 2 to the power of m, where $m = 2^n$, we obtained the number of truth-functions of those elementary propositions. Similarly, by raising 2 to the power of $\beth_0$, (which will be the cardinal number of ω, the first transfinite ordinal) we obtain the transfinite number $\beth_1$. While in the case of truth-tables, our base was 2 because there were 2 truth-values, in this case our base is 2 because, with regards to any member of the series of natural numbers which we have defined ω as, we must decide "yes" or "no" with regards to whether to select it in any particular sub-selection of the natural numbers, when defining $\beth_1$ as the number of all such subselections (Steinhart 2009, 169).

As we saw above in Section 4, Cantor (1891) devised a digital code in which subsets of linear coordinates were encoded by a sequence of *m*'s and *w*'s, which we can likewise think of as a series of yes/no choices. This method could also be applied to encode subsets of the natural numbers as infinitely, but countably long digital sequences. The operation specified in Wittgenstein's $\sum$

notation was designed, as we saw, to count the number of selections of digital sequences of a given power or size. For instance, as the number of truth-functions of 2 elementary propositions, $\sum$ tells us that there are 16 digital sequences of 4 entries each. Each of these sequences could easily be construed of as encoding a subset of a set of natural numbers. For instance, if we start with the set {1, 2, 3, 4}, then (F, F, F, F) encodes the empty set $\emptyset$, while (T, F, T, T) encodes the subset {1, 3, 4}, and (T, T, T, F) encodes the subset {1, 2, 3}. This same procedure could then be extended to the infinite sequence of natural numbers, subselections of which would be encoded by infinite digital sequences.

The operation specified in Wittgenstein's $\sum$ notation would thus involve something akin to, but subtly, and importantly distinct from a "quasi-combinatorial" conception (Bernays 1935, 259–60, Ferreirós 2011) of such numeric, and digitial sequences. While the quasi-combinatorial view conceives of such infinite sequences, or subsets of natural numbers as objects which exist independently of our attempts to encode them, Wittgenstein views the *possibility* of generating, and constructing such sequences, or selections of either natural numbers or digitial code as *subsisting* independently of our actual generation, or construction of them *via* series of iterative operations. Such possibilities are given along with the logical structure of the world, and subsist regardless of whether we exploit these possibilities to do mathematics, or not. Witggenstein is thus a Platonist not about numbers, but about *possibilities*. We saw this to an extent already in Section 3 where it was argued that Wittgenstein adheres to a fixed domain of objects that are "modally real" in the sense that they *subsist* in all possible worlds, and thereby make possible the existence therein, of any atomic facts composed of them. Anticipating Benacerraf (1965), Wittgenstein was apt to characterize numbers not as objects, but as iterations, or exponents of operations which embody abstract structures of a certain nature. In the case of the natural numbers, for example, Wittgenstein means successive application of his Ω operation to embody the abstract structure known as a "progression". Wittgenstein's construction of the natural numbers thus involves neither ur elements, nor sets, but only an operation with abstract, structural features which mirror the abstract structure of logical space.

At TLP 6.03, Wittgenstein provides a symbol meant to capture the general form of a progression and thus the general form of an integer (or natural number), specifically $[0, \xi, \xi + 1]$. Building on this notation, and on that which gives the general form of a proposition, the general form of transfinite number could be symbolized as $[\overline{\omega}, \overline{\xi}, \sum'(\overline{\xi})]$. Like $N'(\overline{\xi})$, $\sum'(\overline{\xi})$ would be an instance of the general form of an operation $\Omega'(\overline{\eta})$, in which the metavariable $(\overline{\eta})$ stands for a selection, or choice operation performed on some type of arguments η, whether they be numbers or elementary propositions (ω stands for the infinity totality of natural numbers while p stands for the infinite totality of elementary propositions). In the symbol for the general form of a transfinite number given above, $\overline{\omega}$, again, represents all digitally encoded subselections of natural numbers, $\overline{\xi}$ represents any arbitrary sub-selection of bases or results of $\sum(\overline{\xi})$, and $\sum$ represents an operation which calculates numbers of subselections of digital sequences which themselves encode subselections of natural numbers. Through successive applications of this procedure, $\sum$ could be used to calculate the number of subselections of natural numbers, yielding $\beth_1 (= 2^{\omega})$, then calculate the number of subselections of the original subselections, yielding $\beth_2 (= 2^{2^{\omega}})$, and so on.

