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## **Book Symposium: *Frege's Conception of Logic* by Patricia A. Blanchette**

Roy T. Cook

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## Book Symposium: *Frege's Conception of Logic*

Patricia A. Blanchette

Roy T. Cook

### 1. Introduction

I'll begin by noting that Professor Blanchette's *Frege's Logic* is a substantial and important contribution to our understanding of Frege's views on logic. In the book, Blanchette focuses on two central interpretative tasks. The first is to provide a detailed explication of Frege's understanding of conceptual analysis, and of how this notion plays out in the development of his logicism. The second is to work out the details of Frege's logic and, perhaps more importantly, to work out in detail his views on the philosophical and mathematical role that logic should play in foundational work on mathematics. In addition to investigating these topics individually, Blanchette rather convincingly shows that there exist deep interconnections between these two superficially rather distinct issues, and then uses these insights to shed light on a number of important questions in Frege scholarship.

Although this is neither the time nor place to attempt a summary of all of the insights to be found in Blanchette's work, I will recommend one part of the text as particularly interesting and insightful for practitioners of logic and historians of the same. In Chapters 5 — 7 of the text Blanchette undertakes a detailed analysis of Frege's views on metamathematics, consistency proofs, and the manner in which these issues play out in the famous Frege-Hilbert correspondence. Out of context it is all too easy to read Frege's disagreement with Hilbert as a mere confusion, an irrational rejection of metatheory tout court, or of his clinging irra-

tionally to an out-dated, old-fashioned understanding of logical truth, logical entailment, and contradiction. In these chapters, however, Blanchette digs deep into the primary texts and demonstrates convincingly that, on the contrary, Frege's disagreement with Hilbert regarding the role of (what were, in effect) model-theoretic consistency proofs stemmed from a deeper disagreement regarding the objects and purposes of logic itself. Of course, historically speaking, the Hilbertian/Tarskian model-theoretic approach to consistency seems to have won the day. But Blanchette's discussion of these issues suggests that it might be worthwhile to take a second look at abandoned (Fregean) alternatives.

I'll also note, rather happily, that I think that the vast majority of what Blanchette has to say about Frege's views on conceptual analysis, logic, and metalogic is correct. But the purpose of this symposium is author-meets critics session, not author-meets-fawning-Fregean-fanboy, and so it would be rather inappropriate if I didn't have anything critical to say. And of course I do. There is one aspect of Blanchette's understanding and explication of Frege's views on logic that I strongly disagree with — the claim that Frege allowed for varying domains in the logics of *Grundgesetze*.

### 2. Frege and Quantification

In Chapter 3, titled "Thoughts and Sharp Boundaries", Blanchette examines Frege's controversial claim that every:

... first-level function of one argument must always be such as to yield an object as its value, whatever object we may take as its argument... (Frege 2013: Vol. II, §63)

Blanchette carefully distinguishes between two claims that Frege might be making here. First, we might interpret Frege as claiming that any function expression must be defined on any argument

whatsoever—that is, that every function (or, at the very least, every function named by a function symbol in the formal language of *Grundgesetze*) must be defined on every object whatsoever. Second, we might interpret Frege as claiming merely that:

... for a given language ... each of its well-formed concatenations of symbols has a determinate reference. (Blanchette 2012: 60)

On this reading, in order to add a function symbol to a language, we need not determine the value that the function symbol takes on all possible arguments, but merely on those arguments that fall in the range of the quantifiers of that language (of course, if we ‘expand’ the domain of the language, we shall have to ‘expand’ the range of the function symbols as well). Blanchette calls the former, stronger, reading the *totality requirement*, and the latter, weaker interpretation the *linguistic completeness requirement*.

Frege’s endorsement (or not) of the principle that Blanchette calls the *totality requirement* is often run together with the question of whether Frege understood the quantifiers of the logic of *Grundgesetze* to be absolutely general (where the quantifiers must be understood as always ranging over absolutely every object whatsoever), or whether Frege allowed a varying domain understanding of his logic (where different ‘applications’ might involve different non-universal domains). Nevertheless, it is important to notice that acceptance of the *totality requirement* and acceptance of an absolutely general reading of the quantifiers are not equivalent. Of course, if the quantifiers of *Grundgesetze* are (and must be) absolutely general, then this (plus Frege’s insistence that all functions—including, notably, logical operators—are total) entails the *totality requirement*. But the converse does not hold: Frege could conceivably have required that any legitimate function be defined on all possible arguments—perhaps, as a means to securing a completely determinate, domain-independent *sense*<sup>1</sup> for the corre-

sponding function symbols—as a pre-requisite to the use of such functions on any domain, even on the varying domains interpretation of *Grundgesetze*. We shall return to this possibility a bit later.

Blanchette spends the majority of this chapter arguing that nothing in the Fregean corpus forces us to accept the *totality requirement*, and that instead we can do justice both to Frege’s actual arguments and to (our reconstructions of) his reasons by adopting only the *linguistic completeness* requirement. The discussion is wide-ranging, covering Frege’s views on piece-meal definitions, the Caesar Problem, quantification, and our use of function symbols in everyday (informal) discourse. Blanchette’s argument is compelling, insofar as it attempts to show that none of these considerations forces Frege (or us, as Frege interpreters) to accept the *totality requirement* (although see Kai Wehmeier’s contribution to this symposium for an opposing view!). But there is at least one other consideration, not taken into account by Blanchette, that does weigh heavily in favor of interpreting the formal system of *Grundgesetze* as requiring acceptance of the *totality requirement*, since this consideration weighs heavily in favor of reading the quantifiers of *Grundgesetze* as always absolutely general. This additional consideration stems from the fact that the technical details of *Grundgesetze* have some rather odd and counterintuitive consequences unless the domain of discourse is constant (and hence absolutely unrestricted), and unless function symbols are always defined for every argument. But these consequences are not merely odd and counterintuitive. In addition, I will suggest that they go against the very nature of the project undertaken in *Grundgesetze*, and as a result force us to adopt an absolutely general reading of the quantifiers and hence accept the *totality requirement* on Frege’s behalf. In order to see the point clearly, it will be helpful to work through a simple example.

Frege’s most important philosophical analyses —that is, those codified in definitions within *Grundgesetze*— reduce the concepts in question to complex constructions involving only logical notions. For example, Frege’s defines the ordered pair of  $\alpha$  and  $\beta$  —that is, “ $\alpha; \beta$ ” in the language of *Grundgesetze*— as the value-range of the concept that holds of all and only the double value-ranges of those binary functions that map  $\alpha$  and  $\beta$  to the True (in that order).

$$\| \! - \dot{\varepsilon}(o \wedge (a \wedge \varepsilon)) = o; a \quad (\Xi)$$

We can gloss this definition in more familiar, modern notation as something like:

$$\langle \alpha, \beta \rangle =_{df} \{ \mathcal{F} : \mathcal{F}(\alpha, \beta) = \top \}$$

noting, however, that this modern formulation obscures the fact that, for Frege, the ordered pair is not a ‘collection’ of binary functions but is rather a collection of object-level surrogates (i.e. double value-ranges) of binary functions.

Before examining the problems that this definition poses for the weaker, *linguistic completeness* reading of the logic of *Grundgesetze*, it is worth highlighting a methodological issue lurking hereabouts. Frege often provides informal prose explications of his defined notions in terms of concepts and relations. As a result, these informal prose elucidations (as he warns us explicitly, more than once) are sometimes misleading, since the higher-order free variables (of both the Fraktur and Roman variety) of *Grundgesetze* range over unary and binary functions, not over concepts and relations (the latter are, of course, special cases of the former). As a result, Frege’s definitions have been almost uniformly misread and misunderstood. Contrary to popular belief, Frege’s mature *Grundgesetze* definition of cardinal number does not identify numbers with ‘collections’ of value-ranges of equinumerous concepts,

but rather with (more ‘inclusive’) ‘collections’ of value-ranges of functions (including concepts) that map equinumerous collections of objects to the True. Along similar lines, the ordered pair of  $\alpha$  and  $\beta$  is the ‘collection’ of all functions that map  $\alpha$  and  $\beta$  (in that order) to the True, and not the (less-inclusive) ‘collection’ of relations that relate  $\alpha$  to  $\beta$  (in that order).<sup>2</sup>