The cardinality of the selection of all such subselections of natural numbers, $\beth_1$ will thus be greater than that of ω by a factor of $2^{\beth_0}$. By raising 2 to the power of $\beth_1$ we may then obtain $\beth_2$ and so on. $\beth_2$ will be the cardinality of the number of possible subselections, within the selection of all subselections of the natural numbers that we used to construct $\beth_1$. This new selection will be greater than the cardinality of the previous selection by a factor of $2^{\beth_1}$. By iteratively applying a power operation based on this procedure of diagonalization, which takes 2 as a base and a transfinite number as an exponent, we may obtain the infinite series of transfinite, Beth numbers (Steinhart 2009, 176). This construction will be structurally analogous to the one we followed when we obtained the number of truth-functions of elementary propositions, by applying an iterative operation which takes 2 as its base, and a (finite or infinite) number first of elementary propositions, and then of truth-possibilities, as an exponent. It will also cohere nicely with Wittgenstein's characterization of number as "the exponent of an operation" (TLP 6.021).

The reason to explicate the Beth series as opposed to the Aleph series, is that it nicely correlates with the structure, and results of the power operation $\sum(\overline{\xi})$, which can be constructed out of Wittgenstein's Σ and $\overline{\xi}$ operations. Notably, however, even though every Beth number is also an Aleph number, whether the Beth numbers are identical to the Aleph numbers, and just how we should go about correlating the Aleph series with the Beth series, are each outstanding philosophical and mathematical problems and controversies, associated with the so-called "continuum problem".[11] While it is well beyond the scope of this paper to fully explicate the problem, and to adjudicate the scholarly controversy surrounding it, it may be worthwhile to explain briefly what it is, and consider what it might imply about Wittgenstein's logical system. The "continuum problem" is simply the question of whether there is a transfinite number of intermediate size, or cardinality, between the cardinality of the natural numbers and that of the real numbers. (It is called the "continuum" problem because the real numbers are equivalent in size to the number of points on a continuous line.) In other words, just as we can ask whether the even numbers are equivalent in size to the natural numbers, we can ask of any infinite subset of the reals, whether it is equivalent in size to the totality of reals. If there is an infinite subset of the reals which is not equivalent in size to the reals, this would suggest the existence of a transfinite number intermediate in cardinality between the naturals and the reals. Cantor's continuum hypothesis just says that there is no such intermediate number. In other words, $2^{\aleph_0} = \aleph_1$. If the continuum hypothesis is true, then $\aleph_1$ is thus equivalent to $\beth_1$ ($\aleph_1 = \beth_1$). In its generalized form, the continuum hypothesis says that all Alephs share this same basic feature, and thus that the Aleph series is equivalent to the Beth series. Raising 2 to the power of any Aleph, according to the generalized continuum hypothesis, will always yield the next largest Aleph, just as was the case in our explication of the Beth series, above.