Returning to the issue at hand, consider what happens to the definition of ordered pair on the varying domains interpretation of the logic of *Grundgesetze*. Let  $\top$  be the True (i.e. the object to which any ‘correct’ *Grundgesetze* sentence, such as  $\ulcorner \alpha = \alpha \urcorner$ , refers) and  $\perp$  be the False (i.e. the object to which any ‘incorrect’ *Grundgesetze* sentence, such as  $\ulcorner \top = \alpha \urcorner$ , refers). For (the mature, *Grundgesetze*-era) Frege,  $\langle \top, \perp \rangle$  —that is:

$$(\ulcorner \alpha = \alpha \urcorner); (\ulcorner \top = \alpha \urcorner)$$

must be in any domain (since the term for this object is in any adequate formal language, or at the very least, is in any formal language containing all logical expressions). Applying definition  $\Xi$ , this means that:

$$\dot{\varepsilon}((\ulcorner \alpha = \alpha \urcorner) \wedge ((\ulcorner \top = \alpha \urcorner) \wedge \varepsilon))$$

must be in every domain. But, if the domain of quantification of *Grundgesetze* is allowed to vary, then the reference of the offset expression above — that is, the reference of the purely logical singular term “ $\langle \top, \perp \rangle$ ” —will vary from domain to domain, in virtue of the fact that the ‘collection’ of functions that map  $\top$  and  $\perp$  (in that order) to  $\top$  will vary from one such domain to another. Note that these is nothing special about this example other than its simplicity— on the varying domains approach the identity of the cardinal

numbers (i.e. which ‘collections’ of value-ranges of functions are identified with 1, 2, . . . ) will also vary from domain to domain.

This already seems to be in some tension with standard interpretations of Frege’s logicist project, since presumably the point of that project is to identify certain logical objects as *the* referents of everyday mathematical vocabulary (not *as one of a varying range of possible* referents of that vocabulary). But, of course, we need not mindlessly pledge our allegiance to standard interpretations. And Blanchette certainly does not do so. On the contrary, one of the themes of other portions of *Frege’s Logic* is that Frege allowed that a particular informal or semiformal mathematical concept (such as CARDINAL NUMBER or ORDERED PAIR) might allow for multiple, distinct formal conceptual analyses. For example, Frege might have defined the ordered pair of  $\alpha$  and  $\beta$  as the ‘collection’ of all functions that map  $\alpha$  and  $\beta$  to the False, rather than as the ‘collection’ of all functions that map  $\alpha$  and  $\beta$  to the True.

While nothing in Frege’s methodology rules out these alternative definitions of mathematical notions such as cardinal number and ordered pair, and Frege was surely aware of the resulting ‘arbitrariness’ in his definitions, it would be a mistake to conflate the following:

1. The fact (if it is such) that there are multiple constructions that could serve as the single correct definition of ordered pair.
2. The fact (if it is such) that, once such a unique construction is chosen to so serve, the definition in question might refer to different things in different contexts.

In short, just because we (and Frege) accept that there are other potential definitions that, if chosen, would have provided the referents of ordered pair singular terms in any context or domain, it does not follow that any such acceptable definition would allow

the referent of those terms to vary from context to context or domain to domain. The former claim seems unavoidable for Frege (although not *any* extensionally adequate definition will do). The latter, however, seems to saddle Frege with an unattractive (for him) sort of relativism: on such a reading, the identity of logical objects such as  $\langle \top, \perp \rangle$  will vary as the domain varies.