The construction of a transfinite number line in terms of the Beth series provided in the previous paragraph, is thus certainly consistent with the truth of Cantor's continuum hypothesis, though it seems irresponsible to represent Wittgenstein as taking any definitive stand on whether the continuum hypothesis is true. Possibly, like Hilbert he simply accepted it as a "plausible theorem... which no one has succeeded in proving" (1900, 1103). Had Wittgenstein undertaken a more robust treatment of transfinite number in the *Tractatus*, perhaps we would have a better sense of where he stood on this matter at the time. But unfortunately, and as noted above in Section 1 and by Russell in his introduction to TLP, he did not. It might help matters if we knew more about Russell's own views on the continuum problem, but that is a topic worthy of careful study in its own right, and which is beyond the scope of this already lengthy paper. Likewise, Wittgenstein's views on continuity and real numbers are themselves a topic worthy of careful, critical and expository research in their own right and, unfortunately, also beyond the scope of this already rather lengthy discussion.[12] If the continuum hypothesis turns out to be false, then we have still provided

[11] For a more detailed exposition of the continuum problem, see Steinhart (2009, 178) and Koellner (2016).

[12] See Rodych (2018) for further discussion of this rich, historically complex, and fascinating topic.

a way of constructing a transfinite number line that plausibly extends Wittgenstein's theory of number, and of operations, to the transfinite case. It will be recalled that the goal of the paper was not to show that Wittgenstein had solved the continuum problem, but that his system implicates transfinite cardinalities and contains ingredients which could plausibly be deployed to construct them.

Within his development of the Tractarian logical system, admittedly, Wittgenstein never explicitly identifies or explicates the Cantorian power operation which I have employed above to construct a transfinite number line. Indeed, it is possible to construe this as the very "lacuna" which Russell says it is not impossible for Wittgenstein to fill. Yet, Wittgenstein does identify at least two distinct operations which, as we have seen, could plausibly be deployed to implement such an operation, one of which is associated with his $\overline{\xi}$ notation, and the other of which is associated with his $\sum$ notation. The former *generates* selections of elementary propositions, while the latter *calculates* the number of truth-possibilities and truth-functions of such elementary propositions. The first of these operations generates all unordered subselections of elementary propositions, which may be presented to the N-operator in the construction of all meaningful propositions. In this respect, the $\overline{\xi}$ operation bears obvious affinities to a Cantorian power operation which generates the power set of a set, by collecting together all subsets of the set into a set.

Notice that, there are obviously significantly many more possible selections of elementary propositions for presentation to the N operator, than those which can be determined by any of the methods identified at TLP 5.501 (that is, either by 1) direct enumeration, or *via* 2) a propositional function, or 3) a formal law). For example, N could easily operate upon a random selection of elementary propositions whose corresponding propositional functions differ in structure and arity from one another, to generate a truth-function of those elementary propositions. Such a random selection might even be infinite in length, and could be characterized as an "arbitrary subselection" of elementary propositions analagous to an "arbitrary subset" of natural numbers of the sort presupposed by the quasi-combinatorial conception of a set embraced implicitly by Cantor, Dedekind, and others (Ferreirós 2011). Such subsets are "arbitrary" in the sense that they exist whether or not they are definable. Analogously, subselections of elementary propositions can be "arbitrary" in the sense that they lie within the range of $\overline{\xi}$ whether they share any defining feature or not. ξ is a propositional variable which takes selections of elementary propositions as its arguments, but over and above that, $\overline{\xi}$ could be construed as representing an operation which generates all subselections of the elementary propositions within the domain of ξ, whether those subselections share any defining feature or not. This fact partly explains why Wittgenstein insists at TLP 5.501, that how we go about determining or describing such subselections of elementary propositions is an "inessential" matter. His $\overline{\xi}$ operation ensures, in advance, that all possible selections of elementary propositions, whether definable or not, are available for the construction of any conceivable truth-function. The expressive completeness of N is thus supposed to be stipulated already within the $N(\overline{\xi})$ symbol contained within Wittgenstein's broader symbol for the general form of a proposition, or truth-function. (TLP 6). In other words, $\overline{\xi}$ is part of the operation specified by $N(\overline{\xi})$. In fact, Wittgenstein never actually mentions an "N" operation in the *Tractatus*, but does identify an $N'(\overline{\xi})$ operation at TLP 6.001.

The expressive completeness of $N(\overline{\xi})$ therefore has little to do with the technical feasibility of the illustrative methods identified at TLP 5.501, of selecting elementary propositions for presentation to N. These are merely specific instances of what is accomplished in general by the $\overline{\xi}$ operation, that are identified by Wittgenstein because they are of assistance, psychologically, in that they help the reader understand how to translate from N-expressions into Russellian notation.

As we have seen, moreover, the second of the two operations specified by Wittgenstein, associated with his $\sum$ notation (given at TLP 4.27 and 4.42), calculates the number of truth-possibilities and truth-functions of n elementary propositions by summing the binomial coefficients which correspond to ways that digital characters, such as "T's" and "F's", can be combined in sequences of a given length. What Cantor's diagonal argument shows, and what Wittgenstein's $\sum$ notation is designed to recover, is that there are always more subselections of such sequences than members of the sequence. Specifically, there will always be more by a power of 2^n, where n is the number of members in the sequence. As we have seen, this same operation could obviously be used to count digital sequences which encode selections, or subsets of the natural numbers. The number of such subsets of natural numbers would be greater than that of the natural numbers by a power of 2^n.

Once the general idea of how to integrate these two operations within an iterative, Cantorian power operation is in place, it becomes clear that Frascolla was right to characterize Ramsey as "misdirected" when the latter claimed that Wittgenstein's view of mathematics was "ridiculously narrow". Wittgenstein's philosophy of mathematics is thus not "narrow" so much as it is underdeveloped, relative to the ideal of an exhaustive, formal exposition. It is clear, however, that Wittgenstein did not aspire to provide such an exposition within the *Tractatus*. Again, the *Tractatus* is not a textbook so much as a treatise on logical philosophy, which aims to provide a general framework which it invites others to develop and flesh out, in some cases formally.

We cast Wittgenstein's operations as selection-theoretic as opposed to set-theoretic, simply because that is how Wittgenstein wants to frame the internal relations both between elementary propositions and truth-functions, as well as between numbers and their successors. Hence at TLP 6.031 Wittgenstein insists that set theory is "completely superfluous". He is emboldened to make this assertion, in part, because he thinks he can generate all truth-functions and all numbers iteratively, *via* distinct, but structurally analogous operations, each instances of the general form of an operation, and without reference to sets. Instead of an "axiom of the power set" (Ferreirós 2007, 322), Wittgenstein can appeal to a power operation which counts "power selections" of the natural numbers, the cardinalities of which are equivalent to transfinite, Beth numbers. The relative cardinalities of these Beth numbers are themselves, in turn, defined by an iterative power operation.

An added motivation Wittgenstein has to avoid appeal to sets is to be found in his *Grundgedanke*, or "fundamental thought" (TLP 4.032), according to which logical and mathematical propositions are purely formal and thus do not refer to, or represent relations between logical or mathematical objects (such as sets). Both logic and mathematics are, on this view, pure formal calculi and thus lack substantive content. Logical and mathematical propositions are true in virtue of their structure alone, and this explains their *a prioricity* as well as necessity.

6. Conclusion

In this paper I have endeavored to explore and address Russell's concern, that Wittgenstein's theory of number stands in need of further technical development specifically with respect to the case of transfinite number. Regarding this "lacuna" in Wittgenstein's system identified by Russell, I set out in pursuit of two distinct, but interrelated goals. The first was to show that Wittgenstein's philosophy of language and logic contains an implicit commitment to transfinite cardinalities, while the second was to sketch the general lines upon which one might extend Wittgenstein's theory of number to the transfinite case, given ingredients already present within his system. In Section 2, we probed Wittgenstein's extensional construal of language, delved into the mechanics of Wittgenstein's N-operator, and explored how language was supposed to be built up truth-functionally by successive applications of N. In Section 3, we then reflected on the size of the Tractarian domain of quantification and defended the claim that it should be construed as infinite. In Section 4, we then saw that Wittgenstein's conception of logic as embodied within internal relations of truth-preservation between atomic and molecular propositions displayed in truth-tables, involved an implied commitment to transfinite, ordinal numbers of increasing cardinality. In Section 5, we then observed the existence of an inner connection, or internal relation between the general form of a truth-function embodied in successive applications of Wittgenstein's N operator, and the general form of a number as embodied in successive iterations of Wittgenstein's Ω operator. We saw that this number line could then be extended into the realm of the transfinite by specifying a Cantorian power operation $\sum(\overline{\xi})$, that takes digitally encoded subselections of the natural numbers as its base $(\overline{\omega})$, and yields as its results, in succession, a series of ordinal numbers of higher, transfinite cardinalities $(\beth_1 - \beth_\omega)$.