There is a response one might make here: one could argue that Frege doesn’t actually *need* “ $\langle \alpha, \beta \rangle$ ” to refer to the same object from context to context, or language to language, or domain to domain. Instead, one might argue that Frege only needs “ $\langle \top, \perp \rangle$ ” to make the ‘same’ contribution to the truth conditions of expressions in which it occurs in each language or context. Throughout *Frege’s Logic* we find a constant and consistent de-emphasis on the role of reference in Frege’s logicism in favor of an emphasis on truth conditions and inferential role, so such a move would certainly be in the spirit of Blanchette’s interpretation. But the consequences of adopting the varying domains interpretation, and rejecting both the absolute generality of the quantifiers of *Grundgesetze* and the totality requirement, should not be underestimated. It is not just that the reference of purely logical terms such as “ $\langle \top, \perp \rangle$ ” *might* vary from domain to domain. Rather, if we accept the varying domains picture, then it seems that the referent of such terms must vary (with the possible exception of the truth values), since *no purely logical object occurs in more than one distinct domain of objects!*

Let  $\Delta_1$  and  $\Delta_2$  be two distinct domains over which the quantifiers of the logic of *Grundgesetze* might range. Since  $\Delta_1$  and  $\Delta_2$  are distinct, there is at least one object  $\alpha$  such that  $\alpha \in \Delta_1$  and  $\alpha \notin \Delta_2$  (or vice versa). Since functions, for Frege, are total, it follows that no function on  $\Delta_1$  is identical to any function on  $\Delta_2$  (they have distinct domains: only functions on  $\Delta_2$  are defined on  $\alpha$ ). Now, the only logical objects in *Grundgesetze* (other than the truth values) are the

value-ranges, and these are individuated in terms of the identity of the functions whose value-ranges they are. Thus, no value-range in  $\Delta_1$  is identical to any value-range in  $\Delta_2$ . If, in addition, we take Frege's suggestion in *Grundgesetze* §10 seriously—that the truth values should be identified with their 'singleton's— then no logical object whatsoever, truth value or value-range, is in both  $\Delta_1$  and  $\Delta_2$ . As a result, we have the following consequences:

1. There are no domains  $\Delta_1$  and  $\Delta_2$  such that  $\Delta_1 \subseteq \Delta_2$ .
2. For any domains  $\Delta_1$  and  $\Delta_2$ , no logical object is in both  $\Delta_1$  and  $\Delta_2$ .
3. There is no universal domain  $\Delta_U$  containing all objects whatsoever.

Two further consequences of the varying domains interpretation of Frege's *Grundgesetze* are worth noting in a bit more detail.

First, if (as is informally suggested in *Grundgesetze* §10) truth values are value-ranges (in particular, are their own 'singleton's), then there is no unique pair of all-purpose truth values: *the* True and *the* False. Rather, each distinct domain  $\Delta$  has its own unique truth values  $\top_\Delta$  and  $\perp_\Delta$ , ontologically distinct from the truth values corresponding to any other domain. This saddles Frege with a very odd, and very un-Fregean-feeling, sort of truth pluralism. If each domain has its own truth values, then we are left wondering what true claims in one domain, and true claims in another, have in common. In short, the varying domains account fails to provide any *uniform* account of the nature of truth that explains truth in a non-discourse-relative manner.

Second, it should be noted that the third claim, regarding the impossibility of a universal domain, does not stem from any worries regarding the size, or 'indefinite extensibility,' etc., of the domain of all objects.<sup>3</sup> It is not that this domain, were it to exist,

would be too big or otherwise badly-behaved in terms of cardinality or the like. Rather, the non-existence of such a universal domain stems from the fact that logical objects existing in one domain can *only* exist in that domain, and in no other. Thus, if there exists more than one domain (and this, presumably, is the heart of the varying domains view), then there cannot be a universal domain.

This brings us to another point. As we have seen, for Frege the varying domains approach seems to entail that no logical object appears in more than one domain—rather, each distinct domain contains its own 'local' version of the truth values, the empty value-range (the Fregean analogue of the empty set), the universal value-range (the Fregean analogue of the universal set), the cardinal numbers, etc. But there is nothing to block consideration of what we might call the *non-logically universal domain*  $\Delta_{NLU}$ — that is, a domain of objects that contains all non-logical objects (even if it fails to contain many—even most— of the 'possible' logical objects).  $\Delta_{NLU}$  contains all non-logical objects, and for each logical object in any other domain  $\Delta$ ,  $\Delta_{NLU}$  would contain, not that logical object, but a surrogate of it (in fact, it would contain many such surrogates that behave like the logical object in question on the non-logical sub-domain shared by  $\Delta_{NLU}$  and  $\Delta$  but behave differently elsewhere). As a result, the truth values  $\top_{\Delta_{NLU}}$  and  $\perp_{\Delta_{NLU}}$  can be thought of as surrogates for the truth values in any other domain and, generalizing a bit, we can conclude that any construction in any domain  $\Delta$  has a 'surrogate' in  $\Delta_{NLU}$ .

Further, one of the recurring themes in Frege's reconstruction of arithmetic is that the numbers are completely general and applicable to everything—in short, everything that exists can be counted, and vice versa (and similar comments hold for the host of other mathematical notions Frege reconstructed, or planned to reconstruct, in *Grundgesetze*). But if this is right (and it is!), then it

looks like the cardinal numbers in  $\Delta_{NLU}$  —even if they are not, technically speaking, the ‘only’ cardinal numbers— have some claim to being the ‘genuine’ cardinal numbers (or, at the very least, the ‘best’ version of the cardinal numbers) since only the cardinal numbers in  $\Delta_{NLU}$  can be used to count every non-logical object (and additionally, to count surrogates for each logical object). But then this non-logically universal domain  $\Delta_{NLU}$  is, even if not the only domain over which the quantifiers of *Grundgesetze* can range, a particularly privileged one, and is arguably the domain within which Frege’s reconstruction of arithmetic and real analysis ought to take place. Given that Frege’s primary goal was to carry out this reconstruction (regardless of what purposes we might have put his work to since), this observation takes much of the ‘oomph’ out of the varying-interpretation reading of *Grundgesetze*.

Of course, the past few paragraphs depend on their being no overlap between the logical objects in one domain and the logical objects in another domain. And the informal ‘proof’ of this fact, given above, depends on our individuating functions ‘internally’ —a function was understood as a total function on a particular domain of discourse, so no two functions from distinct domains of discourse could be identical, since they take different collections of objects as arguments. But perhaps we can do better here, and find some way to provide cross-domain identity conditions on logical objects. Since (again, taking on board Frege’s *Grundgesetze* §10 suggestion) all logical objects are value-ranges of functions, this reduces to finding some way to provide cross-domain identity conditions on functions. In short, given two distinct domains  $\Delta_1$  and  $\Delta_2$ , and two functions  $F_{\Delta_1}$  and  $F_{\Delta_2}$  initially given to us as defined on those domains respectively, we need some criterion for determining whether or not  $F_{\Delta_1} = F_{\Delta_2}$ . One obvious necessary condition is that  $F_{\Delta_1}$  and  $F_{\Delta_2}$  must agree on  $\Delta_1 \cup \Delta_2$ . But knowing this

requires that we know what values  $F_{\Delta_1}$  and  $F_{\Delta_2}$  take, not only on their own domains of discourse, but on other objects in other domains. Further, if this criterion (whatever form it eventually takes, so long as the stated necessary condition is respected) is to allow us to settle cross-domain identity questions for any pair of functions on any pair of distinct domains, this requires us to know what values each function takes on *any* domain of discourse. In other words, formulating an adequate cross-domain identity criterion for functions, which is what we would need to dodge claims (1) through (3) and their rather odd consequences, requires us to accept the totality requirement, even on the varying domains interpretation of the logic of *Grundgesetze* (see, I promised I would come back to this possibility!).