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References

Black, Max, 1964. *A Companion to Wittgenstein's Tractatus*. Cambridge, UK: Cambridge University Press.

Bergmann, Merrie, James Moor and Jack Nelson, 2014. *The Logic Book*, Sixth Edition. New York, NY: McGraw-Hill.

Bizarro, Sara, 2010. "A Hertzian Interpretation of Wittgenstein's Tractatus." *Eidos* 13: 150–65.

Benacerraf, Paul and Hilary Putnam, eds., 1983. *In Philosophy of Mathematics: Selected Readings*, Second Edition. Cambridge, UK: Cambridge University Press.

Benacerraf, Paul, 1965. "What Numbers Could not Be." In Benacerraf and Putnam (1983), pp. 272–94.

Bernays, Paul, 1935. "On Platonism in Mathematics." In Benacerraf and Putnam (1983), pp. 258–71.

Bostock, David, 2012. *Russell's Logical Atomism*. Oxford, UK: Oxford University Press.

Brauldi, Richard A, 2009. *Introductory Combinatorics*, Fifth Edition. Upper Saddle River, NJ: Prentice-Hall.

Burgess, John. P, 2008. "Cats, Dogs, and so on." In *Oxford Studies in Metaphysics*, edited by Dean Zimmerman, pp. 4–56. Oxford, UK: Oxford University Press.

William Ewald, 1996. *From Kant to Hilbert: A Source Book in the Foundations of Mathematics, Volume 2*. Oxford, UK: Clarendon Press.

Cantor, Georg, 1891. "On an Elementary Question in the Theory of Manifolds." In William Ewald (1996), pp. 920–22.

———, 1895/1995. *Contributions to the Founding of the Theory of Transfinite Numbers*, edited by tr. P. E. B. Jourdain. New York, NY: Dover.

Connelly, James, 2015. *Wittgenstein and Early Analytic Semantics: Toward a Phenomenology of Truth*. Lanham, MD: Lexington Books.

———, 2017. "On Operator N and Wittgenstein's Logical Philosophy." *Journal for the History of Analytic Philosophy* 5(4): 1–26.

Church, Alonzo, 1936. "A Note on the Entscheidungsproblem." *Journal of Symbolic Logic* 1: 40–41.

Dunham, Douglas, 2010. M.C. Escher's Use of the Poincaré Models of Hyperbolic Geometry., http://www.math-art.eu/Documents/pdfs/Dunham.pdf, accessed August 2, 2020.

Eisenthal, Joshua, 2018. Models and Multiplicities: Logical Pictures in Hertz and Wittgenstein. Doctoral Dissertation, University of Pittsburgh.

Ferreirós, José, 2007. *Labyrinth of Thought: A History of Set Theory and Its Role in Modern Mathematics*, Second Revised Edition. Berlin: Birkhäuser Verlag.

———, 2011. "On Arbitrary Sets And ZFC." *The Bulletin of Symbolic Logic* 17(3): 361–93.

Fogelin, R. J., 1982. "Wittgenstein's Operator N." *Analysis* 42: 124–27.

———, 1987. *Wittgenstein*, Second Edition. London, UK: Routledge and Kegan Paul.