### 3. Where This Leaves Us

To be completely fair, the points made above do not, perhaps, constitute a knockdown, completely compelling argument that Frege accepted the *totality requirement* and an absolutely general reading of the quantifiers, rather than merely accepting the *linguistic completeness* requirement. But they do show that adopting the latter reading does not come without significant costs. In particular, if the varying domains approach —at least, if it is to support the rejection of the totality requirement— is the right reading of Frege’s *Grundgesetze*, then this means that Frege did not, and never intended, to provide a unique —that is, having a unique domain-independent reference— definition of cardinal numbers, truth values, etc. outright. Rather, Frege’s view is on this interpretation is better read as a sort of relativism, where the nature and identity of logical objects is relative to a particular domain. Expand, contract, or otherwise modify the domain of discourse in question, and one obtains a completely new collection of logical

objects, including new ordered pairs (even of 'old' objects!) and new cardinal numbers.

I see no reason to think that such a reading of Frege is incoherent, and encourage those sympathetic to such an interpretation to further develop it, if only so that we can get clearer on how it differs —both philosophically and technically— from a reading where the quantifiers of *Grundgesetze* are absolutely general and the *totality requirement* is as a result accepted. But it is worth emphasizing that much development is needed here: An account where both the identity of the cardinal numbers (and other logical objects) and the truths that hold of them changes (and must change!) from one domain to another seems to be in significant tension with much of what has been taken to be standard aspects of Frege's logicism, including the uniqueness of the cardinal numbers and other logical objects and their universality and everywhere applicability.

As a result, while I encourage the development of such a view further, I am not going to further develop it myself, nor am I particularly optimistic that it is the right way to understand Frege. Given the discussion above, it seems to me that accepting the totality requirement, and an absolutely general reading of the quantifiers of *Grundgesetze*, on Frege's behalf (as a matter of technical detail regarding how Frege sets up the formal language of *Grundgesetze*, and not as a matter of any deep philosophical argument) seems the simpler route, and the one likely to be more faithful to Frege's intentions to provide a univocal analysis of concepts such as ORDERED PAIR and CARDINAL NUMBER.<sup>4</sup>

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## Notes

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<sup>1</sup> Further fleshing out this idea: One could perhaps argue that, even if we allow for varying domains, functions must be defined on all objects even when applied to domains not containing all of these objects since otherwise they would not have a complete sense: If sense determines reference, then a functional expression whose reference has not been determined for all possible arguments is arguably a functional expression whose sense has not been completely determined.

<sup>2</sup> An embarrassing *mea culpa*: I first observed that Frege's definition of cardinal number identifies cardinal numbers with 'collections' of value-ranges of functions, not 'collections' of value-ranges of concepts, in (Cook 2014). In that review I also present a preliminary (and, in retrospect, deeply flawed) version of the objections developed here. In doing so, I incorrectly claim that Frege's definition of ordered pair amounts to identifying ordered pairs with 'collections' of double value-ranges of relations, rather than with 'collections' of double value-ranges of binary functions more generally. The lesson, of course, is that it is all too easy to slip into reading Frege's notations along modern lines — as quantifying over concepts and relations rather than over functions — even when one supposedly knows better!

<sup>3</sup> For a number of different approaches that reject the existence of a universal domain because of indefinite extensibility type worries, the reader is encouraged to consult the essays in (Rayo & Uzquiano 2006).

<sup>4</sup> Some of the ideas in this essay appeared, in briefer form, in (Cook 2014). Thanks are due to Marcus Rossberg, Kai Wehmeier, Richard Zach, and especially to Patricia Blanchette for helpful comments and feedback.

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