Frascolla, Pasquale, 1994. *Wittgenstein's Philosophy of Mathematics*. New York, NY: Routledge.

———, 1997. "The Tractatus System of Arithmetic." *Synthese* 112: 353–78.

———, 2017. "Wittgenstein's Early Philosophy of Mathematics." In *A Companion to Wittgenstein: Blackwell Companions to Philosophy*, edited by Hans-Johann Glock and John Hyman, pp. 305–18. Malden, MA: Wiley-Blackwell.

Geach, P. T., 1981. "Wittgenstein's Operator N." *Analysis* 41: 168–70.

———, 1982. "More on Wittgenstein's Operator N." *Analysis* 42: 127–28.

Hilbert, David, 1900. "From Mathematical Problems." In William Ewald (1996), pp. 1096–105.

King, John and Desmond Lee, 1980. *Wittgenstein's Lectures, Cambridge 1930-32*. Oxford, UK: Blackwell.

Koellner, Peter, 2016. The Continuum Hypothesis. The Stanford Encyclopedia of Philosophy, https://plato.stanford.edu/archives/win2016/entries/continuum-hypothesis/.

Landini, Gregory, 2007. *Wittgenstein's Apprenticeship with Russell*. Cambridge, UK: Cambridge University Press. Check this.

Li, Jing, 2018. "The Hidden Set-Theoretical Paradox of the Tractatus." *Philosophia* 46: 159–64.

McGray, James W., 2006. "The Power and the Limit of Wittgenstein's N Operator." *History and Philosophy of Logic* 27: 143–69.

Ramsey, Frank, 1923. "Review: Tractatus Logico-Philosophicus." *Mind* 32(128): 465–78

———, 1931. *The Foundations of Mathematics and other Logical Essays*. London, UK: Routledge and Keegan Paul.

Rodych, Victor, 2000. "Wittgenstein's Critique of Set Theory." *Southern Journal of Philosophy* 38(2): 281–319.

———, 2018. Wittgenstein's Philosophy of Mathematics. The Stanford Encyclopedia of Philosophy, https://plato.stanford.edu/archives/spr2018/entries/wittgenstein-mathematics/.

Rogers, Brian and Kai F. Wehmeier, 2012. "Tractarian First-Order Logic: Identity and the N-Operator." *The Review of Symbolic Logic* 5(4): 538–73.

Russell, Bertrand, 1919. *Introduction to Mathematical Philosophy*. New York, NY: Dover Publications.

Schroeder, Severin, 2006. *Wittgenstein: The Way out of the Fly Bottle*. Cambridge, UK: Polity Press.

Soames, Scott, 1983. "Generality, Truth-Functions, and Expressive Capacity in the Tractatus." *The Philosophical Review* 92(4): 573–89.

Steinhart, Eric, 2009. *More Precisely: The Math You Need to do Philosophy*. Peterborough, ON: Broadview Press.

Stern, David G., Brian Rogers and Gabriel Citron, eds., 2016. *Wittgenstein Letures, Cambridge 1930-33: From the Notes of G. E. Moore*. Cambridge, UK: Cambridge University Press.

Wittgenstein, Ludwig, 1974. *Tractatus Logico-Philosophicus, with an Introduction by Bertrand Russell*, translated by D. F. Pears and B. F. McGuiness. London and New York: Routledge.

———, 1978. *Remarks on the Foundations of Mathematics*, Revised Edition, edited by G. E. M. Anscombe, Rush Rhees and G. H. von Wright. Cambridge, MA: MIT Press.

———, 1995. *Cambridge Letters: Correspondence with Russell, Keynes, Moore, Ramsey, and Sraffa*, edited by Brian McGuiness and G. H. von Wright. Oxford, UK: Basil Blackwell.

———, 2005. *The Big Typescript: TS 213*, German-English Scholar's Edition, edited by C. Grant Luckhardt and Maximilian A. E. Aue. Malden, MA: Blackwell.